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GENERAL DRAWDOWN-BASED EXIT IDENTITIES FOR LÉVY PROCESSES OBSERVED AT POISSON ARRIVAL TIME

JINGCHAO LI,* *Shenzhen University*
SHUANMING LI,** *University of Melbourne*
WENYUAN WANG,*** *Fujian Normal University*
KAIXIN YAN,**** *Xiamen University*

Abstract

This paper derives explicit expressions for drawdown-based two-sided exit identities involving the overshoots and undershoots at the exit times under Poisson observation times for spectrally negative Lévy risk processes by using fluctuation theory. All resulting Laplace transforms of the risk quantities of interest are expressed in terms of the scale functions of the spectrally negative Lévy processes.

Keywords: Spectrally negative Lévy process; Poisson observation; drawdown time; fluctuation identity

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1. Introduction

Actuarial mathematics plays an important role in quantifying and managing financial risk, advancing the use of stochastic processes to model market dynamics more accurately. Among these processes, Lévy processes, which are characterized by continuous paths and jump components, can effectively capture the abrupt and unpredictable changes inherent in financial assets and liabilities. A special subclass, the spectrally negative Lévy process (SNLP) with no positive jumps, has attracted many scholars' attention since the 2000s. In actuarial applications, SNLPs model surplus processes that combine Brownian motion (representing investment-related fluctuations) with downward jumps (representing claim losses). These processes align well with empirical data and are supported by extensive results in fluctuation theory. SNLPs encompass a wide range of insurance and financial models, including the perturbed and non-perturbed Cramér–Lundberg process, the Gamma process, and the inverse Gaussian process (see [8, 13, 22, 24, 29]).

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* Postal address: School of Mathematical Sciences, Shenzhen University, Shenzhen, Guangdong 518060, P.R. China. Email: jingchaoli@szu.edu.cn
** Postal address: Department of Economics, Centre for Actuarial Studies, University of Melbourne, Melbourne, VIC 3010, Australia. Email: shli@unimelb.edu.au
*** Postal address: School of Mathematics and Statistics, Fujian Normal University, Fuzhou, Fujian 350117, P.R. China. Email: wwywang@xmu.edu.cn
**** Postal address: School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, P.R. China. Email: kaixinyan@stu.xmu.edu.cn

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A key analytical tool in the study of SNLPs is the scale function, which serves as a central component in describing fluctuation identities and exit problems. Although only semi-explicit, potential measures and exit formulas can be elegantly expressed through these functions, making them a fundamental element in SNLP theory. A comprehensive review by [12] provides detailed analysis and explicit forms of known scale functions. Landriault and Willmot [14] adopted a nonstandard analytic approach to derive new explicit expressions for the Cramér–Lundberg risk process and its perturbed version, whereas [11] presented a novel matrix-based formulation for processes with negative phase-type jumps. For further developments, the review [7] offers an extensive overview of recent advances in this field.

Beyond fluctuation theory, drawdown analysis has emerged as another valuable framework for understanding the risk dynamics of insurance surplus processes driven by Lévy models. The drawdown, defined as the decline from a historical peak, is particularly relevant in insurance mathematics, where assessing risk exposure is crucial for maintaining solvency. For Lévy-driven surplus models, drawdown analysis offers refined insights into potential vulnerabilities and recovery patterns, reflecting the combined effects of jumps and continuous paths. Key drawdown-related quantities studied in the literature include the drawdown time, the running maximum and minimum before drawdown, and the undershoot and overshoot at the drawdown moment. Relevant studies include [15, 25, 26], among others. Alongside drawdown, Parisian ruin has been studied extensively as a complementary, more realistic alternative to classical ruin, typically by requiring the process to stay below zero for a fixed duration before ruin is triggered (e.g. [16, 17, 20, 21]). Recently, researchers have integrated these two concepts by introducing the 'drawdown Parisian ruin' problem, where ruin occurs only when the surplus process remains continuously below its dynamic drawdown level for a fixed amount of time (e.g. [18, 27]).

Another active research direction involves Poisson observation models, initially was introduced in [1, 2] and later generalized in [3, 6] to more flexible schemes with surplus-dependent observation rates. The Poisson observation framework can model the monitoring frequency of an insurer's surplus by an external regulatory authority and serves as a conceptual bridge between continuous-time and discrete-time observation, enabling explicit and elegant formulations.

Building on the growing interest in Poisson observation models, in [5] several elegant identities for exit problems of Lévy processes under Poisson observation were derived, which have proven to be valuable analytical tools. Motivated by this work, we establish new formulas for fluctuation problems involving exit times under Poisson or hybrid (continuous–Poisson) observation, along with the corresponding overshoots and undershoots at exit (see Theorems 3.1 and 3.2).

This paper makes four major contributions. First, in contrast to the classical-ruin-based fluctuation identities presented in [5], our drawdown-based counterparts provide deeper insights into the historical dynamics of the underlying process. Second, incorporating drawdown concepts into SNLPs presents methodological challenges. To address these, an excursion approach that is relatively uncommon in the actuarial science is extensively adopted (see, for example, (38)–(40), and (54)). Third, since the drawdown time is a stopping time that depends on the running maximum, it is essential to continuously track this dynamically changing quantity during calculations and proofs (see, for example, (34), (42), (51), (58), and (64)). In more detail, a key insight underlying the proofs is as follows: standing at the earlier of the drawdown time (under continuous observation) and the first Poisson observation time, one must check whether the surplus process is able to recover to its running maximum in a timely manner. If the surplus

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process does succeed in climbing back to its running maximum in time, there will exist a time interval during which the running maximum remains constant. Consequently, the up- and down-crossing times of the drawdown level are reduced to those of a fixed level. As a result, the drawdown-related Poisson-continuous two-sided exit problem within this interval simplifies to a classical Poisson-continuous two-sided exit problem with fixed crossing levels, for which established solutions are available. From the moment the surplus returns to its running maximum onward, the problem regains its drawdown character, thereby allowing the strong Markov property to come into effect. Conversely, if the surplus process fails to recover to its running maximum in time, the original drawdown involved Poisson-continuous two-sided exit problem directly reduces to a classical exit problem with fixed crossing levels.

In this way, a crucial step in deriving the fluctuation identities is to decompose the target exit problem into a sequence of local subexit problems with fixed (rather than dynamically changing) and drawdown crossing levels. This decomposition calls for careful analysis to ensure a mathematically rigorous transition between the original problem and these sequential local subproblems. Fourth, owing to the intricate nature of the solution expressions for the target drawdown-based Poisson-continuous two-sided exit problems, Section 4 is dedicated exclusively to obtaining simplified solutions for the case of a linear drawdown function (see Theorems 4.1–4.3). Specifically, when the drawdown function vanishes (i.e. reduces to 0), our findings are in exact agreement with those of [5] (see Remarks 4.1–4.2). In addition, Theorem 3.3 shows that a fluctuation identity of Poisson observation converges to that of continuous observation as $\lambda \to \infty$.

The remainder of this paper is organized as follows. In Section 2, some preliminaries and notation of SNLPs are presented. The main results are provided in Section 3. Simplified solutions for the target exit problems with linear drawdown function are presented in Section 4. All proofs are postponed to Section 5.

2. Preliminaries and notation

On a probability space with probability laws $\{\mathrm{P}_x; x \in (-\infty, \infty)\}$ and natural filtration $\{\mathcal{F}_t; t \geq 0\}$, let $X = \{X(t); t \geq 0\}$ be a SNLP which is not a purely increasing linear drift or the negative of a subordinator. The Laplace exponent of X is given by

$$\psi(\theta) := \log\left(\mathrm{E}_x\left(\mathrm{e}^{\theta(X(1)-x)}\right)\right) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{(0,1)}(x)\right)\Pi(\mathrm{d}x),$$

where Π is the Lévy measure satisfying $\int_{(0,\infty)} \left(1 \wedge x^2\right) \Pi(\mathrm{d}x) < \infty$. The function $\psi(\theta)$ is finite, strictly convex, and infinitely differentiable in $\theta \in [0, \infty)$. As in [9], the scale function $\{W_\lambda; \lambda \geq 0\}$ of X is defined as follows. For each $\lambda \geq 0$, $W_\lambda \colon [0, \infty) \to [0, \infty)$ is the unique strictly increasing and continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W_\lambda(x)\mathrm{d}x = \frac{1}{\psi(\theta) - \lambda}, \quad \theta > \Phi_\lambda,$$

where Φ_λ is the largest solution of the equation $\psi(\theta) = \lambda$ (there are at most two). Throughout this paper, the scale function is assumed to be continuously differentiable on $(0, \infty)$, where discussions on the differentiability of the scale function W_λ can be found in [12]. For example, $W_\lambda \in C^2(0, \infty)$ when $\sigma > 0$, i.e. the Lévy process has a nontrivial Brownian component. For convenience, we extend the domain of W_λ to the whole real line by setting $W_\lambda(x) = 0$ for all

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$x < 0$. Let $W(x) = W_{\lambda=0}(x)$. For later use, we define

$$Z_\lambda(x) := 1 + \lambda \int_0^x W_\lambda(z)\mathrm{d}z, \quad \lambda \geq 0,\ x \geq 0,$$

and

$$Z_\lambda(x, \theta) := \mathrm{e}^{\theta x}\left(1 + (\lambda - \psi(\theta)) \int_0^x \mathrm{e}^{-\theta z} W_\lambda(z)\mathrm{d}z\right), \quad \lambda, \theta \geq 0,\ x \geq 0,$$

with $Z_\lambda(x) = 1$ and $Z_\lambda(x, \theta) = \mathrm{e}^{\theta x}$ for $x < 0$. Further define

$$W_a^{(p,q)}(x) := W_q(x) + (p - q) \int_0^a W_q(x - y) W_p(y)\mathrm{d}y, \quad p, q \geq 0,\ x \in (-\infty, \infty), \tag{1}$$

and

$$\begin{aligned} Z_a^{(p,q)}(x, \theta) &:= Z_p(x, \theta) + (q - p) \int_a^x W_q(x - y) Z_p(y, \theta)\mathrm{d}y \\ &= Z_q(x, \theta) + (p - q) \int_0^a W_q(x - y) Z_p(y, \theta)\mathrm{d}y, \quad p, q \geq 0,\ 0 \leq a \leq x, \end{aligned} \tag{2}$$

with $Z_a^{(p,q)}(x, \theta) = Z_p(x, \theta)$ for $x < a$. By [22], we have

$$W_q(x) - W_p(x) = (q - p) \int_0^x W_p(x - y) W_q(y)\mathrm{d}y, \quad p, q \geq 0,\ x \in (-\infty, \infty). \tag{3}$$

Using the Laplace transform method, we can easily show that

$$Z_p(x, \theta) - Z_q(x, \theta) = (p - q) \int_0^x W_q(x - y) Z_p(y, \theta)\mathrm{d}y, \quad p, q \geq 0,\ x \geq 0.$$

In addition, we define

$$\widetilde{Z}(x, p, q) = \frac{\psi(p) Z(x, q) - \psi(q) Z(x, p)}{p - q}, \quad p, q \geq 0,\ p \neq q. \tag{4}$$

Note that for $p = q$, $\widetilde{Z}(x, p, q)$ in (4) simplifies to

$$\widetilde{Z}(x, p, p) = \psi'(p) Z(x, p) - \psi(p) Z'(x, p),$$

where the differentiation of Z is with respect to the second argument.

The first up-crossing and down-crossing time of the process X are defined by

$$\tau_b^+ := \inf\{t \geq 0; X(t) > b\} \quad \text{and} \quad \tau_c^- := \inf\{t \geq 0; X(t) < c\},$$

for $b, c \in (-\infty, \infty)$. The general drawdown time of the process X is defined as

$$\tau_\xi := \inf\{t \geq 0; X(t) < \xi(\overline{X}(t))\},$$

where $\overline{X}(t) := \sup_{s \in [0,t]} X(s)$ is the running maximum process of X and ξ is a drawdown function with $\xi(x) < x$ for all $x \in \mathbb{R}$. By (3.1) in Proposition 3.1 of [15], we have

$$\mathrm{E}_x\left(\mathrm{e}^{-\lambda \tau_a^+} \mathbf{1}_{\{\tau_a^+ < \tau_\xi\}}\right) = \exp\left\{-\int_x^a \frac{W_\lambda'(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}\,\mathrm{d}z\right\}, \quad 0 \leq x < a < \infty,\ \lambda \geq 0, \tag{5}$$

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where $\overline{\xi}(z) := z - \xi(z) > 0$. In addition, according to Proposition 3.2 of [15], we have the following joint probability density function of the state of X and the running maximum of X at an independent exponential random time e_λ on the event that e_λ precedes the drawdown time:

$$\begin{aligned}\mathrm{P}_x(X(e_\lambda) \in \mathrm{d}y, \overline{X}(e_\lambda) \in \mathrm{d}z, e_\lambda < \tau_\xi) &= \lambda W(0) \mathrm{e}^{-\int_x^y \frac{W'_\lambda(\overline{\xi}(w))}{W_\lambda(\overline{\xi}(w))}\mathrm{d}w} \mathbf{1}_{\{y>x\}}\,\mathrm{d}y\,\delta_y(\mathrm{d}z)\\ &+\lambda \mathrm{e}^{-\int_x^z \frac{W'_\lambda(\overline{\xi}(w))}{W_\lambda(\overline{\xi}(w))}\mathrm{d}w}\left(W'_\lambda(z-y) - \frac{W'_\lambda(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)\mathbf{1}_{(\xi(z),z)}(y)\mathrm{d}z\,\mathrm{d}y, \end{aligned} \tag{6}$$

for $x \vee y \le z,\ \lambda \ge 0$, where $\delta_x(\mathrm{d}z)$ is the Dirac measure which assigns unit mass to the point x and e_λ is an exponentially distributed random variable (independent of X) with mean $1/\lambda$. In particular, we have

$$\begin{aligned}\mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y, \overline{X}(e_\lambda) \in \mathrm{d}z\right) &= \lambda W(0)\mathrm{e}^{-\Phi_\lambda(y-x)}\mathbf{1}_{\{y>x\}}\,\delta_y(\mathrm{d}z)\mathrm{d}y\\ &+\lambda\,\mathrm{e}^{-\Phi_\lambda(z-x)}\big(-\Phi_\lambda W_\lambda(z-y) + W'_\lambda(z-y)\big)\mathbf{1}_{\{z>y\}}\,\mathrm{d}z\,\mathrm{d}y, \quad x \vee y \le z,\ \lambda \ge 0. \end{aligned} \tag{7}$$

Furthermore, let T_i be the ith arrival time of an independent Poisson process with rate $\lambda > 0$, and define the following stopping times

$$T_b^+ := \min\{T_i; X(T_i) > b\}, \quad T_c^- := \min\{T_i; X(T_i) < c\},$$

and

$$T_\xi := \min\{T_i; X(T_i) < \xi(\overline{X}(T_i))\}.$$

We shall briefly recall the concepts in excursion theory for the reflected process $\{\overline{X}(t) - X(t); t \ge 0\}$, and we refer to [9] for more details. For $x \in \mathbb{R}$, the process $\{L(t) := \overline{X}(t) - x; t \ge 0\}$ serves as a local time accumulated by the reflected process $\{\overline{X}(t) - X(t); t \ge 0\}$ under P_x. Let the corresponding inverse local time be

$$L^{-1}(t) =: \inf\{s \ge 0; L(s) > t\} = \sup\{s \ge 0; L(s) \le t\}.$$

Furthermore, let $L^{-1}(t-) = \lim_{s \uparrow t} L^{-1}(s)$. The Poisson point process of excursions indexed by this local time is denoted by $\{(t, \varepsilon_t); t \ge 0\}$ (see Section 5 of [15]), where

$$\varepsilon_t(s) := X(L^{-1}(t-)) - X(L^{-1}(t-) + s), \quad s \in \left[0, L^{-1}(t) - L^{-1}(t-)\right),$$

whenever $L^{-1}(t) - L^{-1}(t-) > 0$. For the case of $L^{-1}(t) - L^{-1}(t-) = 0$, define $\varepsilon_t = \Upsilon$ with Υ being an additional isolated point. Accordingly, we denote a generic excursion as $\varepsilon(\cdot)$ (or ε for short) which belongs to the space $\mathcal{E}$ of canonical excursions. The intensity measure of the process $\{(t, \varepsilon_t); t \ge 0\}$ is given by $\mathrm{d}t \times \mathrm{d}n$ where n is a measure on the space of excursions. The lifetime of a canonical excursion ε is denoted by ζ, and its excursion height is denoted by $\overline{\varepsilon} = \sup_{t \in [0,\zeta]} \varepsilon(t)$. The first passage time of a canonical excursion ε is defined as

$$\chi_b^+ := \inf\{t \in [0, \zeta]; \varepsilon(t) > b\}$$

with the convention $\inf \emptyset = \zeta$. From (31) of [23], we have the following joint Laplace transform (under the excursion measure n) of the first up-crossing time and the overshoot at the level s:

$$n\left(\mathrm{e}^{-\lambda\chi_s^+ + \theta(s-\varepsilon(\chi_s^+))}; \overline{\varepsilon} > s\right) = \frac{W'_\lambda(s)}{W_\lambda(s)}Z_\lambda(s,\theta) - \theta Z_\lambda(s,\theta) - (\lambda - \psi(\theta))W_\lambda(s), \tag{8}$$

for $\lambda \ge 0$, $\theta \ge 0$, and $s > 0$.

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3. Main results

This section aims at presenting several drawdown-based Poisson observation-type two-sided exit identities involving the overshoots and undershoots at the exit times of the spectrally negative Lévy process. The main results are stated in Theorems 3.1 and 3.2. To begin with, define

$$\begin{aligned}
f_1(x;\theta,a) &:= \mathrm{E}_x\big(\mathrm{e}^{\theta(X(T_\xi)-\xi(\overline{X}(T_\xi)))};T_\xi<\tau_a^+\big), \quad x\in[0,a],\\
f_1(x;\theta,\infty) &:= \mathrm{E}_x\big(\mathrm{e}^{\theta(X(T_\xi)-\xi(\overline{X}(T_\xi)))};T_\xi<\infty\big), \quad x\in[0,\infty),\\
f_2(x;\theta,a) &:= \mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_\xi\big), \quad x\in[0,\infty),\\
f_3(x;\theta,a) &:= \mathrm{E}_x\big(\mathrm{e}^{\theta(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi)))};\tau_\xi<T_a^+\big), \quad x\in[0,\infty).
\end{aligned}$$

The functions f_1–f_3 are Laplace transforms of the undershoot at the Poisson drawdown time T_ξ or the overshoot at the Poisson exceeding time T_a^+, under hybrid (continuous and Poisson) observation. The following Theorem 3.1 provides closed-form expressions for the functions f_1–f_3 in terms of the scale functions of the spectrally negative Lévy process X. Their detailed proofs are postponed to Section 5.

Theorem 3.1. *Recall* $\overline{\xi}(z) := z-\xi(z)$. *Suppose that* $\lim_{z\to\infty}\overline{\xi}(z)=\infty$, *then, for* $a,\theta\geq 0$, *we have*

$$f_1(x;\theta,\infty)=\mathrm{e}^{\int_0^x h_1(z)\mathrm{d}z}\int_x^\infty g_1(z)\,\mathrm{e}^{-\int_0^z h_1(w)\mathrm{d}w}\,\mathrm{d}z, \quad x\in[0,\infty), \tag{9}$$

$$f_1(x;\theta,a)=\mathrm{e}^{\int_0^x h_1(z)\mathrm{d}z}\int_x^a g_1(z)\,\mathrm{e}^{-\int_0^z h_1(w)\mathrm{d}w}\,\mathrm{d}z, \quad x\in[0,a], \tag{10}$$

$$f_2(x;\theta,a)=\mathrm{e}^{-\int_x^a \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z}f_2(a;\theta,a), \quad x\in[0,a], \tag{11}$$

$$f_2(x;\theta,a)=-\mathrm{e}^{\int_a^x h_2(z)\mathrm{d}z}\int_x^\infty g_2(z)\,\mathrm{e}^{-\int_a^z h_2(w)\mathrm{d}w}\,\mathrm{d}z, \quad x\in(a,\infty), \tag{12}$$

$$\begin{aligned}
f_3(x;\theta,a)&=\mathrm{e}^{-\int_x^a \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z}f_3(a;\theta,a)+\int_x^a \mathrm{e}^{\theta\xi(z)}\mathrm{e}^{-\int_x^z \frac{W'(\overline{\xi}(w))}{W(\overline{\xi}(w))}\mathrm{d}w}\\
&\quad\times\left(\left(-\theta+\frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\right)Z(\overline{\xi}(z),\theta)+\psi(\theta)W(\overline{\xi}(z))\right)\mathrm{d}z, \quad x\in[0,a],
\end{aligned} \tag{13}$$

$$f_3(x;\theta,a)=-\mathrm{e}^{\int_a^x h_2(z)\mathrm{d}z}\int_x^\infty g_3(z)\,\mathrm{e}^{-\int_a^z h_2(w)\mathrm{d}w}\,\mathrm{d}z, \quad x\in(a,\infty), \tag{14}$$

where

$$\begin{aligned}
h_1(x)&=\Phi_\lambda-\frac{\lambda W(\overline{\xi}(x))}{Z(\overline{\xi}(x),\Phi_\lambda)}=\frac{Z'(\overline{\xi}(x),\Phi_\lambda)}{Z(\overline{\xi}(x),\Phi_\lambda)},\\
g_1(x)&=\frac{\lambda}{\lambda-\psi(\theta)}\left(\frac{Z'(\overline{\xi}(z),\Phi_\lambda)}{Z(\overline{\xi}(z),\Phi_\lambda)}Z(\overline{\xi}(z),\theta)-Z'(\overline{\xi}(z),\theta)\right),
\end{aligned}$$

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$$h_2(x) = \frac{W'_\lambda(\overline{\xi}(x)) - \lambda \int_0^{a-\xi(x)} W(y) W'_\lambda(\overline{\xi}(x) - y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))},$$

$$g_2(x) = -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \int_{a\vee\xi(x)}^{x} \mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} W_\lambda(x-y)\right)\mathrm{d}y$$
$$+\lambda \mathrm{e}^{-\theta(\overline{\xi}(x)-a)} \int_0^{\overline{\xi}(x)-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \left(\frac{(W^{(0,\lambda)}_{a-\xi(x)})'(\overline{\xi}(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\right)\mathbf{1}_{\{a > \xi(x)\}},$$

$$g_3(x) = -\left(\left(-\theta + \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\right) Z_\lambda(\overline{\xi}(x), \theta) - (\lambda - \psi(\theta)) W_\lambda(\overline{\xi}(x))\right)$$
$$-\lambda \int_{\xi(x)\wedge a}^{a} Z(y - \xi(x), \theta) W'_\lambda(x-y)\mathrm{d}y + Z_\lambda(\overline{\xi}(x), \theta)\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\mathbf{1}_{\{\xi(x)\le a\}}$$
$$+\lambda \frac{Z^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x), \theta) \int_{\xi(x)\wedge a}^{a} W(y - \xi(x)) W'_\lambda(x-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} - \frac{Z^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x), \theta) W'_\lambda(\overline{\xi}(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}\mathbf{1}_{\{\xi(x)\le a\}}.$$

When $\xi(x) \equiv \xi(a)$ for all $x \in [a, \infty)$, the following corollary is a simplified version of Theorem 3.1 as follows. Its proof is given in Section 5.

Corollary 3.1. *If we assume that $\xi(x) \equiv \xi(a)$ for all $x \in [a, \infty)$, we have*

$$f_2(x; \theta, a) = \mathrm{e}^{-\int_x^a \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z} \frac{\lambda W(\overline{\xi}(a))}{(\theta + \Phi_\lambda) Z(\overline{\xi}(a), \Phi_\lambda)}, \quad x \in [0, a),$$
$$f_2(x; \theta, a) = -\lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y + \frac{\lambda W^{(0,\lambda)}_{\overline{\xi}(a)}(x - \xi(a))}{(\theta + \Phi_\lambda) Z(\overline{\xi}(a), \Phi_\lambda)}, \quad x \in [a, \infty), \tag{15}$$

$$f_3(x; \theta, a) = \mathrm{e}^{-\int_x^a \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z}\left(Z(\overline{\xi}(a), \theta) + \frac{W(\overline{\xi}(a))}{\theta - \Phi_\lambda}\left(\lambda \frac{Z(\overline{\xi}(a), \theta)}{Z(\overline{\xi}(a), \Phi_\lambda)} - \psi(\theta)\right)\right)$$
$$+ \int_x^a \mathrm{e}^{-\int_x^z \frac{W'(\overline{\xi}(w))}{W(\overline{\xi}(w))}\mathrm{d}w}\left(\left(-\theta + \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\right) Z(\overline{\xi}(z), \theta) + \psi(\theta) W(\overline{\xi}(z))\right)\mathrm{d}z, \quad x \in [0, a),$$
$$f_3(x; \theta, a) = Z^{(0,\lambda)}_{\overline{\xi}(a)}(x - \xi(a), \theta) + \frac{W^{(0,\lambda)}_{\overline{\xi}(a)}(x - \xi(a))}{\theta - \Phi_\lambda}\left(\lambda \frac{Z(\overline{\xi}(a), \theta)}{Z(\overline{\xi}(a), \Phi_\lambda)} - \psi(\theta)\right), \quad x \in [a, \infty). \tag{16}$$

Next, we present the second main result of our paper stated in Theorem 3.2. Define

$$f_4(x; \theta, a) := \mathrm{E}_x\left(\mathrm{e}^{\theta(X(T_\xi) - \xi(\overline{X}(T_\xi)))}; T_\xi < T_a^+\right), \quad x \in [0, \infty),$$
$$f_5(x; \theta, a) := \mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+) - a)}; T_a^+ < T_\xi\right), \quad x \in [0, \infty).$$

The functions f_4 and f_5 are Laplace transforms of the undershoot at the drawdown time T_ξ or the overshoot at the exceeding time T_a^+, under purely Poisson observation. The following Theorem 3.2 provides closed-form expressions for the functions f_4 through f_5 in terms of the scale functions of the spectrally negative Lévy process X. Their detailed proofs are deferred to Section 5.

