Math. Proc. Camb. Phil. Soc. (1983), **93**, 467–475
Printed in Great Britain

Localization in enveloping algebras

BY A. I. LICHTMAN
Ben Gurion University of the Negev,
Beer-Sheva, Israel

(*Received* 10 *December* 1981, *revised* 10 *November* 1982)

1. Let L be a finite-dimensional Lie algebra and $U(L)$ its universal envelope. It is known that $U(L)$ is a Noetherian domain (see (5), theorem v. 3·4) and therefore $U(L)$ has a field of fractions. (Throughout the paper we use the term 'field' in the sense of skew field.) We prove in this article the following theorem.

THEOREM 4·1. *Let L be a finite-dimensional solvable Lie algebra over a field of characteristic zero, Δ the field of fractions of the universal envelope $U(L)$. Then Δ is the universal field of fractions of $U(L)$.*

We recall the definition and some simple properties of the universal field of fractions in § 4 of the article and refer the reader to (1), chapter 7, for a detailed exposition.

Theorem 4·1 is an analogue of Passman's theorem in (11), which states that the field of fractions of the group ring RG of a poly-{infinite cyclic} group G is the universal field of fractions of RG.

One of the ideas used in Passman's proof is to combine Roseblade's result for the lifting of the AR-property in cross product (see (12)) together with the transitivity property of specializations. To apply this idea in our case we first prove the following analogue of Roseblade's theorem, which we hope is of independent interest.

PROPOSITION 3·1. *Let S be a right Noetherian ring with a unit and $R = S\langle x_1, x_2, \ldots, x_n\rangle$, where the elements x_α satisfy the conditions $[x_\alpha, x_\beta] \in S + x_1 S + \ldots + x_n S$ $(1 \leqslant \alpha, \beta \leqslant n)$, $[x_\alpha, s] \in S$ $(s \in S;\ \alpha = 1, 2, \ldots, n)$. Let A be a polycentral ideal of S such that*

$$[a, x_\alpha] \in A \quad (a \in A;\ \alpha = 1, 2, \ldots, n).$$

Then AR is an ideal of R which satisfies the weak AR-property. ∎

It is worth remarking that the conditions of Proposition 3·1 are satisfied if $R = U(L)$ is the universal envelope of a solvable finite-dimensional Lie algebra over a field of characteristic zero and A is an L-invariant ideal of $S = U(L')$, where L' is the derived subalgebra of L. (See Corollary of Proposition 3·1.)

2. *Preliminaries.* We recall in this section some concepts and some known results from localization theory. (See, for instance, (10), 11·2.)

An ideal of a ring R is said to be centrally generated if it has a finite generating set consisting of central elements of R. An ideal $I \subseteq R$ is polycentral of height m if there exists a finite series

$$I = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_m = 0 \qquad (2\cdot1)$$

of ideals of R such that, for each $0 \leqslant i \leqslant m-1$, Y_i/Y_{i+1} is a centrally generated ideal of R/Y_{i+1}.

The work on the article was completed when the author was visiting The University of Texas during the Fall term of 1982.

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

An ideal A has the weak AR-property if for any right ideal U there exists an integer n such that

$$U \cap A^n \subseteq UA. \tag{2·2}$$

Some applications of the AR-property depends on the study of the subring $\tilde{R}(A)$ of the polynomial ring $R[u]$:

$$R(A) = \langle R, Y_i u, Y_i u^2, \ldots, Y_i u^{2^i} (i = 0, 1, \ldots, m)\rangle. \tag{2·3}$$

We need two following assertions due to Roseblade. (See (12) or (10).)

LEMMA 2·1. *Let R be a right Noetherian ring. If A is polycentral then $\tilde{R}(A)$ is right Notherian.*

LEMMA 2·2. *If $\tilde{R}(A)$ is right Noetherian then A has the weak AR-property.*

3. 3·1. Let R be a ring and X be a set of derivations in R. A subset $U \subseteq R$ is X-invariant if $D(U) \subseteq U$ for any $D \in X$.

