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BIPOLAR FUZZY KNOWLEDGE-BASED ACCURACY MEASURE WITH APPLICATIONS IN MEDICAL DIAGNOSIS, PATTERN DETECTION AND CLUSTER ANALYSIS

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Abstract

The fuzzy set theory has several types of extensions. Bipolar fuzzy sets are fuzzy set extensions that have been developed by several researchers and applied in various settings. The satisfaction levels of a property and its counter-property define the membership degrees in bipolar fuzzy sets. These sets are useful in expert systems and decision-making, because they provide a refined representation of uncertainty by enabling both positive and negative membership degrees to exist simultaneously. This paper deals with calculating the knowledge passed by a bipolar fuzzy set; a knowledge-measure in the bipolar fuzzy framework is proposed here. Its validity is examined together with its mathematical characteristics and its performance is assessed with different examples. In addition, novel dissimilarity, similarity and accuracy measures are derived from the proposed measure in the bipolar fuzzy framework. The basic properties of the derived measures are outlined and their validity is evaluated. The proposed accuracy measure based on a new approach is discussed for solving cluster analysis issues. Furthermore, a case study about air pollution in different regions of the world in the year 2022 is examined. The proposed approach uses the information gathered from this investigation to generate clusters. In addition, medical diagnosis and pattern detection issues are addressed using the proposed measures in the bipolar fuzzy framework.

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1. Introduction

After the initial development of fuzzy set theory [38, 39], there have been several new concepts proposed for more advanced fuzzy sets like picture fuzzy set [28], extended picture fuzzy set or type-2 picture fuzzy set [30], neutrosophic fuzzy set [31], complex

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fuzzy set [32], interval-valued fuzzy set [29], and so forth. Among these, Atanassov [3] expanded this concept to intuitionistic fuzzy sets, which provide a flexible tool to address resistance arising from ambiguity and incorrect information. Two characteristic functions are used to represent the membership degree ($\mathcal{M}_D$) and nonmembership degree ($\mathcal{N}_D$) of universe elements to the intuitionistic fuzzy sets. Thus, $\mathcal{M}_D$ and $\mathcal{N}_D$ form intuitionistic fuzzy sets. Because this idea naturally occurs in real-world human language and behaviour, Psychology, alongside the other social science disciplines, has studied and used it extensively. Even so, fuzzy and intuitionistic fuzzy sets have been implemented to explain uncertainty-related real-world issues [26]. However, there are instances in which a counter property is necessary to describe an object's information that corresponds to an attribute. Zhang [41] introduced bipolar fuzzy ($\mathcal{B}^{\mathcal{F}}$) sets to address these problems, which is an extension of the fuzzy sets. In $\mathcal{B}^{\mathcal{F}}$-sets, the $\mathcal{M}_D$ lies in the interval $[-1, 1]$. Every element in $\mathcal{B}^{\mathcal{F}}$-sets are mixed with two values. One value denotes the fuzzy set's particular property and lies in the interval $[0, 1]$. The other one lies in the interval $[-1, 0]$ and denotes the degree to which a counter-property of the relevant fuzzy set is satisfied. $\mathcal{B}^{\mathcal{F}}$-sets appear to have been investigated and used rather more frequently and passionately in recent years. Singh [33] stated that every day of our lives uses bipolar information, which is used to measure the information. The rationale is that binary values (0 and 1) do not correspond to the fundamental units of the human mind. Human cognition often uses the boundaries of the information to choose what parts of it to accept and reject [19]. According to Pedrycz and Chen [20], bipolarity is an inevitable situation that deals with ambiguity and uncertainty in fuzzy characteristics and varies at each stage of time. Bipolar information may generally be categorized as pass or fail (True or False). As per Dubois and Prade's report [7], the primary factor is the simultaneous existence of these two areas in the specified $\mathcal{B}^{\mathcal{F}}$-regions $[-1, 0) \times (0, 1]$. In this instance, processing the bipolar requests while simultaneously identifying the positive or negative areas is a difficult task [6]. The cause is bipolarity, which can have varying definitions depending on the stage of life. It is preferable to depict them using $\mathcal{B}^{\mathcal{F}}$-properties. The positive $\mathcal{M}_D$ of a $\mathcal{B}^{\mathcal{F}}$-set denotes a degree of partial attribute fulfilment, whereas the negative $\mathcal{M}_D$ denotes an implicit counter characteristic. In decision science, the $\mathcal{B}^{\mathcal{F}}$-set is a sophisticated method for managing uncertainty.

Zhang and Zhang [44] gave YinYang bipolar logic and logical reasoning in the $\mathcal{B}^{\mathcal{F}}$-framework. Zhang et al. [43] extended the YinYang bipolar logic proposed by Zhang and Zhang [44] into bipolar linaer algebra and YinYang-N-element cellular networks in a $\mathcal{B}^{\mathcal{F}}$-framework. The problems related to neurobiological data mining, diagnostic analysis and psychiatric disorder classification were studied by Zhang et al. [42] by using vector machine in the $\mathcal{B}^{\mathcal{F}}$-framework. The regulation of collective emotions in any organisation was studied in the $\mathcal{B}^{\mathcal{F}}$-framework by Fink and Yolles [8]. Wei et al. [36] proposed hesitant $\mathcal{B}^{\mathcal{F}}$-weighted geometric and averaging operators, and applied them to MADM (multi-attribute decision-making). Some $\mathcal{B}^{\mathcal{F}}$-Hamacher aggregation operators were designed by Wei et al. [37]. Jana et al. [13] studied some MADM problems by the use of $\mathcal{B}^{\mathcal{F}}$-Dombi aggregation operators. Medical

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diagnosis issues are solved by Akram et al. [2] with the help of a technique which is a combination of both ELECTRE (elimination and choice expressing reality) and TOPSIS (technique for order preference by similarity to ideal solutions) methods in the $\mathcal{BF}$-framework. With the help of connection numbers of SPA theory based metric spaces, Riaz and Tehrim [23] extended the traditional VIKOR (Vlsekriterijumska Optimizacija I Kompromisno Resenje) method to the $\mathcal{BF}$-framework. Gulistan et al. [9] combined the complex fuzzy set and $\mathcal{BF}$-set to select a best transport bicycle for a company. The traditional TOPSIS method with $\mathcal{BF}$-N-soft information is studied by Akram et al. [1] to resolve the decision-making problems. A bipolar–complex fuzzy decision-making method is designed with the help of Hamcher aggregation operators by Mahmood et al. [16]. In the interval-valued $\mathcal{BF}$ framework, a CPT-TODIM (TOmada de Decisao Iterativa Multicriterio) technique is designed by Zhao et al. [46] and this technique is used to select a best industrial control security service provider. Dombi aggregation operators are designed by Mahmood and Rehman [15] in the bipolar–complex fuzzy framework to handle MADM problems. Selection of the best vaccine for the treatment of COVID-19 is achieved by Zararsiz and Riaz [40] with the help of a decision-making model in the $\mathcal{BF}$-framework. Based on prospect theory, the probabilistic uncertain linguistic TODIM method was designed by Zhao et al. [45] in the $\mathcal{BF}$-framework and this model is used in selecting the best service provider for network security. The Human–Computer interaction was studied by Jan et al. [12] with the help of a robust hybrid decision-making model for bipolar–complex picture fuzzy soft sets. The solution of problems related to integral equations was carried out by Mani et al. [17] with the help of intuitionistic fuzzy bipolar metric spaces. A hybrid MCDM (multi-criteria decision-making) model was studied by Jana et al. [14] in the $\mathcal{BF}$-framework and this model is used for determining the effect of COVID-19 on different Indian economic sectors.

However, several other investigators shared their perspectives on $\mathcal{BF}$-sets. According to what we know, several researchers did their work in the $\mathcal{BF}$-framework specified in the literature. A $\mathcal{BF}$-knowledge-measure, however, has not yet been suggested in any manuscript. A knowledge-measure in the $\mathcal{BF}$-framework is proposed in the current work and shown to be useful for resolving cluster analysis, pattern detection and medical diagnosis issues. The following points motivate us to do this research.

(1) Most of the researchers who contribute to $\mathcal{BF}$-theory have proposed aggregation operators in the $\mathcal{BF}$-framework only. However, in this paper, a knowledge-measure in the $\mathcal{BF}$-framework is proposed. Some properties of the proposed measures are given with proof.
(2) There is no dissimilarity, similarity and accuracy measure available in the $\mathcal{BF}$-framework to date. So, the proposed knowledge-measure is used to derive a dissimilarity measure, a similarity measure and an accuracy measure in the $\mathcal{BF}$-framework.
(3) The cluster analysis, medical diagnosis and pattern detection problems in the $\mathcal{BF}$-framework are solved by only a few researchers. So, a novel accuracy-based

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clustering approach is proposed to solve cluster analysis issues. Further, the problems of pattern detection as well as medical diagnosis are solved by the proposed dissimilarity, similarity and accuracy measures in the $\mathcal{B}^{\mathcal{F}}$-framework (see Examples 5.6, 5.7, 5.8 and 5.9).

Further, the problems of medical diagnosis, cluster analysis and pattern detection are solved with the help of the proposed measures. In linguistic comparisons, the suggested measure yields a satisfying outcome. There are the following basic points of the contribution of the proposed study.

(1) A $\mathcal{B}^{\mathcal{F}}$ knowledge-measure is proposed and some basic properties are discussed with their proofs.
(2) A dissimilarity measure, a similarity measure and an accuracy measure are proposed in the $\mathcal{B}^{\mathcal{F}}$-framework.
(3) An accuracy-based clustering approach is provided to solve cluster analysis issues in the $\mathcal{B}^{\mathcal{F}}$-framework.
(4) The proposed accuracy measure is applied to solve the problem of cluster analysis for real data. It also handles the medical diagnosis and pattern detection issues.
(5) The proposed dissimilarity and similarity measures also handle cluster analysis and pattern detection problems.

This study is organized as follows. Section 1 examines the broader past contributions to the specified subject made by distinguished scientists, as well as the objectives and motives of the research. Section 2 comprises some key definitions in the field. Section 3 discusses earlier studies. In this section, a knowledge-measure in the $\mathcal{B}^{\mathcal{F}}$-framework is defined and is checked for its reliability. This section also includes an overview of the proposed measure's properties for the comparison of the structured linguistic variables. From the proposed knowledge-measure's definitions, an accuracy measure for $\mathcal{B}^{\mathcal{F}}$-sets is deduced in Section 4. Furthermore, a dissimilarity as well as a similarity measure are proposed in the $\mathcal{B}^{\mathcal{F}}$-framework. Applications of these derived measures to pattern detection, cluster analysis and medical diagnosis issues are provided in Section 5. Additionally, actual data on air pollution in different regions are obtained from the official website. Using the suggested accuracy measure, clusters are created from this data. Furthermore, the accuracy measure that has been suggested is used to address the medical diagnosis problem. Section 6 presents the conclusions and potential future directions.

In the next section, some basic definitions related to $\mathcal{B}^{\mathcal{F}}$-sets are given.

2. Definitions

This section contains some definitions associated with $\mathcal{B}^{\mathcal{F}}$-sets. In the whole manuscript, $\mathcal{B}(K)$ is used to represent the collection of $\mathcal{B}^{\mathcal{F}}$-sets for universal set $K(\neq \phi)$.

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DEFINITION 2.1 [38]. For $W \in \{2, 3, \ldots\}$, let us consider a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}$. For this set K, a fuzzy set F can be defined as follows:

$$F = \{\langle \kappa_\omega, \mu_F(\kappa_\omega)\rangle \mid \kappa_\omega \in K\},$$

where μ_F: $K \to [0, 1]$ is the $\mathcal{M}_D$ and fulfils the condition $0 \leq \mu_F(\kappa_\omega) \leq 1$ for all $\kappa_\omega \in K$.

DEFINITION 2.2 [38]. For $W \in \{2, 3, \ldots\}$, let us consider a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}$. For this set K, a $\mathcal{B}^{\mathcal{F}}$-set B can be defined as follows:

$$B = \{\langle \kappa_\omega, (\mu_B^P(\kappa_\omega), \mu_B^N(\kappa_\omega))\rangle \mid \kappa_\omega \in K\},$$

where $\mu_B^P : K \to [0, 1]$ and $\mu_B^N : K \to [-1, 0]$ represent the positive and negative $\mathcal{M}_D$, respectively. Here, $\mu_B^P(\kappa_\omega)$ represents the extent to which an element κ_ω satisfies the property with respect to any $\mathcal{B}^{\mathcal{F}}$-set B and $\mu_B^N(\kappa_\omega)$ represents the extent to which an element κ_ω satisfies the implicit counter-property with respect to any $\mathcal{B}^{\mathcal{F}}$-set B.

A $\mathcal{B}^{\mathcal{F}}$-set B in the reduced form can be written as follows:

$$B = \{\langle \kappa_\omega, \mu_B(\kappa_\omega)\rangle \mid \kappa_\omega \in K\},$$

where $\mu_B(\kappa_\omega) : K \to [-1, 1]$ is the reduced $\mathcal{M}_D$ and is defined as follows:

$$\mu_B(\kappa_\omega) = \begin{cases} \mu_B^P(\kappa_\omega) & \text{if } \mu_B^N(\kappa_\omega) = 0, \\ \mu_B^N(\kappa_\omega) & \text{if } \mu_B^P(\kappa_\omega) = 0, \\ f(\mu_B^P(\kappa_\omega), \mu_B^N(\kappa_\omega)) & \text{otherwise} \end{cases}$$

for some aggregation function f which combines the values of positive and negative $\mathcal{M}_D$ into a single value. A $\mathcal{B}^{\mathcal{F}}$-set B can also be represented by a $\mathcal{B}^{\mathcal{F}}$-number $B = \langle \mu_B^P, \mu_B^N\rangle$.

DEFINITION 2.3 [37]. Let $B, C \in \mathcal{B}(K)$ and n be any natural number, then basic operations on $\mathcal{B}^{\mathcal{F}}$-sets are defined as follows:

(I) $B \cup C = \{\langle \kappa_\omega, (\max\{\mu_B^P(\kappa_\omega), \mu_B^P(\kappa_\omega)\},\ \min\{\mu_B^N(\kappa_\omega), \mu_B^N(\kappa_\omega)\})\rangle \mid \kappa_\omega \in K\}$;
(II) $B \cap C = \{\langle \kappa_\omega, (\min\{\mu_B^P(\kappa_\omega), \mu_B^P(\kappa_\omega)\},\ \max\{\mu_B^N(\kappa_\omega), \mu_B^N(\kappa_\omega)\})\rangle \mid \kappa_\omega \in K\}$;
(III) $B^o = \{\langle \kappa_\omega, (1 - \mu_B^P(\kappa_\omega), |\mu_B^N(\kappa_\omega)| - 1)\rangle \mid \kappa_\omega \in K\}$;
(IV) $B \subseteq C \Leftrightarrow \mu_B^P(\kappa_\omega) \leq \mu_C^P(\kappa_\omega)$ and $\mu_B^N(\kappa_\omega) \geq \mu_C^N(\kappa_\omega)$ for all $\kappa_\omega \in K$;
(V) $nB = \{\langle \kappa_\omega, (1 - (1 - \mu_B^P(\kappa_\omega))^n, -|\mu_B^N(\kappa_\omega)|^n)\rangle : \kappa_\omega \in K\}$;
(VI) $B^n = \{\langle \kappa_\omega, ((\mu_B^P(\kappa_\omega))^n, |1 + \mu_B^N(\kappa_\omega)|^n - 1)\rangle : \kappa_\omega \in K\}$.

DEFINITION 2.4 [13]. For $W \in \{2, 3, \ldots\}$, take a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}$. Let $B_1, B_2, \ldots B_W$ be $\mathcal{B}^{\mathcal{F}}$-numbers and $\{a_1, a_2, \ldots, a_T\}$ represent criteria weights. The $\mathcal{B}^{\mathcal{F}}$-weighted arithmetic (BFWA) and $\mathcal{B}^{\mathcal{F}}$-weighted geometric (BFWG) operators for $\mathcal{B}^{\mathcal{F}}$-numbers $B_1, B_2, \ldots B_W$ are defined as follows:

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$$BFWA_{a_\tau}(B_1, B_2, \ldots B_W) = \oplus_{\tau=1}^{T} a_\tau \times B_\omega$$
$$= \left\langle 1 - \prod_{\tau=1}^{T}(1 - \mu_{B_\omega}^{P}(\kappa_\omega))^{a_\tau}, -\prod_{\tau=1}^{T}|\mu_{B_\omega}^{N}(\kappa_\omega)|^{a_\tau} \right\rangle,$$
$$BFWG_{a_\tau}(B_1, B_2, \ldots B_W) = \otimes_{\tau=1}^{T}(B_\omega)^{a_\tau}$$
$$= \left\langle \prod_{\tau=1}^{T}(\mu_{B_\omega}^{P}(\kappa_\omega))^{a_\tau}, \prod_{\tau=1}^{T}(1 + \mu_{B_\omega}^{N}(\kappa_\omega))^{a_\tau} - 1 \right\rangle.$$

DEFINITION 2.5 [37]. Let $B = \langle \mu_B^P, \mu_B^N \rangle$ be any $\mathcal{B}^{\mathcal{F}}$-numbers, then the accuracy function $\mathcal{A}(B)$ and the score functions $\mathcal{S}(B)$ are defined as follows:

$$\mathcal{A}(B) = \frac{\mu_B^P - \mu_B^N}{2} \quad \text{and} \quad \mathcal{S}(B) = \frac{1 + \mu_B^P + \mu_B^N}{2}.$$

The order relations for any two $\mathcal{B}^{\mathcal{F}}$-numbers B and C are defined as follows.

(A) If $\mathcal{S}(B) > \mathcal{S}(C)$, then $B \succ C$.
(B) If $\mathcal{S}(B) < \mathcal{S}(C)$, then $B \prec C$.
(C) If $\mathcal{S}(B) = \mathcal{S}(C)$, then accuracy function is used as follows:

(a) if $\mathcal{A}(B) > \mathcal{A}(C)$, then $B \succ C$;
(b) if $\mathcal{A}(B) < \mathcal{A}(C)$, then $B \prec C$;
(c) if $\mathcal{A}(B) = \mathcal{A}(C)$, then $B \sim C$.

DEFINITION 2.6 [34]. Let $B, C \in \mathcal{B}(K)$, then we can define the normalized Hamming and Euclidean measures as follows.

(A) *Normalized Hamming distance measure:*

$$NH_d(B, C) = \frac{1}{W}\sum_{\omega=1}^{W}[|\mu_B^N(\kappa_\omega) - \mu_C^N(\kappa_\omega)| + |\mu_B^P(\kappa_\omega) - \mu_C^P(\kappa_\omega)|]. \tag{2.1}$$

(B) *Normalized Euclidean distance measure:*

$$NE_e(B, C) = \sqrt{\frac{1}{W}\sum_{\omega=1}^{W}[(\mu_B^N(\kappa_\omega) - \mu_C^N(\kappa_\omega))^2 + (\mu_B^P(\kappa_\omega) - \mu_C^P(\kappa_\omega))^2]}. \tag{2.2}$$

As no researcher has proposed any entropy and knowledge-measure in the $\mathcal{B}^{\mathcal{F}}$-framework to date, we extend the notion of the fuzzy set to the $\mathcal{B}^{\mathcal{F}}$-set, and axiomatically define the axioms for a valid $\mathcal{B}^{\mathcal{F}}$-entropy and knowledge-measure.

