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Resolving zero-dimensional singularities in generalized manifolds

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1. A *generalized m-manifold* is a euclidean neighbourhood retract X that is a homology m-manifold, i.e. for which $H_*(X, X-x; \mathbf{Z}) = H_*(\mathbf{R}^m, \mathbf{R}^m - 0; \mathbf{Z})$ for each point x of X. In a generalized m-manifold X, a point p is called *singular*, or a *singularity of* X, if p has no neighbourhood homeomorphic to $\mathbf{R}^m$. We shall denote the set of singular points of X by $S(X)$.

An *m-resolution* of a space X is a pair (M^m, f), where M^m is a topological m-manifold (without boundary) and f is a proper, cell-like mapping of M^m onto X.

If a finite-dimensional space X admits an m-resolution, or if $X \times \mathbf{R}^k$ is an $(m+k)$-manifold for some k, then X is a generalized m-manifold. In fact, if X is finite-dimensional and $X \times \mathbf{R}^k$ is the image of a proper, cell-like map on a generalized $(m+k)$-manifold, then X is a generalized m-manifold: see § 3·3 of (19). No generalized manifold is known to fail to admit a resolution or fail to be a factor of euclidean space. One of the main conjectures of desingularization asserts that these two conditions are equivalent (cf. § 13 of (19)): A generalized manifold X admits a resolution if and only if $X \times \mathbf{R}$ is a manifold.

This conjecture is true in a special case, which we now describe. Let X be a generalized m-manifold with $\dim S(X) = 0$, $m \geqslant 5$. If (M^m, f) is a resolution of X, then $f^{-1}(x)$ is cellular in M^m for all $x \notin S(X)$. (See § 5·2 of (19).) Over the complement of $S(X)$, the Siebenmann approximation theorem (24) allows us to replace f with a new cell-like map $f_1 : M \to X$ for which $f_1^{-1}(x)$ is a point for each $x \notin S(X)$. Now the main result of R. D. Edwards' work on the double suspension problem (14), (15) says that the decomposition $\{f_1^{-1}(x) \times t | x \in X,\ t \in \mathbf{R}\}$ of $M^m \times \mathbf{R}$ is shrinkable and hence that $X \times \mathbf{R}$ is homeomorphic to $M^m \times \mathbf{R}$ (2). Therefore, if X admits a resolution then $X \times \mathbf{R}$ is a manifold. The converse is proved by Bryant and Hollingsworth (7). Thus the conjecture is true when $\dim S(X) \leqslant 0$ and $m \geqslant 5$.

The purpose of the present paper is to prove existence results for the same class of spaces. Assuming that X is a generalized m-manifold, we prove:

THEOREM 1·1. *If* $m \geqslant 5$ *and* $\dim S(X) \leqslant 0$ *then* X *admits a resolution.*

THEOREM 1·2. *If* $m \geqslant 5$ *and* $\dim S(X) \leqslant 0$ *then* $X \times \mathbf{R}$ *is a manifold.*

We have already noted that (1·1) implies (1·2), so we concentrate on proving (1·1). This effort is successful when $m \geqslant 6$ and is completed (for isolated singularities) in section 2. In the case $m = 5$, a difficulty arises with the Kirby–Siebenmann obstruction,

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and the full strength of the equivalence of (1·1) and (1·2) must be invoked to get around the difficulty; this case (for isolated singularities) is completed in section 3. Finally, in section 4, we show how the general cases follow from the isolated singularity cases.

Our proof yields the stronger information (also obtainable via Siebenmann's approximation theorem) that the resolution may be chosen to be *conservative*, i.e. to be a homeomorphism over the complement of $S(X)$. The proof also yields uniqueness:

THEOREM 1·3. *Suppose* $m \geqslant 5$ *and* $\dim S(X) \leqslant 0$. *If* (M_1^m, f_1) *and* (M_2^m, f_2) *are resolutions of* X *then* M_1^m *and* M_2^m *are homeomorphic. Moreover, if* f_1 *and* f_2 *are conservative then the homeomorphism may be chosen to agree with* $f_2^{-1}f_1$ *over the complement of any pre-assigned neighbourhood of* $S(X)$.

