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A FUNCTIONAL LIMIT THEOREM FOR RANDOM WALK CONDITIONED TO STAY NON-NEGATIVE

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Abstract

In this paper we consider an aperiodic integer-valued random walk S and a process S^* that is a harmonic transform of S killed when it first enters the negative half; informally, S^* is 'S conditioned to stay non-negative'. If S is in the domain of attraction of the standard normal law, without centring, a suitably normed and linearly interpolated version of S converges weakly to standard Brownian motion, and our main result is that under the same assumptions a corresponding statement holds for S^*, the limit of course being the three-dimensional Bessel process. As this process can be thought of as Brownian motion conditioned to stay non-negative, in essence our result shows that the interchange of the two limit operations is valid. We also establish some related results, including a local limit theorem for S^*, and a bivariate renewal theorem for the ladder time and height process, which may be of independent interest.

1. *Introduction*

There is an increasing use in modelling of processes derived from random walks that are constrained to stay on a half-line; see [**10**] for a recent example of this. However, the phrase 'random walk conditioned to stay non-negative' has at least two different interpretations. In the first, we consider the first n values of a random walk conditioned on the event that all of these values are non-negative; this is a discrete version of a meander, and it has been known for a long time that if the random walk is in the domain of attraction of a standard normal law, a suitably scaled version of this process converges weakly to a Brownian meander. (See [**8**] or [**11**].) The second interpretation involves conditioning on the event that the random walk *never* goes negative, and so can be thought of as a discrete version of the Bessel process. (Of course when the random walk oscillates, the conditioning event has zero probability, and we have to construct the conditioned process by the h-transform technique.) So a natural question, which we consider here, is whether there is an analogous weak convergence result for this version, the limit clearly being the Bessel process. Note that both the random walk meander and the Brownian meander can be represented as sections of the unconditioned process, which simplifies the proofs. In our case this device is not available and we fall back on the standard technique of establishing convergence of finite-dimensional distributions and tightness.

So let $S = \{S_n : n \geqslant 0\}$ denote a random walk on the integer lattice with starting state zero, that is, $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$ where the ξ_i are independent and identically distributed integer-valued random variables. In addition, assume that S is aperiodic and attracted to a standard normal law, without centring, and let a_n be the associated norming sequence. We define a normed and linearly interpolated

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version of S by

$$X_n(t) = \frac{S_{[nt]}}{a_n} + (nt - [nt])\frac{\xi_{[nt]+1}}{a_n}, \quad t \geqslant 0.$$

The weak convergence of the process $X_n = (X_n(t), t \geqslant 0)$ as $n \to \infty$ to a standard Brownian motion is Donsker's invariance principle (see, for example, [**5**, Theorem 10.1]). Let S^* denote 'S conditioned to stay non-negative' (in the sense of an h-transform) and $R_n = (R_n(t), t \geqslant 0)$ the corresponding linear interpolation of the normed version of S^*. The basic question we study here is when does R_n converge weakly as $n \to \infty$ to R, the three-dimensional Bessel process?

Actually, the fact that this happens in the special case that S is a simple symmetric random walk has been known for a long time, and indeed the classic paper on the Bessel process by Pitman [**18**] makes crucial use of this fact. On the other hand, it is not obvious that the convergence of R_n should hold under precisely the same conditions as that of X_n and, in fact, the results in [**15**] seem to suggest that one might need to assume extra moment conditions. It turns out that this is not the case, and our main result, Theorem 2.1 below, shows that X_n^* converges weakly to R exactly when S_1 is in the domain of attraction of the standard normal law. In the final section, we also extend these results to the case that the initial value of S is non-zero.

Actually our method of proof leads to a *local* limit theorem for S_n^*, and we formulate this as Theorem 2.2. As S^* is a h-transform of S killed when it enters the negative half-line, which we denote by $\hat{S}$, we could also formulate Theorem 2.2 as a local limit theorem for $\hat{S}$. This in turn, by duality, gives the asymptotic behaviour of the joint mass function in the bivariate renewal process of descending ladder heights and times in S; in this form, which is stated as Proposition 1 below, it extends earlier results of Keener [**16**] and Alili and Doney [**1**, **2**].

2. *Conditioning S to stay non-negative*

Given a random walk $S = \{S_n : n \geqslant 0\}$, we denote S conditioned to stay non-negative by $S_n^* = \sum_{i=1}^n \xi_i^*$. In our integer-valued, oscillating case this is an honest Markov chain on $\mathbb{Z}^+$, the non-negative integers, with transition probabilities

$$P_x(S_n^* = y) := P(S_n^* = y \mid S_0^* = x) = \frac{V(y)}{V(x)} P_x(\widehat{S}_n = y), \quad x, y \in \mathbb{Z}^+,$$

where we have denoted by $\widehat{S} = \{\widehat{S}_n : n \geqslant 0\}$ the walk S killed at time $T_1^- = \min\{n : S_n < 0\}$ and V is the renewal function for the strict descending ladder height process of the walk S. (See Bertoin and Doney [**3**].) The strict descending ladder variables for S are defined as follows. Let T_k^- denote the kth strict descending ladder time, that is,

