Canad. Math. Bull. Vol. **68** (3), 2025, pp. 729–743
http://dx.doi.org/10.4153/S0008439525000049
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On even K-groups over p-adic Lie extensions of global function fields

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Abstract. Let p be a fixed prime number, and let F be a global function field with characteristic not equal to p. In this article, we shall study the variation properties of the Sylow p-subgroups of the even K-groups in a p-adic Lie extension of F. When the p-adic Lie extension is assumed to contain the cyclotomic $\mathbb{Z}_p$-extension of F, we obtain growth estimate of these groups. We also establish a duality between the direct limit and inverse limit of the even K-groups.

1 Introduction

Throughout this article, we fix a prime p and an integer $i \geq 2$. Let F be a global function field whose characteristic is not equal to our prime p. Once and for all, we fix a separable closure F^{sep} of F. Let S denote a nonempty finite set of primes of F, and write F_S for the maximal extension of F contained in F^{sep} that is unramified outside the set S. For every finite extension L of F contained in F_S, we write $S(L)$ for the set of primes of L above F, and for brevity, we write $\mathcal{O}_{L,S}$ for the ring of $S(L)$-integers of the field L. We also write $G_S(L) = \mathrm{Gal}(F_S/L)$. The goal of this article is to study the even K-groups $K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]$ as L varies in a given p-adic Lie extension F_∞ of F.

Our approach is via cohomology. Indeed, thanks to the deep work of Rost–Voevodsky [37], the group $K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]$ can be identified with the continuous cohomology group $H^2\big(G_S(L), \mathbb{Z}_p(i)\big)$. Following standard Iwasawa theoretical procedure, we consider the inverse limit of these cohomology groups, and the resulting object is usually called the Iwasawa cohomology group. More precisely, the kth Iwasawa cohomology group of $\mathbb{Z}_p(i)$ over F_∞/F is defined by

$$H^k_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big) := \varprojlim_L H^k\big(G_S(L), \mathbb{Z}_p(i)\big),$$

where L runs through all finite extensions of F contained in F_∞ and the transition maps are given by corestriction on cohomology. These Iwasawa cohomology groups come naturally equipped with a $\mathbb{Z}_p[\![G]\!]$-module structure, where $G = \mathrm{Gal}(F_\infty/F)$. Furthermore, it can be shown that they are finitely generated over $\mathbb{Z}_p[\![G]\!]$ (for instance, see [25, Proposition 4.1.3]) and that $H^k_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big) = 0$ for $k \neq 1, 2$. For

Received by the editors July 25, 2024; revised January 13, 2025; accepted January 14, 2025.
Published online on Cambridge Core January 22, 2025.
AMS subject classification: **11R23**, **11R70**, **11R34**.
Keywords: Global function fields, even K-groups, p-adic Lie extensions.

https://doi.org/10.4153/S0008439525000049 Published online by Cambridge University Press

the first and second Iwasawa cohomology groups, we can even say more and this is the content of the following theorem.

Theorem 1.1 (Theorem 4.1) *Retain the setting as above. In particular, we let $i \geq 2$. Then the following statements are valid.*

(a) *The module $H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ is torsion over $\mathbb{Z}_p[\![G]\!]$. In the event that F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F, the module $H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ is finitely generated over $\mathbb{Z}_p[\![H]\!]$, where $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$.*

(b) *$H^1_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ is a torsion $\mathbb{Z}_p[\![G]\!]$-module. Furthermore, in the event that G has dimension ≥ 2, one has*

$$H^1_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i)) = 0.$$

We recall that the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F is the unique $\mathbb{Z}_p$-extension of F contained in the extension of F obtained by adjoining all p-power roots of unity.

The proof of the preceding theorem, which will be given in Section 4, utilizes the fact that the p-cohomological dimension of $\mathrm{Gal}(G_S(F)/F_{\mathrm{cyc}})$ is equal to one (see [28, Theorem 10.1.2(c)]; also see Lemma 4.3 of this article). In [4], an analogous result for the Iwasawa cohomology of the Tate module of an abelian variety is proven by the author and his collaborators. The proof of Theorem 4.1 will follow that in [4] with some slight modification.

The remainder of the article is concerned with applying the above theorem to the study of even K-groups. As a start, the vanishing of the first Iwasawa cohomology group is crucial in the following evaluation of the G-Euler characteristics of the second Iwasawa cohomology group.

Theorem 1.2 (Theorem 4.5) *Notation as before. Suppose further that $G = \mathrm{Gal}(F_\infty/F)$ has dimension ≥ 2 and has no p-torsion. Then one has*

$$\chi\big(G, H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\big) = \frac{\#K_{2i-2}(\mathcal{O}_{F,S})[p^\infty]}{\#K_{2i-1}(\mathcal{O}_{F,S})[p^\infty]}.$$

Note that it is a consequence of a theorem of Harder (see [11, Korollar 3.2.3] or [39, Chapter VI, Corollary 6.2]) that the group $K_n(\mathcal{O}_{F,S})$ is finite for $n \geq 2$, and so the above expression on the right makes sense.

The next application of Theorem 4.1 is concerned with the study of growth formula of even K-groups as done in [23]. However, note that a global function field F with $\mathrm{char}(F) \neq p$ has only one $\mathbb{Z}_p$-extension, namely, the cyclotomic one (see [28, Proposition 10.3.20]). Therefore, a p-adic Lie extension of dimension ≥ 2 is necessarily noncommutative. Due to a lack of a precise structural theorem for modules over a noncommutative Iwasawa algebra, this causes an extra complication when dealing with p-adic Lie extension with dimension ≥ 2. Thankfully, in the event that the module is finitely generated over $\mathbb{Z}_p[\![H]\!]$, the author has obtained growth estimates for such modules (see [22, 23]) which in turn are variants of previous results of Perbet [30] and Lei [20] (also see Propositions 5.3 and 5.4, and the references given). When F_∞ contains F_{cyc}, the extra finite generation property of $H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ as seen

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in Theorem 4.1 allows to apply the results mentioned to obtain growth estimate. As illustration, we present here the case when $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p^r \rtimes \mathbb{Z}_p$.

