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# EFFECTIVE INSEPARABILITY, LATTICES, AND PREORDERING RELATIONS

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**Abstract.** We study effectively inseparable (abbreviated as e.i.) prelattices (i.e., structures of the form $L = \langle \omega, \wedge, \vee, 0, 1, \leq_L \rangle$ where $\omega$ denotes the set of natural numbers and the following four conditions hold: (1) $\wedge, \vee$ are binary computable operations; (2) $\leq_L$ is a computably enumerable preordering relation, with $0 \leq_L x \leq_L 1$ for every $x$; (3) the equivalence relation $\equiv_L$ originated by $\leq_L$ is a congruence on $L$ such that the corresponding quotient structure is a nontrivial bounded lattice; (4) the $\equiv_L$-equivalence classes of 0 and 1 form an effectively inseparable pair of sets). Solving a problem in (Montagna & Sorbi, 1985) we show (Theorem 4.2), that if $L$ is an e.i. prelattice then $\leq_L$ is universal with respect to all c.e. preordering relations, i.e., for every c.e. preordering relation $R$ there exists a computable function $f$ reducing $R$ to $\leq_L$, i.e., $x \; R \; y$ if and only if $f(x) \leq_L f(y)$, for all $x, y$. In fact (Corollary 5.3) $\leq_L$ is locally universal, i.e., for every pair $a <_L b$ and every c.e. preordering relation $R$ one can find a reducing function $f$ from $R$ to $\leq_L$ such that the range of $f$ is contained in the interval $\{x : a \leq_L x \leq_L b\}$. Also (Theorem 5.7) $\leq_L$ is uniformly dense, i.e., there exists a computable function $f$ such that for every $a, b$ if $a <_L b$ then $a <_L f(a, b) <_L b$, and if $a \equiv_L a'$ and $b \equiv_L b'$ then $f(a, b) \equiv_L f(a', b')$. Some consequences and applications of these results are discussed: in particular (Corollary 7.2) for $n \geq 1$ the c.e. preordering relation on $\Sigma_n$ sentences yielded by the relation of provable implication of any c.e. consistent extension of Robinson's system $R$ or $Q$ is locally universal and uniformly dense; and (Corollary 7.3) the c.e. preordering relation yielded by provable implication of any c.e. consistent extension of Heyting Arithmetic is locally universal and uniformly dense.

**§1. Introduction.** Given binary relations $R, S$ on the set $\omega$ of natural numbers, we say that $R$ is *reducible* to $S$ (notation: $R \preceq S$) if there is a computable function $f$ (called a *computable embedding* of $R$ into $S$) such that

$$(\forall x, y)[x \; R \; y \Leftrightarrow f(x) \; S \; f(y)].$$

Moreover, given a relation $R$ and a class $\mathcal{A}$ of relations on $\omega$, we say that $R$ is *$\mathcal{A}$-universal* if $R \in \mathcal{A}$ and $S \preceq R$ for every $S \in \mathcal{A}$.

An interesting case, to which a rapidly growing amount of literature is being devoted, is when one takes $\mathcal{A}$ to be a class of equivalence relations on $\omega$ in some fixed complexity

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measure: the notion of universality becomes in this case a useful tool to measure the computational complexity of classification problems in computable mathematics (problems which are in fact equivalence relations): for instance, it is shown in (Fokina, Friedman, Harizanov, Knight, McCoy, & Montalbán, 2012) that the isomorphism relation for various familiar classes of computable groups is $\Sigma^1_1$-universal. An even more restricted case is when one considers only computably enumerable (henceforth abbreviated as c.e.) equivalence relations (c.e. equivalence relations will be abbreviated as *ceers* throughout the article), see e.g., (Andrews, Lempp, Miller, Ng, San Mauro, & Sorbi, 2014). Interest in ceers is motivated by their importance in logic and algebra: for instance, modulo suitable Gödel numberings identifying the various objects with numbers, the relation of provable equivalence between formulas of a formal system, and word problems for finitely presented groups and semigroups, are ceers. When studying ceers, particular attention has been given to $\Sigma^0_1$-universal ceers: see for instance, (Andrews, Badaev, & Sorbi, 2017) for a survey.