$$T_k^- = \min\{n > T_{k-1}^- : S_n < S_{T_{k-1}^-}\}$$

for $k \geqslant 1$ where $T_0^- \equiv 0$. We let $H_k^- = -S_{T_k^-}$ denote the kth descending ladder height so that the ladder process, $(T^-, H^-) = \{(T_k^-, H_k^-) : k \geqslant 0\}$, is a bivariate renewal process. The renewal function associated with H_1^- is defined for $x \geqslant 0$ by

$$V(x) = \sum_{k \geqslant 0} P(H_k^- \leqslant x),$$

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with the renewal mass function for the descending ladder time and height process given by

$$v(n,x)=\sum_k P(T_k^-=n,H_k^-=x).$$

In a similar vein, the weak ascending ladder variables are denoted by $(T^+,H^+)=\{(T_k^+,H_k^+):k\geqslant 0\}$ and are defined similarly, the kth weak ascending ladder time being

$$T_k^+=\min\{n>T_{k-1}^+:S_n\geqslant S_{T_{k-1}^+}\}$$

for $k\geqslant 1$, again letting $T_0^+\equiv 0$ and $H_k^+=S_{T_k^+}$. The renewal function associated with H_1^+ is denoted throughout by $U(\cdot)$, and we define the renewal mass function for the ascending ladder time and height process to be

$$u(n,x)=\sum_k P(T_k^+=n,H_k^+=x).$$

We can now formulate our main results.

THEOREM 2.1. *Let S be an integer-valued, aperiodic random walk that starts at 0 and is in the domain of attraction of a standard normal law with norming sequence a_n. Then the process R_n constructed from the conditioned random walk S^* by*

$$R_n(t)=\frac{S^*_{[nt]}}{a_n}+(nt-[nt])\frac{\xi^*_{[nt]+1}}{a_n},\quad t\geqslant 0, \tag{2.1}$$

converges weakly as $n\to\infty$ to the three-dimensional Bessel process R.

REMARK 1. Let

$$A_x=\sum_{n\geqslant 0}\mathbf{1}_{\{S_n^*\leqslant x\}},\quad x>0,$$

denote the total time that S^* spends below the level x. Then a simple consequence of Theorem 2.1 is that, as $x\to\infty$,

$$\frac{A_x}{a^{-1}(x)}\xrightarrow{d}\mathcal{A},$$

where a^{-1} is the inverse of a continuous interpolation of the norming sequence a, and $\mathcal{A}$ is the total time spent by the Bessel process below 1. This answers a question raised in [**4**].

The three-dimensional Bessel process R is the diffusion on the positive half line with transition density

$$P_x(R_t\in dy)=\frac{1}{\sqrt{t}}r\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}}\right)dy,\quad x>0$$
$$\text{where } r(u,v)=\sqrt{\frac{2}{\pi}}\frac{v}{u}\sinh(uv)\exp\left\{-\frac{u^2+v^2}{2}\right\},$$

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and entrance law with density

$$P_0(R_t \in dy) = \frac{1}{\sqrt{t}} r_0\left(\frac{y}{\sqrt{t}}\right) dy$$
$$\text{where } r_0(v) = \sqrt{\frac{2}{\pi}} v^2 \exp\left\{-\frac{v^2}{2}\right\}.$$

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *uniformly in* $0 \leqslant z \leqslant Ka_n$, *for any* K,

$$a_n P(S_n^* = z) = r_0\left(\frac{z}{a_n}\right) + o(1) \quad \text{as } n \to \infty.$$

We prove our main technical lemma in Section 3, deduce Theorem 2.2 and Theorem 2.1 in Section 4, and in the final section we use a discrete version of Williams' path decomposition of the Bessel process to establish a version of Theorem 2.1 for a sequence of random walks that have the same step distribution but possibly different non-zero starting points.

3. *A bivariate renewal theorem*

Throughout this section, we assume that S is an integer-valued aperiodic random walk that starts at zero.

PROPOSITION 1. *Assume that* S *is attracted to a standard normal law with norming sequence* a_n. *Then we have*

$$P(\hat{S}_n = z) = u(n, z) \backsim n^{-1} U(z) P(S_n = z) \tag{3.1}$$

uniformly as $n \to \infty$ *for* $0 \leqslant z \leqslant Ka_n$ *for any* $K > 0$.

REMARK 2. We can deduce from this that, uniformly in $0 \leqslant z \leqslant Ka_n$,

$$P(S_m \geqslant 0, m \leqslant n \mid S_n = z) \sim \frac{U(z)}{n}.$$

This result was first proved in [**16**] under heavy assumptions on S for a restricted range of values of z.

In preparation for the proof of Proposition 1 we state some known results concerning the ladder variables. The first is a version of the Wiener–Hopf factorization taken from [**1**].

LEMMA 1. *Denote the first time at which the* (*weak, ascending*) *ladder height process exceeds the level* z *by* $N_z = \min\{k : H_k > z\}$; *then for* $k, z \in \mathbb{Z}^+$,

$$P(T_k = n, H_k = z) = \frac{k}{n} P(H_{k-1} \leqslant S_n < H_k, S_n = z)$$
$$= \frac{k}{n} P(N_z = k, S_n = z),$$

and

$$n\, u(n, z) = E(N_z, S_n = z). \tag{3.2}$$

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For the next result we write $\rho_n = P(S_n \geqslant 0)$, and define sequences $\{b_n\}$ and $\{c_n\}$ satisfying

$$\log n = \sum_{m=1}^{\infty} \frac{1}{m} \rho_m \exp\left\{-\frac{m}{b_n}\right\} \quad \text{and} \quad c_n = a(b_n), \tag{3.3}$$

where $a(\cdot)$ is any continuous increasing function such that $a(n) = a_n$. (That the definition of $\{b_n\}$ is legitimate is shown in [**12**].) Then, specializing results in [**12**] to the case $\rho = \frac{1}{2}$ gives the following lemma.