The following simple fact will be useful in our arguments.

LEMMA 3·1. *Let A be an X-invariant polycentral ideal of height m. Then there exists a series*

$$A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_k = 0,$$

with $k \leqslant m$ where all the terms A_j $(j = 0, 1, \ldots, k)$ are X-invariant and A_j/A_{j+1} is centrally generated $(j = 0, 1, \ldots, k-1)$.

Proof. Let Z be the centre of R. It is easy to see that Z is X-invariant. We consider now the series (2·1) and the subring $Z_{m-1} = A \cap Z$. Clearly, Z_{m-1} is X-invariant and therefore $A_{m-1} = Z_{m-1}R$ is an X-invariant ideal of R. The routine argument shows that one can find a finite number of elements $z_1, z_2, \ldots, z_l \in Z_{m-1}$ such that $Z_{m-1}R = z_1R + z_2R + \ldots + z_lR$. The definition of A_{m-1} and Y_{m-1} implies that $A_{m-1} \supseteq Y_{m-1}$.

Consider now the quotient ring R/A_{m-1} and its ideal A/A_{m-1}. The natural homomorphism $R \to R/A_{m-1}$ can be factored through the natural homomorphism $R \to R/Y_{m-1}$ and therefore the ideal $A/A_{m-1} \subseteq R/A_{m-1}$ is a homomorphic image of the ideal $A/Y_{m-1} \subseteq R/Y_{m-1}$ which is polycentral of height $(m-1)$. We conclude therefore that A/A_{m-1} is a polycentral ideal in R/A_{m-1} of height less or equal than $m-1$ and the proof is completed by induction on m. |

3·2. We prove in this section Proposition 3·1 which an analogue of Roseblade's theorem on the ideals in a cross product $T*G$ of a right Noetherian ring T and a polycyclic-by-finite group G. Our proof is based on Roseblade's method and in particular makes an essential use of Lemmas 2·1 and 2·2.

LEMMA 3·2. *Let S be a right Noetherian ring with unit and $R = S\langle x_1, x_2, \ldots, x_n\rangle$ where*

$$[s, x_\alpha] \in S \quad (s \in S,\ \alpha = 1, 2, \ldots, n), \tag{3·1}$$

and

$$[x_\alpha, x_\beta] \in S + x_1S + \ldots + x_nS \quad (1 \leqslant \alpha,\ \beta \leqslant n). \tag{3·2}$$

If S is right Noetherian then R is right Noetherian too.

Proof. We obtain from (3·1) and (3·2)

$$R = S + \sum_{i \leqslant \alpha \leqslant n} x_\alpha S + \sum_{1 \leqslant \alpha, \beta \leqslant n} x_\alpha x_\beta S + \ldots. \tag{3·3}$$

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

Now let $R^{(-1)} = 0$, $R^0 = S$, and $R^{(i)}$ $(i = 1, 2, \ldots)$ be the sum of the first $(i+1)$ S-submodules in (3·3). The conditions (3·1) and (3·2) imply that the system of submodules $R^{(i)}$ $(i = 0, 1, \ldots)$ defines a filtration in R, i.e.

$$\text{(I)} \quad R^{(i)} \subseteq R^{(j)} \text{ if } i \leqslant j, \qquad \text{(II)} \quad \bigcup_i R^{(i)} = R, \qquad \text{(III)} \quad R^{(i)}R^{(j)} \subseteq R^{i+j}.$$

The idea of the proof of theorem v, 3·6 of (5) about the universal envelope $U(L)$ of a finite-dimensional Lie algebra L is applied in our case too. We consider the associated graded ring $G(R)$ of the filtration R^i $(i = 0, 1, \ldots)$ and obtain as in (5) that $G(R)$ is generated by S and by the classes $\bar{x}_\alpha = x_\alpha + S$ $(\alpha = 1, 2, \ldots, n)$ and satisfies relations

$$\bar{x}_\alpha \bar{x}_\beta - \bar{x}_\beta \bar{x}_\alpha = (x_\alpha x_\beta + R^1) - (x_\beta x_\alpha + R^1) = (x_\alpha x_\beta - x_\beta x_\alpha) + R^{(1)},$$

which gives via (3·2)

$$\bar{x}_\alpha \bar{x}_\beta = \bar{x}_\beta \bar{x}_\alpha \quad (1 \leqslant \alpha,\ \beta \leqslant n). \tag{3·4}$$