Basic references for generalized manifolds are (32) and (3), (4). The recent papers of Cannon (8), (10) and Ancel and Cannon (1) represent major advances in the geometric theory of generalized manifolds. In particular, (8) contains the only other resolution theorem known for generalized manifolds. (A complete theory of PL resolutions of PL homology manifolds was given by Cohen and Sullivan (11), (12).) Cannon's special result on resolving the double suspension of homology 3-spheres (9) plays a major role in his solution to the double suspension problem (10). The main technical result of (10) is a criterion for a space with a resolution to be itself a manifold. Combining this criterion with our (1·1) above, Cannon shows that a generalized m-manifold X, $m \geqslant 5$, $\dim S(X) \leqslant 0$, *is* a manifold if pairs of singular 2-cells in X can be adjusted slightly so as to be disjoint. In partial answer to another question of (10), we show:

THEOREM 1·4. *Suppose* $m \geqslant 5$ *and* $\dim S(X) \leqslant 0$. *If* $S(X)$ *is* $1-\mathrm{LCC}$ *in* X *then* X *is a manifold.*

Proof. The $1-\mathrm{LCC}$ condition is the same as asserting X is $1-\mathrm{LC}$ mod $S(X)$; see (19). Let $f: M^m \to X$ be a conservative resolution. Since $S(X)$ is $1-\mathrm{LCC}$ in X, each end of $X - S(X)$ (hence $M - f^{-1}S(X)$) is simply connected. It follows that $f^{-1}S(X)$ satisfies McMillan's cellularity criterion (20), (25) in M^m and hence that $f^{-1}S(X)$ has arbitrarily small neighbourhoods whose components are m-cells: $f^{-1}S(X)$ is 'definable by cells'. Therefore the decomposition $\{f^{-1}(x) | x \in X\}$ is shrinkable, and X is homeomorphic to M. ∎

Theorem 1·1 can be used to structure certain 5-manifolds. Suppose K^4 is a finite complex with the homology of S^4 and W^5 is an open 5-manifold proper homotopy equivalent to $K \times \mathbf{R}$. Then the end-point compactification X of W is a generalized 5-manifold, so there exists a conservative resolution $f: M^5 \to X$. The homology sphere M^5 has a PL structure by (18), so $W \subset M^5$ does as well. If, in addition, K^4 is simply connected, then $M^5 \approx S^5$, and the two non-degenerate point-inverses of f are cellular in M^5 (20), so W is homeomorphic to $S^4 \times \mathbf{R}$.

2. In this section we will ultimately prove Theorem 1·1 in the case restricted by $m \geqslant 6$ and $S(X) =$ point. Thus we assume throughout section 2 that X is a generalized m-manifold and $S(X) = \{p\}$ for some point p. At this juncture we make no standing assumption about m.

Since X is a locally compact ANR, one gets

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PROPOSITION 2·1. *Given a neighbourhood U of p there exists a compact, connected neighbourhood V of p such that V is contractible to a point in U.*

A neighbourhood V of p as described in (2·1) will be referred to as a *trivial neighbourhood of p with respect to U.*

PROPOSITION 2·2. *If $m \geqslant 6$, there exists a neighbourhood U of p such that* int $U - \{p\}$ *is a* PL *m-manifold.*

Proof. Let U be a trivial neighbourhood of p (with respect to X). Since X is a homology m-manifold, the inclusion map $H^k(X) \to H^k(X-p)$ is an isomorphism for $k < m-1$ and any coefficient group. Since U is contractible to a point in X (so that image $(\tilde{H}^*(X) \to \tilde{H}^*(U)) = 0$), the inclusion j^*: $\tilde{H}^k(X-p) \to \tilde{H}^k(U-p)$ is zero for $k < m-1$. (Compare the exact sequences of the pairs $(X, X-p)$ and $(U, U-p)$.)

Now apply Kirby–Siebenmann(18). The obstructions $o_X \in H^4(X-p; Z_2)$ and $o_U \in H^4(U-p; Z_2)$ to triangulating $X-p$ and $U-p$, respectively, as PL manifolds satisfy the naturality condition $j^* o_X = o_U$. Since $m \geqslant 6$, $j^* o_X = 0$. ∎

We will *assume for the rest of this section that $X-p$ is a* PL *m-manifold and that $m \geqslant 5$.*

PROPOSITION 2·3. *Under the above assumption, given a neighbourhood U of p, there exists a trivial neighbourhood V of p with respect to U such that $V-p$ is a* PL *submanifold of $X-p$ with compact, connected boundary.*

Proof. We can pipe the boundary components together since $\tilde{H}_0(V-p) = 0$. ∎

A neighbourhood V such as that described in (2·3) will be called a *trivial* PL *neighbourhood of p with respect to U.*

THEOREM 2·4. *A trivial* PL *neighbourhood N of p has the homotopy type of a finite complex. Moreover, N has a well-defined simple homotopy type.*

Proof. Let N_1 be a trivial PL neighbourhood of p with respect to N. Since N_1 is a compact ANR, N_1 is dominated by a finite complex. Since $\overline{N-N_1}$ is a finite complex and N_1 is contractible to a point in N, the sum theorem for finiteness obstructions (23), (28) applies to prove that N has finite homotopy type. (West's theorem (31) of course applies here, but the more elementary argument suffices.)