LEMMA 2. *When S_1 is attracted to a standard normal law with norming sequence a_n:*

(i) *we have*

$$\left\{\left(\frac{T_{[kt]}}{b_k}, \frac{H_{[kt]}}{c_k}\right), t \geqslant 0\right\} \xrightarrow{d} \{(B_t, t), t \geqslant 0\}; \tag{3.4}$$

(ii) *for any $\delta > 0$,*

$$P((1-\delta)c_m \leqslant H_m \leqslant (1+\delta)c_m \mid T_m = s) \to 1 \tag{3.5}$$

as $m \to \infty$ uniformly for $d \leqslant s/b_m \leqslant D$ where $0 < d < D < \infty$ are constants.

Proof of Proposition 1. The equality in (3.1) is a consequence of duality, and we also note that it suffices to prove that the asymptotic relation holds uniformly in the range

$$\delta a_n \leqslant z \leqslant K a_n \tag{3.6}$$

for any $K > 1 > \delta > 0$. This is because a slight extension of [**1**, Theorem 7], where it was assumed that $\operatorname{Var} S_1$ was finite, proves the result in the situation that $z/a_n \to 0$.

A consequence of the assumption that S_1 is in the domain of attraction of the normal law is that both H_1 and H_1^- are relatively stable and, hence, using for example Theorem 8.8.1 of [**6**],

$$U(z) \sim c^{-1}(z) = \frac{z}{l^+(z)}, \quad V(z) \sim d^{-1}(z) = \frac{z}{l^-(z)}, \tag{3.7}$$

as $z \to \infty$, where $l^{\pm}$ are slowly varying functions, and c^{-1} and d^{-1} are inverses of the norming sequences for H_n and H_n^-, respectively. Using this in the representation

$$n\, u(n, z) = E(N_z, S_n = z)$$

from (3.2) we can write the asymptotic relation in (3.1) as

$$\begin{aligned} \sum_k k P(N_z = k, S_n = z) &\sim U(z) \sum_k P(N_z = k, S_n = z) \\ &\sim \frac{z}{l^+(z)} \sum_k P(N_z = k, S_n = z). \end{aligned}$$

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Given $\varepsilon > 0$, we split these sums over the following ranges

$$A_1 : k < m_1 := \left[(1-\varepsilon)\frac{z}{l^+(z)}\right],$$
$$A_2 : m_1 \leqslant k \leqslant m_2 = \left[(1+\varepsilon)\frac{z}{l^+(z)}\right],$$
$$A_3 : k > m_2,$$

and define for $i = 1, 2, 3$

$$\Sigma_i = \frac{l^+(z)}{z}\sum_{k\in A_i} kP(N_z = k,\, S_n = z)$$
$$P_i = \sum_{k\in A_i} P(N_z = k, S_n = z),$$

so that it suffices to prove that

$$\Sigma_1 + \Sigma_2 + \Sigma_3 \sim P_1 + P_2 + P_3 = P(S_n = z).$$

Clearly

$$(1-\varepsilon)P_2 \leqslant \Sigma_2 \leqslant (1+\varepsilon)P_2$$

and since $P_1 \geqslant \Sigma_1$ and $\Sigma_3 \geqslant P_3$, the proof of the proposition lies in showing that Σ_3 and P_1 are both of the order of $o(P(S_n = z)) = o(1/a_n)$. (Here, and later, we use the standard bound

$$c \leqslant a_n P(S_n = z) \leqslant d \quad \text{for } 0 \leqslant z \leqslant Ka_n, \tag{3.8}$$

without further comment. We will also write $c, d, \ldots$ for finite positive constants whose values may change from line to line.)

By summation by parts

$$\begin{aligned}\Sigma_3 &= \frac{l^+(z)}{z}\sum_{k\in A_3} kP(N_z = k, S_n = z)\\ &= m_2\frac{l^+(z)}{z}P(N_z \geqslant m_2, S_n = z) + \frac{l^+(z)}{z}\sum_{k>m_2} P(N_z \geqslant k, S_n = z)\\ &:= \Sigma_3^{(1)} + \Sigma_3^{(2)}.\end{aligned}$$

Note that from the definition of m_2

$$\begin{aligned}\Sigma_3^{(1)} &\leqslant cP(N_z \geqslant m_2, S_n = z)\\ &= cP(H_{m_2} \leqslant z, S_n = z)\\ &= c\sum_{x\leqslant z}\sum_{s\leqslant n} P(T_{m_2} = s, H_{m_2} = x)P(S_{n-s} = z - x)\\ &\leqslant c\sum_{s\leqslant n} P(T_{m_2} = s, H_{m_2} \leqslant z)\frac{1}{a_{n-s}}.\end{aligned}$$