Theorem 1.3 (Theorem 5.5) *Let $i \geq 2$ be given. Suppose that F_∞ is a p-adic Lie extension of F containing F^{cyc} with $G := \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}}) \cong \mathbb{Z}_p^{d-1}$. Let F_n be the intermediate subfield of F_∞/F with $\mathrm{Gal}(F_\infty/F_n) = G_n$. Then we have*

$$\mathrm{ord}_p\Big(K_{2i-2}(\mathcal{O}_{F_n,S})[p^\infty]\Big) = \mathrm{rank}_{\mathbb{Z}_p[\![H]\!]}\left(H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\right) np^{(d-1)n} + O(p^{(d-1)n}).$$

Here G_n denotes the lower p-series of G in the sense of [5] (see Section 5 for the precise definition). For more detailed discussion on growth formula, we refer readers to Section 5.

Finally, we come to study the direct limit and inverse limit

$$\varinjlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty] \quad \text{and} \quad \varprojlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty],$$

where the transition maps in the direct limit are induced by the inclusion $\mathcal{O}_{L,S} \longrightarrow \mathcal{O}_{L',S}$ for $L \subseteq L'$, and the transition maps in the inverse limit are the norm maps (also called the transfer maps). Our result is as follow.

Theorem 1.4 (Theorem 6.1) *Let $i \geq 2$ and let F_∞ be a p-adic Lie extension of F which is contained in F_S. Suppose that $G = \mathrm{Gal}(F_\infty/F)$ has no p-torsion. Then there is an isomorphism*

$$\left(\varinjlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]\right)^\vee \cong \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}\left(\varprojlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty], \mathbb{Z}_p[\![G]\!]\right)$$

of $\mathbb{Z}_p[\![G]\!]$-modules.

The above result can be thought as an analog of a classical theorem of Iwasawa [14] which compares the direct limit and inverse of class groups over a $\mathbb{Z}_p$-extension of number fields. This result of Iwasawa has been generalized to the context of a $\mathbb{Z}_p^d$-extension (see the works of Nekovář [27], Vauclair [35] and, more recently, that of Lai and Tan [18]). Motivated by these development, the author proved an analogous result for even K-groups over a $\mathbb{Z}_p^d$-extension of a number field (see [24]). A crucial input in the author's proof is the utilization of an exact sequence of Jannsen's (cf. [15, Theorem 2.1] or [28, Theorem 5.4.13]) which yields the following short exact sequence

$$0 \longrightarrow \left(\varinjlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]\right)^\vee \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}\left(\varprojlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty], \mathbb{Z}_p[\![G]\!]\right)$$
$$\longrightarrow \left(\varinjlim_U H_1\big(U, H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))\big) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \longrightarrow 0$$

(see Section 6 for details). Therefore, in view of the above exact sequence, it remains to show that the rightmost term vanishes. For the latter, it suffices to show that $H_1\big(U, H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))\big)$ is finite. Had U been a commutative group,

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the finiteness of $H_1\big(U, H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))\big)$ then follows from the finiteness of $H_0\big(U, H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))\big) \cong K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]$ and a standard commutative algebra result (see [33, pp. 56–57]; also see [24, Proof of Theorem 2.3]). However, in the function field context, every p-adic Lie extension of dimension ≥ 2 is necessarily noncommutative, and so our proof will take a different route. We shall refer readers to Theorem 6.1 for details but we mention here that an important ingredient toward our proof is the finiteness of higher odd K-groups. Note that this latter finiteness is not true in general in the number field context.

2 Iwasawa algebras and modules

Throughout the article, p will always denote a fixed prime. Let G be a compact p-adic Lie group. The Iwasawa algebra of G over $\mathbb{Z}_p$ is defined by

$$\mathbb{Z}_p[\![G]\!] = \varprojlim_U \mathbb{Z}_p[G/U],$$

where U runs over the open normal subgroups of G and the inverse limit is taken with respect to the canonical projection maps.

For now, let us assume that the group G is pro-p and has no p-torsion. It is then well-known that $\mathbb{Z}_p[\![G]\!]$ is a Noetherian Auslander regular ring (cf. [36, Theorem 3.26]) with no zero divisors (cf. [29]). As a consequence, it admits a skew field $Q(G)$ which is flat over $\mathbb{Z}_p[\![G]\!]$ (see [10, Chapters 6 and 10] or [19, Chapter 4, Sections 9 and 10]). This in turn enables us to define the $\mathbb{Z}_p[\![G]\!]$-rank of a finitely generated $\mathbb{Z}_p[\![G]\!]$-module M, which is given by

$$\mathrm{rank}_{\mathbb{Z}_p[\![G]\!]}(M) = \dim_{Q(G)}(Q(G) \otimes_{\mathbb{Z}_p[\![G]\!]} M).$$

The $\mathbb{Z}_p[\![G]\!]$-module M is said to be a torsion $\mathbb{Z}_p[\![G]\!]$-module if $\mathrm{rank}_{\mathbb{Z}_p[\![G]\!]}(M) = 0$. We also recall a useful equivalent formulation of a torsion $\mathbb{Z}_p[\![G]\!]$-module: $\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) = 0$ (cf. [36, Remark 3.7]). In the event that the torsion $\mathbb{Z}_p[\![G]\!]$-module M satisfies $\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) = 0$, we say that M is a pseudo-null $\mathbb{Z}_p[\![G]\!]$-module. For a finitely generated $\mathbb{Z}_p[\![G]\!]$-module M, denote by $M[p^\infty]$ the $\mathbb{Z}_p[\![G]\!]$-submodule of M consisting of elements of M which are annihilated by some power of p. Howson [13, Proposition 1.11], and independently Venjakob [36, Theorem 3.40], showed that there is a $\mathbb{Z}_p[\![G]\!]$-homomorphism

$$\varphi : M[p^\infty] \longrightarrow \bigoplus_{i=1}^{s} \mathbb{Z}_p[\![G]\!]/p^{\alpha_i},$$

whose kernel and cokernel are pseudo-null $\mathbb{Z}_p[\![G]\!]$-modules, and where the integers s and α_i are uniquely determined. The μ_G-invariant of M is then defined to be $\mu_G(M) = \sum_{i=1}^{s} \alpha_i$.