DEFINITION 2.7. Let $B, C \in \mathcal{B}(K)$, then any valid $\mathcal{B}^{\mathcal{F}}$-entropy measure $IM : \mathcal{B}(K) \to [0, 1]$ have to fulfil the following axioms axiomatically.

(N1) $IM(B) = 1 \Leftrightarrow \mu_B^P(\kappa_\omega) = \frac{1}{2}$ and $\mu_B^N(\kappa_\omega) = -\frac{1}{2}$ for all $\kappa_\omega \in K$.
(N2) $IM(B) = 0$ if $\mu_B^P(\kappa_\omega) = 0 = \mu_B^N(\kappa_\omega)$ or $\mu_B^P(\kappa_\omega) = 1$ and $\mu_B^N(\kappa_\omega) = -1$ for all $\kappa_\omega \in K$.

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(N3) $IM(B) \leq IM(C)$ if B is crisper than C, that is,

$$\begin{cases} \mu_B^P(\kappa_\omega) \leq \mu_C^P(\kappa_\omega) & \text{if } \mu_C^P(\kappa_\omega) \leq \frac{1}{2}, \\ \mu_B^N(\kappa_\omega) \leq \mu_C^N(\kappa_\omega) & \text{if } \mu_C^N(\kappa_\omega) \leq -\frac{1}{2}, \end{cases} \quad \text{or}$$

$$\begin{cases} \mu_B^P(\kappa_\omega) \geq \mu_C^P(\kappa_\omega) & \text{if } \mu_C^P(\kappa_\omega) \geq \frac{1}{2}, \\ \mu_B^N(\kappa_\omega) \geq \mu_C^N(\kappa_\omega) & \text{if } \mu_C^N(\kappa_\omega) \geq -\frac{1}{2}, \end{cases} \quad \text{for all } \kappa_\omega \in K.$$

(N4) For the complement B^o of the $\mathcal{B}^{\mathcal{F}}$-set B, we have $IM(B) = IM(B^o)$.

DEFINITION 2.8. Let $B, C \in \mathcal{B}(K)$, then any valid $\mathcal{B}^{\mathcal{F}}$ knowledge-measure $KM : \mathcal{B}(K) \to [0, 1]$ must fulfil the following axioms axiomatically.

(N1) $KM(B) = 0 \Leftrightarrow \mu_B^P(\kappa_\omega) = \frac{1}{2}$ and $\mu_B^N(\kappa_\omega) = -\frac{1}{2}$ for all $\kappa_\omega \in K$.
(N2) $KM(B) = 1$ if $\mu_B^P(\kappa_\omega) = 0 = \mu_B^N(\kappa_\omega)$ or $\mu_B^P(\kappa_\omega) = 1$ and $\mu_B^N(\kappa_\omega) = -1$ for all $\kappa_\omega \in K$.
(N3) $KM(B) \geq KM(C)$ if B is crisper than C, that is,

$$\begin{cases} \mu_B^P(\kappa_\omega) \leq \mu_C^P(\kappa_\omega) & \text{if } \mu_C^P(\kappa_\omega) \leq \frac{1}{2}, \\ \mu_B^N(\kappa_\omega) \leq \mu_C^N(\kappa_\omega) & \text{if } \mu_C^N(\kappa_\omega) \leq -\frac{1}{2}, \end{cases} \quad \text{or}$$

$$\begin{cases} \mu_B^P(\kappa_\omega) \geq \mu_C^P(\kappa_\omega) & \text{if } \mu_C^P(\kappa_\omega) \geq \frac{1}{2}, \\ \mu_B^N(\kappa_\omega) \geq \mu_C^N(\kappa_\omega) & \text{if } \mu_C^N(\kappa_\omega) \geq -\frac{1}{2}, \end{cases} \quad \text{for all } \kappa_\omega \text{ in } K.$$

(N4) For the complement B^o of the $\mathcal{B}^{\mathcal{F}}$-set B, we have $KM(B) = KM(B^o)$.

DEFINITION 2.9. Let $B, C \in \mathcal{B}(K)$, then any valid $\mathcal{B}^{\mathcal{F}}$-accuracy measure $AM : \mathcal{B}(K) \times \mathcal{B}(K) \to [0, 1]$ must fulfil the following axioms:

(1) $0 \leq AM(B, C) \leq 1$ for all $B, C \in \mathcal{B}(K)$;
(2) $AM(B, C) = 0 \Leftrightarrow B = C = \langle \frac{1}{2}, -\frac{1}{2} \rangle$;
(3) $AM(B, C) = 1 \Leftrightarrow B = C = \langle 1, -1 \rangle$ or $B = C = \langle 0, 0 \rangle$;
(4) $AM(B, C) = KM(B)$ for $B = C$, where KM is the knowledge-measure.

A novel $\mathcal{B}^{\mathcal{F}}$ knowledge-measure and its basic properties are introduced in the upcoming section. A structured linguistic comparison is also given in the following section.

3. A novel bipolar fuzzy knowledge-measure

Let $R_Q = \{R = (r_1, r_2, \ldots, r_Q) | \ \sum_{\omega=1}^{Q} r_\omega = 1; 0 \leq r_\omega \leq 1 \text{ for all } \omega = 1, 2, 3, \ldots, Q\}$ be a collection of all the probability distributions for $Q \in \{2, 3, \ldots\}$. Shannon [25] proposed an entropy measure for $R \in R_Q$ as follows:

$$IM(R) = -\sum_{\omega=1}^{Q} r_\omega \log r_\omega. \tag{3.1}$$

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Many authors extended the measure $IM(R)$. The ϕ-norm entropy is the generalization of (3.1) proposed by Boekee and Van Der Lubbe [4], which is defined as

$$IM_\phi(R) = \frac{\phi}{1-\phi}\left[\left(\sum_{\omega=1}^{Q} r_\omega^\phi\right)^{1/\phi} - 1\right], \quad \phi \in \mathbb{R}^+, \phi \neq 0 \text{ and } \phi \neq 1.$$

Also, $\lim_{\phi\to 1} IM_\phi(R) = IM(R)$. Following the journey of the theory of fuzzy set [38, 39], many authors generalized the ϕ-norm entropy. Hooda [11] generalized the ϕ-norm entropy as follows:

$$IM_\phi^H(F) = \frac{\phi}{\phi-1}\sum_{\omega=1}^{W}[1 - ((\mu_F(\kappa_\omega))^\phi + (1-\mu_F(\kappa_\omega))^\phi)^{1/\phi}],$$
$$\phi \in \mathbb{R}^+, \phi \neq 0 \quad \text{and} \quad \phi \neq 1. \tag{3.2}$$

The notions of entropy and knowledge-measures are complementary concepts. Depending upon the entropy measure [11] given in (3.2), a measure is recently suggested by Singh and Kumar [27] in the fuzzy framework. Based on the notion of measure suggested by Singh and Kumar [27], a novel $\mathcal{B}^\mathcal{F}$ knowledge-measure is proposed in the following subsection.

3.1. Definition For a set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}, W \geq 2$, let $B \in \mathcal{B}(K)$ be a $\mathcal{B}^\mathcal{F}$-set defined by $B = \{\langle \kappa_\omega, (\mu_B^P(\kappa_\omega), \mu_B^N(\kappa_\omega))\rangle \mid \kappa_\omega \in K\}$. We axiomatically proposed a knowledge-measure in the $\mathcal{B}^\mathcal{F}$-framework as follows:

$$KM_\mathcal{B}(B) = \frac{(\sqrt[3]{4}-1)^{-1}}{2W}\sum_{\omega=1}^{W}\left[\left(\begin{array}{l}\sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)} \\ +\sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)}\end{array}\right) - 2\right]. \tag{3.3}$$

Now, the validity of the proposed measure is checked.

THEOREM 3.1. *Let $B, C \in \mathcal{B}(K)$, then any valid $\mathcal{B}^\mathcal{F}$ knowledge-measure $KM_\mathcal{B} : \mathcal{B}(K) \to [0,1]$ given in (3.3) has to fulfil the following axioms axiomatically.*

(N1) $KM_\mathcal{B}(B) = 0 \Leftrightarrow \mu_B^P(\kappa_\omega) = \frac{1}{2}$ *and* $\mu_B^N(\kappa_\omega) = -\frac{1}{2}$ *for all* $\kappa_\omega \in K$.

(N2) $KM_\mathcal{B}(B) = 1$ *if* $\mu_B^P(\kappa_\omega) = 0 = \mu_B^N(\kappa_\omega)$ *or* $\mu_B^P(\kappa_\omega) = 1$ *and* $\mu_B^N(\kappa_\omega) = -1$ *for all* $\kappa_\omega \in K$.

(N3) $KM_\mathcal{B}(B) \geq KM_\mathcal{B}(C)$ *if B is crisper than C, that is,*

$$\begin{cases}\mu_B^P(\kappa_\omega) \leq \mu_C^P(\kappa_\omega) & \text{if } \mu_C^P(\kappa_\omega) \leq \frac{1}{2}, \\ \mu_B^N(\kappa_\omega) \leq \mu_C^N(\kappa_\omega) & \text{if } \mu_C^N(\kappa_\omega) \leq -\frac{1}{2},\end{cases} \quad \text{or}$$

$$\begin{cases}\mu_B^P(\kappa_\omega) \geq \mu_C^P(\kappa_\omega) & \text{if } \mu_C^P(\kappa_\omega) \geq \frac{1}{2}, \\ \mu_B^N(\kappa_\omega) \geq \mu_C^N(\kappa_\omega) & \text{if } \mu_C^N(\kappa_\omega) \geq -\frac{1}{2},\end{cases} \quad \text{for all } \kappa_\omega \in K.$$

(N4) *For the complement B^o of the $\mathcal{B}^\mathcal{F}$-set B, we have $KM_\mathcal{B}(B) = KM_\mathcal{B}(B^o)$.*

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PROOF. (N1). For $i \in (0,1)$ and $j \in (-1,0)$, we define two functions

$$f_1(i) = (\sqrt[3]{4}-1)^{-1} \times (\sqrt[3]{4(i^3+(1-i)^3)}-1),$$
$$f_2(j) = (\sqrt[3]{4}-1)^{-1} \times (\sqrt[3]{4((1+j)^3-j^3)}-1).$$

Differentiating f_1 and f_2 with respect to i and j, respectively, we get

$$\frac{df_1}{di} = \frac{4Z(i^2-(1-i)^2)}{[4(i^3+(1-i)^3)]^{2/3}}, \quad \frac{df_2}{dj} = \frac{4Z((1+j)^2-j^2)}{[4((1+j)^3-j^3)]^{2/3}}, \tag{3.4}$$

where $Z = (\sqrt[3]{4}-1)^{-1}$. To find critical points, put $df_1/di = 0$, $df_2/dj = 0$. On solving, we obtained $i = 1/2$, $j = -1/2$. Differentiating again the functions f_1 and f_2 with respect to i and j, respectively, we have

$$\frac{d^2f_1}{di^2} = \frac{8Z}{[4(i^3+(1-i)^3)]^{2/3}} - \frac{32Z(i^2-(1-i)^2)^2}{[4(i^3+(1-i)^3)]^{5/3}},$$
$$\frac{d^2f_2}{dj^2} = \frac{8Z}{[4((1+j)^3-j^3)]^{2/3}} - \frac{32Z((1+j)^2-j^2)^2}{[4((1+j)^3-j^3)]^{5/3}}.$$

From these values, we have

$$\frac{d^2f_1}{di^2} > 0 \quad \text{for } i = \frac{1}{2} \quad \text{and} \quad \frac{d^2f_2}{dj^2} > 0 \quad \text{for } j = -\frac{1}{2}.$$

It implies that functions f_1 and f_2 take their minimum value at points $i = 1/2$ and $j = -1/2$, respectively. The proposed measure, defined in (3.3), can be written as follows:

$$KM_{\mathcal{B}}(B) = \frac{1}{2W}\sum_{\omega=1}^{W}[f_1(\mu_B^P(\kappa_\omega)) + f_2(\mu_B^N(\kappa_\omega))].$$

Thus, $KM_{\mathcal{B}}(B) = 0 \Leftrightarrow \mu_B^P(\kappa_\omega) = \frac{1}{2}$ and $\mu_B^N(\kappa_\omega) = -\frac{1}{2}$ for all $\kappa_\omega \in K$.

(N2). Consider $B \in \mathcal{B}(K)$ such that $\mu_B^P(\kappa_\omega) = 0 = \mu_B^N(\kappa_\omega)$ or $\mu_B^P(\kappa_\omega) = 1$ and $\mu_B^N(\kappa_\omega) = -1$ for all $\kappa_\omega \in K$. From (3.3), it is easy to see that $KM_{\mathcal{B}}(B) = 1$.

(N3). From (3.4), it is easy to see that

$$\frac{df_1}{di} = \begin{cases} +ve & \text{if } i > \dfrac{1}{2}, \\ -ve & \text{if } i < \dfrac{1}{2}, \end{cases} \quad \text{and} \quad \frac{df_2}{dj} = \begin{cases} +ve & \text{if } j > -\dfrac{1}{2}, \\ -ve & \text{if } j < -\dfrac{1}{2}, \end{cases} \tag{3.5}$$

that is, function f_1 decreases for $i < \frac{1}{2}$ and increases for $i > \frac{1}{2}$. Similarly, function f_2 decreases for $j < -\frac{1}{2}$ and increases for $j > -\frac{1}{2}$. Hence, if B is crisper than C, then (3.3) and (3.5) imply that $KM_{\mathcal{B}}(C) \le KM_{\mathcal{B}}(B)$.

(N4). Consider $B \in \mathcal{B}(K)$. From the definition of complement, we have $B^o = \{\langle \kappa_\omega, (1-\mu_B^P(\kappa_\omega), \mu_B^N(\kappa_\omega)-1)\rangle \mid \kappa_\omega \in K\}$. Thus, from (3.3), we have $KM_{\mathcal{B}}(B) = KM_{\mathcal{B}}(B^o)$.

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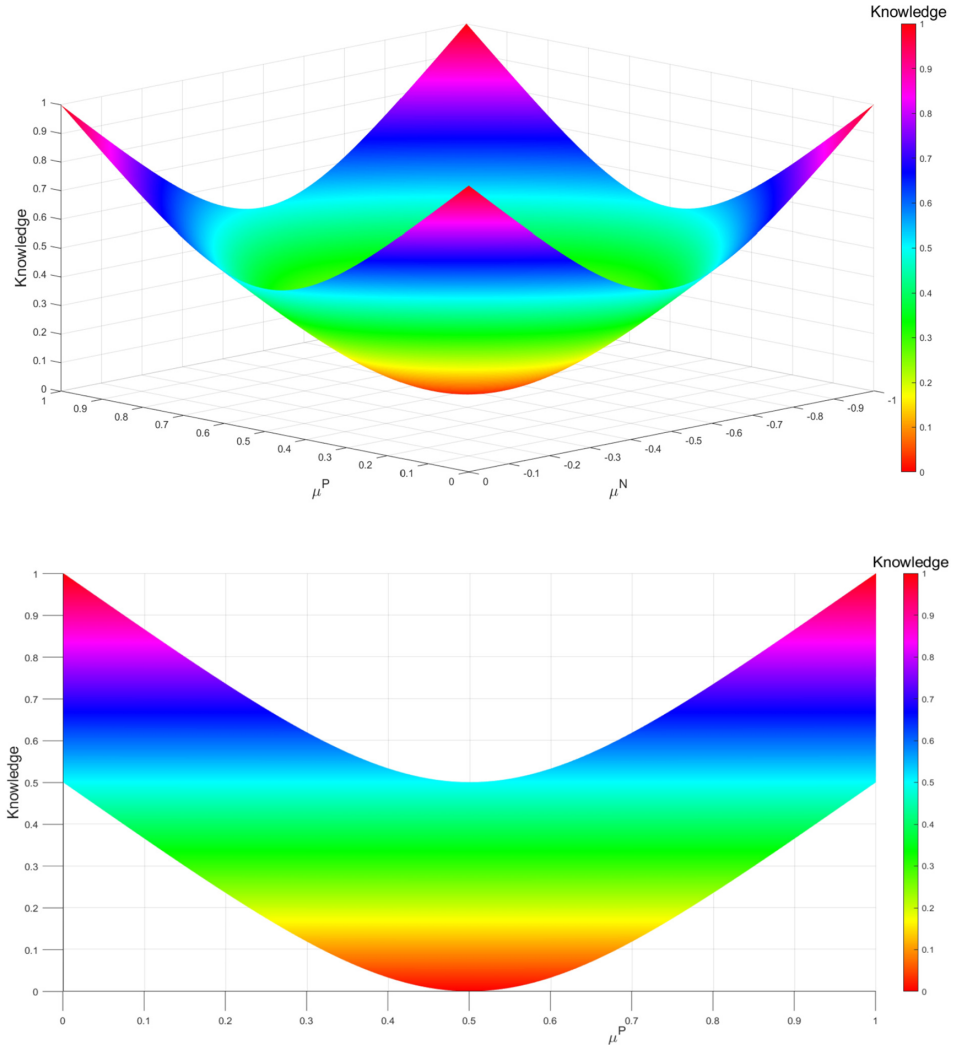


FIGURE 1. Proposed $\mathcal{B}^{\mathcal{F}}$ knowledge-measure.

Hence, $KM_{\mathcal{B}}$ is a valid knowledge-measure in the $\mathcal{B}^{\mathcal{F}}$-framework. This completes the proof. □

EXAMPLE 3.2. Consider $B \in \mathcal{B}(K)$ for $K = \{\kappa\}$. Total knowledge calculated by the proposed measure is shown in Figure 1 for each $\mu_B^P(\kappa) \in [0, 1]$ and $\mu_B^N(\kappa) \in [-1, 0]$.

Now, in the following subsection, the properties of the proposed measure are proved.

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3.2. Properties

THEOREM 3.3. *The values of the proposed measure $KM_{\mathcal{B}}$ lie in* $[0, 1]$.