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Next take $\eta \in (0,1)$, and write

$$a_n \sum_{s \leqslant \eta n} P(T_{m_2} = s, H_{m_2} \leqslant z) \frac{1}{a_{n-s}} \leqslant \frac{a_n}{a_{n(1-\eta)}} P(T_{m_2} < n\eta) = \frac{a_n}{a_{n(1-\eta)}} P\left(\frac{T_{m_2}}{b_{m_2}} < \frac{n\eta}{b_{m_2}} \right).$$

The bounds on z in (3.6) and the fact that $c^{-1}(\cdot) = b^{-1}(a^{-1}(\cdot))$ give

$$b(m_2) \backsim b((1+\varepsilon)c^{-1}(z)) \backsim (1+\varepsilon)^2 a^{-1}(z)$$

and we conclude that for all sufficiently large n

$$cn \leqslant b_{m_2} \leqslant dn, \tag{3.9}$$

so that the last expression is asymptotically bounded by $(1/\sqrt{(1-\eta)})G(\eta/c)$, where we have denoted by G the stable $\frac{1}{2}$ distribution function. On the other hand, (3.9) allows us to use (3.5) of Lemma 2 to see that

$$\begin{aligned}
&a_n \sum_{\eta n < s \leqslant n} P(T_{m_2} = s, H_{m_2} \leqslant z) \frac{1}{a_{n-s}} \\
&\quad = \sum_{n\eta \leqslant s \leqslant n} P(H_{m_2} \leqslant z \mid T_{m_2} = s) P(T_{m_2} = s) \frac{a_n}{a_{n-s}} \\
&\quad = o(1) \sum_{n\eta \leqslant s \leqslant n} P(T_{m_2} = s) \frac{a_n}{a_{n-s}} \\
&\quad = o(1) \sum_{n\eta \leqslant s \leqslant n} \left(\frac{1}{b_{m_2}} g\left(\frac{s}{b_{m_2}} \right) + o\left(\frac{1}{b_{m_2}} \right) \right) \frac{a_n}{a_{n-s}} \\
&\quad \leqslant o(1) \frac{1}{n} a_n \sum_{n\eta \leqslant s \leqslant n} \frac{1}{a_{n-s}} \\
&\quad \sim o(1) \frac{1}{n} a_n \frac{n(1-\eta)}{a_{n(1-\eta)}} \sim o(1)\sqrt{(1-\eta)},
\end{aligned}$$

in which g denotes the density function for the stable $\frac{1}{2}$ distribution and we have used the local limit theorem and standard properties of regularly varying functions. As we are able to choose η as small as we wish, $a_n \Sigma_3^{(1)}$ is $o(1)$, and it is left to demonstrate that $a_n \Sigma_3^{(2)}$ is also $o(1)$.

Note the following upper bound:

$$\begin{aligned}
\Sigma_3^{(2)} &= \frac{l^+(z)}{z} \sum_{k \geqslant m_2} P(N_z > k, S_n = z) \\
&\leqslant c \frac{l^+(z)}{z} \sum_{s \leqslant n} \sum_{k \geqslant m_2} P(T_k = s,\, H_k \leqslant z) \frac{1}{a_{n-s}}\, dz \\
&\leqslant c(\sigma_1 + \sigma_2),
\end{aligned}$$

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where, for some $\eta > 0$,

$$\sigma_1 = \frac{l^+(z)}{z} \sum_{s<n\eta} \sum_{k\geqslant m_2} P(T_k = s, H_k \leqslant z) \frac{1}{a_{n-s}},$$

$$\sigma_2 = \frac{l^+(z)}{z} \sum_{n\eta\leqslant s\leqslant n} \sum_{k\geqslant m_2} P(T_k = s,\ H_k \leqslant z) \frac{1}{a_{n-s}}.$$

Denoting the renewal function for the increasing ladder times by $\Theta(\cdot)$, we have

$$\begin{aligned} a_n\sigma_1 &\leqslant \frac{l^+(z)}{z} \sum_{k\geqslant m_2} P(T_k < n\eta, H_k \leqslant z) \frac{a_n}{a_{n(1-\eta)}} \\ &\leqslant \frac{l^+(z)}{z} \Theta(n\eta) \frac{a_n}{a_{n(1-\eta)}}. \end{aligned}$$

Since the ladder times are in the domain of attraction of a stable $\frac{1}{2}$ law with norming sequence b, the asymptotic behaviour of the function $\Theta(\cdot)$ is known (see, for example, [**6**, p. 361]) to be given by

$$\Theta(n) \sim c b_n^{-1}.$$

The argument that established (3.9) shows that $z/l^+(z)$ is bounded above and below by constant multiples of the sequence b_n^{-1} so

$$\begin{aligned} a_n\sigma_1 &\leqslant \frac{c}{b_n^{-1}} \Theta(n\eta) \frac{a_n}{a_{n(1-\eta)}} \\ &\sim c \frac{b_{n\eta}^{-1}}{b_n^{-1}\sqrt{1-\eta}} \backsim \frac{c\sqrt{\eta}}{\sqrt{1-\eta}}, \end{aligned}$$

and this can be made arbitrarily small by choice of η.

Next we take an arbitrary γ and write σ_2 as the sum of the terms

$$\sigma_2^{(1)} = \frac{l^+(z)}{z} \sum_{n\eta\leqslant s\leqslant n} \sum_{m_2\leqslant k\leqslant \gamma m_2} P(T_k = s, H_k \leqslant z) \frac{1}{a_{n-s}}$$

$$\sigma_2^{(2)} = \frac{l^+(z)}{z} \sum_{n\eta\leqslant s\leqslant n} \sum_{k>\gamma m_2} P(T_k = s, H_k \leqslant z) \frac{1}{a_{n-s}}.$$

Both of these terms can bounded using similar arguments to the above, but the details are omitted; this completes the proof that $a_n\Sigma_3 \to 0$.