We now extend the notion of torsion modules and pseudo-null modules to the case when G is a general compact p-adic Lie group. A well-known theorem of Lazard

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asserts that G contains an open normal subgroup G_0 which is pro-p with no p-torsion (cf. [5, Theorem 8.32]). By [28, Proposition 5.4.17], we have

$$\mathrm{Ext}^i_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \cong \mathrm{Ext}^i_{\mathbb{Z}_p[\![G_0]\!]}(M, \mathbb{Z}_p[\![G_0]\!])$$

for every finitely generated $\mathbb{Z}_p[\![G]\!]$-module M. In view of this, we shall say that M is a torsion $\mathbb{Z}_p[\![G]\!]$-module (resp., psuedo-null $\mathbb{Z}_p[\![G]\!]$-module) if $\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!])$ $= 0$ (resp., $\mathrm{Ext}^i_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) = 0$ for $i = 0, 1$). Equivalently, this is saying that M is a torsion $\mathbb{Z}_p[\![G]\!]$-module (resp., pseudo-null $\mathbb{Z}_p[\![G]\!]$-module), whenever M is a torsion $\mathbb{Z}_p[\![G_0]\!]$-module (resp., pseudo-null $\mathbb{Z}_p[\![G_0]\!]$-module) as in the preceding paragraph. (Also, compare with [36, Discussion after Definition 2.6]).

3 *K*-groups and Galois cohomology

We now describe a relation between the algebraic K-groups and Galois cohomology. To begin with, F will always denote a global function field of characteristic ℓ. We shall, once and for all, fix a finite nonempty set S of primes of F. Let F_S be the maximal algebraic and separable extension of F unramified outside S. For every extension L of F contained in F_S, write $G_S(L)$ for the Galois group $\mathrm{Gal}(F_S/L)$. Finally, we denote by $S(L)$ the set of primes of L above S, and for brevity, we shall write $\mathcal{O}_{L,S}$ for the ring of $S(L)$-integers of L.

For a ring R with identity, write $K_n(R)$ for the algebraic K-groups of R in the sense of Quillen [31] (also see [17, 39]). In this article, we are interested in the group $K_n(\mathcal{O}_{F,S})$. A theorem of Harder [11, Korollar 3.2.3] asserts that the group $K_n(\mathcal{O}_{F,S})$ is finite for every $n \geq 2$. On the other hand, the group $K_n(F)$ is p-divisible by a result of Geisser–Levine [9, Theorem 8.1]. By appealing to this and the localization sequence, one can then show that the group $K_n(\mathcal{O}_{F,S})$ has order coprime to ℓ (we refer readers to [39, Chapter VI, Theorem 6.1 and Corollary 6.2] for details). Note that when $n = 2$, this was earlier established by Bass–Tate [1, Chapter II, Theorem 2.1]).

For the remainder of the article, p will always denote a prime $\neq \ell$. Denoting by μ_{p^n} the cyclic group generated by a primitive p^n-root of unity, we write μ_{p^∞} for the direct limit of the groups μ_{p^n}. The natural action of $G_S(F)$ on μ_{p^∞} induces the cyclotomic character

$$\chi : G_S(F) \longrightarrow \mathrm{Aut}(\mu_{p^\infty}) \cong \mathbb{Z}_p^\times.$$

For a discrete or compact $G_S(F)$-module X, we denote by $X(i)$ the i-fold Tate twist of X. More precisely, $X(i)$ is the $G_S(F)$-module which is X as a $\mathbb{Z}_p$-module but with a $G_S(F)$-action given by

$$\sigma \cdot x = \chi(\sigma)^i \sigma x,$$

where the action on the right is the original action of $G_S(F)$ on X. Plainly, we have $X(0) = X$ and $\mu_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p(1)$. In [34], Soulé connected the higher K-groups with (continuous) Galois cohomology groups via the p-adic Chern class maps

$$\mathrm{ch}^{(p)}_{i,k} : K_{2i-k}(\mathcal{O}_{F,S}) \otimes \mathbb{Z}_p \longrightarrow H^k(G_S(F), \mathbb{Z}_p(i))$$

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for $i \geq 2$ and $k = 1, 2$. (For the precise definition of these maps, we refer readers to loc. cit.) Soulé also proved that these maps are surjective (see [34, Théorème 6(iii)]; also see the work of Dwyer–Friedlander [6]). It is folklore (see, for instance, [17, Theorem 2.7]) that the bijectivity of the p-adic Chern class maps follows as a consequence of the norm residue isomorphism theorem (previously also known as the Bloch–Kato (-Milnor) conjecture; see [2, 26]), and the latter was proven by Rost and Voevodsky [37] with the aid of a patch from Weibel [38]. As a consequence of these developments, we have the following identification between the Sylow p-subgroup of K-groups and Galois cohomology.

Proposition 3.1 *For $i \geq 2$, one has the following isomorphisms of finite abelian groups:*

$$K_{2i-1}(\mathcal{O}_{F,S})[p^\infty] \cong H^1\big(G_S(F), \mathbb{Z}_p(i)\big),$$

$$K_{2i-2}(\mathcal{O}_{F,S})[p^\infty] \cong H^2\big(G_S(F), \mathbb{Z}_p(i)\big).$$

4 Iwasawa cohomology groups

We now introduce the Iwasawa cohomology groups which will be an important object for our study of K-groups. As before, F will denote a global function field of characteristic $\neq p$, and we let $i \geq 2$. For every extension $\mathcal{L}$ of F contained in F_S, the Iwasawa cohomology groups are defined by

$$H^k_{\mathrm{Iw},S}\big(\mathcal{L}/F, \mathbb{Z}_p(i)\big) := \varprojlim_L H^k\big(G_S(L), \mathbb{Z}_p(i)\big),$$

where L runs through all finite extensions of F contained in $\mathcal{L}$ and the transition maps are given by the corestriction maps. In the event that $\mathcal{L}/F$ is a finite extension, we plainly have $H^k_{\mathrm{Iw},S}\big(\mathcal{L}/F, \mathbb{Z}_p(i)\big) = H^k\big(G_S(\mathcal{L}), \mathbb{Z}_p(i)\big)$.