It is very natural to move from equivalence relations to the wider class consisting of the preordering relations. Indeed, many interesting classification problems in computable mathematics occur as preordering relations (for instance, embedding problems, instead of isomorphism problems, for various classes of computable groups). Restriction to c.e. preordering relations (i.e., to preorderings $R$ such that the set $\{\langle x, y\rangle : x \ R \ y\}$ is c.e.) is again mostly motivated by their importance in logic: for instance, if one considers a strong enough formal system, then the relation of provable implication between formulas of a formal system is a $\Sigma^0_1$-universal preordering relation by (Montagna & Sorbi, 1985), i.e., it is c.e. and every c.e. preordering relation is reducible to it.

In this article we investigate some classes of $\Sigma^0_1$-universal (henceforth called simply *universal*) preordering relations. Our quest for interesting universal c.e. preordering relations is based on the notion of effective inseparability for lattices (or, better, prelattices, as we will call them in this article). In this regard, this article can be viewed as a generalization of some aspects of the work in (Montagna & Sorbi, 1985; Nies, 2000; Shavrukov, 2010), concerning effectively inseparable Boolean algebras.

To make more sense of this, we first need some definitions. In general, given a type $\tau = \langle \{f_i\}_{i\in I}, \{R_j\}_{j\in J}\rangle$ for operations and relations, then a *prestructure of type* $\tau$ is a structure $A = \langle \omega, \{f^A_i\}_{i\in I}, \{R^A_j\}_{j\in I}, \leq_A\rangle$ with operations and relations interpreting the type, such that $\leq_A$ is a preordering and the equivalence relation $\equiv_A$ induced by $\leq_A$ (i.e., $x \equiv_A y$ if $x \leq_A y$ and $y \leq_A y$) is a congruence for the operations and the relations. If in addition operations and relations are finitary, $\leq_A$ is c.e., each operation $f^A_i$ (interpreting $f_i$) is computable uniformly in $i$, and each relation $R^A_j$ (interpreting $R_j$) is c.e. uniformly in $j$, then we say that the prestructure is a *c.e. prestructure of type* $\tau$.

DEFINITION 1.1. *If $A$ is a prestructure then the quotient $A_{/\equiv_A}$ will be called the* quotient structure associated with $A$. (*Notice that $A_{/\equiv_A}$ is partially ordered by the partial ordering relation on $\equiv_A$-equivalence classes originated by $\leq_A$.*)

For instance, a preordering relation can be just viewed as a prestructure $R = \langle \omega, \leq_R\rangle$ (called a *preorder*) with empty type, whose associated quotient structure is a partial order. It will be customary in the following to identify preordering relations with their corresponding preorders, so that the terms "preordering relation" and "preorder" will be treated as synonymous.

DEFINITION 1.2. *A* c.e. prelattice *is a c.e. prestructure $L = \langle \omega, \wedge, \vee, \leq_L\rangle$ such that the associated quotient structure is a lattice* (*with $\wedge$ and $\vee$ inducing in the quotient the operations of meet and join, respectively*).

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Having in mind this definition, it is now clear what *c.e. Boolean prealgebras*, *c.e. distributive prelattices*, *c.e. Heyting prealgebras*, *c.e. presemi-lattices*, etc., are. In particular a *c.e. bounded prelattice* is a c.e. prestructure $L = \langle \omega, \wedge, \vee, 0, 1, \leq_L \rangle$ such that the associated quotient structure is a bounded lattice for which the $\equiv_L$-equivalence classes $[0]_L$ and $[1]_L$, of 0 and 1, respectively, are the least element and the greatest element. (It is clear that up to a computable permutation of $\omega$ we may assume that in a c.e. bounded prelattice $L$ one has that $0 \leq_L x \leq_L 1$ for every $x$, so that our choice of 0 and 1 as representatives of the least element and the greatest element, respectively, is no loss of generality.)