The final step in the proof is to show that P_1 is $o(1/a_n)$. The proof again relies on splitting the sum into several parts and showing that each is $o(1/a_n)$. We start by writing $P_1 = P_1^{(1)} + P_1^{(2)}$, where

$$P_1^{(1)} = P(H_{m_1} > z, T_{m_1} \leqslant n, S_n = z),$$

$$P_1^{(2)} = P(H_{m_1} > z, T_{m_1} > n, S_n = z).$$

The first of these terms is shown to be sufficiently small by splitting it into two parts, the first being when the m_1th ladder time is close to zero and the second when it is bounded away from zero. Each part is then shown to be of the required order by a similar method to that used to bound σ_1; the details of this are also omitted. Letting $\nu_n = \max\{k : T_k \leqslant n\}$ denote the number of ladder epochs which occur by time n, we have

$$P_1^{(2)} = P(S_n = z, \nu_n < m_1).$$

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To bound this probability we again need to consider two separate cases depending on the value of T_{ν_n}. Thus, for $\delta \in (0,1)$, we first write

$$P(T_{\nu_n} < n(1-\delta), H_{\nu_n} \geqslant z, S_n = z, \nu_n < m_1) = \sum_{\substack{w \geqslant z, k < m_1 \\ l < n(1-\delta)}} P(T_k = l, H_k = w, v_n = k) P\Big(S_{n-l} = z - w, \max_{j \leqslant n-l} S_j = 0\Big). \tag{3.10}$$

We can bound the final term in the above product by

$$\sum_{y<0} P\left(S_j \leqslant 0 \text{ for } j \leqslant \frac{(n-l)}{2}, S_{(n-l)/2} = y\right) P(S_{(n-l)/2} = z - \omega - y) \leqslant P\left(T_1^- > \frac{(n-l)}{2}\right) \frac{c}{a_{(n-l)/2}},$$

and using this we see that the expression (3.10) is bounded above by a multiple of

$$P(T_{\nu_n < n(1-\delta)}, H_{\nu_n} \geqslant z) P\left(T_1^- > \frac{n\delta}{2}\right) \frac{1}{a_{n\delta/2}} \backsim \frac{1}{a_n \sqrt{\delta/2}} P\left(T_1^- > \frac{n\delta}{2}\right) = o\left(\frac{1}{a_n}\right)$$

as required. An obvious duality argument shows that

$$P(T_{\nu_n} \geqslant n(1-\delta), S_n \in dz) = P(l_n < n\delta, S_n \in dz)$$

where $l_n = \max\{k : S_k \leqslant 0\}$. Taking $\delta < \frac{1}{2}$, l_n is no less than the number of non-positive partial sums up until time $[n/2]$, which we will denote by Γ_n. The arc sine theorem (see, for example, Feller [**14**, p. 418]) gives

$$P\left(\Gamma_n < \frac{n\delta}{2}\right) \to \frac{2}{\pi} \arcsin \sqrt{\delta},$$

and this, in turn, allows us the bound

$$P(T_{\nu_n} \geqslant n(1-\delta), S_n = z) \leqslant a_n P(\Gamma_n < n\delta, S_{[n/2]} = y, S_n = z) \leqslant c P(\Gamma_n < n\delta) \to c \frac{2}{\pi} \arcsin \sqrt{2\delta}.$$

Since this can be made arbitrarily small by choice of δ, we have $a_n P_1 \to 0$, and the result is proved. □

The proof of Theorem 2.2 also requires the following result.

LEMMA 3. *Uniformly in* $0 \leqslant z \leqslant K a_n$ *and* $0 \leqslant y \leqslant K a_n$, *for any* K,

$$\frac{U(z)V(y)}{n} = 2 \frac{z}{a_n} \cdot \frac{y}{a_n} + o(1) \quad \text{as } n \to \infty.$$

Proof. First we show that the result holds uniformly in the range

$$\varepsilon a_n \leqslant z, y \leqslant K a_n, \tag{3.11}$$

for any $\varepsilon > 0$. To see this, recall from (3.7) that we have

$$U(x) \backsim \frac{x}{l^+(x)} \quad \text{and} \quad V(x) \backsim \frac{x}{l^-(x)} \quad \text{as } x \to \infty,$$

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where l^+ and l^- are slowly varying, and from this it follows that the characteristic functions $\psi^\pm$ of $H^\pm$ satisfy

$$1-\psi^\pm(s) \backsim \mp i s l^\pm(1/s) \quad \text{as } s \downarrow 0.$$

From this and the Wiener–Hopf factorization, we obtain, uniformly for $y \in [\varepsilon, K]$, that

$$\begin{aligned} U(a_n z)V(a_n y) &\backsim \frac{a_n z}{l^+(a_n)} \cdot \frac{a_n y}{l^-(a_n)} \\ &\backsim \frac{zy}{1-\psi(1/a_n)}, \end{aligned}$$

where $\psi(s) = E(e^{isS_1})$. Since S_n/a_n converges in distribution to the standard Normal law, a simple calculation gives

$$n\{1-\psi(1/a_n)\} \to \tfrac{1}{2},$$

and this proves the result when (3.11) holds. However, for $z \leqslant \varepsilon a_n$ we have

$$\frac{U(z)V(y)}{n} \leqslant \frac{U(\varepsilon a_n)V(Ka_n)}{n} \leqslant 3K\varepsilon^2 \quad \text{for all large enough } n,$$

and the result follows. □

In the proof of Theorem 2.1 we need the following estimate for the transition function $P_x(\hat{S}_n = y)$.