Similarly, the relation (3·1) implies that

$$[S, \bar{x}_\alpha] = 0 \quad (\alpha = 1, 2, \ldots, n), \tag{3·5}$$

and if S is right Noetherian then (3·4) and (3·5) imply easily that $G(R)$ is Noetherian too. It is known (see (5), theorem v, 3·4) that this implies that R is right Noetherian and the assertion follows. ∎

A special case of the situation considered in Lemma 3·2 is obtained when the condition (3·2) is replaced by the stronger one

$$[x_\alpha, x_\beta] \in S \quad (1 \leqslant \alpha,\ \beta \leqslant n), \tag{3·2′}$$

and all the standard monomials

$$x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_m} \quad (1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m \leqslant n),$$

are right linearly independent over S. In this case every subring $S_i = S\langle x_1, x_2, \ldots, x_i\rangle$ is a differential polynomial ring over S_{i-1}:

$$S_i = S_{i-1}[x_i] \quad (S_0 = S;\ i = 1, 2, \ldots, n). \tag{3·6}$$

Thus in this case R is *a differential polynomial ring in* $x_1, x_2, \ldots, x_n$ *over* S and we denote it by $S[x_1, x_2, \ldots, x_n]$.

We will need in the proof of Theorem 4·1 the following well-known fact.

LEMMA 3·3. *Let L be a finite-dimensional Lie algebra and L' be the derived subalgebra and let $e_1, e_2, \ldots, e_n$ be elements of L which give a basis of the quotient algebra L/L'. Let $U(L)$ and $U(L')$ denote the universal envelopes of L and L' respectively, $A \subseteq U(L')$ be an ideal of $U(L')$ such that*

$$[a, e_\alpha] \in A \quad (a \in A;\ \alpha = 1, 2, \ldots, n),$$

and let $\bar{X}$ be the image of a subset $X \subseteq U(L)$ under the natural homomorphism $U(L) \to U(L)/AU(L)$; then $\overline{U(L)}$ is a differential polynomial ring in $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ over $\overline{U(L')}$ with relations:

$$\overline{U(L)} = \overline{U(L')}[\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n], \tag{3·7}$$

$$[\bar{x}, \bar{e}_\alpha] \in \overline{U(L')} \quad (\bar{x} \in \overline{U(L')};\ \alpha = 1, 2, \ldots, n), \tag{3·8}$$

$$[\bar{e}_\alpha, \bar{e}_\beta] \in \overline{U(L')} \quad (1 \leqslant \alpha,\ \beta \leqslant n). \tag{3·9}$$

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

Proof. We have the following relations in $U(L)$:

$$U(L) = \langle U(L'), e_1, e_2, \ldots, e_n \rangle, \tag{3·7'}$$

$$[x, e_\alpha] \in U(L') \quad (x \in U(L'),\ \alpha = 1, 2, \ldots, n), \tag{3·8'}$$

$$[e_\alpha, e_\beta] \in U(L') \quad (1 \leqslant \alpha,\ \beta \leqslant n). \tag{3·9'}$$

These relations imply relations (3·7)–(3·9). The standard monomials on the elements $e_1, e_2, \ldots, e_n$ are linearly independent over $U(L')$, this follows from the Poincaré–Birkhoff–Witt Theorem; this implies easily the linear independence of the standard monomials in $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ over $\overline{U(L')}$ and the assertion follows. ∎

PROPOSITION 3·1. *Let S be right Noetherian with unit and $R = S\langle x_1, x_2, \ldots, x_n \rangle$, where the elements x_α satisfy* (3·1) *and* (3·2).