DEFINITION 1.3. *By a* computable isomorphism *between prestructures $A, B$ we mean a computable function $f$ which reduces $\leq_A \preceq \leq_B$, and the assignment on equivalence classes $[x]_A \mapsto [f(x)]_B$ is a well defined isomorphism between the corresponding quotient structures of A and B.*

REMARK 1.4. *Our terminology follows largely* (*Montagna & Sorbi, 1985*) *except for the fact that what we call here "c.e. prestructures" are therein called* positive prestructures. *In the Russian literature, a* positively numbered structure *is a pair $\langle A^-, \nu \rangle$, where $A^-$ has universe $\omega$ with computable operations and c.e. relations such that $\nu : \omega \to A$ is a surjective function and the relation "$\nu(x) = \nu(y)$" is a c.e. equivalence relation inducing a congruence on $A^-$. So, what we call in this article "c.e. prelattices," "c.e. Boolean prealgebras" etc., in the Russian literature would be certain "positively numbered lattices," "positively numbered Boolean algebras," etc. Our "e.i. Boolean prealgebras" are e.i. Boolean algebras in the sense of* (*Nerode & Remmel, 1985; Shavrukov, 2010*). *See also* (*Fokina, Khoussainov, Semukhin, & Turetsky, 2016; Gavryushkin, Jain, Khoussainov, & Stephan, 2014; Gavryushkin, Khoussainov, & Stephan, 2016; Selivanov, 2003*) *for an approach to c.e structures similar to the one taken in this article.*

REMARK 1.5. *One could give a seemingly more general definition of a c.e. prestructure of type $\tau$, by asking that the universe of the structure be just a c.e. set instead of $\omega$. Up to computable isomorphisms, nothing is however gained in generality by doing so. Let us briefly give a justification of this claim in the case of c.e. prelattices. Let $L = \langle U \wedge, \vee, \leq \rangle$ be a c.e. prelattice, where U is a c.e. set. Let $h : \omega \longrightarrow U$ be a computable surjection. This induces a c.e. prelattice $L' = \langle \omega, \wedge', \vee', \leq_{L'} \rangle$ where $x \leq_{L'} y$ if and only if $h(x) \leq_L h(y)$, and* (*letting $h^{-1}$ be any partial computable function such that $h(h^{-1}(x)) = x$ for every $x \in U$*) *$x \wedge' y = h^{-1}(h(x) \wedge h(y))$, $x \vee' y = h^{-1}(h(x) \vee h(y))$. The assignment on equivalence classes $[x]_A \mapsto [h(x)]_B$ is clearly a well-defined isomorphism between the corresponding quotient structures of A and B. Notice that if U is infinite, then we can take h to be a computable permutation of $\omega$. On the other hand, if all equivalence classes of $\equiv_L$ are infinite then from any computable isomorphism we can effectively go* (*using a standard back-and-forth argument as in the classical Myhill isomorphism theorem*) *to a computable isomorphism induced by a computable permutation.*

The following notion from computability theory will play a fundamental role in this article. We recall that a disjoint pair $(A, B)$ of sets of numbers is called an *effectively inseparable* (or simply, *e.i.*) pair if there exists a partial computable function $\varphi(u, v)$ (called a *productive function* for the pair) such that

$$(\forall u, v)[A \subseteq W_u \,\&\, B \subseteq W_v \,\&\, W_u \cap W_v = \emptyset \Rightarrow \varphi(u, v) \downarrow \notin W_u \cup W_v].$$

It is known, see, e.g., (Soare, 1987, p. 44), that every e.i. pair of c.e. sets has a total productive function: in fact, one can uniformly go from any index of a productive function

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to an index of a total productive function. Also if $(A_0, A_1)$, $(B_0, B_1)$ are disjoint pairs of c.e. sets, with $(A_0, A_1)$ e.i., and either $(A_0, B_1) \subseteq (B_0, B_1)$ (meaning $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$) or $(A_0, A_1) \leq_m (B_0, B_1)$ (meaning that there is a computable function simultaneously $m$-reducing $A_0 \leq_m B_0$ and $A_1 \leq_m B_1$), then $(B_0, B_1)$ is e.i. as well.