Now suppose that F_∞ is a p-adic Lie extension of F contained in F_S. In other words, F_∞ is a Galois extension of F, whose Galois group $G = \mathrm{Gal}(F_\infty/F)$ is a compact p-adic Lie group. We can now state and prove the first theorem of the article.

Theorem 4.1 *Let $i \geq 2$. Suppose that F_∞ is a p-adic Lie extension of F which is contained in F_S. Then the following statements are valid.*

(a) *The module $H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ is torsion over $\mathbb{Z}_p[\![G]\!]$. In the event that F_∞ contains F_{cyc}, we have $H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ being finitely generated over $\mathbb{Z}_p[\![H]\!]$, where $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$.*

(b) *$H^1_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$ is a torsion $\mathbb{Z}_p[\![G]\!]$-module. Furthermore, in the event that the group G has dimension ≥ 2, one has*

$$H^1_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i)) = 0.$$

The remainder of the section is devoted to the proof of this theorem. As a start, we record the following version of Tate's descent spectral sequence for Iwasawa cohomology.

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Proposition 4.2 *Suppose that $i \geq 2$. Let U be a closed normal subgroup of $G = \mathrm{Gal}(F_\infty/F)$ and write $\mathcal{L}$ for the fixed field of U. Then we have a homological spectral sequence*

$$H_r\big(U, H^{-s}_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))\big) \Longrightarrow H^{-r-s}_{\mathrm{Iw},S}\big(L_\infty/F, \mathbb{Z}_p(i)\big)$$

and an isomorphism

$$H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_U \cong H^2_{\mathrm{Iw},S}\big(\mathcal{L}/F, \mathbb{Z}_p(i)\big).$$

Furthermore, if U is an open subgroup of G, then $\mathcal{L}$ is a finite extension of F and we have

$$H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_U \cong H^2\big(G_S(\mathcal{L}), \mathbb{Z}_p(i)\big).$$

Proof When the group G is commutative, the spectral sequence is a result of Nekovář [27, Proposition 4.2.3]. For a noncommutative G, this follows from [8, Proposition 1.6.5] or [25, Theorem 3.1.8]. The isomorphisms of the proposition follows from reading off the initial $(0, -2)$-term of the spectral sequence. ∎

We record another observation pertaining to the p-cohomological dimension of the group $G_S(F_{\mathrm{cyc}})$ which will be an important component to our subsequent discussion.

Lemma 4.3 *The p-cohomological dimension of the group $G_S(F_{\mathrm{cyc}})$ is equal to* 1.

Proof Let k be the constant field of F, and write $\bar{k}$ for the algebraic closure of k. Since the set S is nonempty and $\mathrm{char}(F) \neq p$, it follows from [28, Theorem 10.1.2(i)(c)] that the group $G_S(\bar{k}F)$ is of p-cohomological dimension 1. Since $G_S(\bar{k}F)$ is a closed subgroup of $G_S(F_{\mathrm{cyc}})$ with quotient $\mathrm{Gal}(\bar{k}F/F_{\mathrm{cyc}})$ whose profinite order is coprime to p, the two groups $G_S(F_{\mathrm{cyc}})$ and $G_S(\bar{k}F)$ have the same p-cohomological dimension (see [28, Proposition 3.3.5(i)]). The conclusion of the lemma now follows. ∎

We are in position to prove Theorem 4.1.

Proof (Proof of Theorem 4.1) We first prove statement (a). The torsionness can be proved via a similar argument to that in [23, Proposition 4.1.1]. We will focus on the finite generation assertion. By base changing F, we may assume that G is pro-p with no p-torsion. Since

$$H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_H \cong H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}_p(i)\big)$$

by Proposition 4.2, a Nakayama's lemma argument reduces us to showing that $H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}_p(i)\big)$ is finitely generated over $\mathbb{Z}_p$. But since we also have

$$H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}_p(i)\big)/p \cong H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}/p(i)\big),$$

this in turn reduces us to showing that $H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}/p(i)\big)$ is finite, or equivalently, $H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}/p(i)\big)$ is torsion over $\mathbb{F}_p[\![\Gamma]\!]$, where $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. By considering the initial term of the spectral sequence

$$\mathrm{Ext}^r_{\mathbb{F}_p[\![\Gamma]\!]}\Big(H^{3-s}_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F, \mathbb{Z}/p(i)\big), \mathbb{F}_p[\![\Gamma]\!]\Big) \Longrightarrow H^{3-r-s}\big(G_S(F_{\mathrm{cyc}}), \mathbb{Z}/p(i)\big)^\vee$$

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(see [8, Subsection 1.6.12], [25, Theorem 4.5.1] or [27, Theorem 5.4.5]), we obtain an isomorphism

$$\mathrm{Ext}^0_{\mathbb{F}_p[\![\Gamma]\!]}\left(H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F,\mathbb{Z}/p(i)\big),\mathbb{F}_p[\![\Gamma]\!]\right)\cong H^2\big(G_S(F_{\mathrm{cyc}}),\mathbb{Z}/p(i)\big)^\vee$$

where we note that the latter vanishes by virtue of Lemma 4.3. Hence it follows that $H^2_{\mathrm{Iw},S}\big(F_{\mathrm{cyc}}/F,\mathbb{Z}_p(i)\big)$ is torsion over $\mathbb{F}_p[\![\Gamma]\!]$ as required. The proof of statement (a) is therefore complete.

We now turn to the verification of statement (b). As before, we may assume that G is pro-p with no p-torsion. Since the first assertion of statement (a) is telling us that

$$\mathrm{rank}_{\mathbb{Z}_p[\![G]\!]}\left(H^2_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)\right)=0,$$

we therefore have

$$-\mathrm{rank}_{\mathbb{Z}_p[\![G]\!]}\left(H^1_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)\right)=\sum_{s\geq 0}(-1)^s\,\mathrm{rank}_{\mathbb{Z}_p[\![G]\!]}\left(H^s_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)\right).$$

By a formula of Howson [12, Theorem 1.1], the latter sum can be expressed as

$$\sum_{r,s\geq 0}(-1)^{r+s}\,\mathrm{rank}_{\mathbb{Z}_p}\left(H_r\big(G,H^s_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)\big)\right).$$

Taking the spectral sequence of Proposition 4.2 into account, this sum is equal to

$$\sum_{m\geq 0}(-1)^m\,\mathrm{rank}_{\mathbb{Z}_p}\left(H^m\big(G_S(F),\mathbb{Z}_p(i)\big)\right)=0,$$

where the final zero is a consequence of the cohomology groups appearing in the sum being finite. We have therefore established the first assertion of statement (b).