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Theorem 3.2. *Recall* $\overline{\xi}(z) := z - \xi(z)$. *Assume that* $\lim_{z\to\infty} \overline{\xi}(z) = \infty$, *then, for* $a, \theta \geq 0$, *we have*

$$f_4(x;\theta,a) = (1 - f_1(x;0,a))\, f_4(a;\theta,a) + f_1(x;\theta,a), \quad x \in [0,a], \tag{17}$$

$$f_4(x;\theta,a) = -\mathrm{e}^{\int_a^x h_4(z)\mathrm{d}z} \int_x^\infty g_4(z)\, \mathrm{e}^{-\int_a^z h_4(w)\mathrm{d}w}\, \mathrm{d}z, \quad x \in (a,\infty), \tag{18}$$

$$f_5(x;\theta,a) = (1 - f_1(x;0,a))\, f_5(a;\theta,a), \quad x \in [0,a], \tag{19}$$

$$f_5(x;\theta,a) = \mathrm{e}^{\int_a^x h_4(z)\mathrm{d}z}\left(\mathrm{e}^{-\int_a^\infty h_4(z)\mathrm{d}z}\mathbf{1}_{\{\theta=0\}} - \int_x^\infty g_5(z)\, \mathrm{e}^{-\int_a^z h_4(w)\mathrm{d}w}\, \mathrm{d}z\right), \quad x \in (a,\infty), \tag{20}$$

where

$$\begin{aligned}
h_4(x) &= \Phi_\lambda - \lambda \int_{\xi(x)\wedge a}^a \rho(y-\xi(x);\lambda, a-\xi(x), \overline{\xi}(x))\left(W_\lambda'(x-y) - \Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y,\\
g_4(x) &= -\lambda \int_{-\infty}^{\xi(x)\wedge a} \mathrm{e}^{\theta(y-\xi(x))}\left(-\Phi_\lambda W_\lambda(x-y) + W_\lambda'(x-y)\right)\mathrm{d}y\\
&\quad - \int_{\xi(x)\wedge a}^a \frac{\lambda\left(Z(y-\xi(x),\theta) - Z(y-\xi(x),\Phi_\lambda)\frac{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\theta)}{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\Phi_\lambda)}\right)}{\lambda - \psi(\theta)}\\
&\quad \times \lambda\left(-\Phi_\lambda W_\lambda(x-y) + W_\lambda'(x-y)\right)\mathrm{d}y + \lambda\upsilon(\overline{\xi}(x);\theta, a-\xi(x))\\
&\quad \times \int_{\xi(x)\wedge a}^a \rho(y-\xi(x);\lambda, a-\xi(x), \overline{\xi}(x))\left(-\Phi_\lambda W_\lambda(x-y) + W_\lambda'(x-y)\right)\mathrm{d}y,\\
g_5(x) &= -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda\int_{a\vee\xi(x)}^x \mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(x-y) - \Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad - \int_{\xi(x)\wedge a}^a \frac{\lambda^2}{\Phi_\lambda+\theta}\frac{Z(y-\xi(x),\Phi_\lambda)}{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\Phi_\lambda)}\left(W_\lambda'(x-y) - \Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad + \lambda\int_{\xi(x)\wedge a}^a \rho(y-\xi(x);\lambda, a-\xi(x), \overline{\xi}(x))\vartheta(\overline{\xi}(x);\theta, a-\xi(x))\\
&\quad \times \left(W_\lambda'(x-y) - \Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y,
\end{aligned}$$

and the functions ϑ, υ, *and* ρ *are given in Lemmas 5.3–5.5 (see Section 5).*

In particular, when $\xi(x) \equiv \xi(a)$ for all $x \in [a,\infty)$, we derive the following corollary, which is a simplified form of Theorem 3.2. The proof is provided in Section 5.

Corollary 3.2. *Suppose that* $\xi(x) \equiv \xi(a)$ *on* $[a,\infty)$. *Then, we have*

$$\begin{aligned}
f_4(x;\theta,a) &= (1 - f_1(x;0,a))\, f_4(a;\theta,a) + f_1(x;\theta,a), \quad x \in [0,a),\\
f_4(x;\theta,a) &= \frac{\lambda}{\lambda - \psi(\theta)}\left(Z(\overline{\xi}(a),\theta) - Z(\overline{\xi}(a),\Phi_\lambda)\frac{\widetilde{Z}(\overline{\xi}(a),\Phi_\lambda,\theta)}{\widetilde{Z}(\overline{\xi}(a),\Phi_\lambda,\Phi_\lambda)}\right), \quad x \in [a,\infty),\\
f_5(x;\theta,a) &= (1 - f_1(x;0,a))\,\vartheta(\overline{\xi}(a);\theta,\overline{\xi}(a)), \quad x \in [0,a),\\
f_5(x;\theta,a) &= \vartheta(x-\xi(a);\theta,\overline{\xi}(a)), \quad x \in [a,\infty),
\end{aligned}$$

where the function ϑ *is given in Lemma 5.3 (see Section 5).*

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Theorems 3.1 and 3.2 derive closed-form solutions for several drawdown-based Poisson-continuous two-sided exit problems involving the overshoots and undershoots at exit times for spectrally negative Lévy risk processes. These solutions are formulated in terms of scale functions. By focusing on a special case (i.e. $\xi(x) \equiv \xi(a)$ for all $x \geq a$), Corollaries 3.1 and 3.2 yield more concise forms of these solutions. In Section 4, another special case where $\xi(x) = kx - d$ is considered, leading to further simplified forms of these solutions. Moreover, when the drawdown effect vanishes ($\xi(x) \equiv 0$), our solutions reduce to the benchmark results of [5], demonstrating the consistency and generality of our framework.

We note that, although our main results (that is, the results of Theorems 3.1 and 3.2) appear somewhat more elaborate than those of [5], they offer unique merits of their own. In contrast to classical ruin-based fluctuation identities [5], our drawdown-based formulations provide deeper insights into the historical path dynamics of the underlying process and present potential advantages for risk management in finance and insurance. The incorporation of drawdown concepts into our target problems introduces notable methodological challenges, which we address through an excursion-based approach, an analytical technique that is relatively uncommon in the actuarial literature. In addition, since drawdown times depend on the dynamically evolving running maximum, a key aspect of our analysis involves carefully tracking this running maximum and decomposing the drawdown-modulated two-sided exit problem into a sequence of localized, classical subproblems with fixed (rather than time-varying) crossing levels or drawdown levels.

The following Theorem 3.3 shows that when $\lambda \to \infty$, our fluctuation identity (10) under Poisson observation converges to its existing counterpart under continuous observation; see (9) of [28] or (3.2) of [15] in the case where ξ is a linear function. Although such convergence seems like a very natural conclusion, it is not straightforward to observe this behavior in the other fluctuation identities of Theorems 3.1 and 3.2. The proof of Theorem 3.3 is given in Section 5.

Theorem 3.3. *For $a, \theta \geq 0$, we have*

$$\lim_{\lambda\to\infty} f_1(x;\theta,a) = \int_x^a \mathrm{e}^{-\int_x^z \frac{W'(\bar{\xi}(w))}{W(\bar{\xi}(w))}\mathrm{d}w}\left[\frac{W'(\bar{\xi}(z))}{W(\bar{\xi}(z))}Z(\bar{\xi}(z),\theta) - Z'(\bar{\xi}(z),\theta)\right]\mathrm{d}z. \tag{21}$$

4. Results for the special case of $\xi(x) = kx - d$ with $k < 1$ and $d \geq 0$

To gain more intuitive insight and derive potentially concise closed-form solutions, we now focus on two special cases that are both economically and mathematically meaningful. We first analyze the case of $\xi(x) = kx - d$ with $k < 1$ and $d \geq 0$, for which only the expression for $f_1(x;\theta,a)$ is presented, the expressions for f_2 through f_5 are not included due to their excessive complexity. We then provide the full set of expressions for the case where $\xi(x) = -d$ with $d \geq 0$.

Theorem 4.1. *Let $\xi(x) = kx - d$ with $k < 1$ and $d \geq 0$. For $a, \theta \geq 0$, we have*

$$f_1(x;\theta,a) = \frac{\lambda}{\lambda-\psi(\theta)}\left[Z((1-k)x+d,\theta) - Z((1-k)a+d,\theta)\left(\frac{Z((1-k)a+d,\Phi_\lambda)}{Z((1-k)x+d,\Phi_\lambda)}\right)^{-\frac{1}{1-k}}\right]$$
$$-\frac{\lambda k}{\lambda-\psi(\theta)}\int_x^a Z'((1-k)z+d,\theta)\left(\frac{Z((1-k)z+d,\Phi_\lambda)}{Z((1-k)x+d,\Phi_\lambda)}\right)^{-\frac{1}{1-k}}\mathrm{d}z.$$

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Theorem 4.2. *Let $\xi(x) \equiv -d$ with $d \geq 0$. For $a, \theta \geq 0$, we have*

$$f_1(x;\theta,\infty) = \frac{\lambda}{\lambda-\psi(\theta)}Z(x+d,\theta) - Z(x+d,\Phi_\lambda)\frac{\psi(\theta)(\Phi_\lambda-\Phi)}{(\lambda-\psi(\theta))(\theta-\Phi)},$$

$$f_1(x;\theta,a) = \frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x+d,\theta) - Z(a+d,\theta)\frac{Z(x+d,\Phi_\lambda)}{Z(a+d,\Phi_\lambda)}\right],$$

$$\begin{aligned} f_2(x;\theta,a) = \frac{W(x+d)}{W(a+d)}\int_a^\infty &\left\{\lambda W_\lambda(z-a) - \lambda e^{-\theta(z-a)}\int_0^{z-a} e^{\theta y}W_\lambda(y)dy\left(\frac{W'_\lambda(z+d)}{W_\lambda(z+d)}+\theta\right)\right.\\ &\left.-\lambda e^{-\theta(z+d-a)}\int_0^{z+d-a} e^{\theta y}W_\lambda(y)dy\left(\frac{(W^{(0,\lambda)}_{a+d})'(z+d)}{W^{(0,\lambda)}_{a+d}(z+d)} - \frac{W'_\lambda(z+d)}{W_\lambda(z+d)}\right)\right\}\\ &\times\frac{W^{(0,\lambda)}_{a+d}(a+d)}{W^{(0,\lambda)}_{a+d}(z+d)}\,dz, \quad x\in[0,a], \end{aligned}$$

$$\begin{aligned} f_2(x;\theta,a) = \int_x^\infty &\left\{\lambda W_\lambda(z-a) - \lambda e^{-\theta(z-a)}\int_0^{z-a} e^{\theta y}W_\lambda(y)dy\left(\frac{W'_\lambda(z+d)}{W_\lambda(z+d)}+\theta\right)\right.\\ &\left.-\lambda e^{-\theta(z+d-a)}\int_0^{z+d-a} e^{\theta y}W_\lambda(y)dy\left(\frac{(W^{(0,\lambda)}_{a+d})'(z+d)}{W^{(0,\lambda)}_{a+d}(z+d)} - \frac{W'_\lambda(z+d)}{W_\lambda(z+d)}\right)\right\}\\ &\times\frac{W^{(0,\lambda)}_{a+d}(x+d)}{W^{(0,\lambda)}_{a+d}(z+d)}\,dz, \quad x\in(a,\infty), \end{aligned}$$

$$\begin{aligned} f_3(x;\theta,a) = \frac{W(x+d)}{W(a+d)}&\left(Z(a+d,\theta) - W(a+d)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda} - \frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a+d,\theta)}{Z(a+d,\Phi_\lambda)}\right)\right)\\ &-e^{-\theta d}\frac{W(x+d)}{W(a+d)}Z(a+d,\theta) + e^{-\theta d}Z(x+d,\theta), \quad x\in[0,a], \end{aligned}$$

$$f_3(x;\theta,a) = Z^{(0,\lambda)}_{a+d}(x+d,\theta) - W^{(0,\lambda)}_{a+d}(x+d)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda} - \frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a+d,\theta)}{Z(a+d,\Phi_\lambda)}\right),\ x\in(a,\infty).$$

Remark 4.1. By further setting $d=0$, one can easily verify that all results in Theorem 4.2 coincide with the corresponding results of Theorem 3.1 in [5]. Indeed, by letting $d=0$, we have

$$f_1(x;\theta,\infty) = \frac{\lambda}{\lambda-\psi(\theta)}Z(x,\theta) - Z(x,\Phi_\lambda)\frac{\psi(\theta)(\Phi_\lambda-\Phi)}{(\lambda-\psi(\theta))(\theta-\Phi)}, \quad x\in[0,\infty),$$

and

$$f_1(x;\theta,a) = \frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x,\theta) - Z(a,\theta)\frac{Z(x,\Phi_\lambda)}{Z(a,\Phi_\lambda)}\right], \quad x\in[0,a],$$

which coincide with (14) and (15), respectively, of [5]. In addition, we have

$$\begin{aligned} f_2(x;\theta,a) &= \lambda\int_x^\infty \frac{d}{dz}\left(e^{-\theta(z-a)}\int_0^{z-a} e^{\theta y}W_\lambda(y)dy\, e^{-\int_x^z h_2(w)dw}\right)dz\\ &= \frac{\lambda W^{(0,\lambda)}_a(x)}{(\theta+\Phi_\lambda)Z(a,\Phi_\lambda)} - \lambda e^{-\theta(x-a)}\int_0^{x-a} W_\lambda(y)e^{\theta y}\,dy, \quad x\in(a,\infty), \end{aligned}$$

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and

$$f_2(x;\theta,a)=\frac{W(x)}{W(a)}f_2(a;\theta,a)=\frac{\lambda W(x)}{(\theta+\Phi_\lambda)Z(a,\Phi_\lambda)},\quad x\in[0,a],$$

which coincide with (17) of [5]. Finally, we have

$$f_3(x;\theta,a)=Z_a^{(0,\lambda)}(x,\theta)-W_a^{(0,\lambda)}(x)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda}-\frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right),\quad x\in(a,\infty),$$

and

$$f_3(x;\theta,a)=Z(x,\theta)-W(x)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda}-\frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right),\quad x\in[0,a],$$

which coincide with (18) of [5].

Theorem 4.3. *Let $\xi(x)\equiv -d$ with $d\geq 0$. For $a,\theta\geq 0$, we have*

$$\begin{aligned}f_4(x;\theta,a)=&\frac{Z(x+d,\Phi_\lambda)}{Z(a+d,\Phi_\lambda)}\Bigg(-\lambda(\mathrm{e}^{\theta d}-1)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}\int_a^\infty Z_\lambda(z+d,\theta)\frac{Z_{a+d}^{(0,\lambda)}(a+d,\Phi_\lambda)}{Z_{a+d}^{(0,\lambda)}(z+d,\Phi_\lambda)}\,\mathrm{d}z\\&+\frac{\lambda}{\lambda-\psi(\theta)}\left(Z_{a+d}^{(0,\lambda)}(a+d,\theta)-Z_{a+d}^{(0,\lambda)}(a+d,\Phi_\lambda)\frac{\widetilde{Z}(a+d,\Phi_\lambda,\theta)}{\widetilde{Z}(a+d,\Phi_\lambda,\Phi_\lambda)}\right)\Bigg)\\&+\frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x+d,\theta)-Z(a+d,\theta)\frac{Z(x+d,\Phi_\lambda)}{Z(a+d,\Phi_\lambda)}\right],\quad x\in[0,a],\\f_4(x;\theta,a)=&-\lambda(\mathrm{e}^{\theta d}-1)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}\int_x^\infty Z_\lambda(z+d,\theta)\frac{Z_{a+d}^{(0,\lambda)}(x+d,\Phi_\lambda)}{Z_{a+d}^{(0,\lambda)}(z+d,\Phi_\lambda)}\,\mathrm{d}z\\&+\frac{\lambda}{\lambda-\psi(\theta)}\left(Z_{a+d}^{(0,\lambda)}(x+d,\theta)-Z_{a+d}^{(0,\lambda)}(x+d,\Phi_\lambda)\frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right),\quad x\in(a,\infty),\\f_5(x;\theta,a)=&\frac{\lambda}{\theta+\Phi_\lambda}\frac{Z(x+d,\Phi_\lambda)}{\widetilde{Z}(a+d,\Phi_\lambda,\Phi_\lambda)},\quad x\in[0,a],\\f_5(x;\theta,a)=&\frac{\lambda}{\theta+\Phi_\lambda}\frac{Z_{a+d}^{(0,\lambda)}(x+d,\Phi_\lambda)}{\widetilde{Z}(a+d,\Phi_\lambda,\Phi_\lambda)}-\lambda\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y,\quad x\in(a,\infty).\end{aligned}$$

Remark 4.2. By further setting $d=0$, one can easily verify that all results in Theorem 4.3 coincide with the corresponding results of Theorem 3.2 in [5]. Then, letting $d=0$, we have

$$f_4(x;\theta,a)=\frac{\lambda}{\lambda-\psi(\theta)}\left(Z_a^{(0,\lambda)}(x,\theta)-Z_a^{(0,\lambda)}(x,\Phi_\lambda)\frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right),\quad x\in(a,\infty),$$

and

$$f_4(x;\theta,a)=\frac{\lambda}{\lambda-\psi(\theta)}\left(Z(x,\theta)-Z(x,\Phi_\lambda)\frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right),\quad x\in[0,a],$$

which coincide with (19) of [5]. In addition, we have

$$f_5(x;\theta,a)=\frac{\lambda}{\theta+\Phi_\lambda}\frac{Z_a^{(0,\lambda)}(x,\Phi_\lambda)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}-\lambda\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y,\quad x\in(a,\infty),$$

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and

$$f_5(x;\theta,a)=\frac{\lambda}{\theta+\Phi_\lambda}\frac{Z(x,\Phi_\lambda)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)},\quad x\in[0,a],$$

which coincide with (20) of [5].

5. Proofs of the results of Section 3

This section is specifically dedicated to presenting comprehensive and detailed proofs for the principal results outlined in Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2. To lay a solid foundation for these proofs, we first require several auxiliary technical lemmas, namely Lemmas 5.1–5.5.

5.1. Technical lemmas and their proofs

In this subsection, we introduce five auxiliary technical lemmas. These lemmas yield five new fluctuation identities, which will play a crucial role in the proofs of Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2. To start, for $\theta\geq 0$, $\lambda>0$, and $0\leq a\leq b$, we define the following five functions:

$$\begin{aligned}
\phi(x;\theta,a)&:=\mathrm{E}_x\big(\mathrm{e}^{\theta X(\tau_0^-)};\tau_0^-<T_a^+\big),\quad x\in\mathbb{R},\\
\varphi(x;\theta,a)&:=\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_0^-\big),\quad x\in\mathbb{R},\\
\vartheta(x;\theta,a)&:=\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_0^-\big),\quad x\in\mathbb{R},\\
\upsilon(x;\theta,a)&:=\mathrm{E}_x\big(\mathrm{e}^{\theta X(T_0^-)};T_0^-<T_a^+\big),\quad x\in\mathbb{R},\\
\rho(x;\lambda,a,b)&:=\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^++\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t}\right),\quad x\leq b.
\end{aligned}$$

Recall that an expression for ϕ, φ, ϑ, and υ for $x\leq a$ has been established in Theorems 3.1 and 3.2 of [5]. The upcoming Lemmas 5.1–5.4 give an expression for these four functions when $x>a$, respectively. However, to the best of the authors' knowledge, there is no existing expression for ρ in the literature. An expression for ρ will be presented in the upcoming Lemma 5.5.

The following Lemma 5.1 provides an expression for ϕ, the Laplace transform of the undershoot at the classical ruin time, given that classical ruin occurs prior to the first time that X exceeds the level a under the Poisson observation scheme.

Lemma 5.1. *For $\theta\geq 0$, we have*

$$\phi(x;\theta,a)=Z_a^{(0,\lambda)}(x,\theta)+\frac{W_a^{(0,\lambda)}(x)}{\theta-\Phi_\lambda}\left(\lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}-\psi(\theta)\right),\quad x>a. \tag{22}$$

In particular, we have

$$\lim_{x\downarrow a}\phi(x;\theta,a)=Z(a,\theta)+\frac{W(a)}{\theta-\Phi_\lambda}\left(\lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}-\psi(\theta)\right), \tag{23}$$

which aligns with (18) of [5] at the barrier a.

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Proof. Using the unnumbered equations right above (8), (10), and (18) of [5], we have that

$$\begin{aligned}
\mathrm{P}_x\big(X(e_\lambda) \in \mathrm{d}y,\, e_\lambda < \tau_0^-\big) &= \lambda\big(\mathrm{e}^{-\Phi_\lambda y} W_\lambda(x) - W_\lambda(x-y)\big)\mathrm{d}y, \quad y \in (0,\infty), \\
\mathrm{E}_y\big(\mathrm{e}^{\theta X(\tau_0^-)};\, \tau_0^- < T_a^+\big) &= Z(y,\theta) - \frac{W(y)}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right), \quad y \in [0,a], \\
\mathrm{E}_x\big(\mathrm{e}^{-\lambda\tau_0^- + \theta X(\tau_0^-)}\big) &= Z_\lambda(x,\theta) - \frac{\lambda - \psi(\theta)}{\Phi_\lambda - \theta} W_\lambda(x),
\end{aligned} \tag{24}$$

where $W(x) = W_{\lambda=0}(x)$ and $Z(x,\theta) = Z_{\lambda=0}(x,\theta)$. By conditioning on the first Poisson observation time, we have

$$\begin{aligned}
&\mathrm{E}_x\big(\mathrm{e}^{\theta X(\tau_0^-)};\, \tau_0^- < T_a^+\big) \\
&= \mathrm{E}_x\big(\mathrm{e}^{\theta X(\tau_0^-)};\, \tau_0^- < e_\lambda \le T_a^+\big) + \mathrm{E}_x\big(\mathrm{e}^{\theta X(\tau_0^-)};\, e_\lambda < \tau_0^- < T_a^+\big) \\
&= \mathrm{E}_x\big(\mathrm{e}^{-\lambda\tau_0^- + \theta X(\tau_0^-)}\big) + \int_0^a \mathrm{E}_y\big(\mathrm{e}^{\theta X(\tau_0^-)};\, \tau_0^- < T_a^+\big)\mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y,\, e_\lambda < \tau_0^-\right) \\
&= Z_\lambda(x,\theta) - \frac{\lambda - \psi(\theta)}{\Phi_\lambda - \theta} W_\lambda(x) \\
&\quad + \lambda \int_0^a \left(Z(y,\theta) - \frac{W(y)}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)\right)\left(\mathrm{e}^{-\Phi_\lambda y} W_\lambda(x) - W_\lambda(x-y)\right)\mathrm{d}y \\
&= Z_\lambda(x,\theta) - \frac{\lambda - \psi(\theta)}{\Phi_\lambda - \theta} W_\lambda(x) + \lambda W_\lambda(x)\int_0^a \mathrm{e}^{-\Phi_\lambda y} Z(y,\theta)\,\mathrm{d}y - \lambda\int_0^a Z(y,\theta) W_\lambda(x-y)\mathrm{d}y \\
&\quad - \frac{\lambda}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)\left(W_\lambda(x)\int_0^a \mathrm{e}^{-\Phi_\lambda y} W(y)\mathrm{d}y - \int_0^a W(y) W_\lambda(x-y)\mathrm{d}y\right) \\
&= Z_\lambda(x,\theta) - \frac{\lambda - \psi(\theta)}{\Phi_\lambda - \theta} W_\lambda(x) + W_\lambda(x)\left(\lambda\frac{\mathrm{e}^{-\Phi_\lambda a} Z(a,\theta) - 1}{\theta - \Phi_\lambda} + \frac{\psi(\theta)\left(1 - \mathrm{e}^{-\Phi_\lambda a} Z(a,\Phi_\lambda)\right)}{\theta - \Phi_\lambda}\right) \\
&\quad - \lambda\int_0^a Z(y,\theta) W_\lambda(x-y)\mathrm{d}y \\
&\quad - \frac{1}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)\left(W_\lambda(x)\left(1 - \mathrm{e}^{-\Phi_\lambda a} Z(a,\Phi_\lambda)\right) + W_a^{(0,\lambda)}(x) - W_\lambda(x)\right) \\
&= Z_\lambda(x,\theta) - \frac{\lambda - \psi(\theta)}{\Phi_\lambda - \theta} W_\lambda(x) + \lambda W_\lambda(x)\frac{\mathrm{e}^{-\Phi_\lambda a} Z(a,\theta) - 1}{\theta - \Phi_\lambda} \\
&\quad - \lambda\int_0^a Z(y,\theta) W_\lambda(x-y)\mathrm{d}y \\
&\quad - \frac{1}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)\left(W_a^{(0,\lambda)}(x) - W_\lambda(x)\right) \\
&\quad + \frac{\lambda}{\theta - \Phi_\lambda}\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)} W_\lambda(x)\left(1 - \mathrm{e}^{-\Phi_\lambda a} Z(a,\Phi_\lambda)\right)
\end{aligned}$$

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$$
\begin{aligned}
&= Z_\lambda(x,\theta) - \frac{\lambda-\psi(\theta)}{\Phi_\lambda-\theta}W_\lambda(x) - \frac{1}{\theta-\Phi_\lambda}\left(\psi(\theta)-\lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)\left(W_a^{(0,\lambda)}(x)-W_\lambda(x)\right)\\
&\quad -\lambda\int_0^a Z(y,\theta)W_\lambda(x-y)\mathrm{d}y + \frac{\lambda}{\theta-\Phi_\lambda}W_\lambda(x)\left(\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}-1\right)\\
&= Z_a^{(0,\lambda)}(x,\theta) + \frac{W_a^{(0,\lambda)}(x)}{\theta-\Phi_\lambda}\left(\lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}-\psi(\theta)\right),
\end{aligned}
$$

which is (22). By (1)–(3), $Z_a^{(0,\lambda)}(a,\theta) = Z(a,\theta)$ and $W_a^{(0,\lambda)}(a) = W(a)$. Hence, letting $x \downarrow a$ in (22) yields (23). This completes the proof. □

The following Lemma 5.2 presents an expression for φ, which is the Laplace transform of the overshoot at the first time that the process X exceeds the level a (under a Poisson observation scheme), conditional on this exceedance occurring before the classical ruin time.