DEFINITION 1.6. *An* effectively inseparable (*or* e.i.) prelattice *is a c.e. bounded prelattice* $L = \langle \omega, \wedge, \vee, 0, 1 \leq_L \rangle$ *such that the pair of equivalence classes* $([0]_L, [1]_L)$ *is e.i.* (*The definition of course specializes to c.e. Boolean prealgebras* (*defining* e.i. Boolean prealgebras) *and c.e. Heyting prealgebras* (*defining* e.i. Heyting prealgebras)*, which are examples of c.e. bounded prelattices.*)

It is shown in (Montagna & Sorbi, 1985) (based on (Pour-El & Kripke, 1967); see also (Ianovski, Miller, Nies, & Ng, 2014)) that if $B$ is an e.i. Boolean prealgebra, and $\leq_B$ is the c.e. preordering relation relative to $B$, then $\leq_B$ is universal. Ianovski *et al.*, (2014) have extended this result throughout the arithmetical hierarchy by showing that for every $n \geq 1$, the preordering relation of a $\Sigma^0_n$-effectively inseparable Boolean prealgebra (therein called a $\Sigma^0_n$-*effectively inseparable Boolean algebra*) is a $\Sigma^0_n$-universal preordering relation.

Moving in this article from e.i. Boolean prealgebras to e.i. prelattices, we show that the preordering relation of any e.i. prelattice is universal. In fact, we show that every c.e. preorder can be computably embedded in any nonempty interval of any e.i. prelattice (this property will be called *local universality*: see Corollary 5.3). Moreover every e.i. prelattice $L$ is uniformly dense (Theorem 5.7), meaning that there is a computable function $f(a, b)$ such that for every $a, b$ if $a <_L b$ then $a <_L f(a, b) <_L b$, and if $a \equiv_L a'$ and $b \equiv_L b'$, then $f(a, b) \equiv_L f(a', b')$. Remarkably, these results on universality, local universality and uniform density, do not use distributivity (on the other hand, Example 3.16 shows that e.i. nondistributive prelattices do exist), but do not hold in general if we move from e.i. prelattices to e.i. pre-semi-lattices, i.e., c.e. bounded pre-semi-lattices $S = \langle \omega, \vee, 0, 1, \leq_S \rangle$ such that the pair $([0]_S, [1]_S)$ is e.i. (in Observation 4.7 we exhibit an e.i. upper pre-semi-lattice whose preordering relation is not universal). We also point out (Theorems 8.2 and 8.3) several distinct natural computable isomorphism types of e.i. prelattices.

The greater generality of considering prelattices instead of Boolean prealgebras enables us to extend to c.e. consistent extensions of very weak systems of arithmetic such as Robinson's systems $Q$ or $R$ some of the results in (Shavrukov & Visser, 2014) stated for c.e. consistent of Elementary Arithmetic $EA$ (see Corollary 7.2): in particular for $n \geq 1$ the c.e. preordering relation on $\Sigma_n$ sentences yielded by the relation of provable implication of any c.e. consistent extension of Robinson's $Q$ or $R$ is locally universal and uniformly dense.

Since our results apply to every e.i. lattice, they hold of e.i. Heyting prealgebras as well (for instance, the Heyting prealgebra of the Lindenbaum sentence algebra of any c.e. consistent intuitionistic extension of Heyting Arithmetic, as defined in Definition 3.11) to which one can therefore extend the above mentioned results showing, for instance (Corollary 7.3) that the c.e. preordering relation of provable implication of any c.e. consistent extension of Heyting Arithmetic is locally universal (we believe that this was previously unknown, or at least we have not found any clear reference to it in the literature), and (previously unknown) uniformly dense.

In contrast with what happens for e.i. Boolean prealgebras which are universal with respect to the class of all c.e. Boolean prealgebras (a c.e. Boolean prealgebra $B$ is *universal*

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with respect to the class of all c.e. Boolean prealgebras if for every c.e. Boolean prealgebra $B'$ one has that $\leq_{B'} \preceq \leq_B$ via a reduction $f$ which induces a monomorphism between the associated quotient structures) we show that, although guaranteeing universality of its preordering relation, being an e.i. prelattice does not suffice to guarantee universality with respect to the class of all c.e. prelattices: Theorem 6.5 provides an e.i. distributive prelattice such that not all c.e. distributive prelattices reduce to it via a reduction inducing a monomorphisms between the corresponding quotient structures.