Let A be a polycentral ideal of S such that

$$[a, x_\alpha] \in A \quad (a \in A;\ 1 \leqslant \alpha \leqslant n). \tag{3·10}$$

Then AR is an ideal of R which satisfies the weak AR-property.

Proof. Apply Lemma 3·1 and find a series

$$A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_k = 0,$$

where A_j/A_{j+1} is centrally generated and

$$[a_j, x_\alpha] \in A_j \quad (a_j \in A_j,\ j = 1, 2, \ldots, k). \tag{3·10'}$$

Consider the polynomial ring $R[u]$ and its subring

$$\begin{aligned} \tilde{R}(AR) &= \langle R, A_j Ru, A_j Ru^2, \ldots, A_j Ru^{2j}\ (1 \leqslant j \leqslant k) \rangle \\ &= \langle R, A_j u, A_j u^2, \ldots, A_j u^{2j}\ (1 \leqslant j \leqslant k) \rangle \\ &= \langle S, A_j u, A_j u^2, \ldots, A_j u^{2j}, x_\alpha\ (1 \leqslant j \leqslant k,\ 1 \leqslant \alpha \leqslant n) \rangle \\ &= \langle \tilde{S}(A), x_\alpha\ (1 \leqslant \alpha \leqslant n) \rangle. \end{aligned} \tag{3·11}$$

Since A is polycentral, $\tilde{S}(A)$ is Noetherian by Lemma 2·1. The relations (3·10) and (3·10') now imply that

$$[\tilde{S}(A), x_\alpha] \subseteq \tilde{S}(A).$$

This together with (3·11) implies that all the conditions of Lemma 3·2 are satisfied and therefore $\tilde{R}(AR)$ is Noetherian which completes the proof. ∎

COROLLARY. *Let L be a Lie algebra, H be a nilpotent ideal of L. Let A be an L-invariant ideal of H. Then $AU(L)$ is an ideal of $U(L)$, which satisfies the weak AR-property.*

Proof. The ideal A is polycentral by McConnell's theorem in (7). The assertion follows now from Proposition 3·1.

4. 4·1. Let R be a ring. A multiplicatively closed subset $M \subseteq R$ is a right denominator set if it satisfies the right Ore condition in R, i.e., for given $a \in R$, $b \in M$,

$$aM \cap bR \neq \varnothing. \tag{4·1}$$

If M contains only regular elements of R then (4·1) is a necessary and sufficient condition for the existence of the right ring of fractions RM^{-1} and M is localizable if it satisfies (4·1). A completely prime ideal $A \subseteq R$ is called localizable if its complement $M = R \backslash A$ is localizable.

We will need the following well-known facts.

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

LEMMA 4·1. *Let A be a completely prime ideal of R and $M \subseteq R\backslash A$ be a right denominator set of regular elements in R. Then*

(i) *AM^{-1} is a completely prime ideal in RM^{-1}.*

(ii) *A is localizable in R if and only if AM^{-1} is localizable in RM^{-1}.*

(iii) *The natural homomorphism $R \xrightarrow{\varphi} R/A$ is extended uniquely to the natural homomorphism $RM^{-1} \xrightarrow{\varphi_M} RM^{-1}/AM^{-1}$.*

The proofs are immediate. |

4·2. We recall now a few facts about specializations which will be used later in the proof of Theorem 4·1. The detailed exposition can be found in (1).

A specialization θ from a field Δ_1 to a field Δ_2 is an epimorphism of a local subring $T \subseteq \Delta_1$ on Δ_2. The ring T is the domain of θ.

Let Δ be a field, containing a ring R and generated by it. Then Δ is called the universal field of fractions of R if every homomorphism $R \xrightarrow{\varphi} \Delta_1$, where Δ_1 is a division ring generated by $\varphi(R)$, can be extended to a specialization $\Delta \to \Delta_1$.

We need the following properties of specializations.