In order to give N a simple homotopy type, apply the above paragraph to N_1 to obtain a finite complex K homotopy equivalent to N_1. Define the simple homotopy type of N to be that of $\overline{N-N_1} \underset{\partial N_1}{\cup} K$. (Use a mapping cylinder to get $\partial N_1 \subset K$.) Any two complexes obtained in this way are simply equivalent by the sum theorem for Whitehead torsion (21) since $\pi_1(N_1) \to \pi_1(N)$ is trivial. ∎

THEOREM 2·5. *If N is a trivial* PL *neighbourhood of p, then $(N, \partial N)$ is a simple Poincaré pair of dimension m.*

Proof. Since N is a generalized m-manifold, $(N, \partial N)$ satisfies Poincaré duality in dimension m. In case N is not simply connected, more is required of the duality isomorphism ((29)), §2).

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Let N_1 be a trivial PL neighbourhood of p with respect to $\operatorname{int} N$, and let L be a finite complex homotopy equivalent to N_1 so that

$$K = \overline{N - N_1} \underset{\partial N_1}{\cup} L$$

is a finite complex representing the simple homotopy type of N (as in (2·4)). Then $(K, \partial K) = (K, \partial N)$ is a finite pair satisfying Poincaré duality as are $(\overline{N - N_1}, \partial N \cup \partial N_1)$ and $(L, \partial N_1)$. It is well known that the PL manifold pair $(\overline{N - N_1}, \partial N \cup \partial N_1)$ is a (simple) Poincaré pair (see (29)). Since L is contractible in K, the natural action of $\pi_1(L)$ on $(L, \partial N_1)$ as a $\pi_1(K)$-module is simple so that $(L, \partial N_1)$ is a Poincaré pair (with respect to the simple action). By (the proof of) proposition 2·7 of (29), $(K, \partial K)$ is also. ∎

LEMMA 2·6. *Let N be a trivial* PL *neighbourhood of p, and let η be the* (*stable*) PL *normal bundle of $X - p$. Then $\eta | N - p$ is fibre homotopically trivial.*

Proof. Let N_1 be a trivial PL neighbourhood of p with respect to N, and let L be a finite complex homotopy equivalent to N_1 so that

$$K = \overline{N - N_1} \underset{\partial N_1}{\cup} L$$

represents the simple homotopy type of N. Let N_0 be a compact neighbourhood of p with PL manifold boundary such that N is contractible to a point in $\operatorname{int} N_0$. Then $(J, \partial J) = (\overline{N_0 - N} \underset{\partial N}{\cup} K, \partial N_0)$ is a finite Poincaré pair (proposition 2·7 of (29)). Let ξ be the (stable) Spivak normal fibre space of $(J, \partial J)$ (26). By uniqueness (26), $\xi | \overline{N - N_1}$ is (stably) fibre homotopically equivalent to $\eta_0 | \overline{N - N_1}$, where η_0 is the complement of the 0-section of η. Since K is contractible in J, $\xi | K$ is fibre homotopically trivial, so that $\eta_0 | \overline{N - N_1}$ is also. Since N_1 was arbitrarily small, $\eta_0 | N - p$ (and hence $\eta | N - p$) is fibre homotopically trivial. ∎

LEMMA 2·7. *Let N_1 and N_2 be trivial* PL *neighbourhoods of p with $N_2 \subset \operatorname{int} N_1$ and N_2 contractible in N_1. Let η be the* PL *normal bundle of $X - p$. Then $\eta | \partial N_2$ is trivial as a* PL *bundle.*

Proof. Let f: $(N_1 - p) \to B_{\mathrm{PL}}$ be the classifying map for $\eta | (N_1 - p)$. By (2·6), the composition