To continue, we shall work under the assumption that G has dimension ≥ 2. We now call upon another spectral sequence of Nekovář

$$\mathrm{Ext}^r_{\mathbb{Z}_p[\![G]\!]}\left(H^s\big(G_S(F_\infty),\mathbb{Q}_p/\mathbb{Z}_p(i)\big)^\vee,\mathbb{Z}_p[\![G]\!]\right)\Longrightarrow H^{r+s}_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)$$

(cf. [8, Subsection 1.6.12], [25, Theorem 4.5.1] or [27, Theorem 5.4.5]), whose low degree terms fit into the following exact sequence

$$\begin{aligned}0\longrightarrow\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}&\left(H^0\big(G_S(F_\infty),\mathbb{Q}_p/\mathbb{Z}_p(i)\big)^\vee,\mathbb{Z}_p[\![G]\!]\right)\longrightarrow H^1_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)\\&\longrightarrow\mathrm{Ext}^0_{\mathbb{Z}_p[\![G]\!]}\left(H^1\big(G_S(F_\infty),\mathbb{Q}_p/\mathbb{Z}_p(i)\big)^\vee,\mathbb{Z}_p[\![G]\!]\right).\end{aligned}$$

As the group G is of dimension ≥ 2, it follows from a classical observation of Jannsen (cf. [15, Corollary 2.6(b)(c)]) that $\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}\left(M,\mathbb{Z}_p[\![G]\!]\right)=0$ whenever M is finitely generated over $\mathbb{Z}_p$. Since this is the case for the module $H^0\big(G_S(F_\infty),\mathbb{Q}_p/\mathbb{Z}_p(i)\big)^\vee$, we see that the leftmost term of the above exact sequence is zero. Consequently, the module $H^1_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)$, which is torsion over $\mathbb{Z}_p[\![G]\!]$ as seen in the previous paragraph, injects into $\mathrm{Ext}^0_{\mathbb{Z}_p[\![G]\!]}\left(H^1\big(G_S(F_\infty),\mathbb{Q}_p/\mathbb{Z}_p(i)\big)^\vee,\mathbb{Z}_p[\![G]\!]\right)$ which is a reflexive $\mathbb{Z}_p[\![G]\!]$-module (cf. [36, Proposition 3.11(i)]). Thus, we must have $H^1_{\mathrm{Iw},S}\big(F_\infty/F,\mathbb{Z}_p(i)\big)=0$. This therefore completes the proof of the theorem. ∎

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Remark 4.4 Some part of the assertions in Theorem 4.1 should be generalized to more general Galois representations. Indeed, the theorem is inspired by a work of the author, where, together with his coauthors, the $\mathbb{Z}_p[\![H]\!]$-finite generation of the second Iwasawa cohomology group and vanishing of the first Iwasawa cohomology group for the Tate module of an abelian variety are established (see [4]). For a high-brow approach on the second assertion of Theorem 4.1(a), we refer readers to the work of Witte [40].

We end with a byproduct of the discussion in this section. For the remainder of the section, we assume that G is a compact p-adic Lie group with no p-torsion. The G-Euler characteristics of a $\mathbb{Z}_p[\![G]\!]$-module M is said to be well-defined if $H_j(G, M)$ is finite for every $j \geq 0$. In the event of such, the G-Euler characteristics of M is then defined to be

$$\chi(G, M) = \prod_{j \geq 0} \big(\#H_j(G, M)\big)^{(-1)^j}.$$

Note that since the group G is assumed to have no p-torsion, it has finite p-cohomological dimension (see [32, Corollaire 1]), and therefore, the alternating product is a finite product. We now evaluate the G-Euler characteristics of the second Iwasawa cohomology group for p-adic Lie extension with dimension at least 2.

Theorem 4.5 *Notation as before. Suppose further that G has dimension ≥ 2 and has no p-torsion. Then one has*

$$\chi\big(G, H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\big) = \frac{\#K_{2i-2}(\mathcal{O}_{F,S})[p^\infty]}{\#K_{2i-1}(\mathcal{O}_{F,S})[p^\infty]}.$$

Proof Indeed, from Theorem 4.1(b), we have $H^1_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big) = 0$. Taking this latter observation into account, the spectral sequence

$$H_r\big(G, H^{-s}_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\big) \Longrightarrow H^{-r-s}\big(G_S(F), \mathbb{Z}_p(i)\big)$$

degenerates yielding

$$H_0\big(G, H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\big) \cong K_{2i-2}(\mathcal{O}_{F,S})[p^\infty],$$

$$H_1\big(G, H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\big) \cong K_{2i-1}(\mathcal{O}_{F,S})[p^\infty]$$

and $H_j\big(G, H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\big) = 0$ for $j \geq 2$. The conclusion is now immediate. ∎

Remark 4.6 Note that Proposition 4.5 does not require F_∞ to contain F_{cyc}.

5 Growth of even *K*-groups

In this section, we present growth formulas for even K-groups over certain p-adic Lie extensions of global function fields. For a positive integer m, we write $\mathrm{ord}_p(m)$ for the highest power of p dividing m, i.e., $m = p^{\mathrm{ord}_p(m)}m'$ for some $p \nmid m'$. If N is a finite

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group, we then write $\mathrm{ord}_p(N)$ for $\mathrm{ord}_p(|N|)$. In particular, if N is a finite p-group, one has $|N| = p^{\mathrm{ord}_p(N)}$.

We begin with the cyclotomic $\mathbb{Z}_p$-extension.