Lemma 5.2. *For $\theta \geq 0$, we have*

$$
\varphi(x;\theta,a) = -\lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y + \frac{\lambda W_a^{(0,\lambda)}(x)}{(\theta+\Phi_\lambda)\,Z(a,\Phi_\lambda)}, \quad x>a. \tag{25}
$$

Proof. From (17) of [5], it follows that

$$
\varphi(x;\theta,a) = \mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, T_a^+ < \tau_0^-\right) = \frac{\lambda}{\theta+\Phi_\lambda}\frac{W(x)}{Z(a,\Phi_\lambda)}, \quad x\leq a. \tag{26}
$$

By conditioning on the first Poisson observation time and using (24) and (26), we have

$$
\begin{aligned}
\varphi(x;\theta,a) &= \int_a^\infty \mathrm{e}^{-\theta(y-a)}\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\, e_\lambda<\tau_0^-\right) + \int_0^a \varphi(y;\theta,a)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\, e_\lambda<\tau_0^-\right)\\
&= \lambda\int_a^\infty \mathrm{e}^{-\theta(y-a)}\left(\mathrm{e}^{-\Phi_\lambda y}W_\lambda(x)-W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad + \frac{\lambda^2}{\theta+\Phi_\lambda}\int_0^a \frac{W(y)}{Z(a,\Phi_\lambda)}\left(\mathrm{e}^{-\Phi_\lambda y}W_\lambda(x)-W_\lambda(x-y)\right)\mathrm{d}y\\
&= \lambda\,\mathrm{e}^{\theta a}W_\lambda(x)\int_a^\infty \mathrm{e}^{-(\theta+\Phi_\lambda)y}\,\mathrm{d}y - \lambda\int_a^\infty \mathrm{e}^{-\theta(y-a)}W_\lambda(x-y)\mathrm{d}y\\
&\quad + \frac{\lambda^2}{\theta+\Phi_\lambda}\frac{W_\lambda(x)}{Z(a,\Phi_\lambda)}\int_0^a W(y)\mathrm{e}^{-\Phi_\lambda y}\,\mathrm{d}y - \frac{\lambda^2}{\theta+\Phi_\lambda}\frac{1}{Z(a,\Phi_\lambda)}\int_0^a W(y)W_\lambda(x-y),\,\mathrm{d}y\\
&= \frac{\lambda\,\mathrm{e}^{-\Phi_\lambda a}W_\lambda(x)}{\theta+\Phi_\lambda} - \lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\\
&\quad + \frac{\lambda}{\theta+\Phi_\lambda}\frac{W_\lambda(x)\left(1-\mathrm{e}^{-\Phi_\lambda a}Z(a,\Phi_\lambda)\right)}{Z(a,\Phi_\lambda)} + \frac{\lambda}{\theta+\Phi_\lambda}\frac{W_a^{(0,\lambda)}(x)-W_\lambda(x)}{Z(a,\Phi_\lambda)}\\
&= \frac{\lambda\,\mathrm{e}^{-\Phi_\lambda a}W_\lambda(x)}{\theta+\Phi_\lambda} - \lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\\
&\quad + \frac{\lambda}{\theta+\Phi_\lambda}\frac{W_a^{(0,\lambda)}(x)-\mathrm{e}^{-\Phi_\lambda a}W_\lambda(x)Z(a,\Phi_\lambda)}{Z(a,\Phi_\lambda)}\\
&= \frac{\lambda}{\theta+\Phi_\lambda}\frac{W_a^{(0,\lambda)}(x)}{Z(a,\Phi_\lambda)} - \lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y, \quad x>a,
\end{aligned}
$$

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where we observe that the integral representation on the right-hand side of the first equality arises from conditioning on the first Poisson arrival time, and (24) employed in the derivation may be regarded as a special case of (6). This completes the proof. □

The following Lemma 5.3 derives an expression for ϑ, the Laplace transform of the overshoot at the first time that the process X exceeds the level a (under a Poisson observation scheme), conditional on the exceedance occurring in finite time.

Lemma 5.3. *For $\theta \geq 0$ and $x > a$, we have*

$$\vartheta(x;\theta,a) = \frac{\lambda}{\theta+\Phi_\lambda}\frac{Z_a^{(0,\lambda)}(x,\Phi_\lambda)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)} - \lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y.$$

In particular, we have

$$\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\infty\big) = -\lambda\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y + \frac{\Phi_\lambda-\Phi}{\theta+\Phi_\lambda}Z_\lambda(x-a,\Phi). \quad (27)$$

Proof. It can be found in Theorem 3.2 of [5] that

$$\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_0^-\big) = \frac{\lambda}{\theta+\Phi_\lambda}\frac{Z(x,\Phi_\lambda)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}, \quad x\leq a.$$

For $x > a$, we have

$$\begin{aligned}
&\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_0^-\big)\\
&=\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};\tau_0^-<T_a^+<T_0^-\big)+\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_0^-\big)\\
&=\mathrm{E}_x\big(\mathrm{E}_{X(\tau_0^-)}\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_0^-\big);\tau_0^-<T_a^+\big)+\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_0^-\big)\\
&=\frac{\lambda}{\theta+\Phi_\lambda}\frac{1}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\mathrm{E}_x\big(\mathrm{e}^{\Phi_\lambda X(\tau_0^-)};\tau_0^-<T_a^+\big)+\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_0^-\big).
\end{aligned}$$

By Lemmas 5.1 and 5.2 we have

$$\begin{aligned}
\mathrm{E}_x\big(\mathrm{e}^{\Phi_\lambda X(\tau_0^-)};\tau_0^-<T_a^+\big) &= Z_a^{(0,\lambda)}(x,\Phi_\lambda)+\lim_{\theta\to\Phi_\lambda}\frac{W_a^{(0,\lambda)}(x)}{\theta-\Phi_\lambda}\left(\lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}-\psi(\theta)\right)\\
&= Z_a^{(0,\lambda)}(x,\Phi_\lambda)+\frac{W_a^{(0,\lambda)}(x)}{Z(a,\Phi_\lambda)}\lim_{\theta\to\Phi_\lambda}\frac{\psi(\Phi_\lambda)Z(a,\theta)-\psi(\theta)Z(a,\Phi_\lambda)}{\theta-\Phi_\lambda}\\
&= Z_a^{(0,\lambda)}(x,\Phi_\lambda)-\frac{W_a^{(0,\lambda)}(x)}{Z(a,\Phi_\lambda)}\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda),
\end{aligned}$$

and

$$\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_0^-\big) = -\lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y + \frac{\lambda W_a^{(0,\lambda)}(x)}{(\theta+\Phi_\lambda)\,Z(a,\Phi_\lambda)}.$$

Therefore, for $x > a$ we have

$$\mathrm{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_0^-\big) = \frac{\lambda}{\theta+\Phi_\lambda}\frac{Z_a^{(0,\lambda)}(x,\Phi_\lambda)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)} - \lambda\,\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y.$$

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It remains to prove (27). By (1) and (3) we have

$$\begin{aligned}
&W_{a+u}^{(0,\lambda)}(x+u)\\
&= W_\lambda(x+u) - \lambda\int_0^{a+u} W_\lambda(x+u-y)W(y)\mathrm{d}y\\
&= W_\lambda(x+u) + \lambda\left(\int_0^{x-a} W_\lambda(y)W(x+u-y)\mathrm{d}y - \int_0^{x+u} W_\lambda(y)W(x+u-y)\mathrm{d}y\right)\\
&= W_\lambda(x+u) - (W_\lambda(x+u) - W(x+u)) + \lambda\int_0^{x-a} W(x+u-y)W_\lambda(y)\mathrm{d}y\\
&= W(x+u) + \lambda\int_0^{x-a} W(x+u-y)W_\lambda(y)\mathrm{d}y. \qquad (28)
\end{aligned}$$

It can be found in (6) of [5] and (3) of [10] that

$$\lim_{a\to\infty}\frac{Z(a,\theta)}{W(a)} = \frac{\psi(\theta)}{\theta-\Phi}, \quad \lim_{x\to\infty}\frac{W_\lambda(x+y)}{W_\lambda(x)} = \mathrm{e}^{\Phi_\lambda y},$$

which imply

$$\lim_{u\to\infty}\frac{W(x+u)}{Z(a+u,\Phi_\lambda)} = \frac{\Phi_\lambda-\Phi}{\lambda}\mathrm{e}^{\Phi(x-a)}, \quad \lim_{u\to\infty}\frac{W(x+u-y)}{Z(a+u,\Phi_\lambda)} = \frac{\Phi_\lambda-\Phi}{\lambda}\mathrm{e}^{\Phi(x-a-y)},$$

then it follows from (28) that

$$\begin{aligned}
\lim_{u\to\infty}\frac{W_{a+u}^{(0,\lambda)}(x+u)}{Z(a+u,\Phi_\lambda)} &= \lim_{u\to\infty}\left(\frac{W(x+u)}{Z(a+u,\Phi_\lambda)} + \lambda\int_0^{x-a}\frac{W(x+u-y)}{Z(a+u,\Phi_\lambda)}W_\lambda(y)\mathrm{d}y\right)\\
&= \frac{\Phi_\lambda-\Phi}{\lambda}\mathrm{e}^{\Phi(x-a)} + \lambda\int_0^{x-a}\frac{\Phi_\lambda-\Phi}{\lambda}\mathrm{e}^{\Phi(x-a-y)}W_\lambda(y)\mathrm{d}y,
\end{aligned}$$

which together with Lemma 5.2 further leads to

$$\begin{aligned}
\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)}; T_a^+ < \infty\right) &= \lim_{u\to\infty}\varphi(x+u;\theta,a+u)\\
&= -\lambda\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y + \frac{\Phi_\lambda-\Phi}{\theta+\Phi_\lambda}Z_\lambda(x-a,\Phi).
\end{aligned}$$

This completes the proof. □

The following Lemma 5.4 gives an expression for υ, the Laplace transform of the undershoot at the first time that X down-crosses 0 (under the Poisson observation scheme), conditional on this down-crossing occurring before the first time that X exceeds the level a under the same Poisson observation scheme.

Lemma 5.4. *For $\theta > \Phi_\lambda$ and $x > a$, we have*

$$\upsilon(x;\theta,a) = \frac{\lambda}{\lambda-\psi(\theta)}\left(Z_a^{(0,\lambda)}(x,\theta) - \frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}Z_a^{(0,\lambda)}(x,\Phi_\lambda)\right).$$

Proof. From Theorem 3.2 in [5] it follows that

$$\mathrm{E}_x\left(\mathrm{e}^{\theta X(T_0^-)}; T_0^- < T_a^+\right) = \frac{\lambda}{\lambda-\psi(\theta)}\left(Z(x,\theta) - Z(x,\Phi_\lambda)\frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right), \quad x \le a.$$

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By conditioning on the first Poisson observation time and using (6), for $x > a$, we have

$$\begin{aligned}
&\mathrm{E}_x\big(\mathrm{e}^{\theta X(T_0^-)}; T_0^- < T_a^+\big)\\
&= \mathrm{E}_x\big(\mathrm{e}^{\theta X(T_0^-)}; T_0^- = e_\lambda < T_a^+\big) + \mathrm{E}_x\big(\mathrm{e}^{\theta X(T_0^-)}; e_\lambda < T_0^- < T_a^+\big)\\
&= \int_{-\infty}^{0} \mathrm{e}^{\theta y} \mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y\right) + \int_0^a \mathrm{E}_y\big(\mathrm{e}^{\theta X(T_0^-)}; T_0^- < T_a^+\big) \mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y\right)\\
&= \lambda \int_{-\infty}^{0} \mathrm{e}^{\theta y} \left(\frac{\mathrm{e}^{\Phi_\lambda (x-y)}}{\psi'(\Phi_\lambda)} - W_\lambda(x-y) \right) \mathrm{d}y\\
&\quad + \frac{\lambda^2}{\lambda - \psi(\theta)} \int_0^a \left(Z(y,\theta) - Z(y,\Phi_\lambda) \frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)} \right) \left(\frac{\mathrm{e}^{\Phi_\lambda (x-y)}}{\psi'(\Phi_\lambda)} - W_\lambda(x-y) \right) \mathrm{d}y\\
&= \frac{\lambda}{\psi'(\Phi_\lambda)} \frac{\mathrm{e}^{\Phi_\lambda x}}{\theta - \Phi_\lambda} - \lambda \frac{Z_\lambda(x,\theta)}{\psi(\theta) - \lambda}\\
&\quad + \frac{\lambda^2}{\lambda - \psi(\theta)} \frac{\mathrm{e}^{\Phi_\lambda x}}{\psi'(\Phi_\lambda)} \frac{\mathrm{e}^{-\Phi_\lambda a} Z(a,\theta) - 1}{\theta - \Phi_\lambda} + \frac{\lambda}{\lambda - \psi(\theta)} \frac{\mathrm{e}^{\Phi_\lambda x}}{\psi'(\Phi_\lambda)} \frac{\psi(\theta)\left(1 - \mathrm{e}^{-\Phi_\lambda a} Z(a,\Phi_\lambda)\right)}{\theta - \Phi_\lambda}\\
&\quad + \frac{\lambda}{\lambda - \psi(\theta)} \left(Z_a^{(0,\lambda)}(x,\theta) - Z_\lambda(x,\theta) \right) - \frac{\lambda}{\lambda - \psi(\theta)} \frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)} \left(Z_a^{(0,\lambda)}(x,\Phi_\lambda) - \mathrm{e}^{\Phi_\lambda x} \right)\\
&\quad - \frac{\lambda}{\lambda - \psi(\theta)} \frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)} \frac{\mathrm{e}^{\Phi_\lambda x}}{\psi'(\Phi_\lambda)} \left(-\mathrm{e}^{-\Phi_\lambda a} \widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda) + \psi'(\Phi_\lambda) \right)\\
&= \frac{\lambda}{\lambda - \psi(\theta)} \left(Z_a^{(0,\lambda)}(x,\theta) - \frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)} Z_a^{(0,\lambda)}(x,\Phi_\lambda) \right).
\end{aligned}$$

This completes the proof. □

Using a Poisson technique, the following Lemma 5.5 establishes an expression for ρ, the joint Laplace transform of the first time that X exceeds b and the accumulated time spent by X in the interval $[0, a]$ before exceeding, all under continuous observation.

Lemma 5.5. *For $0 \le a \le b$ and $\lambda \ge 0$, we have*

$$\rho(x;\lambda,a,b) = \frac{Z_a^{(0,\lambda)}(x,\Phi_\lambda)}{Z_a^{(0,\lambda)}(b,\Phi_\lambda)}, \quad x \le b.$$

Proof. For $x \le 0$, we have

$$\rho(x;\lambda,a,b) = \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^+ + \lambda \int_0^{\tau_b^+} \mathbf{1}_{[0,a]}(X(t))\mathrm{d}t} \right) = \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_0^+} \right) \rho(0;\lambda,a,b) = \mathrm{e}^{\Phi_\lambda x} \rho(0;\lambda,a,b).$$

For $x \in [0, a)$,

$$\begin{aligned}
&\rho(x;\lambda,a,b) = \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^+ + \lambda \int_0^{\tau_b^+} \mathbf{1}_{[0,a]}(X(t))\mathrm{d}t} \right)\\
&= \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^+ + \lambda \int_0^{\tau_b^+} \mathbf{1}_{[0,a]}(X(t))\mathrm{d}t}; \tau_b^+ < \tau_0^- \right) + \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^+ + \lambda \int_0^{\tau_b^+} \mathbf{1}_{[0,a]}(X(t))\mathrm{d}t}; \tau_0^- < \tau_b^+ \right)\\
&= \mathrm{P}_x\left(\tau_a^+ < \tau_0^-\right) \mathrm{E}_a\left(\mathrm{e}^{-\lambda\tau_b^+ + \lambda \int_0^{\tau_b^+} \mathbf{1}_{[0,a]}(X(t))\mathrm{d}t}; \tau_b^+ < \tau_0^- \right)
\end{aligned}$$

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$$
\begin{aligned}
&+\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^+ +\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t};\tau_0^- < \tau_a^+\right) + \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_b^+ +\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t};\tau_a^+ < \tau_0^- < \tau_b^+\right)\\
&= \mathrm{P}_x\left(\tau_a^+ < \tau_0^-\right)\mathrm{E}_a\left(\mathrm{e}^{-\lambda\tau_b^+ +\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t};\tau_b^+ < \tau_0^-\right)\\
&\quad+\mathrm{E}_x\left(\rho(X(\tau_0^-);\lambda,a,b);\tau_0^- < \tau_a^+\right) + \mathrm{P}_x\left(\tau_a^+ < \tau_0^-\right)\mathrm{E}_a\left(\mathrm{e}^{-\lambda\tau_b^+ +\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t};\tau_0^- < \tau_b^+\right)\\
&= \mathrm{P}_x\left(\tau_a^+ < \tau_0^-\right)\mathrm{E}_a\left(\mathrm{e}^{-\lambda\tau_b^+ +\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t}\right)\\
&\quad+\mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda X(\tau_0^-)};\tau_0^- < \tau_a^+\right)\rho(0;\lambda,a,b)\\
&= \frac{W(x)}{W(a)}\rho(a;\lambda,a,b) + \left(Z(x,\Phi_\lambda) - \frac{Z(a,\Phi_\lambda)}{W(a)}W(x)\right)\rho(0;\lambda,a,b).
\end{aligned}
$$

Recalling that $Z(x,\Phi_\lambda) = \mathrm{e}^{\Phi_\lambda x}$ and $W(x) = 0$ for $x < 0$, we have for $x \in [a,b]$ that

$$
\begin{aligned}
\rho(x;\lambda,a,b) &= \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_a^-}\rho(X(\tau_a^-);\lambda,a,b);\tau_a^- < \tau_b^+\right) + \frac{W_\lambda(x-a)}{W_\lambda(b-a)}\\
&= \left(\frac{\rho(a;\lambda,a,b)}{W(a)} - \frac{Z(a,\Phi_\lambda)}{W(a)}\rho(0;\lambda,a,b)\right)\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_a^-}W(X(\tau_a^-));\tau_a^- < \tau_b^+\right)\\
&\quad+\rho(0;\lambda,a,b)\,\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_a^-}Z(X(\tau_a^-),\Phi_\lambda);\tau_a^- < \tau_b^+\right) + \frac{W_\lambda(x-a)}{W_\lambda(b-a)}.
\end{aligned}
$$

The scale function $Z(x,\Phi_\lambda)$ can be rewritten as

$$
Z(x,\Phi_\lambda) := \mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda X(\tau_0^-)};\tau_0^- < \infty\right) + \frac{\lambda}{\Phi_\lambda - \Phi}W(x),
$$

with $\mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda X(\tau_0^-)};\tau_0^- < \infty\right)$ being the same form as (14) in [22], and $W(x)$ satisfying (13) in [22]. By Lemma 2.1 in [22], we know that

$$
\begin{aligned}
\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_a^-}Z(X(\tau_a^-),\Phi_\lambda);\tau_a^- < \tau_b^+\right) &= Z(x,\Phi_\lambda) + \lambda\int_a^x W_\lambda(x-y)Z(y,\Phi_\lambda)\mathrm{d}y\\
&\quad-\frac{W_\lambda(x-a)}{W_\lambda(b-a)}\left(Z(b,\Phi_\lambda) + \lambda\int_a^b W_\lambda(b-y)Z(y,\Phi_\lambda)\mathrm{d}y\right)\\
&= Z_a^{(0,\lambda)}(x,\Phi_\lambda) - \frac{W_\lambda(x-a)}{W_\lambda(b-a)}Z_a^{(0,\lambda)}(b,\Phi_\lambda).
\end{aligned}
$$

In addition, it follows from [19] that

$$
\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_a^-}W(X(\tau_a^-));\tau_a^- < \tau_b^+\right) = W_a^{(0,\lambda)}(x) - \frac{W_\lambda(x-a)}{W_\lambda(b-a)}W_a^{(0,\lambda)}(b).
$$

Therefore, for $x \in [a,b]$, we have

$$
\begin{aligned}
\rho(x;\lambda,a,b) &= \left(\frac{\rho(a;\lambda,a,b)}{W(a)} - \frac{Z(a,\Phi_\lambda)}{W(a)}\rho(0;\lambda,a,b)\right)\left(W_a^{(0,\lambda)}(x) - \frac{W_\lambda(x-a)}{W_\lambda(b-a)}W_a^{(0,\lambda)}(b)\right)\\
&\quad+\rho(0;\lambda,a,b)\left(Z_a^{(0,\lambda)}(x,\Phi_\lambda) - \frac{W_\lambda(x-a)}{W_\lambda(b-a)}Z_a^{(0,\lambda)}(b,\Phi_\lambda)\right) + \frac{W_\lambda(x-a)}{W_\lambda(b-a)}.
\end{aligned}
$$