$$(N_1 - p) \to B_{\mathrm{PL}} \to B_G$$

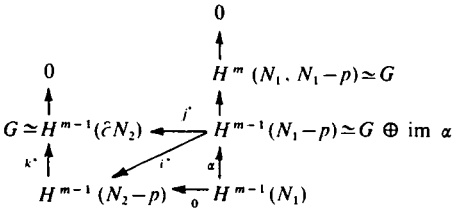
is null-homotopic (where B_{PL} and B_G, as usual, denote the classifying spaces for PL bundles and spherical fibrations, respectively). Thus we may assume f maps $N_1 - p$ into G/PL, the fibre of the map $B_{\mathrm{PL}} \to B_G$. Observe that $f | \partial N_2$ classifies $\eta | \partial N_2$, since ∂N_2 is PL bicollared in $N_1 - p$ ((16), 2.16.2). Moreover, j^*: $\tilde{H}^k(N_1 - p) \to \tilde{H}^k(\partial N_2)$ is zero for $k < m - 1$ (see the proof of (2·2)), so that the only obstruction to obtaining a homotopy of $f | \partial N_2$ in G/PL to a point is an element $o \in H^{m-1}(\partial N_2; \pi_{m-1}(G/\mathrm{PL}))$. Therefore ∂N_2 is parallelizable over the complement of a point, and hence ∂N_2 admits a normal map to S^{m-1}. But ∂N_2 is the boundary of a Poincaré pair, so its index (or Kervaire invariant) is zero. (See, for example, the Remark on p. 40 of (22).) When $m = 5$, the manifold ∂N_2 is smoothable (see, for example, p. 211 of (18)) and the argument for (3·1) of

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(17) shows that ∂N_2 is parallelizable as a smooth manifold. When $m \geqslant 6$, ∂N_2 is normally cobordant to a homotopy equivalence to S^{m-1}, hence ∂N_2 is normally cobordant to S^{m-1}, and hence ∂N_2 bounds a parallelizable manifold. ∎

THEOREM 2·8. *Let N_1 and N_2 be trivial* PL *neighbourhoods of p with $N_2 \subset \operatorname{int} N_1$ and N_2 contractible in N_1. Let η be the (stable) normal bundle of $X - p$. Then $\eta|(N_2 - p)$ is trivial as a* PL *bundle.*

Proof. For any coefficient group G, we have the following commutative diagram:



since $i^*(\operatorname{im} \alpha) = 0$, $k^*|\operatorname{im} i^*$ is one–one. By naturality of obstructions, the obstruction $o \in H^{m-1}(N_2 - p,\ \pi_{m-1}(G/\mathrm{PL}))$ to $\eta|(N_2 - p)$ being PL trivial satisfies: $o \in \operatorname{im} i^*$ and $k^* o = 0$ (the latter by (2·7)). Therefore $o = 0$. ∎

THEOREM 2·9. *Given a neighbourhood U of p, there exists a trivial* PL *neighbourhood N of p with respect to U, a* PL *manifold M, and a simple proper homotopy equivalence $f\colon M \to X$ such that $f|f^{-1}(X - \operatorname{int} N)$ is a* PL *homeomorphism.*

Proof. Given $U = N_0$, choose N_1, N_2 and N_3 such that N_i is a trivial PL neighbourhood of p with respect to N_{i-1} $(i = 1, 2, 3)$. (N_2 will be the desired N.) For large n, $N_3 \times (\mathbf{R}^n - 0)$ represents the Spivak normal fibre space of N_3 and $(N_3 - p) \times \mathbf{R}^n$ is the PL normal bundle of $N_3 - p$ ((2·6) and (2·8)). Then the Thom–Pontrjagin construction yields a PL manifold M_3 with normal bundle ν and a degree one normal map

$$f\colon (M_3, \partial M_3) \to (N_3, \partial N_3)$$

such that $f|\partial M_3$ is a PL homeomorphism, and ν is PL trivial. (See, for example, § 10 of (29).)

Since $m \geqslant 5$, we may assume that f is $[\frac{1}{2}m]$-connected where $[k]$ is the greatest integer $\leqslant k$. Let

$$M_2 = \overline{N_2 - N_3} \underset{f|\partial M_3}{\cup} M_3$$

and extend f (via the identity on $N_2 - N_3$) to an $[\frac{1}{2}m]$-connected degree one normal map $f\colon M_2 \to N_2$. (All normal bundles are PL trivial.) Set $k = [\frac{1}{2}m]$, and let $K_i(\ )$ represent homology kernels in dimension i. Then

$$\ker f_i = K_i(M_2, \partial M_2) \tilde{\leftarrow} K_i(M_2) \tilde{\leftarrow} K_i(M_3) \tilde{\leftarrow} K_i(M_3, \partial M_3) = \ker (f|M_3)_i.$$

These modules (over $\mathbf{Z}\pi_1 M_2$) are zero except possibly in dimension k (if $m = 2k$) or dimensions k and $k+1$ (if $m = 2k+1$). We claim now that through surgery we can kill the remaining kernels making f a simple equivalence.

