Extending the substitution method for integration

Introduction

The famous *substitution method* (see, e.g., [1, p. 384]), if directly applied to evaluate integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx, \tag{1}$$

where $P(x)$ and $Q(x)$ are real polynomials in x, works only under the restrictive condition that *the derivative of the denominator is a nonzero scalar multiple of the numerator*, or more mathematically, if (1) can be written as

$$\int \frac{\alpha f'(x)}{f(x)}\,dx \;=\; \alpha \ln|f(x)| \;+\; c, \tag{2}$$

where $\alpha \in \mathbb{R}$ and f is a differentiable function. An extension can be made by reflecting on the following way to evaluate

$$\int \frac{x^6 + 2}{x^7 - x}\,dx, \tag{3}$$

which, although does not satisfy the condition above, still has the weaker property that the exponent of every term in the numerator is one less than that in the denominator. The basic idea is to notice that dividing the numerator and denominator by x^5 transforms (3) into an integral solvable by (2). Written thoroughly,

$$\int \frac{x^6+2}{x^7-x}\,dx = \int \frac{x+2x^{-5}}{x^2-x^{-4}}\,dx = \int \frac{\frac{1}{2}\left(2x+4x^{-5}\right)}{x^2-x^{-4}}\,dx = \frac{1}{2}\ln\left|x^2-x^{-4}\right| + c.$$

Binomial numerator and denominator

More generally, integrals of the form

$$\int \frac{Ax^{m-1} + B}{Cx^m + Dx}\,dx,$$

where A, B, C, D and m are constants with $AD - BC \neq 0$ and $m \neq 1$, is solvable by the tactic above if, and only if, there exist real numbers k and n such that

$$\frac{d}{dx}\left(\frac{Cx^m + Dx}{x^n}\right) = k\left(\frac{Ax^{m-1} + B}{x^n}\right)$$

$$\Leftrightarrow \left(Cm - Ak - Cn\right)x^{m-1-n} + \left(D - Bk - Dn\right)x^{-n} = 0.$$

That is, if, and only if, the system of equations

$$\begin{cases} Ak + Cn = Cm \\ Bk + Dn = D \end{cases}$$

has a solution (k, n), which is always true since $AD - BC \neq 0$. In fact, the

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solution is unique and has a compact expression in terms of 2×2 determinants, by Cramer's rule. Our conclusion is

Proposition 1: Let A, B, C, D and m be real numbers with $AD - BC \neq 0$ and $m \neq 1$. Then the integral $\int \frac{Ax^{m-1} + B}{Cx^m + Dx}\,dx$ is solvable by substitution after dividing the numerator and denominator by x^n, where $n = \frac{AD - BCm}{AD - BC}$.

To make use of this result, the reader may be interested to verify the following two equalities

$$\int \frac{x^7 + 1}{x^8 + 2x}\,dx = \frac{1}{14}\ln\left|x^{14} + 2x^7\right| + c,$$

$$\int \frac{x^7 + 1}{x^8 - x}\,dx = \frac{2}{7}\ln\left|x^{7/2} - x^{-7/2}\right| + c.$$

Variation

As an effort to extend the tactic as much as possible, let us now consider the integral

$$\int \frac{Ax^p + Bx^q}{(Cx^r + Dx^s)^m}\,dx,$$

where variables other than x are all real numbers with $AD - BC \neq 0$, $m \neq 1$, $p > q$ and $r > s$. Solving this requires the existence of two real numbers k and n such that if we divide the numerator and denominator of the integrand by x^n, the resulting integral

$$\int \frac{Ax^p + Bx^q}{(Cx^r + Dx^s)^m}\,dx = \int \frac{Ax^{p-n} + Bx^{q-n}}{(Cx^{r-n/m} + Dx^{s-n/m})^m}\,dx$$

satisfies

$$\frac{d}{dx}\left(Cx^{r-n/m} + Dx^{s-n/m}\right)^m = k\left(Ax^{p-n} + Bx^{q-n}\right)$$

$$C\left(r - \frac{n}{m}\right)x^{r-n/m-1} + D\left(s - \frac{n}{m}\right)x^{s-n/m-1} = Akx^{p-n} + Bkx^{q-n}. \quad (4)$$

Since $p > q$ and $r > s$, then $r - \frac{n}{m} - 1 > s - \frac{n}{m} - 1$ and $p - n > q - n$. Therefore, (4) holds for all x if, and only if, the system of equations

$$\begin{cases} r - \frac{n}{m} - 1 = p - n \\ s - \frac{n}{m} - 1 = q - n \end{cases}$$

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has a unique solution n, which, if substituted into each equation in the system

$$\begin{cases} Ak + \dfrac{C}{m}n = Cr \\ Bk + \dfrac{D}{m}n = Ds \end{cases}$$

gives the same value of k. It is possible to show that this condition is equivalent to the following.

Proposition 2: Let A, B, C, D, p, q, r, s and m be real numbers satisfying

$$m \neq 1,$$

$$p - q = r - s > 0,$$

$$A \neq 0, B \neq 0 \text{ and } AD - BC \neq 0 \text{ and}$$

$$\frac{C}{A}(rm - p - 1) = \frac{D}{B}(sm - q - 1).$$

Then the integral $\displaystyle\int \frac{Ax^p + Bx^q}{(Cx^r + Dx^s)^m}\,dx$ is solvable by substitution after dividing the numerator and denominator by x^n, where

$$n = \frac{m(p - r + 1)}{m - 1} = \frac{m(q - s + 1)}{m - 1}.$$

By applying this result, one can show that

$$\int \frac{x^3 - 2x}{(x^2 + 2)^3}\,dx = -\frac{1}{2(x + 2x^{-1})^2} + c,$$

and that

$$\int \frac{x^4 + 1}{x^2\sqrt{x^4 - 1}}\,dx = \sqrt{x^2 - x^{-2}} + c.$$

Beyond binomial

In the case where $P(x)$ and $Q(x)$ consist of more than two nonzero terms, the equality

$$\frac{d}{dx}\left[\frac{Q(x)}{x^n}\right] = k\left[\frac{P(x)}{x^n}\right]$$

gives rise to an overdetermined system of equations in two variables k and n. In the fortunate case where the system possesses a solution (k, n), we can solve the integral $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ as above. Moreover, a similar variation as in

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the previous section is certainly realisable. We leave to the reader to verify the following equalities

$$\int \frac{3x^4 - x^3 - 1}{3x^5 - 2x^4 + x}\,dx = \frac{1}{2}\ln\left|3x^2 - 2x + x^{-2}\right| + c$$

$$\int \frac{2x^4 + x^2 - x}{(4x^3 - 2x + 1)^2}\,dx = -\frac{1}{8x - 4x^{-1} + 2x^{-2}} + c$$

$$\int \frac{x^6 - 1}{x^3\sqrt[3]{x^6 + x^3 + 1}}\,dx = \frac{1}{2}\sqrt[3]{(x^3 + 1 + x^{-3})} + c.$$

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