LEMMA 4·2. *Let R and R/A be Ore domains with field of fractions Δ and Δ_1 correspondingly and $M \subseteq R\backslash A$ be a right denominator set of regular elements in R. Then the natural homomorphism $\varphi: R \to R/A$ can be extended to a specialization $\theta: \Delta_1 \to \Delta_2$ if and only if the homomorphism $\varphi_M: RM^{-1} \to RM^{-1}/AM^{-1}$ can be extended.*

The proof is immediate. |

The following transitivity lemma can be read off from § 7·2 of (1) or be proven in a straightforward way as in (11), lemma 2·2.

LEMMA 4·3. *Let $\theta: \Delta_1 \to \Delta_2$ be a specialization with a domain Q_1 and $\beta: \Delta_2 \to \Delta_3$ be a specialization with a domain Q_2. Then $Q_3 = \theta^{-1}(Q_2)$ is a local subring of Δ_1 with a maximal ideal N and the natural homomorphism*

$$Q_3 \to Q_3/N$$

defines a specialization $\beta\theta: \Delta_1 \to \Delta_3$.

4·3. The following assertion makes the first step in the construction of a specialization in the proof of Theorem 4·1.

PROPOSITION 4·2. *Let L be a solvable Lie algebra over a field of characteristic zero, $U(L)$ be its universal envelope, H be a nilpotent ideal of L. Let A be a prime L-invariant ideal of $U(H)$. Then $AU(L)$ is a completely prime localizable ideal in $U(L)$.*

Proof. Recall first that any prime ideal in the universal envelope of a finite-dimensional soluble Lie algebra over a field of characteristic zero is completely prime. (See (3), 3·7·2.)

It is known that the ideal $AU(L)$ must be completely prime too. Indeed, if $x_1, x_2, \ldots, x_n$ is the basis of the quotient algebra L/H then the standard monomials on these elements form a basis of $U(L)$ as a $U(H)$-module. The argument which is similar to the proof of the fact that $U(L)$ is a domain (see Theorem v. 3·6 of (5) can be applied now to show that $U(L)/AU(L)$ is a domain.

The ideal $AU(L)$ has the weak AR-property by the Corollary of Proposition 3·1. Finally, it is known (see (6, 8, 9)) that a prime ideal in the universal envelope of Lie algebra with the weak AR-property can be localized.

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

PROPOSITION 4·3. *Let $R = \langle S, x_i\,(i \in I)\rangle$ be a ring, generated by a subring S and a system of elements x_i, which satisfies the condition*

$$sx_i = x_i\varphi_i(s) + D_i(s), \tag{4·2}$$

where φ_i is an automorphism of S and D_i is a φ_i-derivation, i.e. an endomorphism of $(S, +)$ such that

$$D_i(ab) = D_i(a)\,\varphi_i(b) + aD_i(b) \quad (a, b \in S;\ i \in I). \tag{4·3}$$

Let $M \subseteq S$ be a right denominator set in S, which consists of regular elements of R and satisfies the condition

$$\varphi_i(M) \subseteq M \quad (i \in I).$$

Then M is a right denominator set in R.

Proof. Consider the ring of fractions SM^{-1}. It is easy to verify that every φ_i has a unique extension to SM^{-1} given by

$$\varphi_i(sm^{-1}) = \varphi_i(s)\,\varphi_i(m^{-1}).$$

Furthermore, using the same arguments as in (2), § 1·3 we can show that D_i has a unique extension to SM^{-1}. We reserve for the extensions of φ_i and D_i the original notations.

Consider now for every i the skew polynomial ring $S[t_i, \varphi_i, D_i]$ (see (2), § 1·3) and their coproduct

$$Q = \coprod_S S[t_i, \varphi_i, D_i].$$

Since every $S[t_i, \varphi_i, D_i]$ is free over S we obtain that the subring S is imbedded isomorphically in Q (see (2), § 1).

Clearly, there exists a homomorphism θ of Q on R defined by

$$\theta(t_i) = x_i \quad (i \in I), \quad \theta(s) = s \quad (s \in S),$$

and this implies easily that it is enough to prove that M is a right denominator in Q.