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Case $m = 2k$. Since $f|M_3$ induces an isomorphism from $\pi_1(M_3)$ to $\pi_1(N_3)$, the inclusion $\pi_1(M_3) \to \pi_1(M_2)$ is zero. Hence by the Hurewicz Theorem applied to the universal cover $\tilde{M}_2$ of M_2, a basis for $K_k(M_2)$ can be represented by lifts of maps $\phi_i: S^k \to M_3$. Choose lifts $\tilde{\phi}_i$ of ϕ_i so that the lifts all map into the same components of $\tilde{M}_3 \subset \tilde{M}_2$. Then for any i, j, and any $g \in \pi_1 M_2$, $g \neq 1$, we have the intersection

$$\phi_i[S^k] \cdot g\phi_j[S^k] = 0.$$

That is, the non-degenerate intersection form

$$K_k(M_2) \otimes K_k(M_2) \to \mathbf{Z}\pi_1 M_2$$

has values in $\mathbf{Z}$. By the plumbing construction (17), (5) (performed on M_3) we can assume that the signature (or Arf invariant, depending on whether k is even or odd) of this form is zero.

Since $\pi_1(M_3) \to \pi_1(M_2)$ is zero, the Whitney trick (see, for example, (30)) can be applied to obtain disjoint PL embeddings $\phi_i: S^k \to M_2$ representing half a basis of $K_k(M_2)$ that is annihilated by the intersection form, as in (5).

Case $m = 2k+1$. Just as above, it is possible to find generators $\phi_i[S^k] \in K_k(M_3) \simeq K_k(M_2)$ of $K_k(M_2)$ given by PL embeddings ϕ_i so that the intersection form

$$K_k(M_2) \otimes K_{k+1}(M_2) \to \mathbf{Z}\pi_1 M_2$$

has values in the integers. Thus simply connected surgery techniques can be applied to kill $K_k(M_2)$.

Therefore, in either of the above cases, we can obtain a PL manifold M_2' and a simple homotopy equivalence $f': (M_2', \partial M_2') \to (N_2, \partial N_2)$ that is a PL homeomorphism on $\partial M_2'$. Set

$$M = X - N_2 \bigcup_{f'|\partial M_2'} M_2'$$

and

$$f = \mathrm{id} \bigcup_{\mathrm{id}|\partial N_2} f'$$

to obtain the mapping as desired. ∎

ADDENDUM 2·10. *The* PL *normal bundle of* M_2' *is (stably) trivial.*

Proof. M_2' is contractible in M. ∎

DEFINITION. We shall call a homotopy equivalence $f: M \to X$ such as that constructed in (2·9) a *homotopy resolution of* X *over* N.

THEOREM 2·11. *Given* $N_1, N_2, N_3 \subset X = N_0$ *such that* N_i *is a trivial* PL *neighbourhood of* p *with respect to* N_{i-1} $(i = 1, 2, 3)$; *and given homotopy resolutions* $f_i: M_i \to X$, $i = 1, 2$, *of* X *over* N_3; *there exists a* PL *homeomorphism* $h: M_1 \approx M_2$ *such that* $h = f_2^{-1} \circ f_1$ *over* $f_1^{-1}(X - N_1)$.

Proof. Without loss of generality we may have $M_i - f_i^{-1}(X - N_3) = X - N$ $(i = 1, 2)$ and $f_1 = f_2$ on $X - N_3$. Let $P_i = f_i^{-1}(N_3)$, $g_i = f_i|P_i$, and C_i be the mapping cylinder of g_i, reduced over $\partial P_i = \partial N_3$. Set

$$Y = C_1 \bigcup_{N_3 \times 0} N_3 \times [0, 1] \bigcup_{N_3 \times 1} C_2,$$

and

$$Z = P_1 \bigcup_{\partial N_3 \times 0} \partial N_3 \times [0, 1] \bigcup_{\partial N_3 \times 1} P_2.$$

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Then (Y, Z) is a Poincaré pair with trivial Spivak normal bundle, and Z is a closed PL manifold with PL trivial normal bundle. By resolving over a smaller neighbourhood of p if necessary, we can assume that (Y, Z) is a simple Poincaré pair. Hence there is a PL manifold W_3 with boundary Z and a normal map $F_3: (W_3, Z) \to (Y, Z)$ such that $F_3|Z =$ identity. Set

$$W_2 = \overline{N_2 - N_3} \times [0, 1] \underset{\partial N_3 \times [0,1]}{\cup} W_3$$

and

$$L = \overline{N_2 - N_3} \times [0, 1] \underset{\partial N_3 \times [0,1]}{\cup} K.$$

Then F_3 extends via the identity on $\overline{N_2 - N_3} \times [0, 1]$ to a normal map $F_2: W_2 \to L$; the kernels of F_2 lie in the image of $H_* W_3$ in $H_* W_2$. We can do surgery on W_2 rel ∂ just as in the proof of (2·9) to assume without loss that F_2: $(W_2, \partial W_2) \to (L, \partial L)$ is a homotopy equivalence and $F_2|\partial W_2 =$ identity. Thus W_2 is a PL h-cobordism between $\overline{N_2 - N_3} \cup P_1$, and $\overline{N_2 - N_3} \cup P_2$ that is a PL product over ∂N_2.