Theorem 5.1 *Let $i \geq 2$ be given. Denote by F_n the intermediate subfields of F_{cyc}/F with $|F_n : F| = p^n$. Then we have*

$$\mathrm{ord}_p\Big(K_{2i-2}(\mathcal{O}_{F_n,S})[p^\infty]\Big) = \mathrm{rank}_{\mathbb{Z}_p}\left(H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\right)n + O(1).$$

Proof Write $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. By a combination of Propositions 3.1 and 4.2, we have

$$H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_{\Gamma_n} \cong H^2\big(G_S(F_n), \mathbb{Z}_p(i)\big) \cong K_{2i-2}(\mathcal{O}_{F_n,S})[p^\infty].$$

The conclusion of the theorem now follows from an application of [28, Proposition 5.3.17] and taking Theorem 4.1(a) into account. ∎

Remark 5.2 In the number field context, the above was first proved by Coates [3, Theorem 9] for K_2. Subsequently, Ji–Qin [16, Theorem 3.3(2)] extends Coates's result to the higher even K-groups. In these works, they have shown that the growth has the form of $\mu p^n + \lambda n + O(1)$. It is generally conjectured the leading term μp^n vanishes for the cyclotomic $\mathbb{Z}_p$-extension of a number field (see [14]; also see [23, Section 5]), although this vanishing is only known in the case when the base field is abelian over $\mathbb{Q}$ (see [7]). In the function field context, this vanishing is guaranteed by virtue of Theorem 4.1.

Recall that Lazard's theorem (cf. [5, Corollary 8.34]) tells us that every compact p-adic Lie group contains an open normal subgroup which is a uniform pro-p group in the sense of [5, Definition 4.1]. Therefore, one can always reduce consideration for a general compact p-adic Lie group to the case of a uniform pro-p group which we will do for the remainder of this section. For a uniform pro-p group, we shall write G_n for the lower p-series $P_{n+1}(G)$ which is defined recursively by $P_1(G) = G$, and

$$P_{n+1}(G) = \overline{P_n(G)^p[P_n(G), G]}, \text{ for } n \geq 1.$$

For every n, we have $|G : G_n| = |G : P_{n+1}(G)| = p^{dn}$ (see [5, Sections 3 and 4]), where d is the dimension of the uniform pro-p group G. We now record two results which we require.

Proposition 5.3 *Let G be a uniform pro-p group which contains a closed subgroup $H \cong \mathbb{Z}_p^{d-1}$ such that $G/H \cong \mathbb{Z}_p$. Suppose that M is a $\mathbb{Z}_p[\![G]\!]$-module which is finitely generated over $\mathbb{Z}_p[\![H]\!]$ and that M_{G_n} is finite for every n. Then we have*

$$\mathrm{ord}_p(M_{G_n}[p^\infty]) = \mathrm{rank}_{\mathbb{Z}_p[\![H]\!]}(M)np^{(d-1)n} + O(p^{(d-1)n}).$$

Proof This proposition is implicit in the discussion in [20, 21], and a formal presentation of which is given in [23, Proposition 2.4.1]. ∎

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Proposition 5.4 *Let G be a uniform pro-p group which contains a closed subgroup H such that $G/H \cong \mathbb{Z}_p$. Suppose that M is a $\mathbb{Z}_p[\![G]\!]$-module which is finitely generated over $\mathbb{Z}_p[\![H]\!]$. Then we have*

$$\mathrm{ord}_p(M_{G_n}/p^n) \leq \mathrm{rank}_{\mathbb{Z}_p[\![H]\!]}(M)np^{(d-1)n} + \mu_H(M)p^{(d-1)n} + O(np^{(d-2)n}).$$

Proof This result can be thought as a variant of the results of Perbet in [30], and its idea of proof follows closely to that in Perbet (see [22, Proposition 2.4] for the details). ∎

Building on our Theorem 4.1, we can prove the following asymptotic formula for the growth of even K-groups for a special class of p-adic Lie extensions.

Theorem 5.5 *Let $i \geq 2$ be given. Suppose that F_∞ is a p-adic Lie extension of F containing F_{cyc} with $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}}) \cong \mathbb{Z}_p^{d-1}$. Let F_n be the intermediate subfield of F_∞/F with $\mathrm{Gal}(F_\infty/F_n) = G_n$. Then we have*

$$\begin{aligned}&\mathrm{ord}_p\Big(K_{2i-2}(\mathcal{O}_{F_n,S})[p^\infty]\Big)\\ &\quad = \mathrm{rank}_{\mathbb{Z}_p[\![H]\!]}\Big(H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\Big)np^{(d-1)n} + O(p^{(d-1)n}).\end{aligned}$$

Proof By Theorem 4.1, the module $H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)$ is finitely generated over $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})]\!]$. Therefore, we may combine Propositions 4.2 and 5.3 to obtain the asymptotic formula. ∎

For a general p-adic Lie extension which contains F_{cyc}, we at least have an upper bound.

Proposition 5.6 *Let $i \geq 2$ be given. Suppose that F_∞ is a p-adic Lie extension of F containing F_{cyc} with G being a uniform pro-p group of dimension d. Let F_n be the intermediate subfield of F_∞/F with $\mathrm{Gal}(F_\infty/F_n) = G_n$. Then we have*

$$\begin{aligned}\mathrm{ord}_p\Big(K_{2i-2}(\mathcal{O}_{F_n})[p^n]\Big) &\leq \mathrm{rank}_{\mathbb{Z}_p[\![H]\!]}\Big(H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\Big)np^{(d-1)n}\\ &+ \mu_H\Big(H^2_{\mathrm{Iw},S}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\Big)p^{(d-1)n} + O(np^{(d-2)n}).\end{aligned}$$

Proof In this context, we make use of Proposition 5.4 and note that $\#K_{2i-2}(\mathcal{O}_{F_n})/p^n = \#K_{2i-2}(\mathcal{O}_{F_n})[p^n]$ in view of the finiteness of $K_{2i-2}(\mathcal{O}_{F_n})$. ∎

6 Comparing limits

We come to the final section of the article, whose goal is to prove a relation between the direct limit and the inverse limit of even K-groups. For two finite subextensions $L \subseteq L'$ of F contained in F_∞, the inclusion $\mathcal{O}_{L,S} \longrightarrow \mathcal{O}_{L',S}$ induces a map $\jmath_{L/L'}$: $K_{2i-2}(\mathcal{O}_{L,S})[p^\infty] \longrightarrow K_{2i-2}(\mathcal{O}_{L',S})[p^\infty]$ by functoriality. (Note that the induced map may not be injective in general.) In the other direction, there is the norm map (also called the transfer map) $\mathrm{Tr}_{L'/L} : K_{2i-2}(\mathcal{O}_{L',S})[p^\infty] \longrightarrow K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]$. Consider

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the following direct limit and inverse limit

$$\varinjlim_{L} K_{2i-2}(\mathcal{O}_{L,S})[p^\infty] \quad \text{and} \quad \varprojlim_{L} K_{2i-2}(\mathcal{O}_{L,S})[p^\infty],$$

whose transition maps are given by the maps $\jmath_{L/L'}$ and $\mathrm{Tr}_{L'/L}$ respectively. The main result of this section is as follows.