PROOF. For all $\kappa_\omega \in \mathrm{K}$, we have

$$\begin{aligned}
&\mu_B^P(\kappa_\omega), (1-\mu_B^P(\kappa_\omega)), (1+\mu_B^N(\kappa_\omega)), -\mu_B^N(\kappa_\omega) \in [0,1],\\
&\Rightarrow (\mu_B^P(\kappa_\omega))^3, (1-\mu_B^P(\kappa_\omega))^3, (1+\mu_B^N(\kappa_\omega))^3, -(\mu_B^P(\kappa_\omega))^3 \in [0,1],\\
&\Rightarrow ((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3), ((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3) \in \left[\frac{1}{4}, 1\right],\\
&\Rightarrow 4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3) \le 4,\ 4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3) \in [1,4],\\
&\Rightarrow \sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)},\ \sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} \in [1, \sqrt[3]{4}],\\
&\Rightarrow \sqrt[3]{4\begin{pmatrix}(\mu_B^P(\kappa_\omega))^3 +\\ (1-\mu_B^P(\kappa_\omega))^3\end{pmatrix}} - 1,\ \sqrt[3]{4\begin{pmatrix}(1+\mu_B^N(\kappa_\omega))^3\\ -(\mu_B^N(\kappa_\omega))^3\end{pmatrix}} - 1 \in [0, \sqrt[3]{4}-1],\\
&\Rightarrow \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\omega=1}^{W} \left[\sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)} - 1\right] \in \left[0, \frac{1}{2}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\omega=1}^{W} \left[\sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} - 1\right] \in \left[0, \frac{1}{2}\right],\\
&\Rightarrow \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\omega=1}^{W} \left[\sqrt[3]{4\begin{pmatrix}(\mu_B^P(\kappa_\omega))^3 +\\ (1-\mu_B^P(\kappa_\omega))^3\end{pmatrix}} + \sqrt[3]{4\begin{pmatrix}(1+\mu_B^N(\kappa_\omega))^3\\ -(\mu_B^N(\kappa_\omega))^3\end{pmatrix}} - 2\right] \in [0,1],
\end{aligned}$$

that is, $KM_{\mathcal{B}}(B) \in [0, 1]$, which completes the proof. □

THEOREM 3.4. *The measure $KM_{\mathcal{B}}$ satisfies the following property:*

$$KM_{\mathcal{B}}(B \cup C) + KM_{\mathcal{B}}(B \cap C) = KM_{\mathcal{B}}(B) + KM_{\mathcal{B}}(C) \quad \textit{for all } B, C \in \mathcal{B}(K).$$

PROOF. Take $B, C \in \mathcal{B}(K)$. Take the partition $\{K_1, K_2\}$ of K such that $B \subseteq C$ for the partition K_1 and $C \subseteq B$ for the partition K_2. Now, for all $\kappa_\omega \in K$, we have

$$\begin{aligned}
&KM_{\mathcal{B}}(B \cup C) + KM_{\mathcal{B}}(B \cap C)\\
&= \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\omega=1}^{W} \left[\begin{pmatrix}\sqrt[3]{4((\mu_{B\cup C}^P(\kappa_\omega))^3 + (1-\mu_{B\cup C}^P(\kappa_\omega))^3)}\\ +\sqrt[3]{4((1+\mu_{B\cup C}^N(\kappa_\omega))^3 - (\mu_{B\cup C}^N(\kappa_\omega))^3)}\end{pmatrix} - 2\right]\\
&\quad + \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\omega=1}^{W} \left[\begin{pmatrix}\sqrt[3]{4((\mu_{B\cap C}^P(\kappa_\omega))^3 + (1-\mu_{B\cap C}^P(\kappa_\omega))^3)}\\ +\sqrt[3]{4((1+\mu_{B\cap C}^N(\kappa_\omega))^3 - (\mu_{B\cap C}^N(\kappa_\omega))^3)}\end{pmatrix} - 2\right].
\end{aligned}$$

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For partitions K_1 and K_2, we have

$$\begin{aligned}
&KM_{\mathcal{B}}(B \cup C) + KM_{\mathcal{B}}(B \cap C)\\
&= \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\kappa_\omega \in K_1} \left[\left(\begin{array}{l} \sqrt[3]{4((\mu_C^P(\kappa_\omega))^3 + (1-\mu_C^P(\kappa_\omega))^3)} \\ +\sqrt[3]{4((1+\mu_C^N(\kappa_\omega))^3 - (\mu_C^N(\kappa_\omega))^3)} \end{array} \right) - 2 \right]\\
&\quad + \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\kappa_\omega \in K_1} \left[\left(\begin{array}{l} \sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)+} \\ \sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} \end{array} \right) - 2 \right]\\
&\quad + \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\kappa_\omega \in K_2} \left[\left(\begin{array}{l} \sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)+} \\ \sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} \end{array} \right) - 2 \right]\\
&\quad + \frac{(\sqrt[3]{4}-1)^{-1}}{2W} \sum_{\kappa_\omega \in K_2} \left[\left(\begin{array}{l} \sqrt[3]{4((\mu_C^P(\kappa_\omega))^3 + (1-\mu_C^P(\kappa_\omega))^3)+} \\ \sqrt[3]{4((1+\mu_C^N(\kappa_\omega))^3 - (\mu_C^N(\kappa_\omega))^3)} \end{array} \right) - 2 \right],
\end{aligned}$$

which gives

$$KM_{\mathcal{B}}(B \cup C) + KM_{\mathcal{B}}(B \cap C) = KM_{\mathcal{B}}(B) + KM_{\mathcal{B}}(C).$$

This completes the proof. □

3.3. Structured linguistic comparison Linguistic variables are represented by a $\mathcal{B}^{\mathcal{F}}$-set. Furthermore, the linguistic hedges are represented by any operations that are performed on any $\mathcal{B}^{\mathcal{F}}$-set. On linguistic variables, many linguistic hedges are employed. These linguistic hedges are calculated and apply the proposed measure to them. Let $B \in \mathcal{B}(K)$ be a $\mathcal{B}^{\mathcal{F}}$-set defined by $B = \{\langle \kappa_\omega, (\mu_B^P(\kappa_\omega), \mu_B^N(\kappa_\omega))\rangle \mid \kappa_\omega \in K\}$ for a set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}, W \geq 2$. Let the linguistic variable "Huge" be represented by this $\mathcal{B}^{\mathcal{F}}$-set B. Let $z > 0$ be any number, then the modifier B^z for this $\mathcal{B}^{\mathcal{F}}$-set B is defined as follows:

$$B^z = \{\langle \kappa_\omega, ((\mu_B^P(\kappa_\omega))^z, |1 + \mu_B^N(\kappa_\omega)|^z - 1)\rangle \mid \kappa_\omega \in K\}.$$

Here, $B^{0.5}$ is used to represent the dilatation and B^2 is used to represent the concentration for the $\mathcal{B}^{\mathcal{F}}$-set B. For the comparison of linguistic hedges, we use the following short form: More or Less Huge is written as H_1, Huge is written as H_2, Very Huge is written as H_3, Quite Very Huge is written as H_4, Very Very Huge is written as H_5. Linguistic hedges for this $\mathcal{B}^{\mathcal{F}}$-set B are defined as follows:

$$\begin{cases} H_1 & \text{is written for } B^{1/2}, \\ H_2 & \text{is written for } B, \\ H_3 & \text{is written for } B^2, \\ H_4 & \text{is written for } B^3, \\ H_5 & \text{is written for } B^4. \end{cases} \tag{3.6}$$

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As we go from H_1 to H_5, the vagueness associated with the set decreases, whereas the knowledge associated with the set increases, that is,

$$\begin{aligned} &IM(H_1) > IM(H_2) > IM(H_3) > IM(H_4) > IM(H_5), \\ &KM(H_1) < KM(H_2) < KM(H_3) < KM(H_4) < KM(H_5). \end{aligned} \tag{3.7}$$

Here, the *IM* and *KM* represent the entropy and knowledge-measures, respectively.

Now, for the computation of structured linguistic hedges, two examples are given as follows.

EXAMPLE 3.5. Let $B \in \mathcal{B}(K)$ be a $\mathcal{B}^{\mathcal{F}}$-set defined by

$$B = \begin{Bmatrix} \langle 60, (0.1000, -0.5000)\rangle, & \langle 59, (0.3000, -0.1000)\rangle, \\ \langle 58, (0.2000, -0.6000)\rangle, & \langle 57, (0.7000, -0.9000)\rangle, \\ \langle 56, (0.4000, -0.6000)\rangle, & \langle 55, (0.8000, -0.5000)\rangle, \\ \langle 54, (0.4000, -0.2000)\rangle, & \langle 53, (0.2000, -0.3000)\rangle, \\ \langle 52, (0.4000, -0.9000)\rangle, & \langle 51, (0.6000, -0.2000)\rangle \end{Bmatrix}$$

for a set $K = \{60, 59, 58, 57, 56, 55, 54, 53, 52, 51\}$. Let the linguistic variable "Huge" be represented by this $\mathcal{B}^{\mathcal{F}}$-set B, take linguistic hedges as given in (3.6) and compute the modifiers as follows:

$$B^{0.5} = \begin{Bmatrix} \langle 60, (0.3162, -0.2929)\rangle, & \langle 59, (0.5477, -0.6838)\rangle, \\ \langle 58, (0.4472, -0.2254)\rangle, & \langle 57, (0.8367, -0.0513)\rangle, \\ \langle 56, (0.6325, -0.2254)\rangle, & \langle 55, (0.8944, -0.2929)\rangle, \\ \langle 54, (0.6325, -0.5528)\rangle, & \langle 53, (0.4472, -0.4523)\rangle, \\ \langle 52, (0.6325, -0.0513)\rangle, & \langle 51, (0.7746, -0.5528)\rangle, \end{Bmatrix};$$

$$B = \begin{Bmatrix} \langle 60, (0.1000, -0.5000)\rangle, & \langle 59, (0.3000, -0.1000)\rangle, \\ \langle 58, (0.2000, -0.6000)\rangle, & \langle 57, (0.7000, -0.9000)\rangle, \\ \langle 56, (0.4000, -0.6000)\rangle, & \langle 55, (0.8000, -0.5000)\rangle, \\ \langle 54, (0.4000, -0.2000)\rangle, & \langle 53, (0.2000, -0.3000)\rangle, \\ \langle 52, (0.4000, -0.9000)\rangle, & \langle 51, (0.6000, -0.2000)\rangle \end{Bmatrix};$$

$$B^2 = \begin{Bmatrix} \langle 60, (0.0100, -0.7500)\rangle, & \langle 59, (0.0900, -0.9900)\rangle, \\ \langle 58, (0.0400, -0.6400)\rangle, & \langle 57, (0.4900, -0.1900)\rangle, \\ \langle 56, (0.1600, -0.6400)\rangle, & \langle 55, (0.6400, -0.7500)\rangle, \\ \langle 54, (0.1600, -0.9600)\rangle, & \langle 53, (0.0400, -0.9100)\rangle, \\ \langle 52, (0.1600, -0.1900)\rangle, & \langle 51, (0.3600, -0.9600)\rangle \end{Bmatrix};$$

$$B^3 = \begin{Bmatrix} \langle 60, (0.0010, -0.8750)\rangle, & \langle 59, (0.0270, -0.9990)\rangle, \\ \langle 58, (0.0080, -0.7840)\rangle, & \langle 57, (0.3430, -0.2710)\rangle, \\ \langle 56, (0.0640, -0.7840)\rangle, & \langle 55, (0.5120, -0.8750)\rangle, \\ \langle 54, (0.0640, -0.9920)\rangle, & \langle 53, (0.0080, -0.9730)\rangle, \\ \langle 52, (0.0640, -0.2710)\rangle, & \langle 51, (0.2160, -0.9920)\rangle \end{Bmatrix};$$

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$$B^4 = \begin{Bmatrix} \langle 60, (0.0001, -0.9375)\rangle, & \langle 59, (0.0081, -0.9999)\rangle, \\ \langle 58, (0.0016, -0.8704)\rangle, & \langle 57, (0.2401, -0.3439)\rangle, \\ \langle 56, (0.0256, -0.8704)\rangle, & \langle 55, (0.4096, -0.9375)\rangle, \\ \langle 54, (0.0256, -0.9984)\rangle, & \langle 53, (0.0016, -0.9919)\rangle, \\ \langle 52, (0.0256, -0.3439)\rangle, & \langle 51, (0.1296, -0.9984)\rangle \end{Bmatrix}.$$

Using the proposed measure, the knowledge linked with these hedges is calculated and their values are $KM(H_1) = 0.2791$, $KM(H_2) = 0.3192$, $KM(H_3) = 0.5467$, $KM(H_4) = 0.6811$, $KM(H_5) = 0.7541$. From these values, we get the proposed measure satisfies the sequence

$$KM(H_1) < KM(H_2) < KM(H_3) < KM(H_4) < KM(H_5),$$

which is the same as defined in (3.7). So, the proposed measure performs well.

Consider the second example given as follows.

EXAMPLE 3.6. Let $B \in \mathcal{B}(K)$ be a $\mathcal{B}^{\mathcal{F}}$-set defined by

$$B = \begin{Bmatrix} \langle 95, (0.1000, -0.7500)\rangle, & \langle 94, (0.8400, -0.0300)\rangle, \\ \langle 93, (0.3600, -0.9500)\rangle, & \langle 92, (0.4100, -0.2400)\rangle, \\ \langle 91, (0.7600, -0.5400)\rangle, & \langle 90, (0.2500, -0.4400)\rangle, \\ \langle 89, (0.7500, -0.6800)\rangle, & \langle 88, (0.8600, -0.8900)\rangle, \\ \langle 87, (0.6200, -0.5800)\rangle, & \langle 86, (0.5400, -0.9600)\rangle, \\ \langle 85, (0.0700, -0.3500)\rangle, & \langle 84, (0.8800, -0.7800)\rangle \end{Bmatrix}$$

for a set $K = \{95, 94, 93, 92, 91, 90, 89, 88, 87, 86\}$. Let the linguistic variable "Huge" be represented by this $\mathcal{B}^{\mathcal{F}}$-set B, take linguistic hedges as given in (3.6) and compute the modifiers as follows:

$$B^{0.5} = \begin{Bmatrix} \langle 95, (0.3162, -0.5000)\rangle, & \langle 94, (0.9165, -0.0151)\rangle, \\ \langle 93, (0.6000, -0.7764)\rangle, & \langle 92, (0.6403, -0.1282)\rangle, \\ \langle 91, (0.8718, -0.3218)\rangle, & \langle 90, (0.5000, -0.2517)\rangle, \\ \langle 89, (0.8660, -0.4343)\rangle, & \langle 88, (0.9274, -0.6683)\rangle, \\ \langle 87, (0.7874, -0.3519)\rangle, & \langle 86, (0.7348, -0.8000)\rangle, \\ \langle 85, (0.2646, -0.1938)\rangle, & \langle 84, (0.9381, -0.5310)\rangle \end{Bmatrix};$$

$$B = \begin{Bmatrix} \langle 95, (0.1000, -0.7500)\rangle, & \langle 94, (0.8400, -0.0300)\rangle, \\ \langle 93, (0.3600, -0.9500)\rangle, & \langle 92, (0.4100, -0.2400)\rangle, \\ \langle 91, (0.7600, -0.5400)\rangle, & \langle 90, (0.2500, -0.4400)\rangle, \\ \langle 89, (0.7500, -0.6800)\rangle, & \langle 88, (0.8600, -0.8900)\rangle, \\ \langle 87, (0.6200, -0.5800)\rangle, & \langle 86, (0.5400, -0.9600)\rangle, \\ \langle 85, (0.0700, -0.3500)\rangle, & \langle 84, (0.8800, -0.7800)\rangle \end{Bmatrix};$$

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$$B^2 = \begin{Bmatrix} \langle 95,(0.0100,-0.9375)\rangle, & \langle 94,(0.7056,-0.0591)\rangle, \\ \langle 93,(0.1296,-0.9975)\rangle, & \langle 92,(0.1681,-0.4224)\rangle, \\ \langle 91,(0.5776,-0.7884)\rangle, & \langle 90,(0.0625,-0.6864)\rangle, \\ \langle 89,(0.5625,-0.8976)\rangle, & \langle 88,(0.7396,-0.9879)\rangle, \\ \langle 87,(0.3844,-0.8236)\rangle, & \langle 86,(0.2916,-0.9984)\rangle, \\ \langle 85,(0.0049,-0.5775)\rangle, & \langle 84,(0.7744,-0.9516)\rangle \end{Bmatrix};$$

$$B^3 = \begin{Bmatrix} \langle 95,(0.0010,-0.9844)\rangle, & \langle 94,(0.5927,-0.0873)\rangle, \\ \langle 93,(0.0467,-0.9999)\rangle, & \langle 92,(0.0689,-0.5610)\rangle, \\ \langle 91,(0.4390,-0.9027)\rangle, & \langle 90,(0.0156,-0.8244)\rangle, \\ \langle 89,(0.4219,-0.9672)\rangle, & \langle 88,(0.6361,-0.9987)\rangle, \\ \langle 87,(0.2383,-0.9259)\rangle, & \langle 86,(0.1575,-0.9999)\rangle, \\ \langle 85,(0.0003,-0.7254)\rangle, & \langle 84,(0.6815,-0.9894)\rangle \end{Bmatrix};$$

$$B^4 = \begin{Bmatrix} c\langle 95,(0.0001,-0.9961)\rangle, & \langle 94,(0.4979,-0.1147)\rangle, \\ \langle 93,(0.0168,-1.0000)\rangle, & \langle 92,(0.0283,-0.6664)\rangle, \\ \langle 91,(0.3336,-0.9552)\rangle, & \langle 90,(0.0039,-0.9017)\rangle, \\ \langle 89,(0.3164,-0.9895)\rangle, & \langle 88,(0.5470,-0.9999)\rangle, \\ \langle 87,(0.1478,-0.9689)\rangle, & \langle 86,(0.0850,-1.0000)\rangle, \\ \langle 85,(0.0000,-0.8215)\rangle, & \langle 84,(0.5997,-0.9977)\rangle \end{Bmatrix}.$$

Using the proposed measure, the knowledge linked with these hedges is calculated and their values are $KM(H_1) = 0.3768$, $KM(H_2) = 0.4045$, $KM(H_3) = 0.5362$, $KM(H_4) = 0.6262$, $KM(H_5) = 0.6904$. From these values, we get the proposed measure satisfies the sequence

$$KM(H_1) < KM(H_2) < KM(H_3) < KM(H_4) < KM(H_5),$$

which is the same as defined in (3.7). So, the proposed measure performs well also in this example. Therefore, the proposed measure's performance is outstanding and promising.

By using the proposed measure, a few measures are derived in the next section and some basic properties are also discussed.

4. Deductions of some novel measures

4.1. Accuracy measure in bipolar fuzzy framework and its properties In the framework of $\mathcal{B}^{\mathcal{F}}$-measures, accuracy and knowledge are related. The idea of an accuracy measure is required when comparing the accuracy of two $\mathcal{B}^{\mathcal{F}}$-sets together. From (3.3), a $\mathcal{B}^{\mathcal{F}}$-accuracy measure is deduced as follows.

Let $B, C \in \mathcal{B}(K)$ for a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}, W \geq 2$. The $\mathcal{B}^{\mathcal{F}}$-accuracy measure is proposed as follows:

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$$AM_{\mathcal{B}}(B,C) = \frac{(\sqrt[3]{4}-1)^{-1}}{6W} \times \sum_{\omega=1}^{W} \left[\begin{array}{l} \left(\begin{array}{l} \sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)} \\ +\sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} \end{array} \right) - 2 \\ + \left(\begin{array}{l} \sqrt[3]{4((\mu_C^P(\kappa_\omega))^3 + (1-\mu_C^P(\kappa_\omega))^3)} \\ +\sqrt[3]{4((1+\mu_C^N(\kappa_\omega))^3 - (\mu_C^N(\kappa_\omega))^3)} \end{array} \right) - 2 \\ + \left(\begin{array}{l} \sqrt[3]{4\left(\frac{(\mu_B^P(\kappa_\omega))^2(\mu_C^P(\kappa_\omega)) + (\mu_B^P(\kappa_\omega))(\mu_C^P(\kappa_\omega))^2}{2} + \right.} \\ \left. \frac{(1-\mu_B^P(\kappa_\omega))^2(1-\mu_C^P(\kappa_\omega)) + (1-\mu_B^P(\kappa_\omega))(1-\mu_C^P(\kappa_\omega))^2}{2} \right) \\ +\sqrt[3]{4\left(\frac{(1+\mu_B^N(\kappa_\omega))^2(1+\mu_C^N(\kappa_\omega)) + (1+\mu_B^N(\kappa_\omega))(1+\mu_C^N(\kappa_\omega))^2}{2} - \right.} \\ \left. \frac{(\mu_B^N(\kappa_\omega))^2(\mu_C^N(\kappa_\omega)) + (\mu_B^N(\kappa_\omega))(\mu_C^N(\kappa_\omega))^2}{2} \right) \end{array} \right) - 2 \end{array} \right]. \tag{4.1}$$

We now verify the reliability of the measure $AM_{\mathcal{B}}$.