Now let

$$W_1 = \overline{N_1 - N_2} \times [0, 1] \underset{\partial N_2 \times [0,1]}{\cup} W_2.$$

By the sum theorem for Whitehead torsion (21), W_1 is an s-cobordism between

$$\overline{N_1 - N_3} \cup P_1 \quad \text{and} \quad \overline{N_1 - N_3} \cup P_2$$

that is a product on the boundary. (Remember that $\pi_1 W_2 \to \pi_1 W_1$ is trivial.) The s-cobordism theorem (see (16)) then gives a PL homeomorphism

$$h_1: \overline{N_1 - N_3} \cup P_1 \to \overline{N_1 - N_3} \cup P_2$$

that is the identity on the boundary. Then

$$h = \mathrm{id}_{\overline{X - N_1}} \cup h_1: M_1 \to M_2$$

is the desired homeomorphism. ∎

THEOREM 2·12. *Assume that X is a generalized m-manifold with only one singularity p and that either $m \geqslant 6$ or $m = 5$ and $X - p$ is a* PL *manifold. Then there exists an m-manifold M and a cell-like map $f: M \to X$ such that $f|f^{-1}(X - p)$ is a homeomorphism. If $X - p$ is a* PL *manifold then M has a* PL *structure under which $f|M - f^{-1}(p)$ is a* PL *homeomorphism.*

Proof. The proof is similar to that of theorem 4 of (7). Let $N_1, N_2, \ldots$ be neighbourhoods of p such that

(i) $N_{i+1} \subset \operatorname{int} N_i$.

(ii) N_{i+1} is a trivial PL neighbourhood with respect to N_i, and

(iii) $\cap N_i = \{p\}$.

By passing to subsequences when necessary, we can obtain the following from (2·9) and (2·11):

(iv) PL m-manifolds $M_1, M_2, \ldots$;

(v) homotopy resolutions $f_i: M_i \to X$ of X over N_i; and

(vi) PL homeomorphisms $g_i: M_i \to M_{i+1}$ such that $g_i = f_{i+1}^{-1} f_i$ on $f_i^{-1}(\overline{X - N_i})$.

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Without loss of generality we may assume $f_i^{-1}(\overline{X-N_i}) = \overline{X-N_i}$ and $g_i|\overline{X-N_i} =$ identity. Set $P_i = M_i - (X - N_i)$.

Let $h_i' = g_i \dots g_2 g_1 : M_1 \to M_{i+1}$ and $h_i = f_{i+1} h_i' : M_1 \to X$. Then properties (iii) and (vi) guarantee that $f = \lim_i h_i$ is a map from M_1 to X and that $f|f^{-1}(X-p)$ is a PL homeomorphism. The set

$$f^{-1}(p) = \cap\, (h_1')^{-1}(P_{i+1})$$

$$= \cap\, h_i^{-1}(N_{i+1})$$

is cell-like because $h_i^{-1}(N_{i+1})$ is contractible in $h_{i-1}^{-1}(N_i)$ for $i = 2, 3, \dots,$ (19). |

3. The argument in section 2 breaks down almost immediately (Proposition 2·2) when $m = 5$, since deleted neighbourhoods of the singular point p of X do not *a priori* have vanishing Kirby–Siebenmann obstruction. This is the only place, however, where section 2 fails in an effort to resolve the isolated singularity p; assuming $X - p$ is a PL 5-manifold, the remainder of section 2 is valid.

In this section we prove the analogue of (2·2) in dimension 5. The proof uses the full strength of the equivalence of (1·1) and (1·2) as well as the valid portions of section 2.