Consider for this purpose the system of rings of skew polynomials $SM^{-1}[t, \varphi_i, D_i]$ $(i \in I)$ and their coproduct

$$T = \coprod_S SM^{-1}[t_i, \varphi_i, D_i].$$

The canonical imbedding $Q \subseteq T$ follows from (2), 5·1 and it is easy to verify that any element of T has a form xb^{-1}, $x \in Q$, $b \in M$. Hence, $T = QM^{-1}$ which implies that M is a right denominator set in Q and the assertion follows. ∎

COROLLARY. *Let S be an Ore domain, $R = S\langle x_1, x_2, \ldots, x_n\rangle$, where the elements x_α $(\alpha = 1, 2, \ldots, n)$ satisfy the conditions* (3·1) *and* (3·2). *Then R is an Ore ring provided that it contains no zero divisors.*

Proof. It follows from Proposition 4·3 that S is a right denominator set in R. We can consider therefore the ring

$$RS^{-1} = \Delta\langle x_1, x_2, \ldots, x_n\rangle,$$

where Δ is the field of fractions of S; the ring $\Delta\langle x_1, x_2, \ldots, x_n\rangle$ is Noetherian by Lemma 3·1 and it follows now from Lemma 4·1 that $\Delta\langle x_1, x_2, \ldots, x_n\rangle$ and $S\langle x_1, x_2, \ldots, x_n\rangle$ are Ore domains. ∎

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

4·4. We assume in this section that $R = S[x_1, x_2, \ldots, x_n\rangle$ is a differential polynomial ring over an Ore domain S where the elements $x_1, x_2, \ldots, x_n$ satisfy (3·1) and (3·2′) and as in (3·6) $S_i = S_{i-1}[x_i]$ $(i = 1, 2, \ldots, n)$. Clearly, every S_i $(i = 1, 2, \ldots, n)$ is a differential polynominal ring in $x_1, x_2, \ldots, x_i$ over S and R is a differential polynomial ring in $x_{i+1}, x_{i+2}, \ldots, x_n$ over S_i. The following Lemma follows therefore immediately from Proposition 4·3 and its Corollary.

LEMMA 4·4. *For any $i = 0, 1, \ldots, n$ the set $S_i\backslash 0$ is a right denominator set in R.*

Let Δ be the field of fractions of R. We prove in this section Proposition 4·4 which will make the second main step in the proof of Theorem 5·1.

PROPOSITION 4·4. *Let A be a completely prime ideal of R such that $S \cap A = 0$, T be the field of fractions of R/A.† Then the natural homomorphism $R \to R/A$ can be extended to a specialization $\theta: \Delta \to T$.*

We need the following simple fact, which can be obtained as a corollary of theorem 2·4 in (13). We give a proof for the sake of completeness.

LEMMA 4·5. *Let R be a Noetherian domain. Assume that A is a completely prime ideal in R which is generated by an element $a \in A$ such that*

$$A = aR = Ra.$$

Then A can be localized.

Proof. Lemma 8 in (8) implies that A has the AR-property and hence $\cap_{k=1}^{\infty} A^k = 0$. We obtain therefore that for any $u \in A$ there exists a natural k such that

$$u = a^k r_1 = r_2 a^k, \quad r_1, r_2 \in R\backslash A. \tag{4·4}$$

Now let $r \in R$, $s \in R\backslash A$. Since R is an Ore domain we find $x, y \in R$ such that

$$rx = sy, \quad x \neq 0. \tag{4·5}$$

If now $x \in A$ we find by (4·4) natural k_1, k_2 such that

$$x = x_1 a^{k_1}, \quad x \in R\backslash A,$$

$$y = y_1 a^{k_2}, \quad y_1 \in R\backslash A,$$

and rewrite (4·5) in a form

$$rx_1 a^{k_1} = sy_1 a^{k_2}. \tag{4·5′}$$

Since $S \in R\backslash A$, $y_1 \in R\backslash A$ it follows easily from (4·5′) that $k_1 \leqslant k_2$ and hence