Theorem 6.1 *Let $i \geq 2$ and let F_∞ be a p-adic Lie extension of F which is contained in F_S. Then there is an isomorphism*

$$\left(\varinjlim_{L} K_{2i-2}(\mathcal{O}_{L,S})[p^\infty]\right)^\vee \cong \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}\left(\varprojlim_{L} K_{2i-2}(\mathcal{O}_{L,S})[p^\infty], \mathbb{Z}_p[\![G]\!]\right)$$

of $\mathbb{Z}_p[\![G]\!]$-modules.

The following is an immediate corollary of the theorem.

Corollary 6.2 *Retain settings as in Theorem 6.1. Then we have $\varinjlim_{L} K_{2i-2}(\mathcal{O}_L)[p^\infty] = 0$ if and only if $\varprojlim_{L} K_{2i-2}(\mathcal{O}_L)[p^\infty]$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$.*

We now give the proof of Theorem 6.1.

Proof of Theorem 6.1 Now, if the dimension of G is 1, then by base-changing, we may assume that F_∞/F is a $\mathbb{Z}_p$-extension, which is a commutative p-adic Lie extension. In this situation, the theorem follows from a direct similar argument to that in [24, Theorem 3.1]. Therefore, we may assume that G has dimension ≥ 2. From the discussion in [34, Chapter III], there is a commutative diagram

$$\begin{array}{ccc} K_{2i-2}(\mathcal{O}_{L,S})[p^\infty] & \xrightarrow{\mathrm{ch}_i^L} & H^2\left(G_S(L), \mathbb{Z}_p(i)\right) \\ \downarrow{\scriptstyle \mathrm{Tr}_{L/F}} & & \downarrow{\scriptstyle \mathrm{cor}} \\ K_{2i-2}(\mathcal{O}_{F,S})[p^\infty] & \xrightarrow{\mathrm{ch}_i^F} & H^2\left(G_S(F), \mathbb{Z}_p(i)\right) \end{array}$$

which in turn yields

$$H^2_{\mathrm{Iw}}\left(F_\infty/F, \mathbb{Z}_p(i)\right) \cong \varprojlim_{L} K_{2i-2}(\mathcal{O}_{L,S})[p^\infty].$$

For brevity, write $M = H^2_{\mathrm{Iw}}\left(F_\infty/F, \mathbb{Z}_p(i)\right)$. By Jannsen's result (cf. [15, Theorem 2.1] or [28, Theorem 5.4.13]), there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow \left(\varinjlim_{U}(M_U[p^\infty])\right)^\vee &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \\ \longrightarrow \left(\varinjlim_{U} H_1(U, M) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee &\longrightarrow 0. \end{aligned} \tag{6.1}$$

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where the transition map in the limit is given by norm map given as follow: For two open normal subgroups $V \subseteq U$ of G, the norm map $N_{U/V}$ on M_V factors through $M_U = (M_V)_{U/V}$ to yield a map $M_U \longrightarrow M_V$, which by abuse of notation is also denoted by $N_{U/V}$.

By a similar argument to that in [24, Lemma 3.6], we obtain the following commutative diagram

$$\begin{array}{ccccc} H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_U & \xrightarrow[\sim]{} & H^2\big(G_S(L), \mathbb{Z}_p(i)\big) & \xleftarrow[\sim]{\mathrm{ch}_i^L} & K_{2i-2}(\mathcal{O}_{L,S})[p^\infty] \\ \downarrow{\scriptstyle N_{U/V}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle j_{L/L'}} \\ H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_V & \xrightarrow[\sim]{} & H^2\big(G_S(L'), \mathbb{Z}_p(i)\big) & \xleftarrow[\sim]{\mathrm{ch}_i^{L'}} & K_{2i-2}(\mathcal{O}_{L',S})[p^\infty] \end{array}$$

which yields an identification of the leftmost term in the short exact sequence (6.1) with

$$\varinjlim_L K_{2i-2}(\mathcal{O}_{L,S})[p^\infty].$$

It therefore remains to show the rightmost term in the short exact sequence (6.1) vanishes. For this, it suffices to show that $H_1(U, M)$ is finite for every open subgroup U of G. But as already seen in the proof of Theorem 4.5, if we denote by L the fixed field of U, then one has

$$H_1\Big(U, H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)\Big) \cong K_{2i-1}(\mathcal{O}_{L,S})[p^\infty],$$

where the latter K-group is finite by Harder's theorem. The proof of our theorem is now complete. ∎

Acknowledgements The author would like to thank Li-Tong Deng, Yukako Kezuka and Yongxiong Li for their interest and comments. In particular, the joint work [4] with them has been a great inspiration to the writing of this article. The author also likes to thank Chao Qin for helpful comments and discussions relating to the article. Finally, the author thanks the anonymous referee for the valuable comments and suggestions which helps to improve the clarity of the manuscript.

References

[1] H. Bass and J. Tate, *The Milnor ring of a global field*. In: H. Bass (ed.), Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, WA), Lecture Notes in Mathematics, 342, Springer, Berlin, Heidelberg, 1972, pp. 349–446.

[2] S. Bloch and K. Kato, *p-adic étale cohomology*. Inst. Hautes Études Sci. Publ. Math. No. **63**(1986), 107–152.

[3] J. Coates, *On K_2 and some classical conjectures in algebraic number theory*. Ann. Math. (2) **95**(1972), 99–116.