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It remains to solve $\rho(0;\lambda,a,b)$ and $\rho(a;\lambda,a,b)$. For this purpose, we write $0<e_\lambda(1)<e_\lambda(2)<\cdots$ and $0<\overline{e}_\lambda(1)<\overline{e}_\lambda(2)<\cdots$ for all the arrival times of two independent Poisson processes with the same rate λ. We also assume that both Poisson processes are independent of process X. Then for two independent exponential random variables e_λ and $\overline{e}_\lambda$ with same rate λ, we have

$$
\begin{aligned}
&\rho(0;\lambda,a,b)=\mathrm{E}\left(\mathrm{e}^{-\lambda\tau_b^+ +\lambda\int_0^{\tau_b^+}\mathbf{1}_{[0,a]}(X(t))\mathrm{d}t}\right)\\
&=\mathrm{E}\left(\mathrm{e}^{-\lambda\int_0^{\tau_b^+}\mathbf{1}_{(-\infty,0)}(X(t))\mathrm{d}t-\lambda\int_0^{\tau_b^+}\mathbf{1}_{(a,b]}(X(t))\mathrm{d}t}\right)\\
&=\mathrm{P}\left(\{e_\lambda(i)\}\cap\{s:s<\tau_b^+,X(s)\in(-\infty,0)\}=\{\overline{e}_\lambda(i)\}\cap\{s:s<\tau_b^+,X(s)\in(a,b)\}=\emptyset\right)\\
&=\mathrm{P}\left(\tau_b^+<e_\lambda\wedge\overline{e}_\lambda\right)+\int_0^b\mathrm{P}\left(e_\lambda<\tau_b^+\wedge\overline{e}_\lambda,X(e_\lambda)\in\mathrm{d}y\right)\rho(y;\lambda,a,b)\\
&\quad+\int_{-\infty}^a\mathrm{P}\left(\overline{e}_\lambda<\tau_b^+\wedge e_\lambda,X(\overline{e}_\lambda)\in\mathrm{d}y\right)\rho(y;\lambda,a,b)\\
&=\mathrm{e}^{-\Phi_{2\lambda}b}+\lambda\int_0^b\int_0^\infty\mathrm{e}^{-2\lambda t}\mathrm{P}\left(t<\tau_b^+,X(t)\in\mathrm{d}y\right)\rho(y;\lambda,a,b)\mathrm{d}t\\
&\quad+\lambda\int_{-\infty}^a\int_0^\infty\mathrm{e}^{-2\lambda t}\mathrm{P}\left(t<\tau_b^+,X(t)\in\mathrm{d}y\right)\rho(y;\lambda,a,b)\mathrm{d}t\\
&=\mathrm{e}^{-\Phi_{2\lambda}b}+\frac{1}{2}\int_0^b\mathrm{P}\left(e_{2\lambda}<\tau_b^+,X(e_{2\lambda})\in\mathrm{d}y\right)\rho(y;\lambda,a,b)\\
&\quad+\frac{1}{2}\int_{-\infty}^a\mathrm{P}\left(\overline{e}_{2\lambda}<\tau_b^+,X(\overline{e}_{2\lambda})\in\mathrm{d}y\right)\rho(y;\lambda,a,b)\\
&=\mathrm{e}^{-\Phi_{2\lambda}b}+\lambda\mathrm{e}^{-\Phi_{2\lambda}b}\int_a^b W_{2\lambda}(b-x)\rho(x;\lambda,a,b)\mathrm{d}x\\
&\quad+2\lambda\mathrm{e}^{-\Phi_{2\lambda}b}\int_0^a W_{2\lambda}(b-x)\rho(x;\lambda,a,b)\mathrm{d}x\\
&\quad+\lambda\int_{-\infty}^0\left(\mathrm{e}^{-\Phi_{2\lambda}b}W_{2\lambda}(b-x)-W_{2\lambda}(-x)\right)\rho(x;\lambda,a,b)\mathrm{d}x\\
&=\mathrm{e}^{-\Phi_{2\lambda}b}\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}+\mathrm{e}^{-\Phi_{2\lambda}b}\rho(a;\lambda,a,b)\left(\frac{W_{2\lambda}(b)}{W(a)}-\frac{W_a^{(0,\lambda)}(b)}{W(a)}\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}\right)\\
&\quad+\mathrm{e}^{-\Phi_{2\lambda}b}\rho(0;\lambda,a,b)\left(\mathrm{e}^{\Phi_{2\lambda}b}-Z_a^{(0,\lambda)}(b,\Phi_\lambda)\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}\right)\\
&\quad-\mathrm{e}^{-\Phi_{2\lambda}b}\rho(0;\lambda,a,b)\,Z(a,\Phi_\lambda)\left(\frac{W_{2\lambda}(b)}{W(a)}-\frac{W_a^{(0,\lambda)}(b)}{W(a)}\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}\right),
\end{aligned}
\tag{29}
$$

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and

$$\begin{aligned}
&\rho(a;\lambda,a,b)\\
&=\mathrm{e}^{-\Phi_{2\lambda}b}+\lambda\mathrm{e}^{-\Phi_{2\lambda}b}\int_a^b W_{2\lambda}(b-x)\rho(x;\lambda,a,b)\mathrm{d}x\\
&\quad+2\lambda\int_0^a\left(\mathrm{e}^{-\Phi_{2\lambda}b}W_{2\lambda}(b-x)-W_{2\lambda}(a-x)\right)\rho(x;\lambda,a,b)\mathrm{d}x\\
&\quad+\lambda\int_{-\infty}^0\left(\mathrm{e}^{-\Phi_{2\lambda}b}W_{2\lambda}(b-x)-W_{2\lambda}(a-x)\right)\rho(x;\lambda,a,b)\mathrm{d}x\\
&=\mathrm{e}^{-\Phi_{2\lambda}b}\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}\\
&\quad+\mathrm{e}^{-\Phi_{2\lambda}b}\rho(a;\lambda,a,b)\left(\frac{W_{2\lambda}(b)}{W(a)}-\frac{W_a^{(0,\lambda)}(b)}{W(a)}\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}-\mathrm{e}^{\Phi_{2\lambda}b}\frac{W_{2\lambda}(a)}{W(a)}+\mathrm{e}^{\Phi_{2\lambda}b}\right)\\
&\quad+\mathrm{e}^{-\Phi_{2\lambda}b}\rho(0;\lambda,a,b)\left(-Z_a^{(0,\lambda)}(b,\Phi_\lambda)\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}+\mathrm{e}^{\Phi_{2\lambda}b}\,Z(a,\Phi_\lambda)\frac{W_{2\lambda}(a)}{W(a)}\right)\\
&\quad-\mathrm{e}^{-\Phi_{2\lambda}b}\rho(0;\lambda,a,b)\,Z(a,\Phi_\lambda)\left(\frac{W_{2\lambda}(b)}{W(a)}-\frac{W_a^{(0,\lambda)}(b)}{W(a)}\frac{W_{2\lambda}(b-a)}{W_\lambda(b-a)}\right).
\end{aligned}\tag{30}$$

Solving (29) and (30), we get

$$\begin{aligned}
\rho(a;\lambda,a,b)&=Z(a,\Phi_\lambda)\rho(0;\lambda,a,b),\\
\rho(0;\lambda,a,b)&=\frac{1}{Z_a^{(0,\lambda)}(b,\Phi_\lambda)},\\
\rho(a;\lambda,a,b)&=\frac{Z(a,\Phi_\lambda)}{Z_a^{(0,\lambda)}(b,\Phi_\lambda)}.
\end{aligned}$$

Therefore, for $x\in(-\infty,a)$, we have

$$\rho(x;\lambda,a,b)=\frac{W(x)}{W(a)}\rho(a;\lambda,a,b)+\left(Z(x,\Phi_\lambda)-\frac{Z(a,\Phi_\lambda)}{W(a)}W(x)\right)\rho(0;\lambda,a,b)=\frac{Z(x,\Phi_\lambda)}{Z_a^{(0,\lambda)}(b,\Phi_\lambda)},$$

and, for $x\in[a,b]$, we have

$$\begin{aligned}
\rho(x;\lambda,a,b)&=\left(\frac{\rho(a;\lambda,a,b)}{W(a)}-\frac{Z(a,\Phi_\lambda)}{W(a)}\rho(0;\lambda,a,b)\right)\left(W_a^{(0,\lambda)}(x)-\frac{W_\lambda(x-a)}{W_\lambda(b-a)}W_a^{(0,\lambda)}(b)\right)\\
&\quad+\rho(0;\lambda,a,b)\left(Z_a^{(0,\lambda)}(x,\Phi_\lambda)-\frac{W_\lambda(x-a)}{W_\lambda(b-a)}Z_a^{(0,\lambda)}(b,\Phi_\lambda)\right)+\frac{W_\lambda(x-a)}{W_\lambda(b-a)}\\
&=\frac{Z_a^{(0,\lambda)}(x,\Phi_\lambda)}{Z_a^{(0,\lambda)}(b,\Phi_\lambda)}.
\end{aligned}$$

This completes the proof. □

5.2. Proof of Theorem 3.1

Proof. We first prove that, under the condition $\lim_{z\to\infty}\overline{\xi}(z)=\infty$, we have

$$\lim_{x\to\infty}f_1(x;\theta,a)=0,\tag{31}$$

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$$\lim_{x\to\infty} f_2(x;\theta,a) = \mathbf{1}_{\{\theta=0\}}, \tag{32}$$

$$\lim_{x\to\infty} f_3(x;\theta,a) = 0. \tag{33}$$

In fact, stopping times τ_ξ and T_a^+ can be rewritten as

$$\begin{aligned}
\tau_\xi &= \inf\{t\geq 0; \overline{Y}(t) - Y(t) > \overline{\xi}(x+\overline{Y}(t))\} = \inf\left\{t\geq 0; Y(t) < \zeta_x(\overline{Y}(t))\right\},\\
T_\xi &= \inf\left\{T_i; \overline{Y}(T_i) - Y(T_i) > \overline{\xi}(x+\overline{Y}(T_i))\right\} = \inf\left\{T_i; Y(T_i) < \zeta_x(\overline{Y}(T_i))\right\},\\
T_a^+ &= \inf\left\{T_i; Y(T_i) > a-x\right\},
\end{aligned}$$

with $Y(t) := X(t) - x$, $\overline{Y}(t) := \sup_{s\leq t} Y(s)$, and $\zeta_x(y) := \xi(x+y) - x$. It is not hard to see that $y \mapsto \zeta_x(y)$ is a drawdown function with $\lim_{x\to\infty}\zeta_x(y) = \lim_{x\to\infty}\left(y - \overline{\xi}(x+y)\right) = -\infty$ uniformly for all $y\in\mathbb{R}_+$, and

$$\mathrm{P}_0\left(\lim_{x\to\infty}\tau_{\zeta_x} = \infty,\ \lim_{x\to\infty} T_{\zeta_x} = \infty,\ \lim_{x\to\infty} T_{a-x}^+ = T_1,\ \lim_{x\to\infty}\tau_{a-x}^+ = 0\right) = 1.$$

Hence,

$$\begin{aligned}
\lim_{x\to\infty}\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; T_\xi < \tau_a^+\right) &\leq \lim_{x\to\infty}\mathrm{P}_0\left(T_{\zeta_x} < \tau_{a-x}^+\right) = 0,\\
\lim_{x\to\infty}\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)}; \tau_\xi < T_a^+\right) &\leq \lim_{x\to\infty}\mathrm{P}_0\left(\tau_{\zeta_x} < T_{a-x}^+\right) = 0,\\
\lim_{x\to\infty}\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)}; T_a^+ < \tau_\xi\right) &= \lim_{x\to\infty}\mathrm{e}^{-\theta(x-a)}\mathrm{E}_0\left(\mathrm{e}^{-\theta X(T_{a-x}^+)}; T_{a-x}^+ < \tau_{\zeta_x}\right)\\
&= \begin{cases} 1, & \theta = 0,\\ \lim\limits_{x\to\infty}\mathrm{e}^{-\theta(x-a)}\mathrm{E}_0\left(\mathrm{e}^{-\theta X(T_1)}\right) = 0, & \theta > 0,\end{cases}
\end{aligned}$$

these prove (31), (32), and (33). Now, define κ_b^+ to be the first up-crossing time of level b after the drawdown time τ_ξ as follows:

$$\kappa_b^+ := \inf\{t\geq\tau_\xi; X(t)\geq b\}.$$

Then we have

$$\begin{aligned}
&f_1(x;\theta,a)\\
&:= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; T_\xi < \tau_a^+\right)\\
&= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; \tau_\xi \leq T_\xi < \kappa_{\overline{X}(\tau_\xi)}^+ < \tau_a^+\right)\\
&\quad + \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; \tau_\xi \leq \kappa_{\overline{X}(\tau_\xi)}^+ < T_\xi < \tau_a^+\right)\\
&= \mathrm{E}_x\left(\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; T_\xi < \kappa_{\overline{X}(\tau_\xi)}^+ \middle| \mathcal{F}_{\tau_\xi}\right); \tau_\xi < \tau_a^+\right)\\
&\quad + \mathrm{E}_x\left(\mathrm{E}_x\left(\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; T_\xi < \tau_a^+ \middle| \mathcal{F}_{\kappa_{\overline{X}(\tau_\xi)}^+}\right); \kappa_{\overline{X}(\tau_\xi)}^+ < T_\xi \middle| \mathcal{F}_{\tau_\xi}\right); \tau_\xi < \tau_a^+\right)\\
&= \mathrm{E}_x\left(\left[\mathrm{E}_{X(\tau_\xi)}\left(\mathrm{e}^{\theta\left(X(T_{\xi(b)}^-)-\xi(b)\right)}; T_{\xi(b)}^- < \tau_b^+\right)\right]\Big|_{b=\overline{X}(\tau_\xi)}; \tau_\xi < \tau_a^+\right)\\
&\quad + \mathrm{E}_x\left(\mathrm{E}_x\left(\mathrm{E}_{\overline{X}(\tau_\xi)}\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; T_\xi < \tau_a^+\right); \kappa_{\overline{X}(\tau_\xi)}^+ < T_\xi \middle| \mathcal{F}_{\tau_\xi}\right); \tau_\xi < \tau_a^+\right)
\end{aligned}$$

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$$= \mathrm{E}_x\left(\left[\mathrm{E}_{X(\tau_\xi)}\left(\mathrm{e}^{\theta\left(X(T^-_{\xi(b)})-\xi(b)\right)}; T^-_{\xi(b)} < \tau^+_b\right)\right]\Big|_{b=\overline{X}(\tau_\xi)}; \tau_\xi < \tau^+_a\right)$$
$$+\mathrm{E}_x\left(f_1(\overline{X}(\tau_\xi);\theta,a)\left[\mathrm{P}_{X(\tau_\xi)}\left(\tau^+_b < T^-_{\xi(b)}\right)\right]\Big|_{b=\overline{X}(\tau_\xi)}; \tau_\xi < \tau^+_a\right). \tag{34}$$

As $X(\tau_\xi) - \xi(\overline{X}(\tau_\xi)) \le 0$, then by (3) and (15) in [5], we get

$$\begin{aligned}
&\mathrm{E}_{X(\tau_\xi)}\left(\mathrm{e}^{\theta\left(X(T^-_{\xi(b)})-\xi(b)\right)}; T^-_{\xi(b)} < \tau^+_b\right)\Big|_{b=\overline{X}(\tau_\xi)}\\
&= \mathrm{E}_{X(\tau_\xi)-\xi(b)}\left(\mathrm{e}^{\theta X(T^-_0)}; T^-_0 < \tau^+_{b-\xi(b)}\right)\Big|_{b=\overline{X}(\tau_\xi)}\\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left(\mathrm{e}^{\theta X(\tau_\xi)}\mathrm{e}^{-\theta\xi(\overline{X}(\tau_\xi))} - \mathrm{e}^{\Phi_\lambda X(\tau_\xi)}\mathrm{e}^{-\Phi_\lambda\xi(\overline{X}(\tau_\xi))}\frac{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\theta)}{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\Phi_\lambda)}\right)
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
\mathrm{P}_{X(\tau_\xi)}\left(\tau^+_b < T^-_{\xi(b)}\right)\Big|_{b=\overline{X}(\tau_\xi)} &= 1 - \mathrm{P}_{X(\tau_\xi)-\xi(b)}\left(T^-_0 < \tau^+_{b-\xi(b)}\right)\Big|_{b=\overline{X}(\tau_\xi)}\\
&= 1 - \frac{\lambda}{\lambda-\psi(0)}\left(1 - \mathrm{e}^{\Phi_\lambda X(\tau_\xi)}\mathrm{e}^{-\Phi_\lambda\xi(\overline{X}(\tau_\xi))}\frac{Z(\bar{\xi}(\overline{X}(\tau_\xi)))}{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\Phi_\lambda)}\right)\\
&= \mathrm{e}^{\Phi_\lambda X(\tau_\xi)}\mathrm{e}^{-\Phi_\lambda\xi(\overline{X}(\tau_\xi))}\frac{Z(\bar{\xi}(\overline{X}(\tau_\xi)))}{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\Phi_\lambda)}.
\end{aligned} \tag{36}$$

Combining (34), (35), and (36), we have

$$\begin{aligned}
f_1(x;\theta,a) :=\ & \frac{\lambda}{\lambda-\psi(\theta)}\mathrm{E}_x\left(\mathrm{e}^{\theta(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi)))}; \tau_\xi < \tau^+_a\right)\\
&-\frac{\lambda}{\lambda-\psi(\theta)}\mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi)))}\frac{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\theta)}{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\Phi_\lambda)}; \tau_\xi < \tau^+_a\right)\\
&+\mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda X(\tau_\xi)}f_1(\overline{X}(\tau_\xi);\theta,a)\mathrm{e}^{-\Phi_\lambda\xi(\overline{X}(\tau_\xi))}\frac{Z(\bar{\xi}(\overline{X}(\tau_\xi)))}{Z(\bar{\xi}(\overline{X}(\tau_\xi)),\Phi_\lambda)}; \tau_\xi < \tau^+_a\right).
\end{aligned} \tag{37}$$

By the compensation formula in excursion theory (see Corollary 4.11 of [9], Theorem 4.4 of [13], or Proposition 3.1 of [15]), (5), and (8), we have

$$\begin{aligned}
&\mathrm{E}_x\left(\mathrm{e}^{\theta(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi)))}; \tau_\xi < \tau^+_a\right)\\
&= \mathrm{E}_x\left(\sum_{r\in[0,a-x]}\prod_{s<r}\mathbf{1}_{\{\bar{\varepsilon}_s \le \bar{\xi}(x+s)\}}\mathrm{e}^{\theta(\bar{\xi}(x+r)-\varepsilon_r(\chi^+_{\bar{\xi}(x+r)}))}\mathbf{1}_{\{\bar{\varepsilon}_r > \bar{\xi}(x+r)\}}\right)\\
&= \int_0^{a-x}\mathrm{P}_x(L_r^{-1} < \tau_\xi)\, n\left(\mathrm{e}^{\theta(\bar{\xi}(x+r)-\varepsilon(\chi^+_{\bar{\xi}(x+r)}))}\mathbf{1}_{\{\bar{\varepsilon} > \bar{\xi}(x+r)\}}\right)\mathrm{d}r\\
&= \int_0^{a-x}\mathrm{e}^{-\int_x^{x+r}\frac{W'(\bar{\xi}(z))}{W(\bar{\xi}(z))}\mathrm{d}z} n\left(\mathrm{e}^{\theta(\bar{\xi}(x+r)-\varepsilon(\chi^+_{\bar{\xi}(x+r)}))}\mathbf{1}_{\{\bar{\varepsilon} > \bar{\xi}(x+r)\}}\right)\mathrm{d}r
\end{aligned}$$

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$$= \int_x^a \mathrm{e}^{-\int_x^s \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z} n\left(\mathrm{e}^{\theta(\overline{\xi}(s)-\varepsilon(\chi^+_{\overline{\xi}(s)}))}; \overline{\varepsilon} > \overline{\xi}(s)\right) \mathrm{d}s$$
$$= \int_x^a \mathrm{e}^{-\int_x^s \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z} \left(\frac{W'(\overline{\xi}(s))}{W(\overline{\xi}(s))} Z(\overline{\xi}(s), \theta) - \theta Z(\overline{\xi}(s), \theta) + \psi(\theta) W(\overline{\xi}(s))\right) \mathrm{d}s, \tag{38}$$

$$\mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi)))} \frac{Z(\overline{\xi}(\overline{X}(\tau_\xi)), \theta)}{Z(\overline{\xi}(\overline{X}(\tau_\xi)), \Phi_\lambda)}; \tau_\xi < \tau_a^+\right)$$
$$= \int_x^a \mathrm{e}^{-\int_x^s \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z} \frac{Z(\overline{\xi}(s), \theta)}{Z(\overline{\xi}(s), \Phi_\lambda)}$$
$$\times \left(\frac{W'(\overline{\xi}(s))}{W(\overline{\xi}(s))} Z(\overline{\xi}(s), \Phi_\lambda) - \Phi_\lambda Z(\overline{\xi}(s), \Phi_\lambda) + \lambda W(\overline{\xi}(s))\right) \mathrm{d}s, \tag{39}$$

and

$$\mathrm{E}_x\left(\mathrm{e}^{\Phi_\lambda X(\tau_\xi)} f_1(\overline{X}(\tau_\xi); \theta, a) \mathrm{e}^{-\Phi_\lambda \xi(\overline{X}(\tau_\xi))} \frac{Z(\overline{\xi}(\overline{X}(\tau_\xi)))}{Z(\overline{\xi}(\overline{X}(\tau_\xi)), \Phi_\lambda)}; \tau_\xi < \tau_a^+\right)$$
$$= \int_x^a \mathrm{e}^{-\int_x^s \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z} \mathrm{e}^{-\Phi_\lambda \xi(s)} \frac{Z(\overline{\xi}(s))}{Z(\overline{\xi}(s), \Phi_\lambda)} f_1(s; \theta, a)$$
$$\times \mathrm{e}^{\Phi_\lambda \xi(s)} \left(\frac{W'(\overline{\xi}(s))}{W(\overline{\xi}(s))} Z(\overline{\xi}(s), \Phi_\lambda) - \Phi_\lambda Z(\overline{\xi}(s), \Phi_\lambda) + \lambda W(\overline{\xi}(s))\right) \mathrm{d}s. \tag{40}$$

Plugging (38)–(40) into (37) and then differentiating both sides of the yielding equation with respect to x, we have

$$\begin{aligned}
&f_1'(x; \theta, a) \\
:=& f_1(x; \theta, a)\left(\frac{W'(\overline{\xi}(x))}{W(\overline{\xi}(x))} - \frac{Z(\overline{\xi}(x))}{Z(\overline{\xi}(x), \Phi_\lambda)}\left(\frac{W'(\overline{\xi}(x))}{W(\overline{\xi}(x))} Z(\overline{\xi}(x), \Phi_\lambda) - \Phi_\lambda Z(\overline{\xi}(x), \Phi_\lambda) + \lambda W(\overline{\xi}(x))\right)\right) \\
&- \frac{\lambda}{\lambda - \psi(\theta)}\left(\frac{W'(\overline{\xi}(x))}{W(\overline{\xi}(x))} Z(\overline{\xi}(x), \theta) - \theta Z(\overline{\xi}(x), \theta) + \psi(\theta) W(\overline{\xi}(x))\right) \\
&+ \frac{\lambda}{\lambda - \psi(\theta)} \frac{Z(\overline{\xi}(x), \theta)}{Z(\overline{\xi}(x), \Phi_\lambda)}\left(\frac{W'(\overline{\xi}(x))}{W(\overline{\xi}(x))} Z(\overline{\xi}(x), \Phi_\lambda) - \Phi_\lambda Z(\overline{\xi}(x), \Phi_\lambda) + \lambda W(\overline{\xi}(x))\right) \\
=& f_1(x; \theta, a)\left(\Phi_\lambda - \frac{\lambda W(\overline{\xi}(x))}{Z(\overline{\xi}(x), \Phi_\lambda)}\right) - \frac{\lambda}{\lambda - \psi(\theta)}\left(-\theta Z(\overline{\xi}(x), \theta) + \psi(\theta) W(\overline{\xi}(x))\right) \\
&+ \frac{\lambda}{\lambda - \psi(\theta)} \frac{Z(\overline{\xi}(x), \theta)}{Z(\overline{\xi}(x), \Phi_\lambda)}\left(-\Phi_\lambda Z(\overline{\xi}(x), \Phi_\lambda) + \lambda W(\overline{\xi}(x))\right).
\end{aligned} \tag{41}$$

Solving (41) with boundary condition (31) yields (10).

Identity (9) for $\theta > \Phi$ follows immediately from (10) by letting a tend to ∞; by analytic continuation it is also true for any $\theta \geq 0$.