THEOREM 4.1. *Let $B, C \in \mathcal{B}(K)$ for a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}, W \geq 2$, then any valid $\mathcal{B}^{\mathcal{F}}$-accuracy measure $AM_{\mathcal{B}} : \mathcal{B}(K) \times \mathcal{B}(K) \to [0,1]$ must fulfil the following axioms:*

(M1) $0 \leq AM_{\mathcal{B}}(B,C) \leq 1$ *for all* $B, C \in \mathcal{B}(K)$;
(M2) $AM_{\mathcal{B}}(B,C) = 0 \Leftrightarrow B = \langle \frac{1}{2}, -\frac{1}{2} \rangle = C$;
(M3) $AM_{\mathcal{B}}(B,C) = 1 \Leftrightarrow B = \langle 1, -1 \rangle = C$ *or* $B = \langle 0, 0 \rangle = C$;
(M4) $AM_{\mathcal{B}}(B,C) = KM_{\mathcal{B}}(B)$ *for* $B = C$, *where* $KM_{\mathcal{B}}$ *is the knowledge-measure.*

PROOF. The above-given axioms are easy to prove for the accuracy measure given in (4.1). □

EXAMPLE 4.2. Let $B = \langle \mu_B^P, \mu_B^N \rangle$ be a $\mathcal{B}^{\mathcal{F}}$-number, $C_1 = \langle 0.5, -0.5 \rangle$ be the most $\mathcal{B}^{\mathcal{F}}$-number, and $C_2 = \langle 0, 0 \rangle$ and $C_3 = \langle 1, -1 \rangle$ be the crisp $\mathcal{B}^{\mathcal{F}}$-numbers. Figures 2, 3 and 4 show the accuracy of the $\mathcal{B}^{\mathcal{F}}$-number B with respect to $\mathcal{B}^{\mathcal{F}}$-numbers C_1, C_2 and C_3, respectively.

From Figures 2, 3 and 4, the following points hold true.

1. The value of the measure $AM_{\mathcal{B}}$ lies in between 0 and 1.
2. The measure $AM_{\mathcal{B}}$ is symmetric.
3. For two equal $\mathcal{B}^{\mathcal{F}}$-numbers B and C, the measure $AM_{\mathcal{B}}(B,C)$ retains the same values as retained by the measure $KM_{\mathcal{B}}$.

Now, some properties of the measure $AM_{\mathcal{B}}(B,C)$ are discussed.

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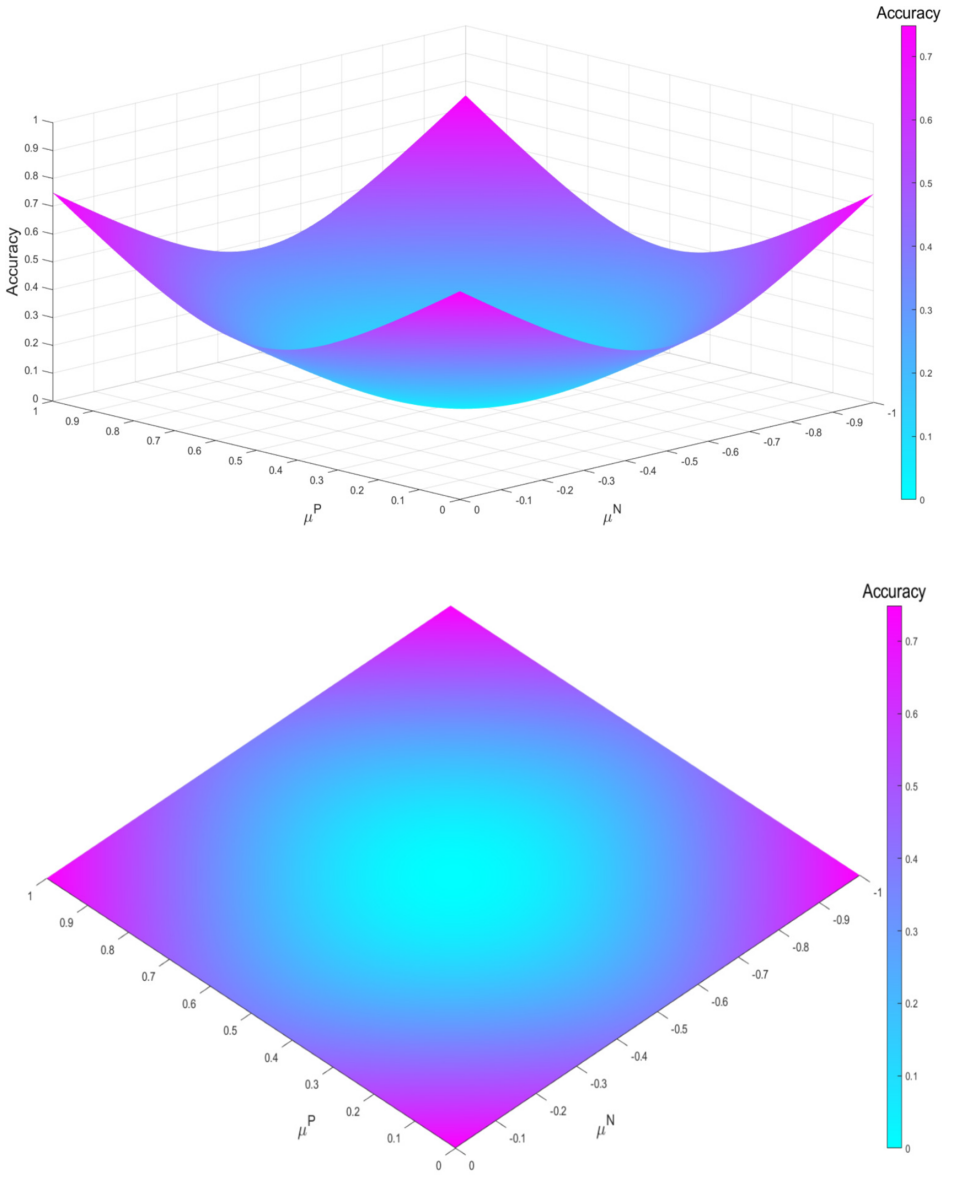


FIGURE 2. Accuracy of bipolar fuzzy number B with respect to the most bipolar fuzzy number $C_1 = \langle 0.5, -0.5 \rangle$.

THEOREM 4.3. *The following points hold good for the proposed $\mathcal{B}^{\mathcal{F}}$-accuracy measure $AM_{\mathcal{B}}$:*

(I) $AM_{\mathcal{B}}(D, B \cap C) + AM_{\mathcal{B}}(D, B \cup C) = AM_{\mathcal{B}}(D, C) + AM_{\mathcal{B}}(D, B)$;
(II) $AM_{\mathcal{B}}(C \cap D, B) + AM_{\mathcal{B}}(C \cup D, B) = AM_{\mathcal{B}}(D, B) + AM_{\mathcal{B}}(C, B)$;

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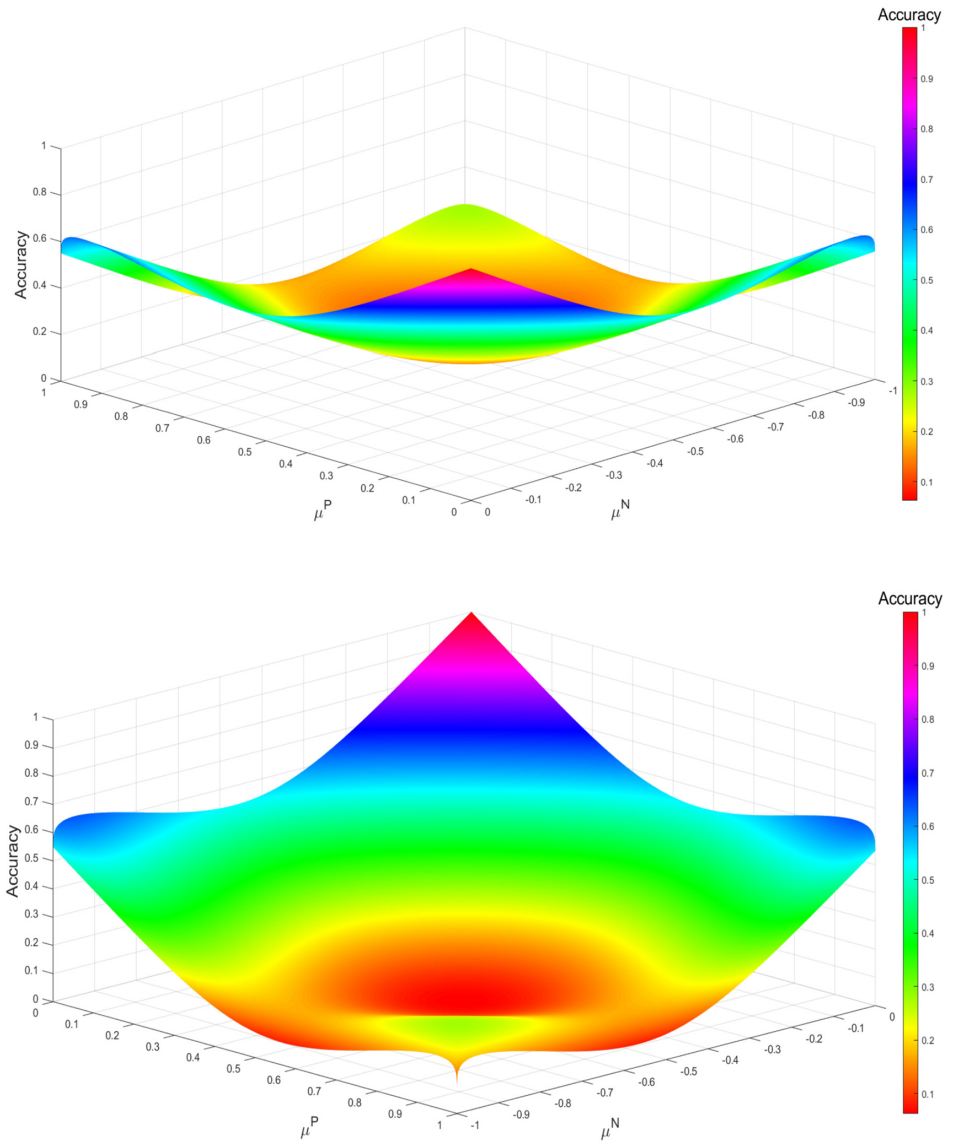


FIGURE 3. Accuracy of the bipolar fuzzy number B with respect to the crisp bipolar fuzzy number $C_1 = \langle 0, -0 \rangle$.

(III) $AM_{\mathcal{B}}(B, B^o) = AM_{\mathcal{B}}(B^o, B)$;
(IV) $AM_{\mathcal{B}}(C, B) + AM_{\mathcal{B}}(C, B^o) = AM_{\mathcal{B}}(C^o, B^o) + AM_{\mathcal{B}}(C^o, B)$,

where $B, C, D \in \mathcal{B}(K)$, and the notation ^{o} is used for complement.

PROOF. The above-given properties are easy to prove for the accuracy measure given in (4.1). □

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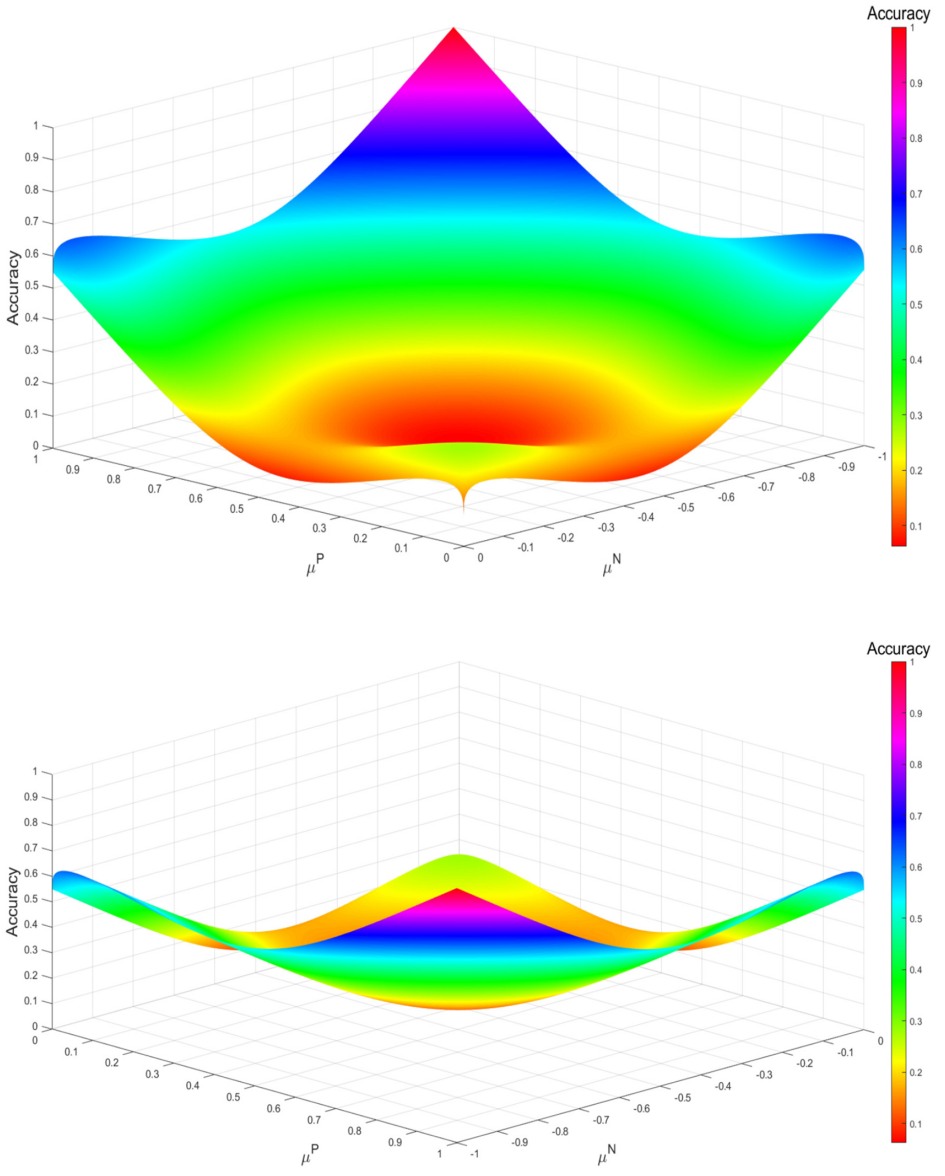


FIGURE 4. Accuracy of the bipolar fuzzy number B with respect to the crisp bipolar fuzzy number $C_1 = \langle 1, -1 \rangle$.

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4.2. Similarity measure in bipolar fuzzy framework and its properties Let $B, C \in \mathcal{B}(K)$; a $\mathcal{B}^{\mathcal{F}}$-similarity measure is proposed as follows:

$$SM_{\mathcal{B}}(B,C) = 1 - \frac{|(M_B^P - M_C^P) + (M_B^N - M_C^N)|}{2W(\sqrt[3]{4}-1)} \quad \text{where}$$

$$\begin{aligned} M_B^P &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)} - 1\right], \\ M_B^N &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} - 1\right], \\ M_C^P &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((\mu_C^P(\kappa_\omega))^3 + (1-\mu_C^P(\kappa_\omega))^3)} - 1\right], \\ M_C^N &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((1+\mu_C^N(\kappa_\omega))^3 - (\mu_C^N(\kappa_\omega))^3)} - 1\right]. \end{aligned} \tag{4.2}$$

We now verify the reliability of the measure $SM_{\mathcal{B}}$.

THEOREM 4.4. *Let $B, C \in \mathcal{B}(K)$ for a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}$, $W \geq 2$, then any valid $\mathcal{B}^{\mathcal{F}}$-similarity measure $SM_{\mathcal{B}} : \mathcal{B}(K) \times \mathcal{B}(K) \to [0,1]$ must fulfil the following axioms:*

(S1) *$SM_{\mathcal{B}}(B,C)$ lies in between 0 and 1 for all $B, C \in \mathcal{B}(K)$;*
(S2) *$SM_{\mathcal{B}}$ is symmetric;*
(S3) *$SM_{\mathcal{B}}(B,C)$ is unity for $B = C$;*
(S4) *if $B \subseteq C$ and $C \subseteq D$, then $SM_{\mathcal{B}}(C,B) \geq SM_{\mathcal{B}}(D,B)$ and $SM_{\mathcal{B}}(D,C) \geq SM_{\mathcal{B}}(D,B)$.*

PROOF. The above-given axioms are easy to prove for the similarity measure given in (4.2). □

EXAMPLE 4.5. Let $B = \langle \mu_B^P, \mu_B^N \rangle$ be a $\mathcal{B}^{\mathcal{F}}$-number, $C_1 = \langle 0.5, -0.5 \rangle$ be the most $\mathcal{B}^{\mathcal{F}}$-number, and $C_2 = \langle 0, 0 \rangle$ and $C_3 = \langle 1, -1 \rangle$ be the crisp $\mathcal{B}^{\mathcal{F}}$-numbers. Figure 5 shows the similarity of $\mathcal{B}^{\mathcal{F}}$-number B with respect to $\mathcal{B}^{\mathcal{F}}$-numbers C_1, C_2 and C_3.

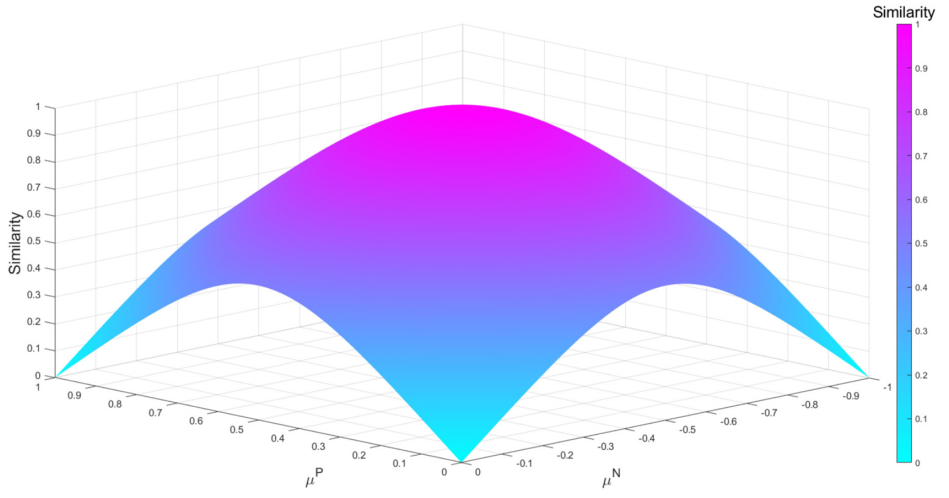
From Figure 5, the following points hold good.