[4] L.-T. Deng, Y. Kezuka, Y.-X. Li, and M. F. Lim, *Non-commutative Iwasawa theory of abelian varieties over global function fields*. Preprint, arXiv:2405.20963 [math.NT].

[5] J. Dixon, M. P. F. Du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*. 2nd ed., Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1999.

https://doi.org/10.4153/S0008439525000049 Published online by Cambridge University Press

[6] W. G. Dwyer and E. Friedlander, *Algebraic and etale K-theory*. Trans. Amer. Math. Soc. **292**(1985), no. 1, 247–280.
[7] B. Ferrero and L. C. Washington, *The Iwasawa invariant μ_p vanishes for abelian number fields*. Ann. Math. (2) **109**(1979), no. 2, 377–395.
[8] T. Fukaya and K. Kato, *A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory*. In: Proceedings of the St. Petersburg mathematical society, *Vol.* 12, American Mathematical Society Translations: Series 2, Ser. 2, 219, American Mathematical Society, Providence, RI, 2006, pp. 1–85.
[9] T. Geisser and M. Levine, *The K-theory of fields in characteristic p*. Invent. Math. **139**(2000), no. 3, 459–493.
[10] K. R. Goodearl and R. B. Warfield, *An introduction to non-commutative Noetherian rings*, London Mathematical Society Student Texts 61, Cambridge University Press, Cambridge, 2004.
[11] G. Harder, *Die Kohomologie S-arithmetischer Gruppen über Funktionenkörpern*. Invent. Math. **42**(1977), 135–175.
[12] S. Howson, *Euler characteristics as invariants of Iwasawa modules*. Proc. London Math. Soc. (3) **85**(2002), no. 3, 634–658.
[13] S. Howson, *Structure of central torsion Iwasawa modules*. Bull. Soc. Math. France **130**(2002), no. 4, 507–535.
[14] K. Iwasawa, *On Zl-extensions of algebraic number fields*. Ann. of Math. (2) **98**(1973), 246–326.
[15] U. Jannsen, *Iwasawa modules up to isomorphism*. In: Algebraic number theory, Advanced Studies in Pure Mathematics, 17, Academic Press, Boston, MA, 1989, pp. 171–207.
[16] Q. Ji and H. Qin, *Iwasawa theory for K_{2nF}*. J. K-Theory **12**(2013), no. 1, 115–123.
[17] M. Kolster, *K-theory and arithmetic*. In: Contemporary developments in algebraic *K*-theory, ICTP Lecture Notes Series, 15, The Abdus Salam International Centre for Theoretical Physics, Trieste, 2004, pp. 191–258.
[18] K. F. Lai and K.-S. Tan, *A generalized Iwasawa's theorem and its application*. Res. Math. Sci. **8**(2021), no. 2, Paper No. 20, 18 pp.
[19] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, 189, Springer, New York, 1999.
[20] A. Lei, *Estimating class numbers over metabelian extensions*. Acta Arith. **180**(2017), 347–364.
[21] D. Liang and M. F. Lim, *On the Iwasawa asymptotic class number formula for $\mathbb{Z}_p^r \rtimes \mathbb{Z}_p$-extensions*. Acta Arith. **189**(2019), no. 2, 191–208.
[22] M. F. Lim, *A note on asymptotic class number upper bounds in p-adic Lie extensions*. Acta Math. Sin. (Engl. Ser.) **35**(2019), no. 9, 1481–1490.
[23] M. F. Lim, *On the growth of even K-groups of rings of integers in p-adic Lie extensions*. Israel J. Math. **249**(2022), no. 2, 735–767.
[24] M. F. Lim, *Comparing direct limit and inverse limit of even K-groups in multiple $\mathbb{Z}_p$-extensions*. J. Théor. Nr. Bordx. **36**(2024) no. 2, pp. 537–555.
[25] M. F. Lim and R. Sharifi, *Nekovář duality over p-adic Lie extensions of global fields*. Doc. Math. **18**(2013), 621–678.
[26] J. Milnor, *Algebraic K-theory and quadratic forms*. Invent. Math. **9**(1969/70), 318–344.
[27] J. Nekovář, *Selmer complexes*. Astérisque **310**(2006), viii+559 pp.
[28] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der mathematischen Wissenschaften, 323, Springer-Verlag, Berlin, 2008.
[29] A. Neumann, *Completed group algebras without zero divisors*. Arch. Math. **51**(1988), no. 6, 496–499.
[30] G. Perbet, *Sur les invariants d'Iwasawa dans les extensions de Lie p-adiques*. Algebra Number Theory 5(2011), 819–848.
[31] D. Quillen, *Higher algebraic K-theory. I*. In: Algebraic *K* -theory, I: Higher *K* -theories (Proc. Conf., Battelle Memorial Inst., Seattle, WA, 1972), Lecture Notes in Mathematics, 341, Springer, Berlin 1973, pp. 85–147.
[32] J. -P. Serre, *Sur la dimension cohomologique des groupes profinis*. Topology 3(1965), pp. 413–420.
[33] J.-P. Serre and L. Algebra, *Springer monographs in mathematics*, Spring-Verlag, Berlin, 2000.
[34] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*. Invent. Math. **55**(1979), no. 3, 251–295.
[35] D. Vauclair, *Sur la dualité et la descente d'Iwasawa*. Ann. Inst. Fourier Grenob. **59**(2009), no. 2, 691–767.

https://doi.org/10.4153/S0008439525000049 Published online by Cambridge University Press

[36] O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*. J. Eur. Math. Soc. **4**(2002), no. 3, 271–311.
[37] V. Voevodsky, *On motivic cohomology with* $\mathbf{Z}/l$*-coefficients*. Ann. of Math. (2) **174**(2011), no. 1, 401–438.
[38] C. Weibel, *The norm residue isomorphism theorem*. J. Topol. **2**(2009), no. 2, 346–372.
[39] C. Weibel, *The K-book. An introduction to algebraic K -theory*, Graduate Studies in Mathematics, 145, American Mathematical Society, Providence, RI, 2013, xii+618 pp.
[40] M. Witte, *Non-commutative Iwasawa main conjecture*. Int. J. Number Theory **16**(2020), no. 9, 2041–2094.

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