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For $x \le a$, by the Markov property of X and (5), we have

$$\begin{aligned} f_2(x;\theta,a) &:= \mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, T_a^+ < \tau_\xi\right) \\ &= \mathrm{E}_x\left(\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, T_a^+ < \tau_\xi \,\middle|\, \mathcal{F}_{\tau_a^+}\right);\, \tau_a^+ < \tau_\xi\right) \\ &= \mathrm{P}_x\left(\tau_a^+ < \tau_\xi\right) f_2(a;\theta,a) \\ &= \mathrm{e}^{-\int_x^a \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\mathrm{d}z} f_2(a;\theta,a), \end{aligned}$$

which is (11). By conditioning on the first Poisson observation time, for $x > a$, we have

$$\begin{aligned} &f_2(x;\theta,a) \\ &= \int_a^\infty \int_{x\vee y}^\infty \mathrm{e}^{-\theta(y-a)} \mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y,\, \overline{X}(e_\lambda) \in \mathrm{d}z,\, e_\lambda < \tau_\xi\right) \\ &\quad + \int_{-\infty}^a \int_x^\infty \mathrm{P}_y\left(\tau_z^+ < T_a^+ \wedge \tau_{\xi(z)}^-\right) f_2(z;\theta,a) \mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y,\, \overline{X}(e_\lambda) \in \mathrm{d}z,\, e_\lambda < \tau_\xi\right) \\ &\quad + \int_{-\infty}^a \int_x^\infty \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, T_a^+ < \tau_{\xi(z)}^-\right) \mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y,\, \overline{X}(e_\lambda) \in \mathrm{d}z,\, e_\lambda < \tau_\xi\right) \\ &\quad - \int_{-\infty}^a \int_x^\infty \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, \tau_z^+ < T_a^+ < \tau_{\xi(z)}^-\right) \\ &\quad \times \mathrm{P}_x\left(X(e_\lambda) \in \mathrm{d}y,\, \overline{X}(e_\lambda) \in \mathrm{d}z,\, e_\lambda < \tau_\xi\right). \end{aligned} \tag{42}$$

By (17) in [5] and (10) in [19], it can be verified that

$$\begin{aligned} \mathrm{P}_y\left(\tau_z^+ < T_a^+ \wedge \tau_{\xi(z)}^-\right) &= \mathrm{P}_{y-\xi(z)}\left(\tau_{\overline{\xi}(z)}^+ < T_{a-\xi(z)}^+ \wedge \tau_0^-\right) \\ &= \mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\lambda\int_0^{\tau_{\overline{\xi}(z)}^+} \mathbf{1}_{[a-\xi(z),\overline{\xi}(z)]}(X(t))\mathrm{d}t};\, \tau_{\overline{\xi}(z)}^+ < \tau_0^-\right) \\ &= \frac{W(y-\xi(z))}{W_{a-\xi(z)}^{(0,\lambda)}(\overline{\xi}(z))}, \end{aligned} \tag{43}$$

$$\begin{aligned} \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, T_a^+ < \tau_{\xi(z)}^-\right) &= \mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\theta(X(T_{a-\xi(z)}^+)-(a-\xi(z)))};\, T_{a-\xi(z)}^+ < \tau_0^-\right) \\ &= \frac{\lambda}{\Phi_\lambda+\theta} \frac{W(y-\xi(z))}{Z(a-\xi(z),\,\Phi_\lambda)}, \end{aligned} \tag{44}$$

for $y \in (\xi(z), a)$, $z \in (x \vee y, \infty)$, and

$$\begin{aligned} &\mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, \tau_z^+ < T_a^+ < \tau_{\xi(z)}^-\right) \\ &= \mathrm{P}_y\left(\tau_z^+ < T_a^+ \wedge \tau_{\xi(z)}^-\right) \mathrm{E}_z\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\, T_a^+ < \tau_{\xi(z)}^-\right) \\ &= \mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\lambda\int_0^{\tau_{\overline{\xi}(z)}^+} \mathbf{1}_{[a-\xi(z),\overline{\xi}(z)]}(X(t))\mathrm{d}t};\, \tau_{\overline{\xi}(z)}^+ < \tau_0^-\right) \\ &\quad \times \mathrm{E}_{\overline{\xi}(z)}\left(\mathrm{e}^{-\theta(X(T_{a-\xi(z)}^+)-(a-\xi(z)))};\, T_{a-\xi(z)}^+ < \tau_0^-\right) \\ &= \frac{W(y-\xi(z))}{W_{a-\xi(z)}^{(0,\lambda)}(\overline{\xi}(z))} \varphi(\overline{\xi}(z);\theta,a-\xi(z)), \quad y \in (\xi(z),a),\ z \in (x\vee y,\infty). \end{aligned} \tag{45}$$

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By (6), (42)–(45), we have

$$
\begin{aligned}
&f_2(x;\theta,a)\\
&=\lambda W(0)\int_x^\infty \mathrm{e}^{-\int_x^y \frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\mathrm{e}^{-\theta(y-a)}\,\mathrm{d}y\\
&+\lambda\int_x^\infty \mathrm{e}^{-\int_x^z \frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\mathrm{d}z\int_{a\vee\xi(z)}^z \mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(z-y)-\frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}W_\lambda(z-y)\right)\mathrm{d}y\\
&+\lambda\int_x^\infty \mathrm{e}^{-\int_x^z \frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}f_2(z;\theta,a)\mathrm{d}z\int_{a\wedge\xi(z)}^a \frac{W(y-\xi(z))\left(W'_\lambda(z-y)-\frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}W_\lambda(z-y)\right)}{W^{(0,\lambda)}_{a-\xi(z)}(\bar\xi(z))}\mathrm{d}y\\
&+\frac{\lambda^2}{\Phi_\lambda+\theta}\int_x^\infty \mathrm{e}^{-\int_x^z \frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\mathrm{d}z\int_{a\wedge\xi(z)}^a \frac{W(y-\xi(z))\left(W'_\lambda(z-y)-\frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}W_\lambda(z-y)\right)}{Z(a-\xi(z),\Phi_\lambda)}\,\mathrm{d}y\\
&-\lambda\int_x^\infty \mathrm{e}^{-\int_x^z \frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\frac{W(y-\xi(z))}{W^{(0,\lambda)}_{a-\xi(z)}(\bar\xi(z))}\varphi(\bar\xi(z);\theta,a-\xi(z))\mathrm{d}z\\
&\times\int_{a\wedge\xi(z)}^a\left(W'_\lambda(z-y)-\frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}W_\lambda(z-y)\right)\mathrm{d}y. \qquad (46)
\end{aligned}
$$

Taking derivative on both sides of (46) with respect to x, we get

$$
\begin{aligned}
&f'_2(x;\theta,a)\\
&=\left(\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}-\lambda\int_{a\wedge\xi(x)}^a\frac{W(y-\xi(x))\left(W'_\lambda(x-y)-\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}W_\lambda(x-y)\right)}{W^{(0,\lambda)}_{a-\xi(x)}(\bar\xi(x))}\,\mathrm{d}y\right)f_2(x;\theta,a)\\
&-\lambda W(0)\mathrm{e}^{-\theta(x-a)}-\lambda\int_{a\vee\xi(x)}^x\mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(x-y)-\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&-\frac{\lambda^2}{\Phi_\lambda+\theta}\int_{a\wedge\xi(x)}^a\frac{W(y-\xi(x))\left(W'_\lambda(x-y)-\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}W_\lambda(x-y)\right)}{Z(a-\xi(x),\Phi_\lambda)}\,\mathrm{d}y\\
&+\lambda\frac{W(y-\xi(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\bar\xi(x))}\varphi(\bar\xi(x);\theta,a-\xi(x))\int_{a\wedge\xi(x)}^a\left(W'_\lambda(x-y)-\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&=\left(\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}+\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}\frac{\lambda}{W^{(0,\lambda)}_{a-\xi(x)}(\bar\xi(x))}\frac{W^{(0,\lambda)}_{a-\xi(x)}(\bar\xi(x))-W_\lambda(\bar\xi(x))}{-\lambda}\right.\\
&\left.-\frac{\lambda}{W^{(0,\lambda)}_{a-\xi(x)}(\bar\xi(x))}\int_0^{a-\xi(x)}W(y)W'_\lambda(\bar\xi(x)-y)\mathrm{d}y\right)f_2(x;\theta,a)\\
&-\lambda W(0)\mathrm{e}^{-\theta(x-a)}-\lambda\int_{a\vee\xi(x)}^x\mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(x-y)-\frac{W'_\lambda(\bar\xi(x))}{W_\lambda(\bar\xi(x))}W_\lambda(x-y)\right)\mathrm{d}y
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{\lambda^2 \mathrm{e}^{-\theta(\overline{\xi}(x)-a)}}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}\int_0^{\overline{\xi}(x)-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\\
&\times\int_{a\wedge\xi(x)}^{a}W(y-\xi(x))\left(W'_\lambda(x-y)-\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
=&\frac{W'_\lambda(\overline{\xi}(x))-\lambda\int_0^{a-\xi(x)}W(y)W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}f_2(x;\theta,a)\\
&-\lambda W(0)\mathrm{e}^{-\theta(x-a)}-\lambda\int_{a\vee\xi(x)}^{x}\mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(x-y)-\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&+\lambda\mathrm{e}^{-\theta(\overline{\xi}(x)-a)}\int_0^{\overline{\xi}(x)-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{(W^{(0,\lambda)}_{a-\xi(x)})'(\overline{\xi}(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}-\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\right)\mathbf{1}_{\{a>\xi(x)\}}. \qquad (47)
\end{aligned}
$$

Solving (47) with boundary condition (32) yields

$$
f_2(x;\theta,a)=\mathrm{e}^{\int_a^x h_2(z)\mathrm{d}z}\left(\mathrm{e}^{-\int_a^\infty h_2(z)\mathrm{d}z}\mathbf{1}_{\{\theta=0\}}-\int_x^\infty g_2(z)\mathrm{e}^{-\int_a^z h_2(w)\mathrm{d}w}\,\mathrm{d}z\right), \qquad (48)
$$

where $h_2(x)$ is given by

$$
h_2(x)=\frac{W'_\lambda(\overline{\xi}(x))-\lambda\int_0^{a-\xi(x)}W(y)W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}.
$$

To proceed with the simplification of f_2, we next examine the behavior of $h_2(x)$ as x approaches infinity. We divide our discussion into two distinct cases: (i) $\xi(x)\geq a$ and (ii) $\xi(x)<a$. Recall that $\lim_{x\to\infty}\overline{\xi}(x)=\infty$. In the case $\xi(x)\geq a$, we have

$$
h_2(x)=\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}, \qquad (49)
$$

which, combined with the facts that $W'_\lambda(x)/W_\lambda(x)$ is decreasing and $\lim_{x\to\infty}W'_\lambda(x)/W_\lambda(x)=\Phi_\lambda$, yields that $h_2(x)\geq\Phi_\lambda>0$ for all large $x\in\{x;\xi(x)\geq a\}$. In the case $\xi(x)<a$, we have

$$
\begin{aligned}
h_2(x)&=\frac{W'_\lambda(\overline{\xi}(x))-\lambda\int_0^{a-\xi(x)}W(y)W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}\\
&=\frac{W'_\lambda(\overline{\xi}(x))-\lambda\int_0^{a-\xi(x)}W(y)W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W_\lambda(\overline{\xi}(x))-\lambda\int_0^{a-\xi(x)}W(y)W_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}\\
&=\frac{\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}-\lambda\int_0^{a-\xi(x)}W(y)\frac{W'_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x)-y)}\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\mathrm{d}y}{1-\lambda\int_0^{a-\xi(x)}W(y)\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\,\mathrm{d}y}\\
&\geq\frac{\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}-\frac{W'_\lambda(x-a)}{W_\lambda(x-a)}\lambda\int_0^{a-\xi(x)}W(y)\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\,\mathrm{d}y}{1-\lambda\int_0^{a-\xi(x)}W(y)\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\,\mathrm{d}y}
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} - \frac{W'_\lambda(x-a)}{W_\lambda(x-a)}}{1-\lambda\int_0^{a-\xi(x)} W(y)\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\,\mathrm{d}y} + \frac{\frac{W'_\lambda(x-a)}{W_\lambda(x-a)}\left(1-\lambda\int_0^{a-\xi(x)} W(y)\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\,\mathrm{d}y\right)}{1-\lambda\int_0^{a-\xi(x)} W(y)\frac{W_\lambda(\overline{\xi}(x)-y)}{W_\lambda(\overline{\xi}(x))}\,\mathrm{d}y} \\
&\geq \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} - \frac{W'_\lambda(x-a)}{W_\lambda(x-a)} + \frac{W'_\lambda(x-a)}{W_\lambda(x-a)} \\
&= \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} \\
&\to \Phi_\lambda > 0,
\end{aligned} \tag{50}
$$

where we have used the facts that $W'_\lambda(x)/W_\lambda(x)$ is decreasing in x, $\lim_{x\to\infty} W'_\lambda(x)/W_\lambda(x) = \Phi_\lambda$, and $\lim_{x\to\infty} W_\lambda(x+y)/W_\lambda(x) = \mathrm{e}^{\Phi_\lambda y}$ for any $y \in \mathbb{R}$. Combining (49) and (50) yields

$$
\mathrm{e}^{-\int_a^\infty h_2(z)\mathrm{d}z} = 0,
$$

which, together with (48), gives (12).

For $x \leq a$, by the Markov property of X, (5), and (38), we have

$$
\begin{aligned}
f_3(x;\theta,a) &:= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi < T_a^+\right) \\
&= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi < \tau_a^+\right) + \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_a^+ < \tau_\xi < T_a^+\right) \\
&= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi < \tau_a^+\right) + \mathrm{P}_x\left(\tau_a^+ < \tau_\xi\right) f_3(a;\theta,a) \\
&= \int_x^a \mathrm{e}^{-\int_x^z \frac{W'(\overline{\xi}(w))}{W(\overline{\xi}(w))}\,\mathrm{d}w}\left(\left(-\theta + \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\right)Z(\overline{\xi}(z),\theta) + \psi(\theta)W(\overline{\xi}(z))\right)\mathrm{d}z \\
&\quad + \mathrm{e}^{-\int_x^a \frac{W'(\overline{\xi}(z))}{W(\overline{\xi}(z))}\,\mathrm{d}z} f_3(a;\theta,a),
\end{aligned}
$$

this gives (13).

Let e_λ denote the first Poisson observation time. For $x > a$ we have

$$
\begin{aligned}
f_3(x;\theta,a) &:= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi < T_a^+\right) \\
&= \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi < T_a^+,\,\tau_\xi < e_\lambda\right) \\
&\quad + \mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};e_\lambda < \tau_\xi < T_a^+\right) \\
&= \mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_\xi+\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)}\right) \\
&\quad + \int_{-\infty}^a\int_x^\infty \mathrm{P}_y\left(\tau_z^+ < T_a^+ \wedge \tau_{\xi(z)}^-\right) f_3(z;\theta,a) \\
&\qquad \times \mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\,\overline{X}(e_\lambda)\in\mathrm{d}z,\,e_\lambda < \tau_\xi\right) \\
&\quad + \int_{-\infty}^a\int_x^\infty \mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(\tau_{\xi(z)}^-)-\xi(z)\right)};\tau_{\xi(z)}^- < T_a^+\right) \\
&\qquad \times \mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\,\overline{X}(e_\lambda)\in\mathrm{d}z,\,e_\lambda < \tau_\xi\right) \\
&\quad - \int_{-\infty}^a\int_x^\infty \mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(\tau_{\xi(z)}^-)-\xi(z)\right)};\tau_z^+ < \tau_{\xi(z)}^- < T_a^+\right) \\
&\quad \times \mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\,\overline{X}(e_\lambda)\in\mathrm{d}z,\,e_\lambda < \tau_\xi\right).
\end{aligned} \tag{51}
$$

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By (18) in [5] and (10) in [19], it can be verified that, for $y \in (\xi(z), a)$ and $z \in (x \vee y, \infty)$,

$$\begin{aligned}
&\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(\tau^-_{\xi(z)})-\xi(z)\right)}; \tau^-_{\xi(z)} < T^+_a\right)\\
&= \mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{\theta X(\tau^-_0)}; \tau^-_0 < T^+_{a-\xi(z)}\right)\\
&= Z(y-\xi(z),\theta) - \frac{W(y-\xi(z))}{\theta-\Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a-\xi(z),\theta)}{Z(a-\xi(z),\Phi_\lambda)}\right),
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(\tau^-_{\bar\xi(z)})-\xi(z)\right)}; \tau^+_z < \tau^-_{\xi(z)} < T^+_a\right)\\
&= \mathrm{P}_y\left(\tau^+_z < T^+_a \wedge \tau^-_{\xi(z)}\right)\mathrm{E}_z\left(\mathrm{e}^{\theta\left(X(\tau^-_{\xi(z)})-\xi(z)\right)}; \tau^-_{\xi(z)} < T^+_a\right)\\
&= \mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\lambda\int_0^{\tau^+_{\bar\xi(z)}}\mathbf{1}_{[a-\xi(z),\bar\xi(z)]}(X(t))\mathrm{d}t}; \tau^+_{\bar\xi(z)} < \tau^-_0\right)\\
&\quad\times\mathrm{E}_{\bar\xi(z)}\left(\mathrm{e}^{\theta X(\tau^-_0)}; \tau^-_0 < T^+_{a-\xi(z)}\right)\\
&= \frac{W(y-\xi(z))}{W^{(0,\lambda)}_{a-\xi(z)}(\bar\xi(z))}\phi(\bar\xi(z);\theta, a-\xi(z)),
\end{aligned} \tag{53}$$

and, using a similar compensation formula-based argument as that of (38), we have

$$\begin{aligned}
&\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau_\xi+\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)}\right)\\
&= \int_x^\infty \mathrm{e}^{-\int_x^z\frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\left(\left(-\theta+\frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}\right)Z_\lambda(\bar\xi(z),\theta) - (\lambda-\psi(\theta))W_\lambda(\bar\xi(z))\right)\mathrm{d}z.
\end{aligned} \tag{54}$$

From (6), (51)–(54), we have

$$\begin{aligned}
f_3(x;\theta,a) = &\int_x^\infty \mathrm{e}^{-\int_x^z\frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\left(\left(-\theta+\frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}\right)Z_\lambda(\bar\xi(z),\theta) - (\lambda-\psi(\theta))W_\lambda(\bar\xi(z))\right)\mathrm{d}z\\
&+\lambda\int_x^\infty \mathrm{e}^{-\int_x^z\frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w} f_3(z;\theta,a)\mathrm{d}z\int_{\xi(z)\wedge a}^a \frac{W(y-\xi(z))}{W^{(0,\lambda)}_{a-\xi(z)}(\bar\xi(z))}\\
&\times\left(W'_\lambda(z-y) - \frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}W_\lambda(z-y)\right)\mathrm{d}y\\
&+\lambda\int_x^\infty \mathrm{e}^{-\int_x^z\frac{W'_\lambda(\bar\xi(w))}{W_\lambda(\bar\xi(w))}\mathrm{d}w}\mathrm{d}z\\
&\times\int_{\xi(z)\wedge a}^a\left(Z(y-\xi(z),\theta) - \frac{W(y-\xi(z))}{\theta-\Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a-\xi(z),\theta)}{Z(a-\xi(z),\Phi_\lambda)}\right)\right)\\
&\times\left(W'_\lambda(z-y) - \frac{W'_\lambda(\bar\xi(z))}{W_\lambda(\bar\xi(z))}W_\lambda(z-y)\right)\mathrm{d}y
\end{aligned}$$

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$$-\lambda \int_x^\infty \mathrm{e}^{-\int_x^z \frac{W'_\lambda(\overline{\xi}(w))}{W_\lambda(\overline{\xi}(w))}\mathrm{d}w}\,\mathrm{d}z \frac{W(y-\xi(z))}{W^{(0,\lambda)}_{a-\xi(z)}(\overline{\xi}(z))}$$
$$\times \int_{\xi(z)\wedge a}^a \phi(\overline{\xi}(z);\theta, a-\xi(z))\left(W'_\lambda(z-y) - \frac{W'_\lambda(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)\mathrm{d}y. \tag{55}$$

Differentiating both sides of (55) with respect to x gives

$$\begin{aligned}
&f'_3(x;\theta,a)\\
&= \left(\frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} - \lambda\int_{\xi(x)\wedge a}^a \frac{W(y-\xi(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\right) f_3(x;\theta,a)\\
&\quad - \left(\left(-\theta + \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\right)Z_\lambda(\overline{\xi}(x),\theta) - (\lambda-\psi(\theta))W_\lambda(\overline{\xi}(x))\right)\\
&\quad -\lambda\int_{\xi(x)\wedge a}^a \left(Z(y-\xi(x),\theta) - \frac{W(y-\xi(x))}{\theta-\Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a-\xi(x),\theta)}{Z(a-\xi(x),\Phi_\lambda)}\right)\right)\\
&\quad \times\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad +\lambda\frac{W(y-\xi(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}\int_{\xi(x)\wedge a}^a \phi(\overline{\xi}(x);\theta,a-\xi(x))\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&= \frac{W'_\lambda(\overline{\xi}(x)) - \lambda\int_0^{a-\xi(x)} W(y)W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} f_3(x;\theta,a)\\
&\quad - \left(\left(-\theta + \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\right)Z_\lambda(\overline{\xi}(x),\theta) - (\lambda-\psi(\theta))W_\lambda(\overline{\xi}(x))\right)\\
&\quad -\lambda\int_{\xi(x)\wedge a}^a Z(y-\xi(x),\theta)\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad +\lambda\int_{\xi(x)\wedge a}^a \frac{W(y-\xi(x))}{\theta-\Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a-\xi(x),\theta)}{Z(a-\xi(x),\Phi_\lambda)}\right)\\
&\quad \times\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad +\lambda\frac{W(y-\xi(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))}\int_{\xi(x)\wedge a}^a Z^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x),\theta)\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad +\frac{\lambda}{\theta-\Phi_\lambda}\left(\lambda\frac{Z(a-\xi(x),\theta)}{Z(a-\xi(x),\Phi_\lambda)} - \psi(\theta)\right)\\
&\quad \times\int_{\xi(x)\wedge a}^a W(y-\xi(x))\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{W'_\lambda(\overline{\xi}(x)) - \lambda \int_0^{a-\xi(x)} W(y) W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} f_3(x;\theta,a) \\
&\quad - \left(\left(-\theta + \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} \right) Z_\lambda(\overline{\xi}(x),\theta) - (\lambda - \psi(\theta)) W_\lambda(\overline{\xi}(x)) \right) \\
&\quad - \lambda \int_{\xi(x)\wedge a}^{a} Z(y-\xi(x),\theta) \left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} W_\lambda(x-y) \right) \mathrm{d}y \\
&\quad + \lambda \frac{Z^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x),\theta)}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} \int_{\xi(x)\wedge a}^{a} W(y-\xi(x)) \left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} W_\lambda(x-y) \right) \mathrm{d}y \\
&= \frac{W'_\lambda(\overline{\xi}(x)) - \lambda \int_0^{a-\xi(x)} W(y) W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} f_3(x;\theta,a) \\
&\quad - \left(\left(-\theta + \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} \right) Z_\lambda(\overline{\xi}(x),\theta) - (\lambda - \psi(\theta)) W_\lambda(\overline{\xi}(x)) \right) \\
&\quad - \lambda \int_{\xi(x)\wedge a}^{a} Z(y-\xi(x),\theta) W'_\lambda(x-y)\mathrm{d}y + Z_\lambda(\overline{\xi}(x),\theta) \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))} \mathbf{1}_{\{\xi(x)\le a\}} \\
&\quad + \lambda \frac{Z^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x),\theta)}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} \int_{\xi(x)\wedge a}^{a} W(y-\xi(x)) W'_\lambda(x-y)\mathrm{d}y \\
&\quad - \frac{Z^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x),\theta)}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} W'_\lambda(\overline{\xi}(x)) \mathbf{1}_{\{\xi(x)\le a\}}. \qquad (56)
\end{aligned}
$$

We then obtain (14) by solving (56) with boundary condition (33). □

5.3. Proof of Corollary 3.1

Proof. We provide the proof for $f_2(x;\theta,a)$ and $f_3(x;\theta,a)$ where $x \in (a,\infty)$; the proof for $x \in [0,a]$ is analogous. From Theorem 3.1, we have

$$
\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)}; T_a^+ < \tau_\xi \right) = -\int_x^\infty g_2(z)\, \mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\, \mathrm{d}z, \quad x \in (a,\infty).
$$

By the assumption $\xi(z) \equiv \xi(a)$ for all $z \in [a,\infty)$, we get

$$
\begin{aligned}
\int_x^z h_2(w)\mathrm{d}w &= \int_x^z \frac{W'_\lambda(\overline{\xi}(w)) - \lambda \int_0^{\overline{\xi}(a)} W(y) W'_\lambda(\overline{\xi}(w)-y)\mathrm{d}y}{W^{(0,\lambda)}_{\overline{\xi}(a)}(\overline{\xi}(w))} \mathrm{d}w \\
&= \int_x^z \frac{\mathrm{d}}{\mathrm{d}w} \ln W^{(0,\lambda)}_{\overline{\xi}(a)}(w-\xi(a))\mathrm{d}w \\
&= \ln \frac{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a))}.
\end{aligned}
$$

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For $x \geq a$, by definition of φ in Section 5, we have

$$\begin{aligned}
&\varphi\left(x-\xi(a);\theta,\overline{\xi}(a)\right)=\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_{\xi(a)}^-\right)\\
&=\int_a^\infty\int_{x\vee y}^\infty \mathrm{e}^{-\theta(y-a)}\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z,e_\lambda<\tau_{\xi(a)}^-\right)\\
&\quad+\int_{-\infty}^a\int_x^\infty \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_{\xi(a)}^-\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z,e_\lambda<\tau_{\xi(a)}^-\right)\\
&=\lambda W(0)\int_x^\infty\frac{W_\lambda(x-\xi(a))}{W_\lambda(y-\xi(a))}\mathrm{e}^{-\theta(y-a)}\,\mathrm{d}y\\
&\quad+\lambda\int_x^\infty\frac{W_\lambda(x-\xi(a))}{W_\lambda(z-\xi(a))}\mathrm{d}z\int_{a\vee\xi(z)}^z\mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(z-y)-\frac{W_\lambda'(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad+\frac{\lambda^2}{\Phi_\lambda+\theta}\int_x^\infty\frac{W_\lambda(x-\xi(a))}{W_\lambda(z-\xi(a))}\,\mathrm{d}z\int_{a\wedge\xi(z)}^a\frac{W(y-\xi(z))\left(W_\lambda'(z-y)-\frac{W_\lambda'(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)}{Z(a-\xi(z),\Phi_\lambda)}\,\mathrm{d}y.
\end{aligned}$$

Differentiate both sides of the above equation and then rearranging terms yields

$$\begin{aligned}
&W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)\\
&=-\lambda W(0)\mathrm{e}^{-\theta(z-a)}-\lambda\int_{a\vee\xi(z)}^z\mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(z-y)-\frac{W_\lambda'(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad-\frac{\lambda^2}{\Phi_\lambda+\theta}\int_{a\wedge\xi(z)}^a\frac{W(y-\xi(z))\left(W_\lambda'(z-y)-\frac{W_\lambda'(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)}{Z(a-\xi(z),\Phi_\lambda)}\,\mathrm{d}y.
\end{aligned}$$

Hence, g_2 can be rewritten as

$$\begin{aligned}
g_2(z)&=W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)\\
&\quad+\lambda\frac{W(y-\xi(z))}{W^{(0,\lambda)}_{a-\xi(z)}(\overline{\xi}(z))}\varphi(\overline{\xi}(z);\theta,a-\xi(z))\int_{a\wedge\xi(z)}^a\left(W_\lambda'(z-y)-\frac{W_\lambda'(\overline{\xi}(z))}{W_\lambda(\overline{\xi}(z))}W_\lambda(z-y)\right)\mathrm{d}y\\
&=W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)+\lambda\frac{\varphi(z-\xi(a);\theta,\overline{\xi}(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\\
&\quad\times\int_0^{\overline{\xi}(a)}W(y)\left(W_\lambda'(z-\xi(a)-y)-\frac{W_\lambda'(z-\xi(a))}{W_\lambda(z-\xi(a))}W_\lambda(z-\xi(a)-y)\right)\mathrm{d}y\\
&=W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)\\
&\quad+\frac{\varphi(z-\xi(a);\theta,\overline{\xi}(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\left(\frac{W_\lambda'(z-\xi(a))}{W_\lambda(z-\xi(a))}W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))-W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))\right)
\end{aligned}$$

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$$= W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)$$

$$+\varphi(z-\xi(a);\theta,\overline{\xi}(a))\left(\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}-\frac{W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\left(\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)\right)-\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)\frac{W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}.$$

Therefore, we have

$$\begin{aligned}
&\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_\xi\right)\\
&=-\int_x^\infty g_2(z)\,\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\,\mathrm{d}z=-W^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a))\\
&\quad\times\int_x^\infty\left(\frac{\frac{\mathrm{d}}{\mathrm{d}z}\left(\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)\right)}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}-\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))}{\left(W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))\right)^2}\right)\mathrm{d}z\\
&=-W^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a))\left(\frac{\varphi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\Bigg|_{z=x}^{\infty}\right)\\
&=\varphi\left(x-\xi(a);\theta,\overline{\xi}(a)\right),
\end{aligned}$$

which, together with (25), implies (15).