(1) The value of the measure $SM_{\mathcal{B}}$ lies in between 0 and 1.
(2) The measure $SM_{\mathcal{B}}$ is symmetric.
(3) The measure $SM_{\mathcal{B}}(B,C)$ is unity for $B = C$.

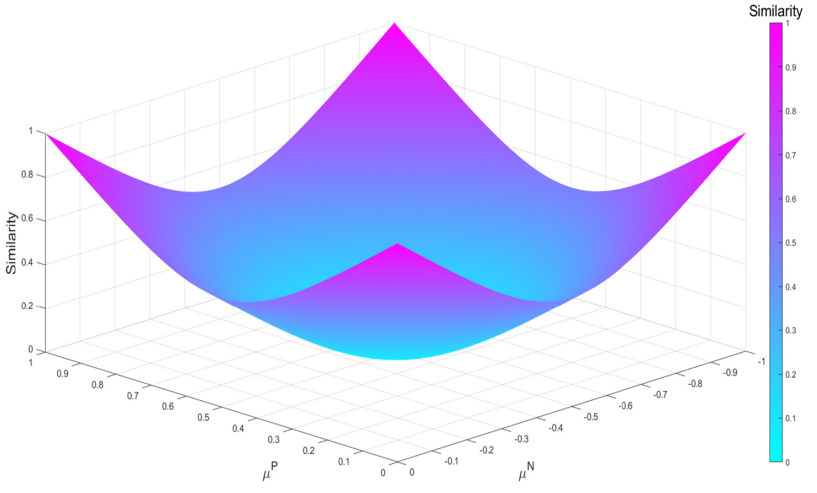
Now, basic properties of the measure $SM_{\mathcal{B}}$ are discussed.

THEOREM 4.6. *The following points hold good for the proposed $\mathcal{B}^{\mathcal{F}}$-similarity measure $SM_{\mathcal{B}}$:*

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(a) $SM_{\mathcal{B}}(B, C_1)$.



(b) $SM_{\mathcal{B}}(B, C_2)$ and $SM_{\mathcal{B}}(B, C_3)$.

FIGURE 5. Proposed $\mathcal{B}^{\mathcal{T}}$-similarity measure.

(I) $SM_{\mathcal{B}}(C, B) = SM_{\mathcal{B}}(C^o, B^o)$;
(II) $SM_{\mathcal{B}}(C^o, B) = SM_{\mathcal{B}}(C, B^o)$;
(III) $\zeta SM^1_{\mathcal{B}} + (1-\zeta)SM^2_{\mathcal{B}}$, $(0 \le \zeta \le 1)$ *is a similarity measure for given similarity measures* $SM^1_{\mathcal{B}}$ *and* $SM^2_{\mathcal{B}}$,

where $B, C \in \mathcal{B}(K)$.

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PROOF. The above-given properties are easy to prove for the similarity measure given in (4.2). □

4.3. Dissimilarity measure in bipolar fuzzy framework and its properties Let $B, C \in \mathcal{B}(K)$. We proposed a $\mathcal{B}^{\mathcal{F}}$-dissimilarity measure as follows:

$$
\begin{aligned}
DM_{\mathcal{B}}(B,C) &= \frac{|(M_B^P - M_C^P) + (M_B^N - M_C^N)|}{2W(\sqrt[3]{4}-1)} \quad \text{where} \\
M_B^P &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((\mu_B^P(\kappa_\omega))^3 + (1-\mu_B^P(\kappa_\omega))^3)} - 1\right], \\
M_B^N &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((1+\mu_B^N(\kappa_\omega))^3 - (\mu_B^N(\kappa_\omega))^3)} - 1\right], \\
M_C^P &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((\mu_C^P(\kappa_\omega))^3 + (1-\mu_C^P(\kappa_\omega))^3)} - 1\right], \\
M_C^N &= \sum_{\omega=1}^{W}\left[\sqrt[3]{4((1+\mu_C^N(\kappa_\omega))^3 - (\mu_C^N(\kappa_\omega))^3)} - 1\right].
\end{aligned} \tag{4.3}
$$

We now verify the reliability of the measure $DM_{\mathcal{B}}$.

THEOREM 4.7. *Let $B, C \in \mathcal{B}(K)$ for a universal set $K = \{\kappa_1, \kappa_2, \ldots, \kappa_W\}$, $W \geq 2$, then any valid $\mathcal{B}^{\mathcal{F}}$-dissimilarity measure $DM_{\mathcal{B}} : \mathcal{B}(K) \times \mathcal{B}(K) \to [0,1]$ has to fulfil the following axioms:*

(S1) *$DM_{\mathcal{B}}(B,C)$ lies in between 0 and 1 for all $B, C \in \mathcal{B}(K)$;*
(S2) *$DM_{\mathcal{B}}$ is symmetric;*
(S3) *$DM_{\mathcal{B}}(B,C)$ vanishes for $B = C$;*
(S4) *if $B \subseteq C$ and $C \subseteq D$, then $DM_{\mathcal{B}}(C,B) \leq DM_{\mathcal{B}}(D,B)$ and $DM_{\mathcal{B}}(D,C) \leq DM_{\mathcal{B}}(D,B)$.*

PROOF. The above-given axioms are easy to prove for the similarity measure given in (4.3). □

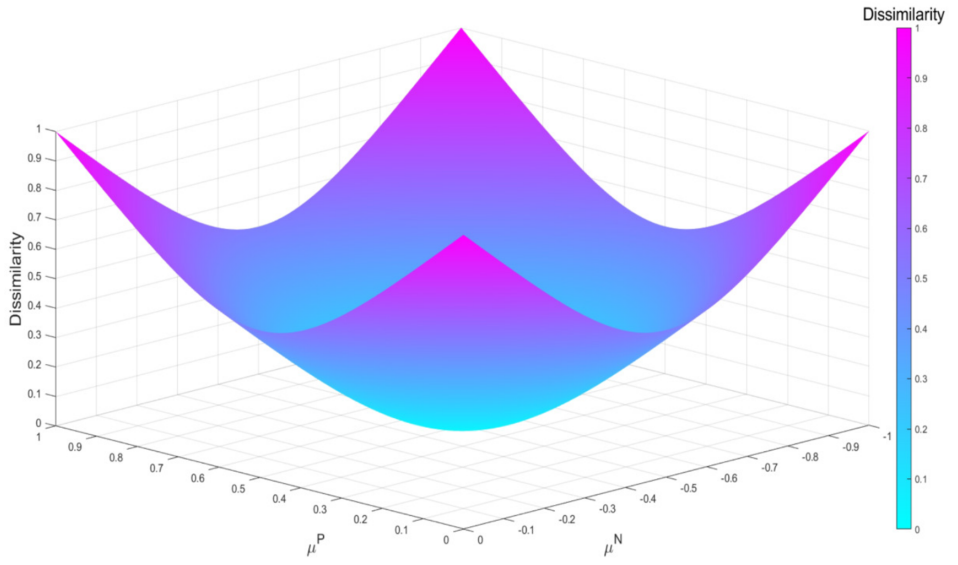
EXAMPLE 4.8. Let $B = \langle \mu_B^P, \mu_B^N \rangle$ be a $\mathcal{B}^{\mathcal{F}}$-number, $C_1 = \langle 0.5, -0.5 \rangle$ be the most $\mathcal{B}^{\mathcal{F}}$-number, and $C_2 = \langle 0, 0 \rangle$ and $C_3 = \langle 1, -1 \rangle$ be the crisp $\mathcal{B}^{\mathcal{F}}$-numbers. Figure 6 shows the dissimilarity of $\mathcal{B}^{\mathcal{F}}$-number B with respect to $\mathcal{B}^{\mathcal{F}}$-numbers C_1, C_2 and C_3.

From Figure 6, the following points hold good.

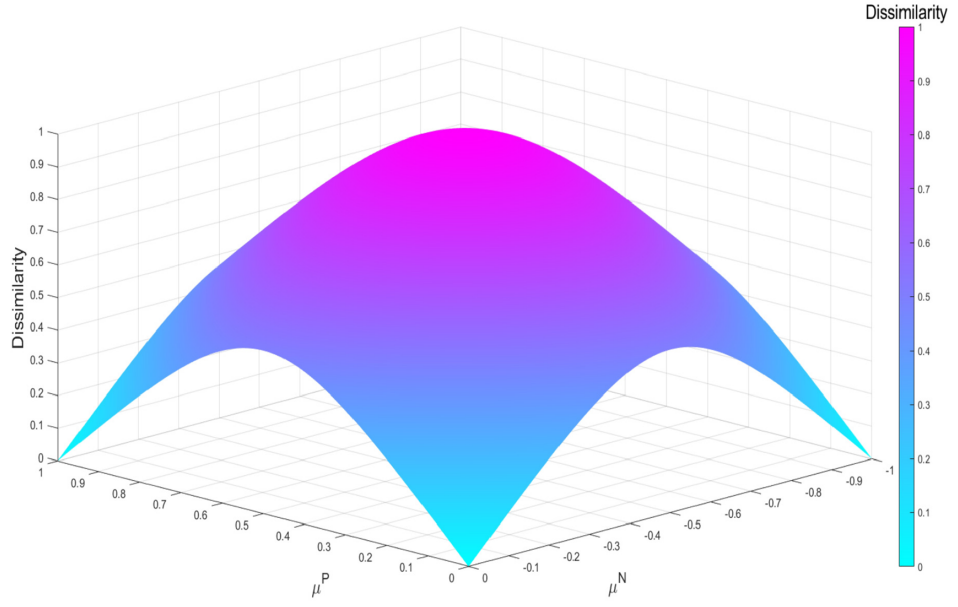
1. The value of the measure $DM_{\mathcal{B}}$ lies in between 0 and 1.
2. The measure $DM_{\mathcal{B}}$ is symmetric.
3. The measure $DM_{\mathcal{B}}(B,C)$ vanishes for $B = C$.

Now, basic properties of the measure $DM_{\mathcal{B}}$ are discussed.

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(a) $DM_{\mathcal{B}}(B, C_1)$.



(b) $DM_{\mathcal{B}}(B, C_2)$ and $DM_{\mathcal{B}}(B, C_3)$.

FIGURE 6. Proposed $\mathcal{B}^{\mathcal{T}}$-dissimilarity measure.

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THEOREM 4.9. *The following points hold good for the proposed $\mathcal{B}^{\mathcal{F}}$-dissimilarity measure $DM_{\mathcal{B}}$:*

(I) $DM_{\mathcal{B}}(C, B) = DM_{\mathcal{B}}(C^o, B^o)$;
(II) $DM_{\mathcal{B}}(C^o, B) = DM_{\mathcal{B}}(C, B^o)$;
(III) $\zeta DM^1_{\mathcal{B}} + (1-\zeta)DM^2_{\mathcal{B}}$, $(0 \le \zeta \le 1)$ *is a dissimilarity measure for given dissimilarity measures $DM^1_{\mathcal{B}}$ and $DM^2_{\mathcal{B}}$,*

where $B, C \in \mathcal{B}(K)$.

PROOF. The above-given properties are easy to prove for the dissimilarity measure given in (4.3). □

Now, the implementation of the derived measures are discussed in pattern detection, cluster analysis and medical diagnosis issues in the next section.

5. Applications of the proposed measures

5.1. Cluster analysis A method for classifying data called cluster analysis divides a given collection of objects into groups or categories based on the similarities between them. In cluster analysis issues, we try to fit the alternates in a cluster so that they have high similarity with each other in that cluster and more dissimilarity from the alternates of other clusters. Consequently, an attempt is made to depict the human ability to arrange similar events into uniform groups. There may be a variety of reasons for choosing and creating lessons in this way. In essence, cluster analysis is a technique for identifying patterns in a group of unsorted things that have never been observed before. There are many researchers [5, 10, 18, 21, 22, 24, 35] who did their work to solve the problem related to cluster analysis. Rico et al. [24] provide a clustering technique under similarity measure. However, the suggested accuracy measure is employed in place of the similarity measure in the current manuscript. The following are these definitions.

DEFINITION 5.1. Let us consider a set of alternatives represented by $\{B_1, B_2, \ldots, B_W\}$, where $B_1, B_2, \ldots, B_W \in \mathcal{B}(K)$. We define a symmetrical accuracy matrix $D = \{d_{\omega\tau}\}_{W\times W}$, where $d_{\omega\tau} = AM_{\mathcal{B}}(B_\omega, B_\tau)$ for $\omega \neq \tau$ and $d_{\omega\tau} = AM_{\mathcal{B}}(B_\omega, B_\tau)/KM_{\mathcal{B}}(B_\omega)$ for $\omega = \tau$. $AM_{\mathcal{B}}$ is used for the proposed accuracy measure and $KM_{\mathcal{B}}(B_\omega)$ is used for the proposed knowledge-measure. The matrix D fulfils the properties given as follows.

(i) Each member $d_{\omega\tau}$ lies in between 0 and 1.
(ii) Matrix D is symmetric.
(iii) Diagonal entries of matrix D are unity.

DEFINITION 5.2. If $D = \{d_{\omega\tau}\}_{W\times W}$ is any symmetrical accuracy matrix, then the symmetrical composition matrix $D^2 = D \otimes D$ for the matrix D is given by $D^2 = \{n_{\omega\tau}\}_{W\times W}$, where $n_{\omega\tau} = \max_\psi\{\min\{d_{\omega\psi}, d_{\psi\tau}\}\}$; $\omega, \tau = 1, 2, \ldots, W$. Any composition of accuracy matrices satisfies the properties satisfied by an accuracy matrix.

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DEFINITION 5.3. An equivalent accuracy matrix $D = \{d_{\omega\tau}\}_{W\times W}$ satisfies the property $D^2 \subseteq D$, that is, $\max_{\psi}\{\min\{d_{\omega\psi}, d_{\psi\tau}\}\} \leq d_{\omega\tau}$; $\omega, \tau = 1, 2, \ldots, W$.

DEFINITION 5.4. After applying successive compositions, that is, $D \to D^2 \to D^4 \to \cdots \to D^{2^z} \to \cdots$ for an accuracy matrix $D = \{d_{\omega\tau}\}_{W\times W}$, we get a stage at which $D^{2^z} = D^{2^{z+1}}$. At this stage, the equivalent accuracy matrix is the matrix D^{2^z} for a number z.

DEFINITION 5.5 [24]. If $D = \{d_{\omega\tau}\}_{W\times W}$ is any equivalent accuracy matrix, then $M_\Psi = \{m_{\omega\tau}\}_{W\times W}$ is the Ψ-cutting matrix, where for any confidence level $\Psi \in [0, 1]$, we have $m_{\omega\tau} = 1$ for $m_{\omega\tau} \geq \Psi$, and $m_{\omega\tau} = 0$ for $m_{\omega\tau} < \Psi$.

5.1.1. The proposed algorithm We can split the proposed algorithm into three basic parts. In the first part, we collect the input. In the second part, the conversion of data into the $\mathcal{B}^{\mathcal{F}}$-framework is carried out, and the accuracy matrix and equivalent accuracy matrix are calculated. In the third part, different clusters are formed for different values of confidence levels. Figure 7 is prepared to understand the proposed algorithm. Take a cluster analysis issue involving the alternatives $\{B_1, B_2, \ldots, B_W\}$ and criteria $\{C_1, C_2, \ldots, C_T\}$. The proposed algorithm contains the following steps.

Step 1. *Collect the data and convert it in $\mathcal{B}^{\mathcal{F}}$-framework.* Let us consider a decision matrix $K = \{k_{\omega\tau}\}_{W\times T}$ involving W alternatives and T criteria. The data are then converted into a matrix $L = \{l_{\omega\tau}\}_{W\times T}$ in the $\mathcal{B}^{\mathcal{F}}$-framework, where

$$l_{\omega\tau} = \left\langle \begin{array}{c} 1 - \left(\exp\left(-\dfrac{k_{\omega\tau} - \min_{\{\tau\}}(k_{\omega\tau})}{\max_{\{\tau\}}(k_{\omega\tau}) - \min_{\{\tau\}}(k_{\omega\tau})}\right)\right)^{\Omega}, \\ -\left(\exp\left(-\dfrac{k_{\omega\tau} - \min_{\{\tau\}}(k_{\omega\tau})}{\max_{\{\tau\}}(k_{\omega\tau}) - \min_{\{\tau\}}(k_{\omega\tau})}\right)\right)^{\Omega} \end{array} \right\rangle \tag{5.1}$$

for all $\omega = 1, 2, \ldots, W$; $\tau = 1, 2, \ldots, T$. Generally, Ω is taken to be $1/2$.

Step 2. *Construction of the accuracy matrix.* With the help of Definition 5.1, an accuracy matrix $D = \{d_{\omega\tau}\}_{W\times W}$ is constructed by using the matrix $L = \{l_{\omega\tau}\}_{W\times T}$. Matrix D is symmetrical and its elements represent the accuracy between two alternatives.

Step 3. *Construction of the equivalent accuracy matrix.* Apply the compositions $D \to D^2 \to D^4 \to \cdots \to D^{2^z} \to \cdots$ for an accuracy matrix $D = \{d_{\omega\tau}\}_{W\times W}$. At a stage, we get $D^{2^z} = D^{2^{z+1}}$ for some z. The resultant matrix D^{2^z} is the equivalent accuracy matrix.

Step 4. *Find the values of Confidence levels and make clusters from different Ψ-cutting matrices.* For different values of $\Psi \in [0, 1]$, generate the Ψ-cutting matrix $M_\Psi = \{m_{\omega\tau}\}_{W\times W}$ by using Definition 5.5. Two given alternatives B_ω and B_τ are from the same cluster if the ωth column or row is equal to the τth column or row in the Ψ-cutting matrices. In the same way, we may group all the other alternatives to form various clusters.