$$rx_1 = sy_1 a^{k_2 - k_1},$$

which proves that $R\backslash A$ is a right denominator set. ∎

Proof of Proposition 4·4. Find among the subrings $S_0 = S$, $S_1, \ldots, S_n$ the subring S_i which is maximal with respect to the property $S_i \cap A = 0$. Then $R = S_i[x_{i+1}, x_{i+2}, \ldots, x_n]$, where the elements $x_{i+1}, x_{i+2}, \ldots, x_n$ satisfy the relations (3·1) and (3·2′) over the subring S_i. Lemma 4·4 implies that the set $M = S_i\backslash 0$ is a right denominator set in R and it is easy to see that $RM^{-1} = S_i M^{-1}[x_{i+1}, x_{i+2}, \ldots, x_n]$, where $S_i M^{-1}$ is the field of fractions of S_i and once again the elements $x_{i+1}, x_{i+2}, \ldots, x_n$ satisfy relations (3·1) and (3·2′) (over $S_i M^{-1}$). Lemma 4·1 implies that the homomorphism φ is extended

† The existence of such a field of fractions follows, for instance, from the Corollary of Proposition 4·3.

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

to a specialization $\theta: \Delta \to T$ if and only if the homomorphism $\varphi_M: RM^{-1} \to RM^{-1}/AM^{-1}$ is extended to such a specialization.

It is worth remarking that

$$AM^{-1} \cap S_i M^{-1} = 0, \tag{4·6}$$

but

$$AM^{-1} \cap S_i M^{-1}[x_{i+1}] \neq 0. \tag{4·7}$$

In order to simplify the notations *we assume now that R itself has over S the properties* (4·6) *and* (4·7), *i.e. S is a subfield of R, $A \cap S = 0$ but*

$$A \cap S[x_i] \neq 0.$$

Let $A_1 = A \cap S[x_1]$. It is known that $S[x_1]$ is a principal ideal ring (see (4), § 3·1), and hence there exists an element $\varphi[x_1] \in S[x_1]$ such that

$$A_1 = \varphi[x_1] S[x_1] = S[x_1] \varphi[x_1].$$

We obtain therefore

$$[\varphi[x_1], x_\alpha] \in A \cap S[x_1] = A_1 \quad (\alpha = 1, 2, \ldots, n),$$

and therefore

$$[\varphi[x_1], x_\alpha] = \varphi[x_1] u_\alpha = v_\alpha \varphi[x_1], \quad u_\alpha, v_\alpha \in S[x_1] \quad (\alpha = 1, 2, \ldots, n).$$

This implies easily that the principal ideal $B_1 = (\varphi[x_1]) R$ is a two-sided ideal of R.

The ideal A_1 is completely prime in $S[x_1]$ and $A_1 \neq 0$; since S is a field we conclude that $S_1 = S[x_1]/A_1$ is a field too. Furthermore, the quotient ring R/B_1 is isomorphic to the ring of differential polynomials $S_1[\bar{x}_2, \ldots, \bar{x}_n]$ with relations

$$[S_1, \bar{x}_\alpha] \subseteq S_1, \quad [\bar{x}_\alpha, \bar{x}_\beta] \in S_1 \quad (2 \leqslant \alpha, \ \beta \leqslant n).$$

This implies in particular that B_1 is completely prime and Lemma 4·5 implies that B_1 is localizable in R; thus, if Δ_1 denotes the field of fractions of $S_1[\bar{x}_2, \ldots, \bar{x}_n]$ then we obtained that the natural homomorphism $R \xrightarrow{\varphi} S_1[\bar{x}_2, \ldots, \bar{x}_n]$ can be extended to specialization $\alpha: \Delta \to \Delta_1$.