We then prove the expression for $\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi<T_a^+\right)$. By Theorem 3.1, we have

$$\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi<T_a^+\right)=-\int_x^\infty g_3(z)\,\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\,\mathrm{d}z,\quad x\in(a,\infty).$$

By the assumption that $\xi(z)\equiv\xi(a)$ for all $z\in[a,\infty)$, we get

$$\int_x^z h_2(w)\mathrm{d}w=\ln\frac{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a))}.$$

For $x\geq a$, from the same arguments as used in (51) we have

$$\phi\left(x-\xi(a);\theta,\overline{\xi}(a)\right)=\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau^-_{\xi(a)})-\xi(a)\right)};\tau^-_{\xi(a)}<T_a^+\right)=\mathrm{E}_x\left(\mathrm{e}^{-\lambda\tau^-_{\xi(a)}+\theta\left(X(\tau^-_{\xi(a)})-\xi(a)\right)}\right)$$

$$+\int_{-\infty}^a\int_x^\infty\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(\tau^-_{\xi(a)})-\xi(a)\right)};\tau^-_{\xi(a)}<T_a^+\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z,e_\lambda<\tau^-_{\xi(a)}\right),$$

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which is equivalent to

$$\begin{aligned}
&\phi\left(x-\xi(a);\theta,\overline{\xi}(a)\right)\\
&=\int_x^\infty\int_0^{\overline{\xi}(a)}\left(Z(y,\theta)-\frac{W(y)}{\theta-\Phi_\lambda}\left(\psi(\theta)-\lambda\frac{Z(\overline{\xi}(a),\theta)}{Z(\overline{\xi}(a),\Phi_\lambda)}\right)\right)\\
&\quad\times\frac{W_\lambda(x-\xi(a))}{W_\lambda(z-\xi(a))}\lambda\left(W'_\lambda(z-\xi(a)-y)-\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}W_\lambda(z-\xi(a)-y)\right)\mathrm{d}z\,\mathrm{d}y\\
&\quad+\int_x^\infty\frac{W_\lambda(x-\xi(a))}{W_\lambda(z-\xi(a))}\left(\left(-\theta+\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}\right)Z_\lambda(z-\xi(a),\theta)-(\lambda-\psi(\theta))W_\lambda(z-\xi(a))\right)\mathrm{d}z.
\end{aligned}$$

Differentiate both sides of the above equation and then rearranging terms yields

$$\begin{aligned}
&W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)\\
&=-\int_0^{\overline{\xi}(a)}\left(Z(y,\theta)-\frac{W(y)}{\theta-\Phi_\lambda}\left(\psi(\theta)-\lambda\frac{Z(\overline{\xi}(a),\theta)}{Z(\overline{\xi}(a),\Phi_\lambda)}\right)\right)\\
&\quad\times\lambda\left(W'_\lambda(z-\xi(a)-y)-\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}W_\lambda(z-\xi(a)-y)\right)\mathrm{d}y\\
&\quad-\left(\left(-\theta+\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}\right)Z_\lambda(z-\xi(a),\theta)-(\lambda-\psi(\theta))W_\lambda(z-\xi(a))\right).
\end{aligned}$$

Hence, g_3 can be rewritten as

$$\begin{aligned}
g_3(z)&=W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)+\frac{\phi(z-\xi(a);\theta,\overline{\xi}(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\\
&\quad\times\lambda\int_0^{\overline{\xi}(a)}W(y)\left(W'_\lambda(z-\xi(a)-y)-\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}W_\lambda(z-\xi(a)-y)\right)\mathrm{d}y\\
&=W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)\\
&\quad+\frac{\phi(z-\xi(a);\theta,\overline{\xi}(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\left(\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))-W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))\right)\\
&=W_\lambda(z-\xi(a))\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_\lambda(z-\xi(a))}\right)\\
&\quad+\phi(z-\xi(a);\theta,\overline{\xi}(a))\left(\frac{W'_\lambda(z-\xi(a))}{W_\lambda(z-\xi(a))}-\frac{W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}\right)\\
&=\frac{\mathrm{d}}{\mathrm{d}z}\left(\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)\right)-\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)\frac{W^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a))}{W^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a))}.
\end{aligned}$$

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Therefore, we have

$$\begin{aligned}
&\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(\tau_\xi)-\xi(\overline{X}(\tau_\xi))\right)};\tau_\xi<T_a^+\right)\\
&=-\int_x^\infty g_3(z)\,\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\,\mathrm{d}z=-W_{\overline{\xi}(a)}^{(0,\lambda)}(x-\xi(a))\\
&\quad\times\int_x^\infty\left(\frac{\frac{\mathrm{d}}{\mathrm{d}z}\left(\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)\right)}{W_{\overline{\xi}(a)}^{(0,\lambda)}(z-\xi(a))}-\frac{\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)W_{\overline{\xi}(a)}^{(0,\lambda)'}(z-\xi(a))}{\left(W_{\overline{\xi}(a)}^{(0,\lambda)}(z-\xi(a))\right)^2}\right)\mathrm{d}z\\
&=-W_{\overline{\xi}(a)}^{(0,\lambda)}(x-\xi(a))\left(\frac{\phi\left(z-\xi(a);\theta,\overline{\xi}(a)\right)}{W_{\overline{\xi}(a)}^{(0,\lambda)}(z-\xi(a))}\Bigg|_{z=x}^{\infty}\right)\\
&=\phi\left(x-\xi(a);\theta,\overline{\xi}(a)\right),
\end{aligned}$$

which, together with (22), implies (16). □

5.4. Proof of Theorem 3.2

Proof. Using similar arguments as those adopted in the proof of (31)–(33), one can deduce that

$$\lim_{x\to\infty}f_4(x,\theta,a)=0,\quad \lim_{x\to\infty}f_5(x,\theta,a)=\mathbf{1}_{\{\theta=0\}},\tag{57}$$

where the assumption of $\lim_{z\to\infty}\overline{\xi}(z)=\infty$ is applied.

For $x\le a$ we have

$$\begin{aligned}
&\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};T_\xi<T_a^+\right)\\
&=\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};\tau_a^+<T_\xi<T_a^+\right)+\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};T_\xi<\tau_a^+\right)\\
&=\mathrm{P}_x\left(\tau_a^+<T_\xi\right)\mathrm{E}_a\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};T_\xi<T_a^+\right)+\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};T_\xi<\tau_a^+\right)\\
&=(1-f_1(x;0,a))\,\mathrm{E}_a\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};T_\xi<T_a^+\right)+f_1(x;\theta,a),
\end{aligned}$$

this leads to (17).

For $x>a$, let e_λ denote the first Poisson observation time, we have

$$\begin{aligned}
&f_4(x;\theta,a):=\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};T_\xi<T_a^+\right)\\
&=\mathrm{E}_x\left(\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)};e_\lambda\le T_\xi<T_a^+\right)\\
&=\int_x^\infty\int_{-\infty}^{\xi(z)\wedge a}\mathrm{e}^{\theta(y-\xi(z))}\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad+\int_x^\infty\int_{\xi(z)\wedge a}^{a}\mathrm{P}_y\left(\tau_z^+<T_a^+\wedge T_{\xi(z)}^-\right)f_4(z;\theta,a)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad+\int_x^\infty\int_{\xi(z)\wedge a}^{a}\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(T_{\xi(z)}^-)-\xi(z)\right)};T_{\xi(z)}^-<T_a^+\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad-\int_x^\infty\int_{\xi(z)\wedge a}^{a}\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(T_{\xi(z)}^-)-\xi(z)\right)};\tau_z^+<T_{\xi(z)}^-<T_a^+\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right),
\end{aligned}\tag{58}$$

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where, in the second equality, we have used the fact that

$$\begin{aligned}
&\mathrm{E}_x\left[\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(T_\xi))\right)}; e_\lambda < T_\xi < T_a^+ \wedge \ell^+_{\overline{X}(e_\lambda)}\right]\\
=&\mathrm{E}_x\left[\mathrm{e}^{\theta\left(X(T_\xi)-\xi(\overline{X}(e_\lambda))\right)}; e_\lambda < T_\xi < T_a^+ \wedge \ell^+_{\overline{X}(e_\lambda)}\right]\\
=&\mathrm{E}_x\left[\mathrm{E}_x\left[\mathrm{e}^{\theta\left(X(T^-_{\xi(\overline{X}(e_\lambda))})-\xi(\overline{X}(e_\lambda))\right)}; e_\lambda < T^-_{\xi(\overline{X}(e_\lambda))} < T_a^+ \wedge \ell^+_{\overline{X}(e_\lambda)}\,\middle|\,\mathcal{F}_{e_\lambda}\right]\right]\\
=&\mathrm{E}_x\left[\left(\mathbf{1}_{\{a>y>\xi(z)\}}\,\mathrm{E}_y\left[\mathrm{e}^{\theta\left(X(T^-_{\xi(z)})-\xi(z)\right)}; T^-_{\xi(z)} < T_a^+ \wedge \tau_z^+\right]\right)\Big|_{y=X(e_\lambda),\, z=\overline{X}(e_\lambda)}\right]\\
=&\mathrm{E}_x\left[\left(\mathbf{1}_{\{a>y>\xi(z)\}}\,\mathrm{E}_y\left[\mathrm{e}^{\theta\left(X(T^-_{\xi(z)})-\xi(z)\right)}; T^-_{\xi(z)} < T_a^+\right]\right)\Big|_{y=X(e_\lambda),\, z=\overline{X}(e_\lambda)}\right]\\
&-\mathrm{E}_x\left[\left(\mathbf{1}_{\{a>y>\xi(z)\}}\,\mathrm{E}_y\left[\mathrm{e}^{\theta\left(X(T^-_{\xi(z)})-\xi(z)\right)}; \tau_z^+ < T^-_{\xi(z)} < T_a^+\right]\right)\Big|_{y=X(e_\lambda),\, z=\overline{X}(e_\lambda)}\right],
\end{aligned}$$

with $\ell_b^+ := \inf\{t \geq e_\lambda; X(t) \geq b\}$. By (19) in [5] and (10) in [19], it can be verified that, for $y \in (\xi(z), a)$ and $z \in (x \vee y, \infty)$,

$$\begin{aligned}
&\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(T^-_{\xi(z)})-\xi(z)\right)}; T^-_{\xi(z)} < T_a^+\right)\\
=&\,\mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{\theta X(T_0^-)}; T_0^- < T^+_{a-\xi(z)}\right)\\
=&\,\frac{\lambda}{\lambda-\psi(\theta)}\left(Z(y-\xi(z),\theta) - Z(y-\xi(z),\Phi_\lambda)\frac{\widetilde{Z}(a-\xi(z),\Phi_\lambda,\theta)}{\widetilde{Z}(a-\xi(z),\Phi_\lambda,\Phi_\lambda)}\right),
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
&\mathrm{P}_y\left(\tau_z^+ < T_a^+ \wedge T^-_{\xi(z)}\right)\\
=&\,\mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\lambda\int_0^{\tau^+_{\overline{\xi}(z)}}\mathbf{1}_{[a-\xi(z),\overline{\xi}(z)]\cup(-\infty,0)}(X(t))\mathrm{d}t}\right)\\
=&\,\mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\lambda\tau^+_{\overline{\xi}(z)}+\lambda\int_0^{\tau^+_{\overline{\xi}(z)}}\mathbf{1}_{[0,a-\xi(z)]}(X(t))\mathrm{d}t}\right)\\
=&\,\rho(y-\xi(z);\lambda, a-\xi(z), \overline{\xi}(z)),
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
&\mathrm{E}_y\left(\mathrm{e}^{\theta\left(X(T^-_{\xi(z)})-\xi(z)\right)}; \tau_z^+ < T^-_{\xi(z)} < T_a^+\right)\\
=&\,\mathrm{P}_y\left(\tau_z^+ < T_a^+ \wedge T^-_{\xi(z)}\right)\mathrm{E}_z\left(\mathrm{e}^{\theta\left(X(T^-_{\xi(z)})-\xi(\overline{X}(T^-_{\xi(z)}))\right)}; T^-_{\xi(z)} < T_a^+\right)\\
=&\,\mathrm{E}_{y-\xi(z)}\left(\mathrm{e}^{-\lambda\int_0^{\tau^+_{\overline{\xi}(z)}}\mathbf{1}_{[a-\xi(z),\overline{\xi}(z)]\cup(-\infty,0)}(X(t))\mathrm{d}t}\right.\\
&\left.\times\mathrm{E}_{\overline{\xi}(z)}\left(\mathrm{e}^{\theta X(T_0^-)}; T_0^- < T^+_{a-\xi(z)}\right)\right)\\
=&\,\rho(y-\xi(z);\lambda, a-\xi(z), \overline{\xi}(z))\upsilon(\overline{\xi}(z);\theta, a-\xi(z)).
\end{aligned} \tag{61}$$

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It follows from (7) and (58)–(61) that

$$\begin{aligned}
f_4(x;\theta,a) &= \lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\,\mathrm{d}z\int_{-\infty}^{\xi(z)\wedge a}\mathrm{e}^{\theta(y-\xi(z))}\left(-\Phi_\lambda W_\lambda(z-y)+W_\lambda'(z-y)\right)\mathrm{d}y\\
&\quad+\lambda\int_x^\infty f_4(z;\theta,a)\mathrm{e}^{-\Phi_\lambda(z-x)}\mathrm{d}z\int_{\xi(z)\wedge a}^{a}\rho(y-\xi(z);\lambda,a-\xi(z),\overline{\xi}(z))\\
&\quad\times\left(-\Phi_\lambda W_\lambda(z-y)+W_\lambda'(z-y)\right)\mathrm{d}y\\
&\quad+\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\,\mathrm{d}z\int_{\xi(z)\wedge a}^{a}\frac{\lambda\left(Z(y-\xi(z),\theta)-Z(y-\xi(z),\Phi_\lambda)\frac{\widetilde{Z}(a-\xi(z),\Phi_\lambda,\theta)}{\widetilde{Z}(a-\xi(z),\Phi_\lambda,\Phi_\lambda)}\right)}{\lambda-\psi(\theta)}\\
&\quad\times\lambda\left(-\Phi_\lambda W_\lambda(z-y)+W_\lambda'(z-y)\right)\mathrm{d}y\\
&\quad-\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\upsilon(\overline{\xi}(z);\theta,a-\xi(z))\\
&\quad\times\int_{\xi(z)\wedge a}^{a}\rho(y-\xi(z);\lambda,a-\xi(z),\overline{\xi}(z))\\
&\quad\times\left(-\Phi_\lambda W_\lambda(z-y)+W_\lambda'(z-y)\right)\mathrm{d}y. \qquad (62)
\end{aligned}$$

Taking derivatives with respect to x on both sides of (62) we get

$$\begin{aligned}
f_4'(x;\theta,a) &= \Bigg(\Phi_\lambda-\lambda\int_{\xi(x)\wedge a}^{a}\rho(y-\xi(x);\lambda,a-\xi(x),\overline{\xi}(x))\\
&\quad\times\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\Bigg)f_4(x;\theta,a)\\
&\quad-\lambda\int_{-\infty}^{\xi(x)\wedge a}\mathrm{e}^{\theta(y-\xi(x))}\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&\quad-\int_{\xi(x)\wedge a}^{a}\frac{\lambda\left(Z(y-\xi(x),\theta)-Z(y-\xi(x),\Phi_\lambda)\frac{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\theta)}{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\Phi_\lambda)}\right)}{\lambda-\psi(\theta)}\\
&\quad\times\lambda\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&\quad+\lambda\upsilon(\overline{\xi}(x);\theta,a-\xi(x))\int_{\xi(x)\wedge a}^{a}\rho(y-\xi(x);\lambda,a-\xi(x),\overline{\xi}(x))\\
&\quad\times\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y. \qquad (63)
\end{aligned}$$

Solving (63) with boundary condition (57) gives (18).

For $x\in[0,a]$,

$$\begin{aligned}
f_5(x;\theta,a) &:= \mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_\xi\right)\\
&= \mathrm{E}_x\left(\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\tau_a^+<T_a^+<T_\xi\,\middle|\,\mathcal{F}_{\tau_a^+}\right)\right)\\
&= \mathrm{E}_x\left(\mathrm{E}_a\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_\xi\right);\tau_a^+<T_\xi\right)\\
&= (1-f_1(x;0,a))\,\mathrm{E}_a\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_\xi\right),
\end{aligned}$$

this gives (19).

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For $x \in (a, \infty)$, by conditioning on the first Poisson observation time, we have

$$\begin{aligned}
&f_5(x;\theta,a)\\
&=\int_a^\infty\int_{x\vee y}^\infty \mathrm{e}^{-\theta(y-a)}\mathbf{1}_{\{y\geq\xi(z)\}}\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad+\int_x^\infty\int_{\xi(z)\wedge a}^a \mathrm{P}_y\left(\tau_z^+<T_a^+\wedge T_{\xi(z)}^-\right)f_5(z;\theta,a)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad+\int_x^\infty\int_{\xi(z)\wedge a}^a \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_{\xi(z)}^-\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad-\int_x^\infty\int_{\xi(z)\wedge a}^a \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\tau_z^+<T_a^+<T_{\xi(z)}^-\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\overline{X}(e_\lambda)\in\mathrm{d}z\right). \quad (64)
\end{aligned}$$

Hence, by (7), one gets

$$\begin{aligned}
&f_5(x;\theta,a)\\
&=\lambda W(0)\int_x^\infty \mathrm{e}^{-\theta(y-a)-\Phi_\lambda(y-x)}\,\mathrm{d}y\\
&\quad+\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\,\mathrm{d}z\int_{a\vee\xi(z)}^z \mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad+\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}f_5(z;\theta,a)\mathrm{d}z\int_{\xi(z)\wedge a}^a \rho(y-\xi(z);\lambda,a-\xi(z),\overline{\xi}(z))\\
&\quad\times\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad+\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\int_{\xi(z)\wedge a}^a \frac{\lambda}{\Phi_\lambda+\theta}\frac{Z(y-\xi(z),\Phi_\lambda)}{\widetilde{Z}(a-\xi(z),\Phi_\lambda,\Phi_\lambda)}\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad-\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\int_{\xi(z)\wedge a}^a \rho(y-\xi(z);\lambda,a-\xi(z),\overline{\xi}(z))\vartheta(\overline{\xi}(z);\theta,a-\xi(z))\\
&\quad\times\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y, \quad (65)
\end{aligned}$$

where we have also used the following result (see Equation (20) of [5])

$$\mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<\tau_{\xi(z)}^-\right)=\frac{\lambda}{\Phi_\lambda+\theta}\frac{Z(y-\xi(z),\Phi_\lambda)}{\widetilde{Z}(a-\xi(z),\Phi_\lambda,\Phi_\lambda)},\quad y\leq a,$$

and

$$\begin{aligned}
&\mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\tau_z^+<T_a^+<T_{\xi(z)}^-\right)\\
&=\mathrm{P}_y\left(\tau_z^+<T_a^+\wedge T_{\xi(z)}^-\right)\mathrm{E}_z\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_{\xi(z)}^-\right)\\
&=\rho(y-\xi(z);\lambda,a-\xi(z),\overline{\xi}(z))\\
&\quad\times\mathrm{E}_z\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_{\xi(z)}^-\right)\\
&=\rho(y-\xi(z);\lambda,a-\xi(z),\overline{\xi}(z))\\
&\quad\times\vartheta(\overline{\xi}(z);\theta,a-\xi(z)),\quad y\geq a.
\end{aligned}$$

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Taking derivatives with respect to x on both sides of (65) we get

$$\begin{aligned}
&f_5'(x;\theta,a)\\
&=\left(\Phi_\lambda-\lambda\int_{\xi(x)\wedge a}^{a}\rho(y-\xi(x);\lambda,a-\xi(x),\overline{\xi}(x))\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\right)\\
&\quad\times f_5(x;\theta,a)-\lambda W(0)\mathrm{e}^{-\theta(x-a)}-\lambda\int_{a\vee\xi(x)}^{x}\mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(x-y)-\Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad-\int_{\xi(x)\wedge a}^{a}\frac{\lambda^2}{\Phi_\lambda+\theta}\frac{Z(y-\xi(x),\Phi_\lambda)}{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\Phi_\lambda)}\left(W_\lambda'(x-y)-\Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y\\
&\quad+\lambda\int_{\xi(x)\wedge a}^{a}\rho(y-\xi(x);\lambda,a-\xi(x),\overline{\xi}(x))\vartheta(\overline{\xi}(x);\theta,a-\xi(x))\\
&\quad\times\left(W_\lambda'(x-y)-\Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y. \qquad (66)
\end{aligned}$$

Solving (66) with boundary condition (57) gives (20). □

5.5. Proof of Corollary 3.2

Proof. We only prove the expression for f_5 when $\theta>0$. By Theorem 3.2, one has

$$\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};T_a^+<T_\xi\right)=-\int_x^\infty g_5(z)\,\mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\,\mathrm{d}z,\quad x\in(a,\infty).$$

By the assumption that $\xi(z)\equiv\xi(a)$ for all $z\in[a,\infty)$, we get

$$\begin{aligned}
\int_x^z h_4(w)\mathrm{d}w&=\int_x^z\left(\Phi_\lambda-\lambda\int_{\xi(a)}^{a}\rho\left(y-\xi(\omega);\lambda,a-\xi(\omega),\overline{\xi}(\omega)\right)\right.\\
&\qquad\left.\times\left(W_\lambda'(\omega-y)-\Phi_\lambda W_\lambda(\omega-y)\right)\mathrm{d}y\right)\mathrm{d}\omega\\
&=(z-x)\Phi_\lambda-\int_{x-\xi(a)}^{z-\xi(a)}\frac{I(\omega)}{Z_{\overline{\xi}(a)}^{(0,\lambda)}(\omega-\xi(a),\Phi_\lambda)}\mathrm{d}\omega,
\end{aligned}$$

where

$$I(\omega)=\lambda\int_0^{\overline{\xi}(a)}Z(y-\xi(a),\Phi_\lambda)\left(W_\lambda'(\omega-y)-\Phi_\lambda W_\lambda(\omega-y)\right)\mathrm{d}y.$$

Assuming that $\xi(a)>0$ (since $\xi(a)\le 0$ is trivial), then we separate the interval $[0,\overline{\xi}(a)]$ into two subintervals, $[0,\xi(a)]$ and $(\xi(a),\overline{\xi}(a)]$. Directly from calculations, we obtain

$$\begin{aligned}
&Z_{\overline{\xi}(a)}^{(0,\lambda)'}(\omega-\xi(a),\Phi_\lambda)-\Phi_\lambda Z_{\overline{\xi}(a)}^{(0,\lambda)}(\omega-\xi(a),\Phi_\lambda)\\
&=\lambda\Phi_\lambda\int_{\xi(a)}^{\overline{\xi}(a)}W_\lambda(\omega-y)Z(y-\xi(a),\Phi_\lambda)\mathrm{d}y-\lambda\int_{\xi(a)}^{\overline{\xi}(a)}W_\lambda'(\omega-y)Z(y-\xi(a),\Phi_\lambda)\mathrm{d}y,
\end{aligned}$$

then

$$I(\omega)=-\left(Z_{\overline{\xi}(a)}^{(0,\lambda)'}(\omega-\xi(a),\Phi_\lambda)-\Phi_\lambda Z_{\overline{\xi}(a)}^{(0,\lambda)}(\omega-\xi(a),\Phi_\lambda)\right)+A(\omega),$$