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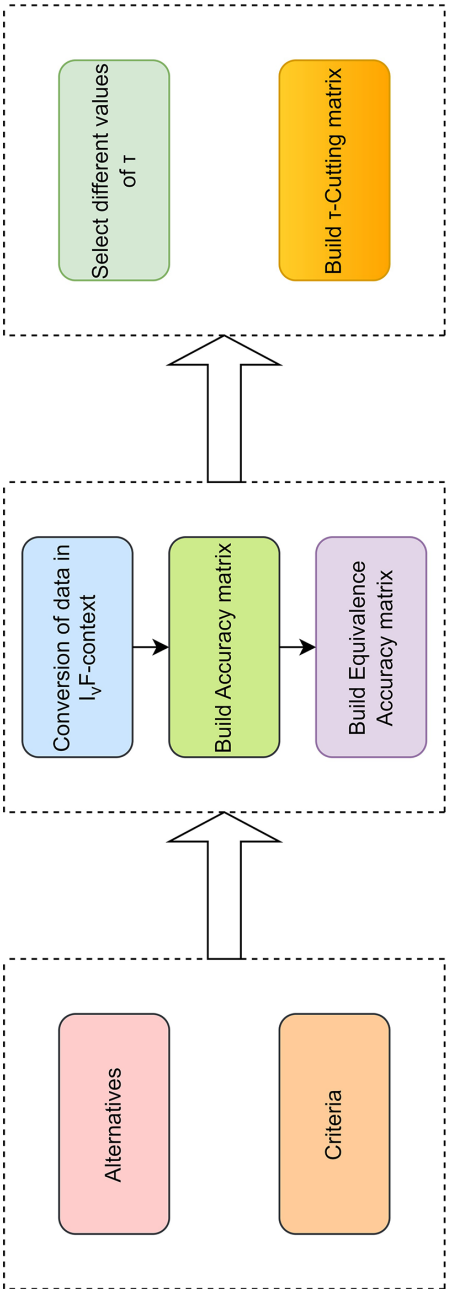


FIGURE 7. Three basic parts of the proposed algorithm.

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5.1.2. Case study Our environment and human health are both badly affected by air pollution. In one year, approximately 7 million people died from outdoor and indoor air pollution as stated by the World Health Organisation (WHO). WHO have stated a range of polluted air that is safe for human breathing. However, at present, more than 99% of people breathe in air that is more polluted than this range. Air pollution is the fourth biggest curse for our atmosphere. As per the Global Air Report 2020, approximately 2.2 million have lost their lives due to indoor air pollution and approximately 4.5 million have lost their lives due to outdoor air pollution. India and China are the most polluted countries in the world as per data. Many other regions of the world are not far behind in the case of air pollution. Air pollution in different continents is a major talking point. Many substances are present in the air such as ammonia, black carbon, nonmethane volatile organic compounds, organic carbon, carbon monoxide, sulphur dioxide, nitrogen oxide, and so forth. A dataset[1] containing these substances in air in different continents is taken. Table 1 and Figure 8 present the actual dataset. Abbreviations used in Table 1 and Figure 8 are Nitrogen oxide (NO), sulphur dioxide (SO_2), carbon monoxide (CO), organic carbon (OC), nonmethane volatile organic compounds (NMVOC), ammonia (NH_3) and black carbon (BC).

We use the proposed algorithm to create clusters for the given data set. For this, take into consideration the following cluster analysis issue.

Consider a cluster analysis issue containing a set of seven alternatives: $B_1, B_2, B_3, B_4, B_5, B_6, B_7$ used for Africa, Asia, Europe, North America, Oceania, South America, Timor, respectively. Let us consider a set of criteria: $C_1, C_2, C_3, C_4, C_5, C_6, C_7$ used for NO, SO_2, CO, OC, NMVOC, BC, NH_3, respectively. The alternatives and criteria are shown in Figure 9.

Let us consider the set of criteria weights $A = \{a_1, a_2, \ldots, a_7\}$, where $\sum_{\tau=1}^{7} a_\tau = 1$ and $0 \leq a_\tau \leq 1$. The data in normalized form is shown in Table 2 by using the formula used by Singh and Kumar [27].

Now, the suggested algorithm is used to create clusters from these data. The steps in the suggested algorithm are as follows.

Step 1. *Collect the data and convert it in the $\mathcal{B}^{\mathcal{F}}$-framework.* First of all, take the data from Table 2. These data are not in the $\mathcal{B}^{\mathcal{F}}$-framework. To convert these data in the $\mathcal{B}^{\mathcal{F}}$-framework, we make use of (5.1). The $\mathcal{B}^{\mathcal{F}}$-matrix L is shown by Tables 3 and 4.

Step 2. *Construction of the accuracy matrix.* With the help of Definition 5.1, an accuracy matrix D is constructed and its value is shown in the following.

[1]The data that support the findings of this study are available at https://www.kaggle.com/datasets/rejeph/air-pollution.

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TABLE 1. Dataset for air pollution for the year 2022.

Regions	NO	SO_2	CO	OC	NMVOC	BC	NH_3
Africa	7431502.5	4995157	87050240	3252728.5	25184732	1191946.5	8922146
Asia	56031916	46772100	259996910	6973148	71669064	3036887.2	35079756
Europe	10775340	7971970	27281258	1088880.8	11475047	415883.88	6337945
North America	10523869	5171649.5	32688806	885297.3	14661643	276148.7	6875147.5
Oceania	1280531.5	1164619	2409977.2	43012.52	1087024.8	35245.816	913411.2
South America	5861211.5	3236210.5	13869350	668478.94	6712245.5	291069.94	5819238
Timor	4958.444	1035.1042	10531.836	1372.14	15570.762	374.58612	8033.6147

*Data in this table is in metric tonnes.

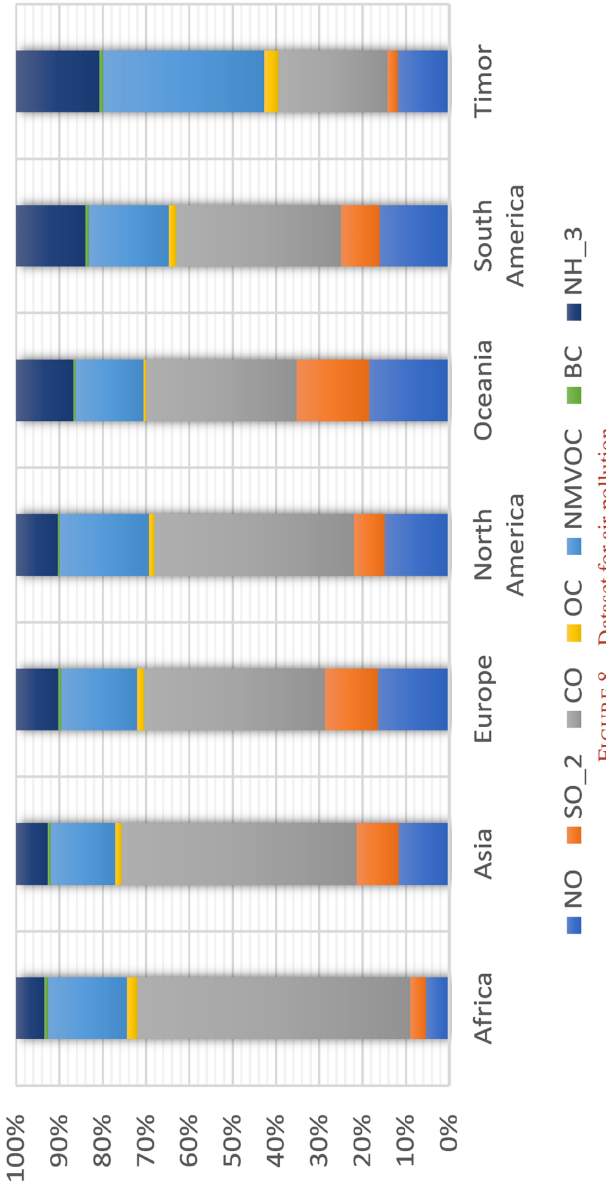


FIGURE 8. Dataset for air pollution.

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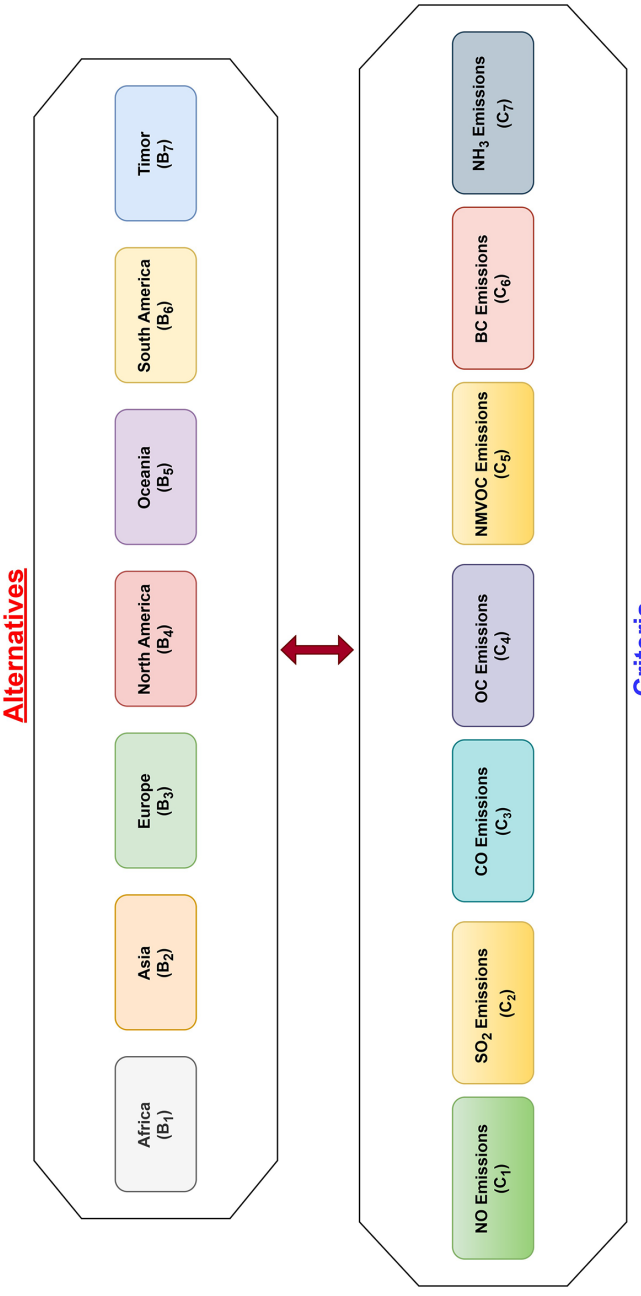


FIGURE 9. Alternatives and criteria.

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TABLE 2. Normalized data.

Regions	NO	SO_2	CO	OC	NMVOC	BC	NH_3
Africa	0.1264	0.1038	0.3133	0.4144	0.3208	0.3597	0.2357
Asia	0.9529	0.9722	0.9358	0.8883	0.9129	0.9165	0.9269
Europe	0.1832	0.1657	0.0982	0.1387	0.1462	0.1255	0.1675
North America	0.1790	0.1075	0.1177	0.1128	0.1868	0.0833	0.1817
Oceania	0.0218	0.0242	0.0087	0.0055	0.0138	0.0106	0.0241
South America	0.0997	0.0673	0.0499	0.0852	0.0855	0.0878	0.1538
Timor	0.0001	0.0000	0.0000	0.0002	0.0002	0.0001	0.0002

Matrix D is symmetrical and every element represents the accuracy between two alternatives,

$$D = \begin{pmatrix} 1 & 0.5616 & 0.6703 & 0.6722 & 0.7033 & 0.6839 & 0.7074 \\ 0.1822 & 1 & 0.2142 & 0.2166 & 0.2540 & 0.2308 & 0.2590 \\ 0.7726 & 0.6958 & 1 & 0.7983 & 0.8277 & 0.8093 & 0.8317 \\ 0.7817 & 0.7054 & 0.8055 & 1 & 0.8367 & 0.8184 & 0.8407 \\ 0.9266 & 0.8566 & 0.9487 & 0.9505 & 1 & 0.9608 & 0.9816 \\ 0.8371 & 0.7634 & 0.8603 & 0.8621 & 0.8907 & 1 & 0.8946 \\ 0.9454 & 0.8763 & 0.9674 & 0.9692 & 0.9963 & 0.9794 & 1 \end{pmatrix}.$$

Step 3. *Construction of the equivalent accuracy matrix.* Apply the successive compositions, that is, $D \to D^2 \to D^4 \to \cdots \to D^{2^z} \to \cdots$ for an accuracy matrix $D = \{d_{\omega\tau}\}_{W\times W}$. The powers of D are shown as follows:

$$D^2 = D \; o \; D$$

$$= \begin{pmatrix} 1.0000 & 0.7074 & 0.7074 & 0.7074 & 0.7074 & 0.7074 & 0.7074 \\ 0.2590 & 1.0000 & 0.2590 & 0.2590 & 0.2590 & 0.2590 & 0.2590 \\ 0.8317 & 0.8317 & 1.0000 & 0.8317 & 0.8317 & 0.8317 & 0.8317 \\ 0.8407 & 0.8407 & 0.8407 & 1.0000 & 0.8407 & 0.8407 & 0.8407 \\ 0.9454 & 0.8763 & 0.9674 & 0.9692 & 1.0000 & 0.9794 & 0.9816 \\ 0.8946 & 0.8763 & 0.8946 & 0.8946 & 0.8946 & 1.0000 & 0.8946 \\ 0.9454 & 0.8763 & 0.9674 & 0.9692 & 0.9963 & 0.9794 & 1.0000 \end{pmatrix},$$

$$D^4 = D^2 \; o \; D^2$$

$$= \begin{pmatrix} 1.0000 & 0.7074 & 0.7074 & 0.7074 & 0.7074 & 0.7074 & 0.7074 \\ 0.2590 & 1.0000 & 0.2590 & 0.2590 & 0.2590 & 0.2590 & 0.2590 \\ 0.8317 & 0.8317 & 1.0000 & 0.8317 & 0.8317 & 0.8317 & 0.8317 \\ 0.8407 & 0.8407 & 0.8407 & 1.0000 & 0.8407 & 0.8407 & 0.8407 \\ 0.9454 & 0.8763 & 0.9674 & 0.9692 & 1.0000 & 0.9794 & 0.9816 \\ 0.8946 & 0.8763 & 0.8946 & 0.8946 & 0.8946 & 1.0000 & 0.8946 \\ 0.9454 & 0.8763 & 0.9674 & 0.9692 & 0.9963 & 0.9794 & 1.0000 \end{pmatrix}.$$

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TABLE 3. Dataset for air pollution in $\mathcal{B}^{\mathcal{T}}$-framework for criteria C_1, C_2, C_3 and C_4.

Region	NO	SO_2	CO	OC
Africa	$\langle 0.0641, -0.9359\rangle$	$\langle 0.0520, -0.9480\rangle$	$\langle 0.1541, -0.8459\rangle$	$\langle 0.2080, -0.7920\rangle$
Asia	$\langle 0.3935, -0.6065\rangle$	$\langle 0.3935, -0.6065\rangle$	$\langle 0.3935, -0.6065\rangle$	$\langle 0.3935, -0.6065\rangle$
Europe	$\langle 0.0916, -0.9084\rangle$	$\langle 0.0817, -0.9183\rangle$	$\langle 0.0511, -0.9489\rangle$	$\langle 0.0750, -0.9250\rangle$
North America	$\langle 0.0896, -0.9104\rangle$	$\langle 0.0538, -0.9462\rangle$	$\langle 0.0609, -0.9391\rangle$	$\langle 0.0614, -0.9386\rangle$
Oceania	$\langle 0.0113, -0.9887\rangle$	$\langle 0.0124, -0.9876\rangle$	$\langle 0.0046, -0.9954\rangle$	$\langle 0.0030, -0.9970\rangle$
South America	$\langle 0.0509, -0.9491\rangle$	$\langle 0.0340, -0.9660\rangle$	$\langle 0.0263, -0.9737\rangle$	$\langle 0.0467, -0.9533\rangle$
Timor	$\langle 0.0000, -1.0000\rangle$	$\langle 0.0000, -1.0000\rangle$	$\langle 0.0000, -1.0000\rangle$	$\langle 0.0000, -1.0000\rangle$

TABLE 4. Dataset for air pollution in $\mathcal{B}^{\mathcal{F}}$-framework for criteria C_5, C_6 and C_7.

Region	NMVOC	BC	NH_3
Africa	$\langle 0.1611, -0.8389\rangle$	$\langle 0.1782, -0.8218\rangle$	$\langle 0.1193, -0.8807\rangle$
Asia	$\langle 0.3935, -0.6065\rangle$	$\langle 0.3935, -0.6065\rangle$	$\langle 0.3935, -0.6065\rangle$
Europe	$\langle 0.0769, -0.9231\rangle$	$\langle 0.0661, -0.9339\rangle$	$\langle 0.0863, -0.9137\rangle$
North America	$\langle 0.0972, -0.9028\rangle$	$\langle 0.0444, -0.9556\rangle$	$\langle 0.0933, -0.9067\rangle$
Oceania	$\langle 0.0074, -0.9926\rangle$	$\langle 0.0057, -0.9943\rangle$	$\langle 0.0128, -0.9872\rangle$
South America	$\langle 0.0457, -0.9543\rangle$	$\langle 0.0467, -0.9533\rangle$	$\langle 0.0795, -0.9205\rangle$
Timor	$\langle 0.0000, -1.0000\rangle$	$\langle 0.0000, -1.0000\rangle$	$\langle 0.0000, -1.0000\rangle$

TABLE 5. Cluster analysis by using proposed accuracy measure.

Class	Confidence level	Clusters
I	$0 < \Psi \leq 0.2590$	$\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$
II	$0.2590 < \Psi \leq 0.7074$	$\{B_1, B_3, B_4, B_5, B_6, B_7\}, \{B_2\}$
III	$0.7074 < \Psi \leq 0.8317$	$\{B_3, B_4, B_5, B_6, B_7\}, \{B_1\}, \{B_2\}$
IV	$0.8317 < \Psi \leq 0.8407$	$\{B_4, B_5, B_6, B_7\}, \{B_1\}, \{B_2\}, \{B_3\}$
V	$0.8407 < \Psi \leq 0.8946$	$\{B_5, B_6, B_7\}, \{B_1\}, \{B_2\}, \{B_3\}, \{B_4\}$
VI	$0.8946 < \Psi \leq 0.9816$	$\{B_5, B_7\}, \{B_1\}, \{B_2\}, \{B_3\}, \{B_4\}, \{B_6\}$
VII	$0.9816 < \Psi \leq 1$	$\{B_1\}, \{B_2\}, \{B_3\}, \{B_4\}, \{B_5\}, \{B_6\}, \{B_7\}$

After successive compositions, we have $D^2 = D^4$. Thus, the resultant matrix D^2 is the equivalent accuracy matrix.

Step 4. Find the values of confidence levels and make clusters from different $\boldsymbol{\Psi}$*-cutting matrices.* For different values of $\Psi \in [0, 1]$, generate the Ψ-cutting matrix M_Ψ by using Definition 5.5. Different values of Ψ in the given example are $\{0.2590, 0.7074, 0.8317, 0.8407, 0.8946, 0.9816, 1\}$. For these values of Ψ, different Ψ-cutting matrices are formed. Two given alternatives B_ω and B_τ are from the same cluster if the ωth column or row is equal to the τth column or row in these Ψ-cutting matrices. In the same way, we may group all the other alternatives to form various clusters. Table 5 shows different clusters.

Table 5 examined that the suggested accuracy measure may effectively address the clustering issues. With the new approach, we get clusters. However, we can also use the proposed similarity measure in the above algorithm, then we get the output in Table 6. With the similarity measure approach, we get clusters. Hence, the new approach offers high clustering accuracy consequently.

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TABLE 6. Cluster analysis by using a proposed similarity measure.