PROPOSITION 2. *Suppose that x_n and y_n are integers such that*

$$\frac{x_n}{a_n} \to u > 0, \quad \frac{y_n}{a_n} \to v > 0 \quad \text{as } n \to \infty.$$

Then

$$a_n P_{x_n}(\hat{S}_n = y_n) \to \frac{u}{v} r(u,v). \tag{3.12}$$

Proof. Writing $I_n = \min_{r \leqslant n} S_r$ and $\sigma_n = \min\{r \leqslant n : S_r = I_n\}$ we have the following identity:

$$\begin{aligned} P_{x_n}(\widehat{S}_n = y_n) &= \sum_{m=0}^{n-1} \sum_{\omega=0}^{x_n \wedge y_n} P_{x_n}(I_n = \omega, \sigma_n = m, S_n = y_n) \\ &= \sum_{m=0}^{n-1} \sum_{\omega=0}^{x_n \wedge y_n} v(m, x_n - \omega) u(n-m, y_n - \omega). \end{aligned} \tag{3.13}$$

(A proof of this, which is a bivariate version of a result originally due to Spitzer [**19**], can be found in [**2**].) We show first that the tails in the m-summation in (3.13) can be ignored, namely

$$\lim_{\eta \downarrow 0} \limsup_{n \to \infty} a_n \sum_{(m \wedge n-m) \leqslant n\eta} \sum_{\omega=0}^{x_n \wedge y_n} v(m, x_n - \omega) u(n-m, y_n - \omega) = 0. \tag{3.14}$$

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To see this, use (3.8) to see that for $\eta \leqslant \frac{1}{2}$

$$\begin{aligned} a_n &\sum_{m \leqslant n\eta} \sum_{\omega=0}^{\infty} v(m, x_n - \omega) u(n-m, y_n - \omega) \\ &= a_n \sum_{m \leqslant n\eta} P_{x_n}(\sigma_n = m, S_n = y_n) = a_n P(\sigma_n \leqslant n\eta, S_n = y_n - x_n) \\ &\leqslant cP(\sigma_n \leqslant n\eta | S_n = y_n - x_n) \\ &= cP(\sigma(X_n(\cdot)) \leqslant \eta | X_n(1) = (y_n - x_n)/a_n) \\ &\to cP(\sigma(\tilde{B}) \leqslant \eta). \end{aligned}$$

Here $\tilde{B}$ denotes a Brownian Bridge of unit length from 0 to $v - u$, which is known (see [**17**]) to be the weak limit of $X_n(\cdot)$ conditioned on $X_n(1) = (y_n - x_n)/a_n$, and $\sigma(\tilde{B})$ the time at which its minimum value occurs. Clearly $P(\sigma(\tilde{B}) \leqslant \eta) \to 0$ as $\eta \downarrow 0$, and since a similar argument applies to the other term, (3.14) follows. Next, arguments almost identical to those of Proposition 1 show that the function $v(\cdot, \cdot)$ has the asymptotic behaviour

$$nv(n, x) \sim V(x-1)P(S_n = -x), \tag{3.15}$$

as $n \to \infty$, uniformly for all $1 \leqslant x \leqslant a_n K$ where K is any finite positive constant. Then applying Proposition 1 and (3.15) to (3.13), we see that (3.12) will follow from

$$\begin{aligned} \lim_{\eta \downarrow 0} \lim_{n \to \infty} &\sum_{n\eta < m < n(1-\eta)} \sum_{\omega=0}^{x_n \wedge y_n} \frac{V(x_n - \omega - 1)}{m} \\ &\times \frac{U(y_n - \omega)}{n-m} \frac{a_n}{a_m} \frac{1}{a_{n-m}} \phi\left(\frac{x_n - \omega}{a_m}\right) \phi\left(\frac{y_n - \omega}{a_{n-m}}\right) = \frac{u}{v} r(u, v). \end{aligned}$$

Using Lemma 3 and then making the substitutions $m = n\tau$ and $\omega = a_n w$, we see that this, in turn, will follow from

$$2 \int_0^{u \wedge v} \int_0^1 \frac{(u-w)(v-w)}{\tau^{3/2}(1-\tau)^{3/2}} \phi\left(\frac{u-w}{\tau}\right) \phi\left(\frac{v-w}{1-\tau}\right) d\tau \, dw = \frac{u}{v} r(u, v). \tag{3.16}$$

However, because

$$r(u, v) \, dv = \frac{v}{u} P_u\Big(B_1 \in dv, \inf_{s \leqslant 1} B_s > 0\Big)$$