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where

$$A(\omega)=\int_0^{\xi(a)}\frac{\mathrm{e}^{\Phi_\lambda(y-\xi(a))}\left(W_\lambda'(\omega-y)-\Phi_\lambda W_\lambda(\omega-y)\right)}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(\omega-\xi(a),\,\Phi_\lambda)}\,\mathrm{d}y,$$

by noticing that $Z(y-\xi(a),\,\Phi_\lambda)=\mathrm{e}^{\Phi_\lambda(y-\xi(a))}$ for $y\le\xi(a)$. Next we have

$$\begin{aligned}
&\vartheta(x-\xi(a);\theta,\,\overline{\xi}(a))\\
&=\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\,T_a^+<T^-_{\xi(a)}\right)\\
&=\int_a^\infty\int_{x\vee y}^\infty \mathrm{e}^{-\theta(y-a)}\mathbf{1}_{\{y\ge\xi(z)\}}\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&\quad+\int_x^\infty\int_{\xi(z)\wedge a}^a \mathrm{E}_y\left(\mathrm{e}^{-\theta(X(T_a^+)-a)};\,T_a^+<T^-_{\xi(a)}\right)\mathrm{P}_x\left(X(e_\lambda)\in\mathrm{d}y,\,\overline{X}(e_\lambda)\in\mathrm{d}z\right)\\
&=\lambda W(0)\int_x^\infty \mathrm{e}^{-\theta(y-a)-\Phi_\lambda(y-x)}\,\mathrm{d}y\\
&\quad+\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\,\mathrm{d}z\int_{a\vee\xi(a)}^z \mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad+\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\int_{\xi(a)\wedge a}^a\frac{\lambda}{\Phi_\lambda+\theta}\frac{Z(y-\xi(z),\,\Phi_\lambda)}{\widetilde{Z}(a-\xi(z),\,\Phi_\lambda,\,\Phi_\lambda)}\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y,
\end{aligned}$$

differentiating both sides of the above equation and then rearranging terms yields

$$\begin{aligned}
&\frac{\mathrm{d}\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))}{\mathrm{d}z}\\
&=\Phi_\lambda\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))-\lambda W(0)\mathrm{e}^{-\theta(z-a)}\\
&\quad-\lambda\int_{a\vee\xi(a)}^z \mathrm{e}^{-\theta(y-a)}\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y\\
&\quad-\lambda\int_{\xi(a)\wedge a}^a\frac{\lambda}{\Phi_\lambda+\theta}\frac{Z(y-\xi(z),\,\Phi_\lambda)}{\widetilde{Z}(a-\xi(z),\,\Phi_\lambda,\,\Phi_\lambda)}\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y.
\end{aligned}$$

Hence, $g_5(z)$ can be rewritten as

$$\begin{aligned}
g_5(z)&=\frac{\mathrm{d}\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))}{\mathrm{d}z}-\Phi_\lambda\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))\\
&\quad-\lambda\int_x^\infty \mathrm{e}^{-\Phi_\lambda(z-x)}\int_{\xi(a)\wedge a}^a\rho(y-\xi(a);\lambda,\,\overline{\xi}(a),\,z-\xi(a))\\
&\quad\times\vartheta(\overline{\xi}(z);\theta,\,\overline{\xi}(a))\left(W_\lambda'(z-y)-\Phi_\lambda W_\lambda(z-y)\right)\mathrm{d}y\\
&=\frac{\mathrm{d}\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))}{\mathrm{d}z}-\Phi_\lambda\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))\\
&\quad+\lambda\int_0^{\overline{\xi}(a)}\frac{Z(y-\xi(a),\,\Phi_\lambda)}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\,\Phi_\lambda)}\vartheta(z-\xi(a);\theta,\,\overline{\xi}(a))\\
&\quad\times\left(W_\lambda'(z-y-\xi(a))-\Phi_\lambda W_\lambda(z-y-\xi(a))\right)\mathrm{d}y.
\end{aligned}$$

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Similarly, separating the interval $[0, \overline{\xi}(a)]$ into two sub-intervals, $[0, \xi(a)]$ and $(\xi(a), \overline{\xi}(a)]$, we obtain

$$\begin{aligned}
&g_5(z)\\
&= \vartheta'(z-\xi(a);\theta,\overline{\xi}(a)) - \Phi_\lambda \vartheta(z-\xi(a);\theta,\overline{\xi}(a))\\
&\quad + \vartheta(\overline{\xi}(z);\theta,\overline{\xi}(a))\lambda \int_0^{\xi(a)} \frac{\left(W_\lambda'(z-y-\xi(a)) - \Phi_\lambda W_\lambda(z-y-\xi(a))\right) \mathrm{e}^{\Phi_\lambda(y-\xi(a))}}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)}\,\mathrm{d}y\\
&\quad - \vartheta(\overline{\xi}(z);\theta,\overline{\xi}(a)) \frac{(Z^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda) - \Phi_\lambda Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda))}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)}\\
&= \vartheta'(z-\xi(a);\theta,\overline{\xi}(a)) - \vartheta(z-\xi(a);\theta,\overline{\xi}(a)) \frac{Z^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)} + \vartheta(z-\xi(a);\theta,\overline{\xi}(a))B(z),
\end{aligned}$$

where

$$B(z) = \int_0^{\xi(a)} \frac{\mathrm{e}^{\Phi_\lambda(y-\xi(a))}\left(W_\lambda'(z-y-\xi(a)) - \Phi_\lambda W_\lambda(z-y-\xi(a))\right)}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)}\,\mathrm{d}y.$$

Therefore, we have

$$\begin{aligned}
&\mathrm{E}_x\left(\mathrm{e}^{-\theta(X(T_a^+)-a)}; T_a^+ < T_\xi\right) = -\int_x^\infty g_5(z)\,\mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\,\mathrm{d}z\\
&= -Z^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a),\Phi_\lambda)\int_x^\infty \left(\frac{\vartheta(\overline{\xi}(z);\theta,\overline{\xi}(a))A(z)}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)\mathrm{e}^{-\int_x^z A(\omega)\mathrm{d}w}}\right.\\
&\quad \left. + \frac{\vartheta'(z-\xi(a);\theta,\overline{\xi}(a))}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)\mathrm{e}^{-\int_x^z A(\omega)\mathrm{d}w}} - \frac{\vartheta(z-\xi(a);\theta,\overline{\xi}(a))Z^{(0,\lambda)'}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)}{\left(Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)\right)^2 \mathrm{e}^{-\int_x^z A(\omega)\mathrm{d}w}}\right)\mathrm{d}z\\
&= -Z^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a),\Phi_\lambda)\int_x^\infty \left(\frac{\vartheta(z-\xi(a);\theta,\overline{\xi}(a))}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)\mathrm{e}^{-\int_x^z A(\omega)\mathrm{d}w}}\right)'_z \mathrm{d}z\\
&= -Z^{(0,\lambda)}_{\overline{\xi}(a)}(x-\xi(a),\Phi_\lambda)\left(\frac{\vartheta(z-\xi(a);\theta,\overline{\xi}(a))}{Z^{(0,\lambda)}_{\overline{\xi}(a)}(z-\xi(a),\Phi_\lambda)\mathrm{e}^{-\int_x^z A(\omega)\mathrm{d}w}}\Bigg|_{z=x}^\infty\right)\\
&= \vartheta(x-\xi(a);\theta,\overline{\xi}(a)).
\end{aligned}$$

This completes the proof. □

5.6. Proof of Theorem 3.3

Proof. We recall from Theorem 3.1 that

$$f_1(x;\theta,a) = \frac{\lambda}{\lambda-\psi(\theta)}\int_x^a \left(\frac{Z'(\overline{\xi}(z),\Phi_\lambda)}{Z(\overline{\xi}(z),\Phi_\lambda)}Z(\overline{\xi}(z),\theta) - Z'(\overline{\xi}(z),\theta)\right)\mathrm{e}^{-\int_x^z \frac{Z'(\overline{\xi}(w),\Phi_\lambda)}{Z(\overline{\xi}(w),\Phi_\lambda)}\mathrm{d}w}\,\mathrm{d}z. \quad (67)$$

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Note that for large $\lambda > 0$, we have

$$\begin{aligned}
\frac{Z'(\bar{\xi}(x),\,\Phi_\lambda)}{Z(\bar{\xi}(x),\,\Phi_\lambda)} =& \frac{\Phi_\lambda Z(\bar{\xi}(x),\,\Phi_\lambda) - \lambda W(\bar{\xi}(x))}{Z(\bar{\xi}(x),\,\Phi_\lambda)}. \\
=& \Phi_\lambda - \frac{\lambda W(\bar{\xi}(x))}{\mathrm{e}^{\Phi_\lambda \bar{\xi}(x)}\left(1 - \lambda \int_0^{\bar{\xi}(x)} \mathrm{e}^{-\Phi_\lambda z} W(z)\mathrm{d}z\right)} \\
=& \Phi_\lambda - \frac{W(\bar{\xi}(x))}{\mathrm{e}^{\Phi_\lambda \bar{\xi}(x)} \int_{\bar{\xi}(x)}^{\infty} \mathrm{e}^{-\Phi_\lambda z} W(z),\,\mathrm{d}z} \\
=& \Phi_\lambda - \frac{\Phi_\lambda W(\bar{\xi}(x))}{\int_0^{\infty} \Phi_\lambda \mathrm{e}^{-\Phi_\lambda u} W(\bar{\xi}(x) + u)\mathrm{d}u} \\
=& \Phi_\lambda - \frac{\Phi_\lambda W(\bar{\xi}(x))}{\int_0^{\infty} \mathrm{e}^{-z} W(\bar{\xi}(x) + \frac{z}{\Phi_\lambda})\mathrm{d}z} \\
=& \Phi_\lambda \frac{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda}) - W(\bar{\xi}(x))\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)} \quad (e_1 \sim \exp(1) \text{ is a random variable}) \\
=& \frac{\mathrm{E}\left(W'(\bar{\xi}(x) + \theta \frac{e_1}{\Phi_\lambda})\, e_1\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)} \quad (\theta \in (0,\,1)) \\
=& \frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x) + \theta \frac{e_1}{\Phi_\lambda})}{W(\bar{\xi}(x) + \theta \frac{e_1}{\Phi_\lambda})} W(\bar{\xi}(x) + \theta \frac{e_1}{\Phi_\lambda}) e_1\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)}.
\end{aligned} \tag{68}$$

Utilizing (68) and recalling the facts that $x \mapsto \frac{W'(x)}{W(x)}$ is decreasing and $x \mapsto W(x)$ is increasing on $(0,\,\infty)$, we have

$$\frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})}{W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})} W(\bar{\xi}(x)) e_1\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)} \le \frac{Z'(\bar{\xi}(x),\,\Phi_\lambda)}{Z(\bar{\xi}(x),\,\Phi_\lambda)} \le \frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x))}{W(\bar{\xi}(x))} W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda}) e_1\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)},$$

which, together with the monotone convergence theorem, yields

$$\begin{aligned}
\frac{W'(\bar{\xi}(x))}{W(\bar{\xi}(x))} =& \frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x))}{W(\bar{\xi}(x))} W(\bar{\xi}(x)) e_1\right)}{W(\bar{\xi}(x))} \\
=& \lim_{\lambda \to \infty} \frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})}{W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})} W(\bar{\xi}(x)) e_1\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)} \\
\le& \lim_{\lambda \to \infty} \frac{Z'(\bar{\xi}(x),\,\Phi_\lambda)}{Z(\bar{\xi}(x),\,\Phi_\lambda)} \\
\le& \lim_{\lambda \to \infty} \frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x))}{W(\bar{\xi}(x))} W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda}) e_1\right)}{\mathrm{E}\left(W(\bar{\xi}(x) + \frac{e_1}{\Phi_\lambda})\right)} = \frac{\mathrm{E}\left(\frac{W'(\bar{\xi}(x))}{W(\bar{\xi}(x))} W(\bar{\xi}(x)) e_1\right)}{W(\bar{\xi}(x))} = \frac{W'(\bar{\xi}(x))}{W(\bar{\xi}(x))}.
\end{aligned}$$

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What follows is the standard application of the dominated convergence theorem to (67) in order to derive (21). We mention that identifying the dominating function is not difficult, since the integrand of (67) is continuous on the closed interval $[x, a]$, and the function $Z'(\bar{\xi}(x), \Phi_\lambda)/Z(\bar{\xi}(x), \Phi_\lambda)$ converges to 0 as $x \to \infty$, i.e.

$$\lim_{x\to\infty} \frac{Z'(x, \Phi_\lambda)}{Z(x, \Phi_\lambda)} = \lim_{x\to\infty} \frac{\lambda e^{\Phi_\lambda x} \int_x^\infty e^{-\Phi_\lambda y} W'(y)\mathrm{d}y}{\lambda e^{\Phi_\lambda x} \int_x^\infty e^{-\Phi_\lambda y} W(y)\mathrm{d}y} = \lim_{x\to\infty} \frac{W'(x)}{W(x)} = 0, \quad \lambda \in (0, \infty).$$

The proof is complete. □

6. Proofs of the results of Section 4

6.1. Proof of Theorem 4.1

Proof. From Theorem 3.1 it follows that

$$\begin{aligned} h_1(x) &= \Phi_\lambda - \lambda \frac{W((1-k)x+d)}{Z((1-k)x+d, \Phi_\lambda)} \\ &= \frac{Z'((1-k)x+d, \Phi_\lambda)}{Z((1-k)x+d, \Phi_\lambda)} = \frac{1}{1-k} \frac{\mathrm{d}}{\mathrm{d}x} \ln Z((1-k)x+d, \Phi_\lambda), \end{aligned}$$

and, hence, we have

$$e^{-\int_0^x h_1(z)\mathrm{d}z} = e^{-\frac{1}{1-k}\int_0^x \frac{\mathrm{d}}{\mathrm{d}z} \ln Z((1-k)z+d, \Phi_\lambda)\,\mathrm{d}z} = \left(\frac{Z((1-k)x+d, \Phi_\lambda)}{Z(d, \Phi_\lambda)}\right)^{-\frac{1}{1-k}}. \tag{69}$$

In addition, we also have

$$\begin{aligned} g_1(x) &= \frac{\lambda}{\lambda - \psi(\theta)} \left(-\theta Z(\bar{\xi}(x), \theta) + \psi(\theta) W(\bar{\xi}(x))\right) \\ &\quad - \frac{\lambda}{\lambda - \psi(\theta)} \frac{Z(\bar{\xi}(x), \theta)}{Z(\bar{\xi}(x), \Phi_\lambda)} \left(-\Phi_\lambda Z(\bar{\xi}(x), \Phi_\lambda) + \lambda W(\bar{\xi}(x))\right) \\ &= -\frac{\lambda}{\lambda - \psi(\theta)} Z'(\bar{\xi}(x), \theta) + \frac{\lambda}{\lambda - \psi(\theta)} \frac{Z'(\bar{\xi}(x), \Phi_\lambda)}{Z(\bar{\xi}(x), \Phi_\lambda)} Z(\bar{\xi}(x), \theta) \\ &= \frac{\lambda}{\lambda - \psi(\theta)} \left(\frac{Z'((1-k)x+d, \Phi_\lambda)}{Z((1-k)x+d, \Phi_\lambda)} Z((1-k)x+d, \theta) - Z'((1-k)x+d, \theta)\right) \\ &= \frac{\lambda}{\lambda - \psi(\theta)} \left(Z((1-k)x+d, \theta) h_1(x) - Z'((1-k)x+d, \theta)\right), \end{aligned}$$

which together with (10) and (69) yields

$$\begin{aligned} &f_1(x; \theta, a) \\ &= \frac{\lambda e^{\int_0^x h_1(z)\mathrm{d}z}}{\lambda - \psi(\theta)} \int_x^a \left(Z((1-k)z+d, \theta) h_1(z) - Z'((1-k)z+d, \theta)\right) e^{-\int_0^z h_1(w)\mathrm{d}w}\,\mathrm{d}z \\ &= \frac{\lambda e^{\int_0^x h_1(z)\mathrm{d}z}}{\lambda - \psi(\theta)} \int_x^a \left(Z((1-k)z+d, \theta) h_1(z) - (1-k) Z'((1-k)z+d, \theta)\right) e^{-\int_0^z h_1(w)\mathrm{d}w}\,\mathrm{d}z \\ &\quad - \frac{\lambda k e^{\int_0^x h_1(z)\mathrm{d}z}}{\lambda - \psi(\theta)} \int_x^a Z'((1-k)z+d, \theta) e^{-\int_0^z h_1(w)\mathrm{d}w}\,\mathrm{d}z \end{aligned}$$

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$$
\begin{aligned}
&= -\frac{\lambda \mathrm{e}^{\int_0^x h_1(z)\mathrm{d}z}}{\lambda-\psi(\theta)} \int_x^a \frac{\mathrm{d}}{\mathrm{d}z}\left[Z((1-k)z+d,\,\theta)\mathrm{e}^{-\int_0^z h_1(w)\mathrm{d}w}\right]\mathrm{d}z \\
&\quad -\frac{\lambda k}{\lambda-\psi(\theta)} \int_x^a Z'((1-k)z+d,\,\theta)\mathrm{e}^{-\int_x^z h_1(w)\mathrm{d}w}\,\mathrm{d}z \\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left[Z((1-k)x+d,\,\theta) - Z((1-k)a+d,\,\theta)\mathrm{e}^{-\int_x^a h_1(z)\mathrm{d}z}\right] \\
&\quad -\frac{\lambda k}{\lambda-\psi(\theta)} \int_x^a Z'((1-k)z+d,\,\theta)\mathrm{e}^{-\int_x^z h_1(w)\mathrm{d}w}\,\mathrm{d}z \\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left[Z((1-k)x+d,\,\theta) - Z((1-k)a+d,\,\theta)\left(\frac{Z((1-k)a+d,\,\Phi_\lambda)}{Z((1-k)x+d,\,\Phi_\lambda)}\right)^{-\frac{1}{1-k}}\right] \\
&\quad -\frac{\lambda k}{\lambda-\psi(\theta)} \int_x^a Z'((1-k)z+d,\,\theta)\left(\frac{Z((1-k)z+d,\,\Phi_\lambda)}{Z((1-k)x+d,\,\Phi_\lambda)}\right)^{-\frac{1}{1-k}}\mathrm{d}z.
\end{aligned}
$$

The proof is complete. □

6.2. Proof of Theorem 4.2

Proof. From Theorem 3.1, we have

$$
h_1(x) = \Phi_\lambda - \frac{\lambda W(\overline{\xi}(x))}{Z(\overline{\xi}(x),\,\Phi_\lambda)} = \Phi_\lambda - \lambda\frac{W(x+d)}{Z(x+d,\,\Phi_\lambda)} = \frac{Z'(x+d,\,\Phi_\lambda)}{Z(x+d,\,\Phi_\lambda)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln Z(x+d,\,\Phi_\lambda),
$$

and, hence, we have

$$
\mathrm{e}^{-\int_0^x h_1(z)\mathrm{d}z} = \mathrm{e}^{-\int_0^x \frac{\mathrm{d}}{\mathrm{d}z}\ln Z(z+d,\Phi_\lambda)\,\mathrm{d}x} = \frac{Z(x+d,\,\Phi_\lambda)}{Z(d,\,\Phi_\lambda)}.
$$

In addition, we also have

$$
\begin{aligned}
g_1(x) &= \frac{\lambda}{\lambda-\psi(\theta)}\left(-\theta Z(\overline{\xi}(x),\,\theta) + \psi(\theta)W(\overline{\xi}(x))\right) \\
&\quad -\frac{\lambda}{\lambda-\psi(\theta)}\frac{Z(\overline{\xi}(x),\,\theta)}{Z(\overline{\xi}(x),\,\Phi_\lambda)}\left(-\Phi_\lambda Z(\overline{\xi}(x),\,\Phi_\lambda) + \lambda W(\overline{\xi}(x))\right) \\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left(\frac{Z'(x+d,\,\Phi_\lambda)}{Z(x+d,\,\Phi_\lambda)}Z(x+d,\,\theta) - Z'(x+d,\,\theta)\right) \\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left(Z(x+d,\,\theta)h_1(x) - Z'(x+d,\,\theta)\right),
\end{aligned}
$$

which together with (10) and (69) yields

$$
\begin{aligned}
f_1(x;\theta,\,a) &= \frac{\lambda \mathrm{e}^{\int_0^x h_1(z)\mathrm{d}z}}{\lambda-\psi(\theta)} \int_x^a \left(Z(z+d,\,\theta)h_1(z) - Z'(z+d,\,\theta)\right)\mathrm{e}^{-\int_0^z h_1(w)\mathrm{d}w}\,\mathrm{d}z \\
&= -\frac{\lambda \mathrm{e}^{\int_0^x h_1(z)\mathrm{d}z}}{\lambda-\psi(\theta)} \int_x^a \frac{\mathrm{d}}{\mathrm{d}z}\left[Z(z+d,\,\theta)\mathrm{e}^{-\int_0^z h_1(w)\mathrm{d}w}\right]\mathrm{d}z \\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x+d,\,\theta) - Z(a+d,\,\theta)\mathrm{e}^{-\int_x^a h_1(z)\mathrm{d}z}\right] \\
&= \frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x+d,\,\theta) - Z(a+d,\,\theta)\frac{Z(x+d,\,\Phi_\lambda)}{Z(a+d,\,\Phi_\lambda)}\right].
\end{aligned}
$$

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Furthermore, from (9) and (70), it follows that

$$\begin{aligned} f_1(x;\theta,\infty) &= \lim_{a\to\infty} f_1(x;\theta,a) \\ &= \lim_{a\to\infty} \frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x+d,\theta) - Z(a+d,\theta)\frac{Z(x+d,\Phi_\lambda)}{Z(a+d,\Phi_\lambda)}\right] \\ &= \frac{\lambda}{\lambda-\psi(\theta)}Z(x+d,\theta) - Z(x+d,\Phi_\lambda)\frac{\psi(\theta)(\Phi_\lambda-\Phi)}{(\lambda-\psi(\theta))(\theta-\Phi)}. \end{aligned}$$

From Theorem 3.1, we have

$$\begin{aligned} h_2(x) &= \frac{W'_\lambda(\overline{\xi}(x)) - \lambda\int_0^{a-\xi(x)} W(y)W'_\lambda(\overline{\xi}(x)-y)\mathrm{d}y}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} \\ &= \frac{(W^{(0,\lambda)}_{a-\xi(x)})'(\overline{\xi}(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} = \frac{\mathrm{d}}{\mathrm{d}x}\ln W^{(0,\lambda)}_{a+d}(x+d), \end{aligned}$$

and, hence, we have

$$\mathrm{e}^{-\int_a^x h_2(z)\mathrm{d}z} = \frac{W^{(0,\lambda)}_{a+d}(a+d)}{W^{(0,\lambda)}_{a+d}(x+d)}. \tag{70}$$

In addition, we also have

$$\begin{aligned} g_2(x) &= -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda\int_{a\vee\xi(x)}^x \mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(x-y) - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}W_\lambda(x-y)\right)\mathrm{d}y \\ &\quad +\lambda\mathrm{e}^{-\theta(\overline{\xi}(x)-a)}\int_0^{\overline{\xi}(x)-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{(W^{(0,\lambda)}_{a-\xi(x)})'(\overline{\xi}(x))}{W^{(0,\lambda)}_{a-\xi(x)}(\overline{\xi}(x))} - \frac{W'_\lambda(\overline{\xi}(x))}{W_\lambda(\overline{\xi}(x))}\right)\mathbf{1}_{\{a>\xi(x)\}} \\ &= -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda\int_a^x \mathrm{e}^{-\theta(y-a)}\left(W'_\lambda(x-y) - \frac{W'_\lambda(x+d)}{W_\lambda(x+d)}W_\lambda(x-y)\right)\mathrm{d}y \\ &\quad +\lambda\mathrm{e}^{-\theta(x+d-a)}\int_0^{x+d-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{(W^{(0,\lambda)}_{a+d})'(x+d)}{W^{(0,\lambda)}_{a+d}(x+d)} - \frac{W'_\lambda(x+d)}{W_\lambda(x+d)}\right) \\ &= -\lambda W_\lambda(x-a) + \lambda\mathrm{e}^{-\theta(x-a)}\int_0^{x-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{W'_\lambda(x+d)}{W_\lambda(x+d)}+\theta\right) \\ &\quad +\lambda\mathrm{e}^{-\theta(x+d-a)}\int_0^{x+d-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{(W^{(0,\lambda)}_{a+d})'(x+d)}{W^{(0,\lambda)}_{a+d}(x+d)} - \frac{W'_\lambda(x+d)}{W_\lambda(x+d)}\right). \end{aligned} \tag{71}$$

It follows from (70) and (71) that, for $x\in(a,\infty)$,

$$\begin{aligned} &f_2(x;\theta,a) \\ &= -\int_x^\infty g_2(z)\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\,\mathrm{d}z \\ &= \int_x^\infty\left\{\lambda W_\lambda(z-a) - \lambda\mathrm{e}^{-\theta(z-a)}\int_0^{z-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{W'_\lambda(z+d)}{W_\lambda(z+d)}+\theta\right)\right. \\ &\quad \left.-\lambda\mathrm{e}^{-\theta(z+d-a)}\int_0^{z+d-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{(W^{(0,\lambda)}_{a+d})'(z+d)}{W^{(0,\lambda)}_{a+d}(z+d)} - \frac{W'_\lambda(z+d)}{W_\lambda(z+d)}\right)\right\}\frac{W^{(0,\lambda)}_{a+d}(x+d)}{W^{(0,\lambda)}_{a+d}(z+d)}\,\mathrm{d}z. \end{aligned}$$