THEOREM 3·1. *If X is a generalized 5-manifold and $S(X) = \{p\}$, then there is a neighbourhood U of p such that* int $(U - \{p\})$ *has the structure of a* PL *manifold.*

Proof. Let α be an arc in X containing p as an end-point such that $\alpha - p$ is locally flat. Since $\alpha - p$ is contractible, it has a PL neighbourhood in $X - p$ (18). Thus there exists an embedding g of the standard 5-ball B^5 into X, taking a radius I to α, such that $g(B^5)$ is locally flat except at p. There is an open neighbourhood W of p such that

$$W \cap g(\partial B^5) = V \approx \mathbf{R}^4,$$

where $g^{-1}(V)$ is an open hemisphere of ∂B^5.

Set $Y = W - g$ (in B^5) and $Z = Y \bigcup_V Y$. Since Y is deformation retract of

$$Y \cup g\,(\text{int}\, B^5),$$

the Mayer–Vietoris theorem shows that Z is a generalized 5-manifold with

$$S(Z) = \{p\} \subset V.$$

By spinning Y about $V \approx \mathbf{R}^4$, we get Z embedded in a generalized 6-manifold

$$Q = Y \times S^1 / \{v \times S^1 | v \in V\},$$

whose only singularity is the point $p = [p \times S^1]$, in such a way that $Z - p$ is bicollared in $Q - p$.

By (2·2), p has an open neighbourhood P in Q such that $P - \{p\}$ is a PL manifold. Then $P \cap (Z - p)$ is bicollared in P so $(P \cap (Z-p)) \times R$ has a PL structure. Therefore $P \cap (Z - P)$ has a PL structure by the Product Structure Theorem of (18). (This is not surprising since we have doubled a $\mathbf{Z}_2$-obstruction.) By (2·12), there is a 5-manifold M and a cell-like map f: $M \to Z$ such that $f|f^{-1}(Z-p)$ is a homeomorphism. A theorem of

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R. D. Edwards(15) (this is the fact that (1·1) implies (1·2)) applies to show that $Z \times \mathbf{R}$ is a manifold.

We now use an argument of Galewski and Stern(33) to show that $X \times \mathbf{R}$ is a manifold. The $1-$LCC approximation techniques of M. Stan'ko(27) can be used in codimension one (as observed first in this case by Bryant, Edwards and Seebeck(6) and later proved in full generality by Ancel and Cannon(17)) to produce an embedding h: $Y \times \mathbf{R} \to Z \times \mathbf{R}$, arbitrarily close to the inclusion

$$Y \times \mathbf{R} \to Z \times \mathbf{R} = Y \times \mathbf{R} \underset{V \times R}{\cup} Y \times \mathbf{R}$$

[onto the first summand] such that $h(Y \times \mathbf{R})$ has a $1-$LC complement in $Z \times \mathbf{R}$ at each point of $h(V \times \mathbf{R})$. By the $1-$LCC codimension one taming theorem (13), $h(Y \times \mathbf{R})$ is collared in $Z \times \mathbf{R}$ (observe that $h(\partial(Y \times \mathbf{R})) \approx R^5$ is a manifold), and so $Y \times \mathbf{R}$ union a collar on $\partial Y \times \mathbf{R}$ is a manifold. But $Y \times \mathbf{R}$ union a collar on $\partial Y \times \mathbf{R}$ is homeomorphic to $(Y \cup g(\operatorname{int} B^5)) \times \mathbf{R}$, and $Y \cup g(\operatorname{int} B^5)$ is a neighbourhood of p in X. Therefore $X \times \mathbf{R}$ is a manifold.

The conclusion of (3·1) now follows from the Product Structure Theorem (18). Since $X \times \mathbf{R}$ is a manifold, there is neighbourhood U of p in X such that $U \times \mathbf{R}$ has a PL manifold structure; therefore $(U-p) \times \mathbf{R}$ has a PL manifold structure, and finally so does $(U-p)$. ∎

Remark. After proving that $X \times \mathbf{R}$ is a manifold, one could apply the main theorem of (7) directly to resolve the singularity of X. (This is the implication (1·2) $\rightrightarrows$ (1·1).) Alternatively, one can combine (3·1) and (2·12) obtaining

THEOREM 3·2. *If X is a generalized m-manifold, $m \geqslant 5$, with $S(X)$ a locally finite set, then there exists a topological manifold M^m and a (proper) cell-like map f: $M^m \to X$ such that $f|f^{-1}(X-S(X))$ is a homeomorphism.*

The generalization to include $S(X)$ locally finite is easily proved using (2·12) to resolve a neighbourhood of each point of $S(X)$ and piecing the various resolutions together. A similar argument, using (2·11), yields uniqueness:

THEOREM 3·3. *Suppose X is a generalized m-manifold, $m \geqslant 5$, with $S(X)$ a locally finite set. Suppose further that (M_1, f_1) and (M_2, f_2) are conservative resolutions of X and that U is a neighbourhood of $S(X)$. Then there exists a homeomorphism h: $M_1 \approx M_2$ such that $h = f_2^{-1} f_1$ on $f_1^{-1}(X-U)$.*

4. It remains to show how the general cases of (1·1) and (1·3) follow from the isolated singularity cases of section 3. Assume then that X is a generalized m-manifold, $S(X)$ is 0-dimensional, and $m \geqslant 5$. Without loss of generality we may assume that $S(X)$ is compact and that X is connected (in general, $S(X)$ is the union of a discrete family of compact, 0-dimensional sets). The next proposition uses only the fact that X is a connected generalized manifold.