Class	Confidence level	Clusters
I	$0 < \Psi \leq 0.4310$	$\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$
II	$0.4310 < \Psi \leq 0.8466$	$\{B_1, B_3, B_4, B_5, B_6, B_7\}, \{B_2\}$
III	$0.8466 < \Psi \leq 0.8949$	$\{B_3, B_4, B_5, B_6, B_7\}, \{B_1\}, \{B_2\}$
IV	$0.8949 < \Psi \leq 0.9344$	$\{B_3, B_4, B_6\}, \{B_5, B_7\}, \{B_1\}, \{B_2\}$
V	$0.9344 < \Psi \leq 0.9891$	$\{B_3, B_4\}, \{B_1\}, \{B_2\}, \{B_5\}, \{B_6\}, \{B_7\}$
VI	$0.9891 < \Psi \leq 1$	$\{B_1\}, \{B_2\}, \{B_3\}, \{B_4\}, \{B_5\}, \{B_6\}, \{B_7\}$

5.2. Medical diagnoses The process of identifying a problem from the symptoms of an individual and indications is known as a medical diagnosis. Many researchers have worked on this approach and have had good results using distance, correlation and similarity measures. Instead of using the provided measures, we use the proposed accuracy measure. Consider two examples given as follows.

EXAMPLE 5.6. Consider a collection U of eight patients, a collection V of symptoms and a collection W of diagnosis, where

$$U = \{B_\omega\}_{\omega=1}^{8},$$

$$V = \left\{\begin{matrix}\text{Weakness or Numbness, Swelling, Skin problem, Cognitive changes,}\\ \text{Appetite problems, Nausea, Dyspnea, Fever, Pain, Fatigue}\end{matrix}\right\},$$

$$W = \left\{\begin{matrix}\text{Genetic disorders, Endocrine disorders, Gastrointestinal disorders,}\\ \text{Cancer, Autoimmune disorders, Mental Health disorders,}\\ \text{Neurological disorders, Cardiovascular diseases, Chronic diseases,}\\ \text{Infectious diseases}\end{matrix}\right\}.$$

The definitions of the symptoms of the patients are shown in Table 7 and the definitions of the symptoms of the diagnosis are shown in Table 8.

The relation $U \to V \to W$ is used to represent the $\mathcal{B}^{\mathcal{F}}$-numbers in Tables 9, 10, 11 and 12.

To develop a trustworthy diagnosis, we assessed the accuracy of patients and diagnoses about symptoms. We follow this procedure for every diagnosis. If the accuracy measure is high, the patient has that specific disease. Table 13 shows the accuracy between patients and diagnoses.

According to Table 13, patients B_1, B_2 and B_7 have Autoimmune disorders, patient B_3 has Endocrine disorders, patient B_4 has Genetic disorders, patient B_5 has Neurological disorders, patient B_6 has Cardiovascular diseases and patient B_8 has Cancer.

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TABLE 7. The definitions of the symptoms of the patients.

Symptoms of the patients	Definitions
Weakness or Numbness	Limbs that are difficult to move, Tingling or Lack of feeling.
Swelling	Edema, or the retention of fluid in different bodily areas, is frequently observed in illnesses that impact the kidneys, heart or lymphatic system.
Skin problems	Skin anomalies such as rashes, lesions and discolouration.
Cognitive changes	Disorientation, Memory issues, Focus issues or Mental state disturbances.
Appetite problems	Hunger or Loss of appetite, which may be a sign of Hormonal or Metabolic abnormalities.
Nausea	Vomiting and feeling sick to your stomach can be brought on by several Systemic or Gastrointestinal disorders.
Dyspnea	Breathlessness or Difficulty breathing, which may be signs of Heart or Respiratory problems.
Fever	High body temperature that is frequently accompanied by perspiration or chills.
Pain	Numerous symptoms, including Headaches, Joint discomfort, Stomach pain, and so forth, may appear as a result.
Fatigue	Feeling worn out or drained, which frequently lasts even after getting enough rest.

EXAMPLE 5.7. Consider a collection U of five patients, a collection V of symptoms and a collection W of diagnosis, where

$$U = \{B_\omega\}_{\omega=1}^{5},$$
$$V = \left\{\begin{array}{c}\text{Irritation, Chest pain, Running nose, Cough, Sore throat,}\\ \text{Stomach pain, Headache, Fever}\end{array}\right\},$$
$$W = \{\text{Malaria, Heart problem, Stomach problem, Viral fever, Typhoid}\}.$$

The relation $U \to V \to W$ is used to represent the $\mathcal{B}^{\mathcal{F}}$-numbers in the Tables 14 and 15.

To develop a trustworthy diagnosis, we assessed the accuracy of patients and diagnoses about symptoms. We follow this procedure for every diagnosis. If the accuracy measure is high, the patient has that specific disease. Table 16 shows the accuracy between patients and diagnoses.

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TABLE 8. The definitions of the symptoms of the diagnosis.

Symptoms of the diagnosis	Definitions
Genetic disorders	Diseases include Down syndrome, Huntington's disease, Sickle cell anaemia and Cystic fibrosis that result from anomalies in a person's genetic composition.
Endocrine disorders	Hormone and Endocrine gland problems, including Pituitary disorders, Adrenal disorders (Addison's disease, Cushing's syndrome) and Thyroid abnormalities (Hypothyroidism, Hyperthyroidism).
Gastrointestinal disorders	Including conditions affecting the digestive tract, like GERD, Ulcerative colitis, Crohn's disease and Irritable bowel syndrome (IBS).
Cancer	Malignant tumours of numerous kinds impact diverse tissues and organs, including Leukaemia, Lung cancer, Breast cancer and Prostate cancer.
Autoimmune disorders	Illnesses like Lupus, Celiac disease, Type 1 diabetes and Rheumatoid arthritis when the immune system unintentionally targets healthy tissues.
Mental Health disorders	include drug abuse disorders, Bipolar disorder, Schizophrenia, Anxiety disorders and Depression.
Neurological disorders	Diseases like epilepsy, Parkinson's disease, Alzheimer's disease and multiple Sclerosis that impact the Brain, Spinal cord and Nerves.
Cardiovascular diseases	Cardiovascular and Blood vessel disorders, such as Stroke, Arrhythmias, Heart failure and Coronary artery disease.
Chronic diseases	Chronic obstructive pulmonary disease, Asthma, Hypertension and Diabetes are examples of long-term conditions that may need continuing care.
Infectious diseases	Infections caused by Viruses, Bacteria, Fungi or Parasites include Influenza, TB, HIV/AIDS, Malaria and COVID-19.

According to Table 16, patients B_1 has Viral fever, patients B_2 and B_5 have Malaria, patient B_3 has Heart problem, and patient B_4 has Typhoid. Hence, the proposed accuracy measure successfully solves the problems related to medical diagnosis.

5.3. Pattern detection The process of categorizing a random pattern into certain recognized patterns is known as pattern detection. Many academics have concentrated on the pattern detection approach and have employed similarity, correlation and

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TABLE 9. Patient's symptoms (1–5).

	Weakness	Swelling	Skin problem	Cognitive changes	Appetite problems
B_1	$\langle 0.5, -0.4\rangle$	$\langle 0.7, -0.6\rangle$	$\langle 0.3, -0.7\rangle$	$\langle 0.6, -0.2\rangle$	$\langle 0.0, -0.5\rangle$
B_2	$\langle 0.5, -0.0\rangle$	$\langle 0.1, -0.6\rangle$	$\langle 0.2, -0.9\rangle$	$\langle 0.6, -0.1\rangle$	$\langle 0.7, -0.5\rangle$
B_3	$\langle 0.0, -0.6\rangle$	$\langle 0.0, -0.6\rangle$	$\langle 0.8, -0.2\rangle$	$\langle 0.5, -0.5\rangle$	$\langle 0.5, -0.8\rangle$
B_4	$\langle 0.5, -0.0\rangle$	$\langle 0.8, -0.0\rangle$	$\langle 0.1, -0.9\rangle$	$\langle 0.0, -0.4\rangle$	$\langle 0.0, -0.6\rangle$
B_5	$\langle 0.7, -0.1\rangle$	$\langle 0.9, -0.7\rangle$	$\langle 0.3, -0.7\rangle$	$\langle 0.3, -0.8\rangle$	$\langle 0.4, -0.1\rangle$
B_6	$\langle 0.6, -0.7\rangle$	$\langle 0.3, -0.1\rangle$	$\langle 0.4, -0.0\rangle$	$\langle 0.8, -0.0\rangle$	$\langle 0.0, -0.0\rangle$
B_7	$\langle 0.5, -0.0\rangle$	$\langle 0.1, -0.1\rangle$	$\langle 0.0, -0.7\rangle$	$\langle 0.5, -0.7\rangle$	$\langle 0.2, -0.8\rangle$
B_8	$\langle 0.9, -0.0\rangle$	$\langle 0.1, -0.8\rangle$	$\langle 0.1, -0.2\rangle$	$\langle 0.3, -0.4\rangle$	$\langle 0.4, -0.7\rangle$

TABLE 10. Patient's symptoms (6–10).

	Nausea	Dyspnea	Fever	Pain	Fatigue
B_1	$\langle 0.6, -0.8\rangle$	$\langle 0.8, -0.0\rangle$	$\langle 0.2, -0.2\rangle$	$\langle 0.6, -0.4\rangle$	$\langle 0.8, -0.6\rangle$
B_2	$\langle 0.9, -0.8\rangle$	$\langle 0.5, -0.5\rangle$	$\langle 0.7, -0.2\rangle$	$\langle 0.9, -0.4\rangle$	$\langle 0.0, -0.6\rangle$
B_3	$\langle 0.1, -0.0\rangle$	$\langle 0.4, -0.6\rangle$	$\langle 0.6, -0.3\rangle$	$\langle 0.9, -0.1\rangle$	$\langle 0.3, -0.1\rangle$
B_4	$\langle 0.6, -0.4\rangle$	$\langle 0.5, -0.1\rangle$	$\langle 0.1, -0.3\rangle$	$\langle 0.2, -0.5\rangle$	$\langle 0.4, -0.3\rangle$
B_5	$\langle 0.5, -0.0\rangle$	$\langle 0.3, -0.6\rangle$	$\langle 0.7, -0.6\rangle$	$\langle 0.4, -0.6\rangle$	$\langle 0.5, -0.4\rangle$
B_6	$\langle 0.1, -0.7\rangle$	$\langle 0.1, -0.5\rangle$	$\langle 0.1, -0.3\rangle$	$\langle 0.4, -0.8\rangle$	$\langle 0.6, -0.9\rangle$
B_7	$\langle 0.8, -0.3\rangle$	$\langle 0.5, -0.3\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.7, -0.1\rangle$	$\langle 0.8, -0.5\rangle$
B_8	$\langle 0.2, -0.2\rangle$	$\langle 0.4, -0.8\rangle$	$\langle 0.8, -0.8\rangle$	$\langle 0.9, -0.8\rangle$	$\langle 0.6, -0.4\rangle$

distance measurements to get effective results. We use the derived measures to solve pattern detection issues. For this, take two examples as follows.

EXAMPLE 5.8. Let $K = \{0, 9, 1, 8, 2, 7, 3, 6, 4, 5\}$ be a finite set containing ten elements. Take the following patterns in the form of $\mathcal{B}^{\mathcal{F}}$-sets as follows:

$$B_1 = \begin{Bmatrix} \langle 0, (0.44, -0.45)\rangle, & \langle 9, (0.74, -0.78)\rangle, & \langle 1, (0.22, -0.52)\rangle, \\ \langle 8, (0.56, -0.36)\rangle, & \langle 2, (0.36, -0.98)\rangle, & \langle 7, (0.45, -0.25)\rangle, \\ \langle 3, (0.54, -0.01)\rangle, & \langle 6, (0.69, -0.26)\rangle, & \langle 4, (0.78, -0.45)\rangle, \\ & \langle 5, (0.42, -0.78)\rangle & \end{Bmatrix},$$

$$B_2 = \begin{Bmatrix} \langle 0, (0.45, -0.45)\rangle, & \langle 9, (0.58, -0.78)\rangle, & \langle 1, (0.20, -0.56)\rangle, \\ \langle 8, (0.32, -0.96)\rangle, & \langle 2, (0.56, -0.82)\rangle, & \langle 7, (0.56, -0.01)\rangle, \\ \langle 3, (0.87, -0.36)\rangle, & \langle 6, (0.45, -0.87)\rangle, & \langle 4, (0.42, -0.45)\rangle, \\ & \langle 5, (0.10, -0.15)\rangle & \end{Bmatrix},$$

$$B_3 = \begin{Bmatrix} \langle 0, (0.56, -0.54)\rangle, & \langle 9, (0.87, -0.69)\rangle, & \langle 1, (0.10, -0.87)\rangle, \\ \langle 8, (0.29, -0.75)\rangle, & \langle 2, (0.92, -0.35)\rangle, & \langle 7, (0.45, -0.60)\rangle, \\ \langle 3, (0.73, -0.20)\rangle, & \langle 6, (0.38, -0.87)\rangle, & \langle 4, (0.87, -0.52)\rangle, \\ & \langle 5, (0.45, -0.36)\rangle & \end{Bmatrix},$$

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TABLE 11. Diagnosis's symptoms (1–5).

	Weakness	Swelling	Skin problem	Cognitive changes	Appetite problems
Genetic disorders	⟨0.5, −0.5⟩	⟨0.7, −0.6⟩	⟨0.0, −0.9⟩	⟨0.1, −0.0⟩	⟨0.5, −0.1⟩
Endocrine disorders	⟨0.0, −0.6⟩	⟨0.7, −0.1⟩	⟨0.4, −0.0⟩	⟨0.3, −0.0⟩	⟨0.2, −0.9⟩
Gastrointestinal disorders	⟨0.5, −0.5⟩	⟨0.0, −0.0⟩	⟨0.9, −0.7⟩	⟨0.1, −0.9⟩	⟨0.2, −0.1⟩
Cancer	⟨0.5, −0.0⟩	⟨0.9, −0.1⟩	⟨0.0, −0.1⟩	⟨0.1, −0.2⟩	⟨0.6, −0.8⟩
Autoimmune disorders	⟨0.6, −0.0⟩	⟨0.5, −0.3⟩	⟨0.1, −0.7⟩	⟨0.7, −0.1⟩	⟨0.3, −0.6⟩
Mental Health disorders	⟨0.5, −0.2⟩	⟨0.4, −0.0⟩	⟨0.7, −0.1⟩	⟨0.9, −0.7⟩	⟨0.5, −0.8⟩
Neurological disorders	⟨0.4, −0.2⟩	⟨0.5, −0.7⟩	⟨0.1, −0.8⟩	⟨0.3, −0.9⟩	⟨0.5, −0.2⟩
Cardiovascular diseases	⟨0.0, −0.0⟩	⟨0.1, −0.2⟩	⟨0.8, −0.8⟩	⟨0.4, −0.7⟩	⟨0.2, −0.4⟩
Chronic diseases	⟨0.6, −0.3⟩	⟨0.9, −0.7⟩	⟨0.0, −0.1⟩	⟨0.0, −0.6⟩	⟨0.6, −0.8⟩
Infectious diseases	⟨0.9, −0.5⟩	⟨0.1, −0.7⟩	⟨0.5, −0.4⟩	⟨0.3, −0.7⟩	⟨0.9, −0.3⟩

TABLE 12. Diagnosis's symptoms (6–10).

	Nausea	Dyspnea	Fever	Pain	Fatigue
Genetic disorders	⟨0.2, −0.1⟩	⟨0.5, −0.2⟩	⟨0.4, −0.0⟩	⟨0.3, −0.7⟩	⟨0.2, −0.3⟩
Endocrine disorders	⟨0.2, −0.1⟩	⟨0.3, −0.6⟩	⟨0.4, −0.2⟩	⟨0.5, −0.4⟩	⟨0.5, −0.6⟩
Gastrointestinal disorders	⟨0.1, −0.5⟩	⟨0.2, −0.3⟩	⟨0.3, −0.8⟩	⟨0.4, −0.9⟩	⟨0.3, −0.3⟩
Cancer	⟨0.2, −0.4⟩	⟨0.8, −0.7⟩	⟨0.5, −0.9⟩	⟨0.7, −0.3⟩	⟨0.4, −0.6⟩
Autoimmune disorders	⟨0.6, −0.9⟩	⟨0.9, −0.0⟩	⟨0.2, −0.0⟩	⟨0.5, −0.6⟩	⟨0.7, −0.4⟩
Mental Health disorders	⟨0.3, −0.4⟩	⟨0.0, −0.2⟩	⟨0.1, −0.3⟩	⟨0.5, −0.6⟩	⟨0.4, −0.5⟩
Neurological disorders	⟨0.7, −0.0⟩	⟨0.8, −0.3⟩	⟨0.2, −0.5⟩	⟨0.0, −0.6⟩	⟨0.1, −0.4⟩
Cardiovascular diseases	⟨0.3, −0.5⟩	⟨0.0, −0.3⟩	⟨0.1, −0.6⟩	⟨0.5, −0.8⟩	⟨0.2, −0.9⟩
Chronic diseases	⟨0.3, −0.4⟩	⟨0.7, −0.2⟩	⟨0.1, −0.1⟩	⟨0.6, −0.2⟩	⟨0.4, −0.8⟩
Infectious diseases	⟨0.2, −0.6⟩	⟨0.0, −0.9⟩	⟨0.3, −0.3⟩	⟨0.3, −0.7⟩	⟨0.8, −0.0⟩

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TABLE 13. Symptoms of the diagnosis by using proposed accuracy measure.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
Genetic disorders	0.2592	0.3159	0.3062	**0.3956**	0.2603	0.3837	0.3051	0.3251
Endocrine disorders	0.2426	0.2764	**0.3569**	0.3600	0.2065	0.4053	0.3233	0.3261
Gastrointestinal disorders	0.2162	0.2741	0.3230	0.3573	0.2390	0.3965	0.2950	0.3270
Cancer	0.2387	0.2966	0.3157	0.3580	0.2337	0.3646	0.3137	**0.3630**
Autoimmune disorders	**0.3032**	**0.3412**	0.2833	0.3916	0.2051	0.3973	**0.3570**	0.3116
Mental Health disorders	0.2288	0.2887	0.3431	0.3368	0.2087	0.4063	0.3325	0.3183
Neurological disorders	0.2379	0.2963	0.2899	0.3847	**0.2684**	0.3366	0.3254	0.3126
Cardiovascular diseases	0.2361	0.3145	0.3218	0.3663	0.2308	**0.4079**	0.3279	0.3131
Chronic diseases	0.2673	0.2993	0.3124	0.3646	0.2336	0.3638	0.3303	0.3291
Infectious diseases	0.1946	0.2898	0.3178	0.3106	0.2428	0.3802	0.2977	0.3593

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TABLE 14. Patient's symptoms.