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In addition, it follows from (11) that, for $x \in [0, a]$,

$$\begin{aligned}
&f_2(x;\theta,a)\\
&= \mathrm{e}^{-\int_x^a \frac{W'(\bar{\xi}(z))}{W(\bar{\xi}(z))}\,\mathrm{d}z} f(a;\theta,a)\\
&= \frac{W(x+d)}{W(a+d)}\int_a^\infty \left\{\lambda W_\lambda(z-a) - \lambda \mathrm{e}^{-\theta(z-a)}\int_0^{z-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{W'_\lambda(z+d)}{W_\lambda(z+d)}+\theta\right)\right.\\
&\quad \left. -\lambda \mathrm{e}^{-\theta(z+d-a)}\int_0^{z+d-a}\mathrm{e}^{\theta y}W_\lambda(y)\mathrm{d}y\left(\frac{(W^{(0,\lambda)}_{a+d})'(z+d)}{W^{(0,\lambda)}_{a+d}(z+d)} - \frac{W'_\lambda(z+d)}{W_\lambda(z+d)}\right)\right\}\frac{W^{(0,\lambda)}_{a+d}(a+d)}{W^{(0,\lambda)}_{a+d}(z+d)}\,\mathrm{d}z.
\end{aligned} \tag{72}$$

Furthermore, from Theorem 3.1, we have

$$\begin{aligned}
&g_3(x)\\
&= -\left(\left(-\theta + \frac{W'_\lambda(\bar{\xi}(x))}{W_\lambda(\bar{\xi}(x))}\right)Z_\lambda(\bar{\xi}(x),\theta) - (\lambda-\psi(\theta))W_\lambda(\bar{\xi}(x))\right)\\
&\quad -\lambda\int_{\xi(x)\wedge a}^a Z(y-\xi(x),\theta)W'_\lambda(x-y)\mathrm{d}y + Z_\lambda(\bar{\xi}(x),\theta)\frac{W'_\lambda(\bar{\xi}(x))}{W_\lambda(\bar{\xi}(x))}\mathbf{1}_{\{\xi(x)\le a\}}\\
&\quad +\lambda\frac{Z^{(0,\lambda)}_{a-\xi(x)}(\bar{\xi}(x),\theta)}{W^{(0,\lambda)}_{a-\xi(x)}(\bar{\xi}(x))}\int_{\xi(x)\wedge a}^a W(y-\xi(x))W'_\lambda(x-y)\mathrm{d}y - \frac{Z^{(0,\lambda)}_{a-\xi(x)}(\bar{\xi}(x),\theta)}{W^{(0,\lambda)}_{a-\xi(x)}(\bar{\xi}(x))}W'_\lambda(\bar{\xi}(x))\mathbf{1}_{\{\xi(x)\le a\}}\\
&= -\left(Z_\lambda(x+d,\theta)\frac{W'_\lambda(x+d)}{W_\lambda(x+d)} - Z'_\lambda(x+d,\theta)\right)\\
&\quad -\lambda\int_0^{a+d} Z(y,\theta)W'_\lambda(x+d-y)\mathrm{d}y + Z_\lambda(x+d,\theta)\frac{W'_\lambda(x+d)}{W_\lambda(x+d)}\\
&\quad +\lambda\frac{Z^{(0,\lambda)}_{a+d}(x+d,\theta)}{W^{(0,\lambda)}_{a+d}(x+d)}\int_0^{a+d} W(y)W'_\lambda(x+d-y)\mathrm{d}y - \frac{Z^{(0,\lambda)}_{a+d}(x+d,\theta)}{W^{(0,\lambda)}_{a+d}(x+d)}W'_\lambda(x+d)\\
&= (Z^{(0,\lambda)}_{a+d})'(x+d,\theta) - \frac{Z^{(0,\lambda)}_{a+d}(x+d,\theta)}{W^{(0,\lambda)}_{a+d}(x+d)}(W^{(0,\lambda)}_{a+d})'(x+d).
\end{aligned} \tag{73}$$

It follows from (14) and (73) yields that for $x \in (a, \infty)$

$$\begin{aligned}
f_3(x;\theta,a) &= -\int_x^\infty g_3(z)\,\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\,\mathrm{d}z\\
&= -\int_x^\infty \left((Z^{(0,\lambda)}_{a+d})'(z+d,\theta) - Z^{(0,\lambda)}_{a+d}(z+d,\theta)h_2(z)\right)\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\,\mathrm{d}z\\
&= -\int_x^\infty \frac{\mathrm{d}}{\mathrm{d}z}\left(Z^{(0,\lambda)}_{a+d}(z+d,\theta)\mathrm{e}^{-\int_x^z h_2(w)\mathrm{d}w}\right)\mathrm{d}z\\
&= Z^{(0,\lambda)}_{a+d}(x+d,\theta) - W^{(0,\lambda)}_{a+d}(x+d)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda} - \frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a+d,\theta)}{Z(a+d,\Phi_\lambda)}\right).
\end{aligned}$$

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Then, it follows from (13) that for $x \in [0, a]$,

$$\begin{aligned}
f_3(x;\theta,a) &= \mathrm{e}^{-\int_x^a \frac{W'(\bar{\xi}(z))}{W(\bar{\xi}(z))}\,\mathrm{d}z} f_3(a;\theta,a) + \int_x^a \mathrm{e}^{\theta\xi(z)}\mathrm{e}^{-\int_x^z \frac{W'(\bar{\xi}(w))}{W(\bar{\xi}(w))}\,\mathrm{d}w} \\
&\quad \times \left(\left(-\theta + \frac{W'(\bar{\xi}(z))}{W(\bar{\xi}(z))}\right) Z(\bar{\xi}(z),\theta) + \psi(\theta) W(\bar{\xi}(z))\right) \mathrm{d}z \\
&= \frac{W(x+d)}{W(a+d)}\left(Z^{(0,\lambda)}_{a+d}(a+d,\theta) - W^{(0,\lambda)}_{a+d}(a+d)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda} - \frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a+d,\theta)}{Z(a+d,\Phi_\lambda)}\right)\right) \\
&\quad + \mathrm{e}^{-\theta d}\int_x^a \mathrm{e}^{-\int_x^z \frac{W'(\bar{\xi}(w))}{W(\bar{\xi}(w))}\,\mathrm{d}w}\left(\frac{W'(\bar{\xi}(w))}{W(\bar{\xi}(w))} Z(\bar{\xi}(z),\theta) - Z'(\bar{\xi}(z),\theta)\right)\mathrm{d}z \\
&= \frac{W(x+d)}{W(a+d)}\left(Z^{(0,\lambda)}_{a+d}(a+d,\theta) - W^{(0,\lambda)}_{a+d}(a+d)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda} - \frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a+d,\theta)}{Z(a+d,\Phi_\lambda)}\right)\right) \\
&\quad - \mathrm{e}^{-\theta d}\int_x^a \frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{e}^{-\int_x^z \frac{W'(w+d)}{W(w+d)}\,\mathrm{d}w} Z(z+d,\theta)\right)\mathrm{d}z \\
&= \frac{W(x+d)}{W(a+d)}\left(Z(a+d,\theta) - W(a+d)\left(\frac{\psi(\theta)}{\theta-\Phi_\lambda} - \frac{\lambda}{\theta-\Phi_\lambda}\frac{Z(a+d,\theta)}{Z(a+d,\Phi_\lambda)}\right)\right) \\
&\quad - \mathrm{e}^{-\theta d}\frac{W(x+d)}{W(a+d)} Z(a+d,\theta) + \mathrm{e}^{-\theta d} Z(x+d,\theta).
\end{aligned}$$

The proof is complete. □

6.3. Proof of Theorem 4.3

Proof. From Theorem 3.2, we have

$$\begin{aligned}
h_4(x) &= \Phi_\lambda - \lambda\int_{\xi(x)\wedge a}^a \rho(y-\xi(x);\lambda, a-\xi(x), \bar{\xi}(x))\left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right)\mathrm{d}y \\
&= \Phi_\lambda - \lambda\int_{-d}^a \frac{Z^{(0,\lambda)}_{a+d}(y+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y))\mathrm{d}y \\
&= \Phi_\lambda - \lambda\int_{-d}^a \frac{Z(y+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y))\mathrm{d}y \\
&= \frac{\Phi_\lambda Z_\lambda(x+d,\Phi_\lambda) - \lambda\Phi_\lambda\int_0^{a+d} W_\lambda(x+d-y)Z(y,\Phi_\lambda)\mathrm{d}y}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)} \\
&\quad - \frac{\lambda\int_0^{a+d} Z(y,\Phi_\lambda)(W'_\lambda(x-y+d) - \Phi_\lambda W_\lambda(x-y+d))\mathrm{d}y}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)} \\
&= \frac{1}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}\left(\Phi_\lambda Z_\lambda(x+d,\Phi_\lambda) - \lambda\int_0^{a+d} Z(y,\Phi_\lambda)W'_\lambda(x-y+d)\mathrm{d}y\right) \\
&= \frac{(Z^{(0,\lambda)}_{a+d})'(x+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\ln Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda), \quad x \in (a,\infty).
\end{aligned}$$

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and, hence, we have

$$\mathrm{e}^{\int_a^x h_4(z)\mathrm{d}z} = \mathrm{e}^{\int_a^x \frac{\mathrm{d}}{\mathrm{d}z}\ln Z_{a+d}^{(0,\lambda)}(z+d,\Phi_\lambda)\,\mathrm{d}z} = \frac{Z_{a+d}^{(0,\lambda)}(x+d,\,\Phi_\lambda)}{Z_{a+d}^{(0,\lambda)}(a+d,\,\Phi_\lambda)}. \tag{74}$$

In addition, we also have

$$\begin{aligned}
g_4(x) &= -\lambda\int_{-\infty}^{\xi(x)\wedge a}\mathrm{e}^{\theta(y-\xi(x))}\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&\quad-\int_{\xi(x)\wedge a}^{a}\frac{\lambda\left(Z(y-\xi(x),\theta)-Z(y-\xi(x),\,\Phi_\lambda)\frac{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\theta)}{\widetilde{Z}(a-\xi(x),\Phi_\lambda,\Phi_\lambda)}\right)}{\lambda-\psi(\theta)}\\
&\quad\times\lambda\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y+\lambda\upsilon(\overline{\xi}(x);\theta,\,a-\xi(x))\\
&\quad\times\int_{\xi(x)\wedge a}^{a}\rho(y-\xi(x);\lambda,\,a-\xi(x),\,\overline{\xi}(x))\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&=-\lambda\int_{-\infty}^{-d}\mathrm{e}^{\theta(y+d)}\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&\quad-\int_{-d}^{a}\frac{\lambda\left(Z(y+d,\,\theta)-Z(y+d,\,\Phi_\lambda)\frac{\widetilde{Z}(a+d,\Phi_\lambda,\theta)}{\widetilde{Z}(a+d,\Phi_\lambda,\Phi_\lambda)}\right)}{\lambda-\psi(\theta)}\\
&\quad\times\lambda\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&\quad+\frac{\lambda^2}{\lambda-\psi(\theta)}\left(Z_{a+d}^{(0,\lambda)}(x+d,\,\theta)-\frac{\widetilde{Z}(a+d,\,\Phi_\lambda,\,\theta)}{\widetilde{Z}(a+d,\,\Phi_\lambda,\,\Phi_\lambda)}Z_{a+d}^{(0,\lambda)}(x+d,\,\Phi_\lambda)\right)\\
&\quad\times\int_{-d}^{a}\frac{Z(y+d,\,\Phi_\lambda)}{Z_{a+d}^{(0,\lambda)}(x+d,\,\Phi_\lambda)}\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&=\lambda\Phi_\lambda\mathrm{e}^{\theta(x+d)}\int_{x+d}^{\infty}\mathrm{e}^{-\theta y}W_\lambda(y)\mathrm{d}y-\lambda\mathrm{e}^{\theta(x+d)}\int_{x+d}^{\infty}\mathrm{e}^{-\theta y}W_\lambda'(y)\mathrm{d}y\\
&\quad-\int_{-d}^{a}\frac{\lambda\,(Z(y+d,\,\theta))}{\lambda-\psi(\theta)}\lambda\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\,\mathrm{d}y\\
&\quad+\frac{\lambda^2}{\lambda-\psi(\theta)}Z_{a+d}^{(0,\lambda)}(x+d,\,\theta)\int_{-d}^{a}\frac{Z(y+d,\,\Phi_\lambda)}{Z_{a+d}^{(0,\lambda)}(x+d,\,\Phi_\lambda)}\left(-\Phi_\lambda W_\lambda(x-y)+W_\lambda'(x-y)\right)\mathrm{d}y\\
&=\lambda\Phi_\lambda\mathrm{e}^{\theta d}\frac{Z_\lambda(x+d,\,\theta)}{\psi(\theta)-\lambda}-\lambda W_\lambda(x+d)+\lambda\theta\mathrm{e}^{\theta y}\frac{Z_\lambda(x+d,\,\theta)}{\psi(\theta)-\lambda}\\
&\quad+\frac{\lambda}{\lambda-\psi(\theta)}\left(\int_0^{a+d}\lambda Z(y,\,\theta)\left(\Phi_\lambda W_\lambda(x+d-y)-W_\lambda'(x+d-y)\right)\mathrm{d}y\right)\\
&\quad-\frac{\lambda}{\lambda-\psi(\theta)}\frac{Z_{a+d}^{(0,\lambda)}(x+d,\,\theta)}{Z_{a+d}^{(0,\lambda)}(x+d,\,\Phi_\lambda)}\left(\Phi_\lambda Z_\lambda(x+d,\,\Phi_\lambda)-\lambda\int_0^{a+d}Z(y,\,\Phi_\lambda)W_\lambda'(x+d-y)\mathrm{d}y\right)\\
&\quad+\frac{\lambda\Phi_\lambda}{\lambda-\psi(\theta)}\frac{Z_{a+d}^{(0,\lambda)}(x+d,\,\theta)}{Z_{a+d}^{(0,\lambda)}(x+d,\,\Phi_\lambda)}\left(Z_\lambda(x+d,\,\Phi_\lambda)-\lambda\int_0^{a+d}Z(y,\,\Phi_\lambda)W_\lambda(x+d-y)\mathrm{d}y\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \lambda \mathrm{e}^{\theta d} Z_\lambda(x+d,\theta)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda} + \lambda W_\lambda(x+d)\\
&\quad + \frac{\lambda Z^{(0,\lambda)}_{a+d}(x+d,\theta)}{\lambda-\psi(\theta)}\left(\Phi_\lambda \frac{Z_\lambda(x+d,\theta)}{Z^{(0,\lambda)}_{a+d}(x+d,\theta)} - \Phi_\lambda + \frac{(Z^{(0,\lambda)}_{a+d})'(x+d,\theta)}{Z^{(0,\lambda)}_{a+d}(x+d,\theta)} - \frac{Z'_\lambda(x+d,\theta)}{Z^{(0,\lambda)}_{a+d}(x+d,\theta)}\right)\\
&\quad + \frac{\lambda Z^{(0,\lambda)}_{a+d}(x+d,\theta)}{\lambda-\psi(\theta)}\left(\Phi_\lambda - \frac{(Z^{(0,\lambda)}_{a+d})'(x+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}\right)\\
&= \lambda Z_\lambda(x+d,\theta)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}(\mathrm{e}^{\theta d}-1)\\
&\quad + \frac{\lambda Z^{(0,\lambda)}_{a+d}(x+d,\theta)}{\lambda-\psi(\theta)}\left(\frac{(Z^{(0,\lambda)}_{a+d})'(x+d,\theta)}{Z^{(0,\lambda)}_{a+d}(x+d,\theta)} - \frac{(Z^{(0,\lambda)}_{a+d})'(x+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}\right). \qquad (75)
\end{aligned}
$$

It follows from (18), (74). and (75) that, for $x \in (a,\infty)$,

$$
\begin{aligned}
f_4(x;\theta,a) &= -\int_x^\infty g_4(z)\,\mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\,\mathrm{d}z\\
&= -\lambda(\mathrm{e}^{\theta d}-1)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}\int_x^\infty Z_\lambda(z+d,\theta)\frac{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(z+d,\Phi_\lambda)}\,\mathrm{d}z\\
&\quad -\frac{\lambda}{\lambda-\psi(\theta)}\int_x^\infty \left((Z^{(0,\lambda)}_{a+d})'(z+d,\theta) - h_4(z)Z^{(0,\lambda)}_{a+d}(z+d,\theta)\right)\mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\,\mathrm{d}z\\
&= -\lambda(\mathrm{e}^{\theta d}-1)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}\int_x^\infty Z_\lambda(z+d,\theta)\frac{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(z+d,\Phi_\lambda)}\,\mathrm{d}z\\
&\quad -\frac{\lambda}{\lambda-\psi(\theta)}\int_x^\infty \frac{\mathrm{d}}{\mathrm{d}z}\left(Z^{(0,\lambda)}_{a+d}(z+d,\theta)\mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\right)\mathrm{d}z\\
&= -\lambda(\mathrm{e}^{\theta d}-1)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}\int_x^\infty Z_\lambda(z+d,\theta)\frac{Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(z+d,\Phi_\lambda)}\,\mathrm{d}z\\
&\quad +\frac{\lambda}{\lambda-\psi(\theta)}\left(Z^{(0,\lambda)}_{a+d}(x+d,\theta) - Z^{(0,\lambda)}_{a+d}(x+d,\Phi_\lambda)\frac{\widetilde{Z}(a,\Phi_\lambda,\theta)}{\widetilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right).
\end{aligned}
$$

Furthermore, for $x \in [0,a]$, we have

$$
\begin{aligned}
f_4(x;\theta,z) &= (1-f_1(x;0,a))\,f_4(a;\theta,a) + f_1(x;\theta,a)\\
&= \frac{Z(x+d,\Phi_\lambda)}{Z(a+d,\Phi_\lambda)}\left(-\lambda(\mathrm{e}^{\theta d}-1)\frac{\Phi_\lambda-\theta}{\psi(\theta)-\lambda}\int_a^\infty Z_\lambda(z+d,\theta)\frac{Z^{(0,\lambda)}_{a+d}(a+d,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(z+d,\Phi_\lambda)}\,\mathrm{d}z\right.\\
&\quad \left. +\frac{\lambda}{\lambda-\psi(\theta)}\left(Z^{(0,\lambda)}_{a+d}(a+d,\theta) - Z^{(0,\lambda)}_{a+d}(a+d,\Phi_\lambda)\frac{\widetilde{Z}(a+d,\Phi_\lambda,\theta)}{\widetilde{Z}(a+d,\Phi_\lambda,\Phi_\lambda)}\right)\right)\\
&\quad +\frac{\lambda}{\lambda-\psi(\theta)}\left[Z(x+d,\theta) - Z(a+d,\theta)\frac{Z(x+d,\Phi_\lambda)}{Z(a+d,\Phi_\lambda)}\right].
\end{aligned}
$$

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Finally, from Theorem 3.2, we have

$$
\begin{aligned}
g_5(x) = & -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \int_{a\vee\xi(x)}^{x} \mathrm{e}^{-\theta(y-a)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& - \int_{\xi(x)\wedge a}^{a} \frac{\lambda^2}{\Phi_\lambda+\theta} \frac{Z(y-\xi(x),\,\Phi_\lambda)}{\widetilde{Z}(a-\xi(x),\,\Phi_\lambda,\,\Phi_\lambda)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& + \lambda \int_{\xi(x)\wedge a}^{a} \rho(y-\xi(x);\lambda,\, a-\xi(x),\, \overline{\xi}(x))\vartheta(\overline{\xi}(x);\theta,\, a-\xi(x)) \\
& \times \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
= & -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \int_{a}^{x} \mathrm{e}^{-\theta(y-a)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& - \int_{-d}^{a} \frac{\lambda^2}{\Phi_\lambda+\theta} \frac{Z(y+d,\,\Phi_\lambda)}{\widetilde{Z}(a+d,\,\Phi_\lambda,\,\Phi_\lambda)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& + \lambda \int_{-d}^{a} \rho(y+d;\lambda,\, a+d,\, x+d)\vartheta(x+d;\theta,\, a+d) \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
= & -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \int_{a}^{x} \mathrm{e}^{-\theta(y-a)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& - \int_{-d}^{a} \frac{\lambda^2}{\Phi_\lambda+\theta} \frac{Z(y+d,\,\Phi_\lambda)}{\widetilde{Z}(a+d,\,\Phi_\lambda,\,\Phi_\lambda)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& + \lambda \int_{-d}^{a} \frac{Z^{(0,\lambda)}_{a+d}(y+d,\,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\,\Phi_\lambda)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& \times \left(\frac{\lambda}{\theta+\Phi_\lambda} \frac{Z^{(0,\lambda)}_{a+d}(x+d,\,\Phi_\lambda)}{\widetilde{Z}(a+d,\,\Phi_\lambda,\,\Phi_\lambda)} - \lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \right) \\
= & -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \int_{a}^{x} \mathrm{e}^{-\theta(y-a)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& - \lambda^2 \int_{-d}^{a} \frac{Z(y+d,\,\Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\,\Phi_\lambda)} \left(W'_\lambda(x-y) - \Phi_\lambda W_\lambda(x-y)\right) \mathrm{d}y \\
& \times \left(\mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \right) \\
= & -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W'_\lambda(y)\mathrm{d}y + \lambda\Phi_\lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \\
& - \lambda^2 \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \int_0^{a+d} \frac{Z(y,\,\Phi_\lambda) W'_\lambda(x+d-y)}{Z^{(0,\lambda)}_{a+d}(x+d,\,\Phi_\lambda)} \mathrm{d}y \\
& + \lambda^2 \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \int_0^{a+d} \frac{\Phi_\lambda Z(y,\,\Phi_\lambda) W_\lambda(x+d-y)}{Z^{(0,\lambda)}_{a+d}(x+d,\,\Phi_\lambda)} \mathrm{d}y
\end{aligned}
$$

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$$= -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W'_\lambda(y)\mathrm{d}y + \lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y$$
$$\times \left(\frac{\Phi_\lambda Z^{(0,\lambda)}_{a+d}(x+d,\, \Phi_\lambda) - \lambda \int_0^{a+d} Z(y,\, \Phi_\lambda) W'_\lambda(x+d-y)\mathrm{d}y}{Z^{(0,\lambda)}_{a+d}(x+d,\, \Phi_\lambda)} \right)$$
$$= -\lambda W(0)\mathrm{e}^{-\theta(x-a)} - \lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W'_\lambda(y)\mathrm{d}y$$
$$+\lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \frac{(Z^{(0,\lambda)}_{a+d})'(x+d,\, \Phi_\lambda)}{Z^{(0,\lambda)}_{a+d}(x+d,\, \Phi_\lambda)},$$

which, together with (20), yields that for $x \in (a,\, \infty)$,

$$f_5(x;\theta,\, a) = -\int_x^\infty g_5(z)\, \mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\, \mathrm{d}z$$
$$= \lambda \int_x^\infty \left(W(0)\mathrm{e}^{-\theta(z-a)} + \mathrm{e}^{-\theta(z-a)} \int_0^{z-a} \mathrm{e}^{\theta y} W'_\lambda(y)\mathrm{d}y \right.$$
$$\left. -\mathrm{e}^{-\theta(z-a)} \int_0^{z-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y h_4(z) \right) \mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\, \mathrm{d}z$$
$$= \lambda \int_x^\infty \left(W_\lambda(z-a) - \theta \mathrm{e}^{-\theta(z-a)} \int_0^{z-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \right.$$
$$\left. -\mathrm{e}^{-\theta(z-a)} \int_0^{z-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y h_4(z) \right) \mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w}\, \mathrm{d}z$$
$$= \lambda \int_x^\infty \frac{\mathrm{d}}{\mathrm{d}z} \left(\mathrm{e}^{-\theta(z-a)} \int_0^{z-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y \mathrm{e}^{-\int_x^z h_4(w)\mathrm{d}w} \right) \mathrm{d}z$$
$$= \frac{\lambda}{\theta + \Phi_\lambda} \frac{Z^{(0,\lambda)}_{a+d}(x+d,\, \Phi_\lambda)}{\widetilde{Z}(a+d,\, \Phi_\lambda,\, \Phi_\lambda)} - \lambda \mathrm{e}^{-\theta(x-a)} \int_0^{x-a} \mathrm{e}^{\theta y} W_\lambda(y)\mathrm{d}y.$$

Hence, from (19), we have for $x \in [0,\, a]$

$$f_5(x;\theta,\, a) = (1 - f_1(x;0,\, a))\, f_5(a;\theta,\, a) = \frac{\lambda}{\theta + \Phi_\lambda} \frac{Z(x+d,\, \Phi_\lambda)}{\widetilde{Z}(a+d,\, \Phi_\lambda,\, \Phi_\lambda)}.$$

The proof is complete. □

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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