Consider now the image $\varphi(A)$ of A in $S_1[\bar{x}_2, \ldots, \bar{x}_n]$. The usual induction argument now implies that the natural homomorphism

$$S_1[\bar{x}_2, \ldots, \bar{x}_n] \xrightarrow{\psi} S_1[\bar{x}_2, \ldots, \bar{x}_n]/\varphi(A) \simeq R/A,$$

can be extended to a specialization $\beta: \Delta \to T$ and Lemma 4·4 implies the existence of a specialization $\theta = \beta\alpha$, which extends the homomorphism $R \xrightarrow{\psi\varphi} R/A$.

4·5. THEOREM 4·1. *Let L be a finite-dimensional solvable Lie algebra over a field of characteristic zero, $U(L)$ be the universal envelope of L and Δ be the Ore field of fractions of $U(L)$. Then Δ is the universal field of fractions of $U(L)$.*

Proof. Let P be a completely prime ideal of $U(L)$, and T be the field of fractions of $U(L)/P$. We have to prove that the natural homomorphism $U(L) \to U(L)/P$ is extended to a specialization $\theta: \Delta \to T$.

Consider first the intersection $P \cap U(L')$. Since L' is nilpotent (see (5), II.7), Proposition 4·2 implies that $(P \cap U(L'))\, U(L)$ is a completely prime localizable ideal in $U(L)$.

Let $\bar{X}$ denote the image of a subset $X \subseteq U(L)$ under the natural homomorphism

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press

$U(L) \to U(L)/(P \cap U(L'))\,U(L)$. Then $\overline{U(L)}$ is a Notherian domain with a field of fractions, say Q, and $\bar{P} \cap \overline{U(L')} = \bar{0}$. If now $e_1, e_2, \ldots, e_n$ be elements of L which give a basis of L/L' then Lemma 3·3 implies that the elements $\bar{e}_\alpha$ $(\alpha = 1, 2, \ldots, n)$ satisfy all the conditions of Proposition 4·4 and hence the natural homomorphism $\overline{U(L)} \to \overline{U(L)}/\bar{P} \simeq U(L)/P$ can be extended to a specialization $\alpha: Q \to T$. Lemma 4·3 gives now the existence of a specialization $\Delta \to T$, which extends the natural homomorphism $U(L) \to U(L)/P$. ∎

I am grateful to the referee for his useful remarks and criticism which helped to improve the exposition of the article and to correct some inaccuracies and errors.

REFERENCES

(1) Cohn, P. M. *Free rings and their relations* (Academic Press, N.Y. 1971).
(2) Cohn, P. M. *Skew field constructions* (Cambridge University Press, 1977).
(3) Dixmier, J. *Algèbres enveloppantes* (Gauthier-Villars, Paris, 1974).
(4) Jacobson, N. *Theory of rings.* (Amer. Math. Soc., Providence, 1943).
(5) Jacobson, N. *Lie algebras* (Wiley, New York and London, 1962).
(6) Lenagan, T. H. Gelfand–Kirillov dimension in enveloping algebras. *Quart. J. Math. Oxford* (2) **32** (1981), 69–80.
(7) McConnell, J. C. The intersection theorem for a class of non commutative rings. *Proc. London Math. Soc.* (3), **17** (1967), 487–498.
(8) McConnell, J. C. Localization in enveloping rings. *J. London Math. Soc.* **43** (1968), 421–428.
(9) McConnell, J. C. Localization in enveloping rings: Erratum and Addendum. *J. London Math. Soc.* (2), **3** (1971), 405–410.
(10) Passman, D. S. *The algebraic structure of group rings* (Wiley, New York, 1977).
(11) Passman, D. S. Universal fields of fractions for polycyclic group algebras. *Glasgow Math. Journal* **23** (1982), 103–113.
(12) Roseblade, J. E. Applications of the Artin–Rees lemma to group rings. *Symp. Math.* **17** (1976).
(13) Smith, P. F. Localization and *AR* property. *Proc. London Math. Soc.* **22** (1971), 39–69.

https://doi.org/10.1017/S0305004100060771 Published online by Cambridge University Press