(3.16) can be rewritten as

$$2 \int_0^{u \wedge v} \int_0^1 g(u-w, \tau) g(v-w, 1-\tau) \, d\tau \, dw \, dv = P_u\Big(B_1 \in dv, \inf_{s \leqslant 1} B_s > 0\Big)$$

where B is a standard Brownian motion and g is its first passage density, and this is immediate by decomposing the right-hand side according to the value of $\inf_{s \leqslant 1} B_s$ and using Lévy's factorization at the maximum. □

4. *Proofs*

4.1. *Proof of Theorem* 2.2

As $P(S_n^* = z) = V(z)P(\hat{S}_n = z)$, Theorem 2.2 follows from Proposition 1, the normal local limit theorem and the case $y = z$ of Lemma 2. □

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4.2. *Proof of Theorem* 2.1

The convergence of the finite-dimensional distributions follows easily from Theorem 2.2, Lemma 3 and Proposition 2, so we are left to prove tightness of the sequence $\{R_n\}$. As $\{R_n\}$ is a linearly interpolated Markov Chain, according to [**5**, Theorem 8.4], tightness will follow if we can show that for each positive ε and K there exist a $\lambda > 1$ and an integer n_0 such that

$$P\Big(\max_{i\leqslant Kn} S_i^* \geqslant \lambda a_n\Big) \leqslant \frac{\varepsilon}{\lambda^2} \tag{4.1}$$

holds for all $n \geqslant n_0$.

From the convergence of the marginal distribution of R_n to that of R and the explicit form of the latter, we can clearly choose λ so that, for all large n,

$$P(S^*_{[Kn]} \geqslant \lambda a_n) \leqslant \frac{\varepsilon}{2\lambda^2};$$

thus, (4.1) will hold if we can also arrange that

$$P\Big(\max_{i\leqslant Kn} S_i^* \geqslant \lambda a_n, S^*_{[Kn]} < \lambda a_n\Big) \leqslant \frac{\varepsilon}{2\lambda^2}. \tag{4.2}$$

To see this, we argue that

$$\begin{aligned}
&P\Big(\max_{i\leqslant Kn} S_i^* \geqslant \lambda a_n, S^*_{[Kn]} < \lambda a_n\Big)\\
&\quad= \sum_{y<\lambda a_n} P\Big(\max_{i\leqslant Kn} \hat{S}_i \geqslant \lambda a_n, \hat{S}_{[Kn]} = y\Big)V(y)\\
&\quad\leqslant P\Big(\max_{i\leqslant Kn} \widehat{S}_i \geqslant \lambda a_n, \widehat{S}_{[Kn]} < \lambda a_n\Big)V(\lambda a_n)\\
&\quad= P\Big(\max_{i\leqslant Kn} S_i \geqslant \lambda a_n, \min_{i\leqslant Kn} S_i \geqslant 0\Big)V(\lambda a_n)\\
&\quad= P\Big(\max_{i\leqslant Kn} S_i \geqslant \lambda a_n \mid T_1^- > Kn\Big)P(T_1^- > Kn)V(\lambda a_n),
\end{aligned}$$

and note that

$$P(T_1^- > Kn)V(\lambda a_n) \backsim \frac{\lambda}{\sqrt{K}} P(T_1^- > n)V(\lambda a_n) \to \frac{c\lambda}{\sqrt{K}}.$$

On the other hand it is known from [**11**, Examples 1 and 3, Corollary 1] that

$$P\Big(\max_{i\leqslant n} S_i \geqslant \lambda a_n \mid T_1^- > n\Big) \to P\Big(\sup_{t\leqslant 1} B_t^+ \geqslant \lambda\Big), \tag{4.3}$$

where $B^+ = \{B_t^+ : t \geqslant 0\}$ denotes standard Brownian meander. The distribution of $\sup_{t\leqslant 1} B_t^+$ is known explicitly (see [**13**]) and it is easy to check that the right-hand side of (4.3) is exponentially small in λ, so the bound (4.2) holds, and the sequence $\{R_n\}$ is tight. □

5. *Starting away from zero*

In this section we consider briefly the situation where we 'condition to stay non-negative' a sequence of random walks that each have the same step distribution, but possibly different starting points. In order to avoid cumbersome formulae, in this section we adopt the canonical notation and write P_x for the measure of a random

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walk with initial value x, and P_x^* for the measure of the corresponding conditioned version. This is specified by

$$P_x^*(A \cap (S_n = y)) = \frac{V(y)}{V(x)} \hat{P}_x(A \cap (S_n = y))$$

for any event $A \in \mathcal{F}_n$, where $\hat{P}_x$ is the measure corresponding to S killed at time $T_1^- = \min\{n : S_n < 0\}$. For our purposes, the following facts about this measure are useful.

PROPOSITION 3. *Write $\mu = \min(n : S_n = j)$ where $j = \inf_{n \geqslant 0}\{S_n\}$ is the all-time minimum of S. Then:*

(i) *under P_x^* the post-minimum process $\{S_{n+\mu} - j, n \geqslant 0\}$ is independent of the pre-minimum process and has measure P_0^*;*

(ii) *the distribution of j under P_x^* is given by*

$$P(j = y) = \frac{v(x-y)}{V(x)}, \quad 0 \leqslant y \leqslant x, \tag{5.1}$$

where v is the renewal mass function in the downgoing ladder height process.

This result is just a discrete-time version of a known result for Lévy processes in [**9**], and since the proof is similar we omit it.

REMARK 3. The cited result also contains a description of the pre-minimum process that can be taken over into our discrete setting, but we do not need this.