PROPOSITION 4·1. *Given an open-closed subset C of $S(X)$, there exists an arc α in X such that $\alpha \cap S(X) = C$.*

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THEOREM 4·2. *There exists an m-manifold M and a cell-like map $f: M \to X$ such that $f|f^{-1}(X - S(X))$ is a homeomorphism.*

Proof. Let $\alpha: [0, 1] \to X$ be an embedding such that $S(X) \subset \alpha[(0, 1])$. Set

$$C = \alpha^{-1}(S(X)).$$

We have the representation $$C = \bigcap_{n=1}^{\infty} J_n,$$

where $J_1 \supset J_2 \supset \ldots$ and each J_n is the union of mutually exclusive intervals $I_{1,n}, \ldots, I_{k_n,n}$ and where

$$\lim_n \max_{1 \leqslant j \leqslant k_n} \{\operatorname{diam} I_{j,n}\} = 0.$$

Set $\alpha_{j,n} = \alpha(I_{j,n})$ and let X_n be X with the arcs $\alpha_{1,n}, \ldots, \alpha_{k_n,n}$ collapsed to distinct points. Then X_n is a generalized m-manifold with a finite number of singularities. Let $\phi_n: X \to X_n$ be the quotient map and let $g_n: X_{n+1} \to X_n$ be the induced quotient map, $n = 1, 2, \ldots$.

Let $N_1, N_2, \ldots$ be neighbourhoods in X such that $\alpha_{j,n} \subset \operatorname{int} N_n$ $(j = 1, \ldots, k_n)$, $N_{n+1} \subset \operatorname{int} N_n$, and each component of N_{n+1} is contractible to a point in N_n. By (3·2) there exist topological m-manifolds $M_1, M_2, \ldots$ and cell-like maps $f_n: M_n \to X_n$ such that $f_n|f_n^{-1}(X_n - S(X_n))$ is a homeomorphism. Since $g_n f_{n+1}: M_{n+1} \to X_n$ is also a resolution of X_n, there exist (by (3·3)) homeomorphisms $h_n': M_n \to M_{n+1}$ such that $h_n' = f_{n+1}^{-1} g_n^{-1} f_n$ over $X_n - \phi_n(N_n)$. Let $\psi_n: X_n \to X$ be a homotopy inverse of ϕ_n such that $\psi_n = \phi_n^{-1}$ on $\phi_n(X - N_n)$, and set $h_n = \psi_{n+1} f_{n+1} h_n' \ldots h_1': M_1 \to X$. Just as in the proof of (2·12), the sequence $\{h_n\}$ converges to a map $f: M_1 \to X$ that is a homeomorphism on

$$f^{-1}(X - S(X)).$$

The homotopy condition on the sequence $\{N_n\}$ implies $f^{-1}(x)$ is cell-like for each

$$x \in S(X). \blacksquare$$

THEOREM 4·3. *Suppose that (M_1, f_1) and (M_2, f_2) are resolutions of X such that f_i is a homeomorphism on $f_i^{-1}(X - S(X))$, $i = 1, 2$. If U is a neighbourhood of $S(X)$, then there exists a homeomorphism $h: M_1 \approx M_2$ such that $h = f_2^{-1} f_1$ on $f_1^{-1}(X - U)$.*

Proof. Let V be a neighbourhood of $S(X)$ in U such that V has only finitely many components, say $V_1, \ldots, V_k$. Let α_i be an arc in V_i such that $(S(X) \cap V_i) \subset \alpha_i$, and denote by Y the generalized manifold obtained by shrinking the arcs α_i to distinct points. Let $g_i: M_i \to Y$ be the composition of f_i and the quotient map q. Then (M_i, g_i) is a resolution of Y, $i = 1, 2$. An application of (3·3) yields a homeomorphism $h: M_1 \to M_2$ such that $h = g_2^{-1} g_1$ on $g_1^{-1}(Y - q(V))$; this is the desired h. $\blacksquare$

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