	Irritation	Chest pain	Running nose	Cough	Sore throat	Stomach pain	Headache	Fever
B_1	$\langle 0.5, -0.6\rangle$	$\langle 0.3, -0.0\rangle$	$\langle 0.8, -0.2\rangle$	$\langle 0.0, -0.1\rangle$	$\langle 0.1, -0.6\rangle$	$\langle 0.5, -0.4\rangle$	$\langle 0.3, -0.9\rangle$	$\langle 0.7, -0.0\rangle$
B_2	$\langle 0.0, -0.6\rangle$	$\langle 0.0, -0.0\rangle$	$\langle 0.2, -0.2\rangle$	$\langle 0.9, -0.8\rangle$	$\langle 0.6, -0.9\rangle$	$\langle 0.8, -0.3\rangle$	$\langle 0.7, -0.5\rangle$	$\langle 0.3, -0.7\rangle$
B_3	$\langle 0.6, -0.6\rangle$	$\langle 0.9, -0.9\rangle$	$\langle 0.0, -0.0\rangle$	$\langle 0.1, -0.2\rangle$	$\langle 0.5, -0.6\rangle$	$\langle 0.3, -0.3\rangle$	$\langle 0.2, -0.5\rangle$	$\langle 0.7, -0.8\rangle$
B_4	$\langle 0.7, -0.5\rangle$	$\langle 0.9, -0.8\rangle$	$\langle 0.2, -0.2\rangle$	$\langle 0.5, -0.7\rangle$	$\langle 0.4, -0.4\rangle$	$\langle 0.6, -0.6\rangle$	$\langle 0.8, -0.7\rangle$	$\langle 0.6, -0.0\rangle$
B_5	$\langle 0.7, -0.5\rangle$	$\langle 0.0, -0.8\rangle$	$\langle 0.2, -0.1\rangle$	$\langle 0.5, -0.9\rangle$	$\langle 0.7, -0.0\rangle$	$\langle 0.2, -0.3\rangle$	$\langle 0.5, -0.6\rangle$	$\langle 0.4, -0.4\rangle$

TABLE 15. Diagnosis's symptoms.

(a) Diagnosis's symptoms (1–4).

	Irritation	Chest pain	Running nose	Cough
Malaria	⟨0.5, −0.7⟩	⟨0.1, −0.9⟩	⟨0.7, −0.1⟩	⟨0.0, −0.8⟩
Heart problem	⟨0.5, −0.5⟩	⟨0.7, −0.9⟩	⟨0.1, −0.0⟩	⟨0.2, −0.1⟩
Stomach problem	⟨0.5, −0.0⟩	⟨0.9, −0.9⟩	⟨0.0, −0.7⟩	⟨0.1, −0.3⟩
Viral fever	⟨0.6, −0.8⟩	⟨0.8, −0.0⟩	⟨0.1, −0.1⟩	⟨0.3, −0.5⟩
Typhoid	⟨0.6, −0.6⟩	⟨0.9, −0.9⟩	⟨0.0, −0.1⟩	⟨0.1, −0.4⟩

(b) Diagnosis's symptoms (5–8).

	Sore throat	Stomach pain	Headache	Fever
Malaria	⟨0.9, −0.7⟩	⟨0.2, −0.0⟩	⟨0.6, −0.9⟩	⟨0.9, −0.4⟩
Heart problem	⟨0.3, −0.6⟩	⟨0.1, −0.3⟩	⟨0.2, −0.5⟩	⟨0.0, −0.9⟩
Stomach problem	⟨0.6, −0.5⟩	⟨0.3, −0.8⟩	⟨0.6, −0.6⟩	⟨0.4, −0.1⟩
Viral fever	⟨0.6, −0.2⟩	⟨0.3, −0.7⟩	⟨0.8, −0.9⟩	⟨0.9, −0.5⟩
Typhoid	⟨0.6, −0.2⟩	⟨0.2, −0.9⟩	⟨0.9, −0.0⟩	⟨0.5, −0.7⟩

TABLE 16. Symptoms of the diagnosis.

	B_1	B_2	B_3	B_4	B_5
Malaria	0.3615	**0.3521**	0.3451	0.2495	**0.3399**
Heart problem	0.3272	0.3349	**0.3964**	0.2283	0.3015
Stomach problem	0.3186	0.2889	0.3390	0.2698	0.2795
Viral fever	**0.3645**	0.3507	0.3360	0.2674	0.2945
Typhoid	0.2901	0.3080	0.3737	**0.2719**	0.3052

$$B_4 = \begin{Bmatrix} \langle 0,(0.56,-0.03)\rangle, & \langle 9,(0.87,-0.68)\rangle, & \langle 1,(0.20,-0.76)\rangle, \\ \langle 8,(0.35,-0.34)\rangle, & \langle 2,(0.69,-0.06)\rangle, & \langle 7,(0.84,-0.20)\rangle, \\ \langle 3,(0.56,-0.34)\rangle, & \langle 6,(0.54,-0.56)\rangle, & \langle 4,(0.20,-0.54)\rangle, \\ & \langle 5,(0.35,-0.08)\rangle & \end{Bmatrix},$$

$$B_5 = \begin{Bmatrix} \langle 0,(0.56,-0.20)\rangle, & \langle 9,(0.31,-0.36)\rangle, & \langle 1,(0.78,-0.87)\rangle, \\ \langle 8,(0.96,-0.92)\rangle, & \langle 2,(0.54,-0.57)\rangle, & \langle 7,(0.52,-0.63)\rangle, \\ \langle 3,(0.10,-0.49)\rangle, & \langle 6,(0.65,-0.28)\rangle, & \langle 4,(0.85,-0.62)\rangle, \\ & \langle 5,(0.65,-0.10)\rangle & \end{Bmatrix},$$

$$B_6 = \begin{Bmatrix} \langle 0,(0.69,-0.54)\rangle, & \langle 9,(0.89,-0.63)\rangle, & \langle 1,(0.65,-0.25)\rangle, \\ \langle 8,(0.35,-0.78)\rangle, & \langle 3,(0.38,-0.96)\rangle, & \langle 7,(0.64,-0.35)\rangle, \\ \langle 4,(0.02,-0.64)\rangle, & \langle 6,(0.65,-0.10)\rangle, & \langle 4,(0.45,-0.36)\rangle, \\ & \langle 5,(0.65,-0.54)\rangle & \end{Bmatrix},$$

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TABLE 17. Pattern detection using proposed measures and previously known measures.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
$AM_{\mathcal{B}}$	0.2207	0.2506	0.2710	0.2650	0.2459	0.2028	0.2296	0.2196	0.1900	0.2396
$SM_{\mathcal{B}}$	0.9576	0.9808	0.9929	0.9707	0.9808	0.9753	0.9718	0.9513	0.9614	0.9562
$DM_{\mathcal{B}}$	0.0424	0.0192	0.0071	0.0293	0.0192	0.0247	0.0282	0.0487	0.0386	0.0438
NH_d	0.4770	0.5670	0.4260	0.5420	0.6270	0.6350	0.5330	0.4520	0.6140	0.7080
NE_e	0.4267	0.5017	0.3750	0.4744	0.5141	0.5312	0.4259	0.3913	0.5221	0.5865

$$B_7 = \begin{Bmatrix} \langle 0,(0.45,-0.25)\rangle, & \langle 9,(0.63,-0.63)\rangle, & \langle 1,(0.57,-0.30)\rangle, \\ \langle 8,(0.89,-0.12)\rangle, & \langle 2,(0.65,-0.45)\rangle, & \langle 7,(0.30,-0.89)\rangle, \\ \langle 3,(0.39,-0.65)\rangle, & \langle 6,(0.67,-0.45)\rangle, & \langle 4,(0.85,-0.22)\rangle, \\ & \langle 5,(0.65,-0.15)\rangle & \end{Bmatrix},$$

$$B_8 = \begin{Bmatrix} \langle 0,(0.20,-0.45)\rangle, & \langle 9,(0.32,-0.63)\rangle, & \langle 1,(0.36,-0.56)\rangle, \\ \langle 8,(0.89,-0.87)\rangle, & \langle 2,(0.72,-0.10)\rangle, & \langle 7,(0.83,-0.24)\rangle, \\ \langle 3,(0.55,-0.56)\rangle, & \langle 6,(0.24,-0.52)\rangle, & \langle 4,(0.55,-0.45)\rangle, \\ & \langle 5,(0.23,-0.56)\rangle & \end{Bmatrix},$$

$$B_9 = \begin{Bmatrix} \langle 0,(0.26,-0.56)\rangle, & \langle 9,(0.35,-0.96)\rangle, & \langle 1,(0.40,-0.13)\rangle, \\ \langle 8,(0.50,-0.54)\rangle, & \langle 2,(0.65,-0.56)\rangle, & \langle 7,(0.45,-0.50)\rangle, \\ \langle 3,(0.52,-0.67)\rangle, & \langle 6,(0.20,-0.98)\rangle, & \langle 4,(0.36,-0.85)\rangle, \\ & \langle 5,(0.69,-0.20)\rangle & \end{Bmatrix},$$

$$B_{10} = \begin{Bmatrix} \langle 0,(0.63,-0.69)\rangle, & \langle 9,(0.21,-0.63)\rangle, & \langle 1,(0.56,-0.32)\rangle, \\ \langle 8,(0.30,-0.45)\rangle, & \langle 2,(0.36,-0.65)\rangle, & \langle 7,(0.94,-0.58)\rangle, \\ \langle 3,(0.85,-0.99)\rangle, & \langle 6,(0.66,-0.95)\rangle, & \langle 4,(0.84,-0.64)\rangle, \\ & \langle 5,(0.09,-0.23)\rangle & \end{Bmatrix}.$$

Let an unknown pattern in the form of the $\mathcal{B}^{\mathcal{F}}$-set be given by

$$B = \begin{Bmatrix} \langle 0,(0.65,-0.56)\rangle, & \langle 9,(0.63,-0.89)\rangle, & \langle 1,(0.23,-0.78)\rangle, \\ \langle 8,(0.78,-0.54)\rangle, & \langle 2,(0.95,-0.22)\rangle, & \langle 7,(0.12,-0.65)\rangle, \\ \langle 3,(0.65,-0.45)\rangle, & \langle 6,(0.45,-0.23)\rangle, & \langle 4,(0.56,-0.25)\rangle, \\ & \langle 5,(0.21,-0.75)\rangle & \end{Bmatrix}.$$

Using the derived measures, we identify which pattern fits within B the best. Using the derived measures and some previously known measures yields the results in Table 17.

From Table 17, the pattern B fits best in the pattern B_3 by using the derived measures. Also, by using the distance measure NH_d in (2.1) and distance measure NH_e in (2.2), the pattern B fits best in the pattern B_3.

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EXAMPLE 5.9. Let $K\{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75\}$ be a finite set containing fifteen elements. Take the following patterns in the form of $\mathcal{B}^{\mathcal{F}}$-sets:

$$B_1 = \begin{Bmatrix} \langle 5,(0.03,-0.45)\rangle, \langle 10,(0.87,-0.63)\rangle, \langle 15,(0.45,-0.21)\rangle, \\ \langle 20,(0.96,-0.02)\rangle, \langle 25,(0.30,-0.89)\rangle, \langle 30,(0.57,-0.78)\rangle, \\ \langle 35,(0.89,-0.54)\rangle, \langle 40,(0.50,-0.62)\rangle, \langle 45,(0.36,-0.36)\rangle, \\ \langle 50,(0.54,-0.54)\rangle, \langle 55,(0.12,-0.87)\rangle, \langle 60,(0.01,-0.80)\rangle, \\ \langle 65,(0.23,-0.25)\rangle, \langle 70,(0.56,-0.60)\rangle, \langle 75,(0.58,-0.89)\rangle \end{Bmatrix},$$

$$B_2 = \begin{Bmatrix} \langle 5,(0.56,-0.15)\rangle, \langle 10,(0.45,-0.96)\rangle, \langle 15,(0.20,-0.35)\rangle, \\ \langle 20,(0.36,-0.78)\rangle, \langle 25,(0.56,-0.95)\rangle, \langle 30,(0.30,-0.32)\rangle, \\ \langle 35,(0.29,-0.06)\rangle, \langle 40,(0.78,-0.98)\rangle, \langle 45,(0.52,-0.70)\rangle, \\ \langle 50,(0.36,-0.29)\rangle, \langle 55,(0.58,-0.52)\rangle, \langle 60,(0.19,-0.01)\rangle, \\ \langle 65,(0.20,-0.23)\rangle, \langle 70,(0.15,-0.56)\rangle, \langle 75,(0.05,-0.58)\rangle \end{Bmatrix},$$

$$B_3 = \begin{Bmatrix} \langle 5,(0.21,-0.56)\rangle, \langle 10,(0.34,-0.82)\rangle, \langle 15,(0.65,-0.03)\rangle, \\ \langle 20,(0.42,-0.68)\rangle, \langle 25,(0.03,-0.56)\rangle, \langle 30,(0.22,-0.45)\rangle, \\ \langle 35,(0.60,-0.21)\rangle, \langle 40,(0.58,-0.65)\rangle, \langle 45,(0.30,-0.78)\rangle, \\ \langle 50,(0.15,-0.52)\rangle, \langle 55,(0.45,-0.02)\rangle, \langle 60,(0.21,-0.15)\rangle, \\ \langle 65,(0.65,-0.45)\rangle, \langle 70,(0.20,-0.56)\rangle, \langle 75,(0.20,-0.62)\rangle \end{Bmatrix},$$

$$B_4 = \begin{Bmatrix} \langle 5,(0.56,-0.90)\rangle, \langle 10,(0.59,-0.36)\rangle, \langle 15,(0.57,-0.98)\rangle, \\ \langle 20,(0.56,-0.97)\rangle, \langle 25,(0.69,-0.54)\rangle, \langle 30,(0.54,-0.96)\rangle, \\ \langle 35,(0.82,-0.54)\rangle, \langle 40,(0.92,-0.58)\rangle, \langle 45,(0.91,-0.78)\rangle, \\ \langle 50,(0.36,-0.52)\rangle, \langle 55,(0.54,-0.72)\rangle, \langle 60,(0.58,-0.54)\rangle, \\ \langle 65,(0.95,-0.65)\rangle, \langle 70,(0.80,-0.85)\rangle, \langle 75,(0.92,-0.78)\rangle \end{Bmatrix},$$

$$B_5 = \begin{Bmatrix} \langle 5,(0.21,-0.30)\rangle, \langle 10,(0.53,-0.56)\rangle, \langle 15,(0.25,-0.12)\rangle, \\ \langle 20,(0.65,-0.00)\rangle, \langle 25,(0.45,-0.10)\rangle, \langle 30,(0.21,-0.59)\rangle, \\ \langle 35,(0.50,-0.54)\rangle, \langle 40,(0.56,-0.85)\rangle, \langle 45,(0.54,-0.30)\rangle, \\ \langle 50,(0.78,-0.23)\rangle, \langle 55,(0.52,-0.65)\rangle, \langle 60,(0.53,-0.02)\rangle, \\ \langle 65,(0.25,-0.12)\rangle, \langle 70,(0.51,-0.78)\rangle, \langle 75,(0.56,-0.51)\rangle \end{Bmatrix},$$

$$B_6 = \begin{Bmatrix} \langle 5,(0.25,-0.52)\rangle, \langle 10,(0.30,-0.56)\rangle, \langle 15,(0.50,-0.84)\rangle, \\ \langle 20,(0.04,-0.65)\rangle, \langle 25,(0.54,-0.32)\rangle, \langle 30,(0.50,-0.02)\rangle, \\ \langle 35,(0.27,-0.54)\rangle, \langle 40,(0.79,-0.87)\rangle, \langle 45,(0.65,-0.49)\rangle, \\ \langle 50,(0.65,-0.65)\rangle, \langle 55,(0.02,-0.20)\rangle, \langle 60,(0.33,-0.32)\rangle, \\ \langle 65,(0.56,-0.45)\rangle, \langle 70,(0.98,-0.45)\rangle, \langle 75,(0.65,-0.85)\rangle \end{Bmatrix}.$$

Let an unknown pattern in the form of the $\mathcal{B}^{\mathcal{F}}$-set be given by

$$B = \begin{Bmatrix} \langle 5,(0.56,-0.20)\rangle, \langle 10,(0.58,-0.36)\rangle, \langle 15,(0.95,-0.90)\rangle, \\ \langle 20,(0.63,-0.87)\rangle, \langle 25,(0.45,-0.45)\rangle, \langle 30,(0.78,-0.65)\rangle, \\ \langle 35,(0.52,-0.20)\rangle, \langle 40,(0.03,-0.30)\rangle, \langle 45,(0.65,-0.56)\rangle, \\ \langle 50,(0.65,-0.87)\rangle, \langle 55,(0.78,-0.89)\rangle, \langle 60,(0.96,-0.66)\rangle, \\ \langle 65,(0.39,-0.20)\rangle, \langle 70,(0.90,-0.32)\rangle, \langle 75,(0.54,-0.15)\rangle \end{Bmatrix}.$$

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TABLE 18. Pattern detection using proposed measures and previously known measures.

	B_1	B_2	B_3	B_4	B_5	B_6
$AM_{\mathcal{B}}$	0.2588	0.2560	0.1972	0.2729	0.1988	0.2120
$SM_{\mathcal{B}}$	0.9880	0.9817	0.9515	0.9937	0.9311	0.9427
$DM_{\mathcal{B}}$	0.0120	0.0183	0.0485	0.0063	0.0689	0.0573
NH_d	0.6967	0.7280	0.7600	0.5807	0.6433	0.6420
NE_e	0.5936	0.6194	0.6147	0.5078	0.5731	0.5619

Using the derived measures, we identify which pattern fits within B the best. Using the derived measures and some previously known measures yields the results in Table 18.

From Table 18, the pattern B fits best in the pattern B_4 by using the derived measures. Also, by using the distance measure NH_d in (2.1) and distance measure NH_e in (2.2), the pattern B fits best in the pattern B_4. Both these examples of pattern detection imply the efficacy of the proposed measures.

6. Conclusion

The ϕ-norm entropy concept is used to propose a $\mathcal{B}^{\mathcal{F}}$ knowledge-measure in this manuscript. The proposed measure's basic properties are investigated. By using numerical illustrations, the utility of the measure is demonstrated. The proposed measure is used to create and validate an accuracy measure. Some properties are discussed in the $\mathcal{B}^{\mathcal{F}}$-framework. Additionally, similarity and dissimilarity measures in the $\mathcal{B}^{\mathcal{F}}$-framework are derived from the proposed measure and validated. Using the proposed measures, the issues with pattern detection, cluster analysis and medical diagnosis are resolved. In the current work, a novel $\mathcal{B}^{\mathcal{F}}$-clustering approach is developed employing the proposed accuracy measure. The official website provides real data on air pollution in different regions in the year 2022. The proposed approach is then used on this data to create clusters. Additionally, two examples of pattern detection as well as medical diagnosis issues are addressed by using the derived measures. In the future, the intended study may be expanded upon in several ways.

(1) The proposed $\mathcal{B}^{\mathcal{F}}$ knowledge-measure may be extended to the Bipolar complex fuzzy set, $\mathcal{B}^{\mathcal{F}}$-Soft set, n-Polar fuzzy sets and so on.
(2) The resulting $\mathcal{B}^{\mathcal{F}}$-accuracy measure may also be used to create a decision-making approach that we can use to resolve decision-making problems in the $\mathcal{B}^{\mathcal{F}}$-framework.
(3) In the future, we may use the derived measures to the issue of green supplier selection and image processing.

We may also use the proposed methods to solve several real-world problems. Furthermore, we will make a concerted effort in the next years to rectify any deficiencies in the proposed study. Thus, the proposed approach has several practical applications.

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