Now given a norming sequence a_k and a sequence of integer starting points i_k we define

$$R_k(t) = \frac{S_{[kt]}}{a_k} + (kt - [kt])\frac{\xi_{[kt]+1}}{a_k}, \quad t \geqslant 0,$$

so that under $P_{i_k}^*$ we have $R_k(0) = i_k/a_k := z_k$.

THEOREM 5.1. *Suppose that the assumptions of Theorem 2.1 hold and $z_k \to z \geqslant 0$ as $k \to \infty$. Then the process R_k under $P_{i_k}^*$ converges weakly to the Bessel process with initial value z.*

Proof. For $z > 0$, convergence of the finite-dimensional distributions follows from Proposition 2.2 and Lemma 2, so we need only establish tightness. However,

$$\begin{aligned} &P_{i_k}^*\Big(\max_{n \leqslant ck} S_n \geqslant \lambda a_k, S_{[ck]} < \lambda a_k\Big) \\ &\quad \leqslant P_{i_k}\Big(\max_{n \leqslant ck} S_n \geqslant \lambda a_k, \min_{n \leqslant ck} S_n \geqslant 0, S_{[ck]} < \lambda a_k\Big)\frac{V(\lambda a_k)}{V(i_k)} \\ &\quad \leqslant P_{i_k}\Big(\max_{n \leqslant ck} S_n \geqslant \lambda a_k,\Big)\frac{V(\lambda a_k)}{V(i_k)} \to P_z\Big(\sup_{t \leqslant c} B_t \geqslant \lambda\Big)\frac{\lambda}{z}. \end{aligned}$$

As $P_z(\sup_{t \leqslant c} B_t \geqslant \lambda)$ is exponentially small in λ, the result follows as in the proof of Theorem 2.1. In the case $z = 0$ this argument fails, and we need to use Proposition 3. This implies that it suffices to show that the pre-minimum process vanishes in probability as $k \to \infty$. The time at which R_k first takes its all-time minimum value is $k^{-1}\mu\mathrm{m}$, and applying the Markov property at time $[k\varepsilon]$ and

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using Theorem 5.1(ii) gives

$$\begin{aligned}
P^*_{i_k}(\mu > k\varepsilon) &= \sum_{x=0}^{i_k} \sum_{y \geqslant x} P^*_{i_k}\Big(S_{[k\varepsilon]} = y,\ \min_{n \leqslant [k\varepsilon]} S_n = x,\ \min_{n > [k\varepsilon]} S_n \leqslant x\Big) \\
&= \sum_{x=0}^{i_k} \sum_{y \geqslant x} \tilde{P}_{i_k}\Big(S_{[k\varepsilon]} = y,\ \min_{n \leqslant [k\varepsilon]} S_n = x\Big) \frac{V(y)}{V(i_k)} \frac{V(y) - V(y-x)}{V(y)} \\
&\leqslant \frac{1}{V(i_k)} \sum_{x=0}^{i_k} \sum_{y \geqslant x} P_{i_k}\Big(S_{[k\varepsilon]} = y,\ \min_{n \leqslant [k\varepsilon]} S_n = x\Big) V(x) \\
&\leqslant P_{i_k}\Big(\min_{n \leqslant [k\varepsilon]} S_n \geqslant 0\Big) = P_0\Big(\min_{n \leqslant [k\varepsilon]} S_n \geqslant -a_k z_k\Big) \to 0.
\end{aligned}$$

In this calculation we have used the fact that the renewal function V is subadditive, and this also plays a role in showing that, under $P^*_{i_k}$,

$$\sup_{t \leqslant k^{-1}\mu} R_k(t) = \frac{\max_{n \leqslant \mu} S_n}{a_k} \xrightarrow{P} 0.$$

We see this by applying the Markov property at $\tau_k := \min\{n : S_n \geqslant \eta a_k\}$ to get, when $z_k < \eta$,

$$\begin{aligned}
P^*_{i_k}\Big(\max_{n \leqslant \mu} S_n \geqslant \eta a_k\Big) &= \sum_{x=0}^{i_k} \sum_{y \geqslant \eta a_k} P^*_{i_k}\Big(S_\tau = y, \min_{n \leqslant \tau} S_n = x, \min_{n > \tau} S_n > x\Big) \\
&= \sum_{x=0}^{i_k} \sum_{y \geqslant \eta a_k} P^*_{i_k}\Big(S_\tau = y, \min_{n \leqslant \tau} S_n = x\Big) \frac{V(y) - V(y-x)}{V(y)}.
\end{aligned}$$

Using the fact that

$$\frac{V(y) - V(y-x)}{V(y)} \leqslant \frac{V(x)}{V(y)} \leqslant \frac{V(i_k)}{V(\eta a_k)} \to 0,$$

because $i_k/a_k \to 0$, gives

$$P^*_{i_k}\Big(\max_{n \leqslant \mu} S_n \geqslant \eta a_k\Big) \to 0, \tag{5.2}$$

for each fixed $\eta > 0$, and the result follows. □

REMARK 4. Although we have considered only the case of random walks taking values on the integer lattice, it is not difficult to extend our results to the non-lattice case; for details see [**7**]. On the other hand, extending the results to random walks in the domain of attraction of stable laws seems quite challenging, if only because we have little explicit knowledge of 'stable processes conditioned to stay non-negative'.

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