***1.1. Further notations and terminology.*** If $R = \langle \omega, \leq_R \rangle$ is a preorder then we use the following notations. We write $x <_R y$ for $x \leq_R y$ but $y \not\leq_R x$; the symbol $|_R$ denotes $\leq_R$-incomparability, i.e., $x |_R y$ if $x \not\leq_R y$ and $y \not\leq_R x$; if $a \in \omega$ and $X \subseteq \omega$ then we write $a \leq_R X$ if $a \leq_R x$, for all $x \in X$ (define $a <_R X$, $X \leq_R a$, and $X <_R a$ in a similar way). We also use interval notation, i.e., $[x, y]_{\leq_R} = \{a : x \leq_R a \leq_R y\}$, and other possible interval variations such as for instance $(x, y)_{\leq_R}$.

In some of the proofs, we will make use of infinite computable lists of fixed points as given by the Recursion Theorem. The use of these infinite lists can be formally justified as follows: we fix a single index $j$ so that we control $\varphi_j$ by the Recursion Theorem, and we then take a computable list $(j_i)_{i\in\omega}$ of indices for the columns $\varphi_{j_i}(k) = \varphi_j(\langle i, k\rangle)$. We can then control the $\varphi_{j_i}$ in any order we wish, as we are simply controlling the single function $\varphi_j$. An alternative formal justification to arguments using infinite lists of fixed points is also provided by the Case Functional Recursion Theorem (Case, 1974): see also (Odifreddi, 1999) for useful comments about this theorem.

All preorders considered in this article are assumed to be nontrivial, i.e., the associated quotient structures do not collapse to a single element.

## §2. Ceers, preorders and effective inseparability.

From the theory of ceers we recall the following definitions (see, e.g., (Andrews *et al.*, 2014)). Particularly important for us is "uniform finite precompleteness," a notion due to Montagna (1982), and widely used in (Bernardi & Montagna, 1984; Shavrukov, 1996).

DEFINITION 2.1. *Let $R$ be a nontrivial ceer* (*i.e., $R$ is a ceer with at least two equivalence classes*):

- *$R$ is* effectively inseparable (*abbreviated as* e.i.) *if every pair of distinct equivalence classes is e.i.*;
- *$R$ is* uniformly effectively inseparable (*abbreviated as* u.e.i.) *if there exists a partial computable function $\chi(x, y, u, v)$ such that if $x \not R y$ then the partial computable function $\psi(u, v) = \chi(x, y, u, v)$ is a productive function for the pair of equivalence classes $([x]_R, [y]_R)$*;
- *$R$ is* precomplete *if there exists a computable function $f(e, x)$ such that for all $e, x$ if $\varphi_e(x) \downarrow$ then $f(e, x)\ R\ \varphi_e(x)$*;
- *$R$ is* uniformly finitely precomplete (*abbreviated as* u.f.p.) *if there exists a computable function $k(D, e, x)$ (called a* totalizer*, where $D$ is a finite set given by its canonical index) such that for all $D, e, x$ if $\varphi_e(x)$ converges to a number $R$-equivalent to some $d \in D$ then $k(D, e, x)\ R\ \varphi_e(x)$.*

REMARK 2.2. *Notice that since for pairs of c.e. sets it is possible to uniformly go from productive functions to total productive functions (see, e.g., (Soare, 1987, p. 44)), it is easy to see that it is possible for ceers to uniformly go from uniform productive functions witnessing u.e.i.-ness to total uniform productive functions witnessing u.e.i.-ness. Thus we*

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*can assume in the following that our productive functions witnessing u.e.i.-ness for ceers are in fact total.*

FACT 2.3. *For ceers we have*: *precomplete* $\Rightarrow$ *u.f.p.* $\Rightarrow$ *u.e.i.* $\Rightarrow$ *e.i.. It is not known whether the implication u.e.i.* $\Rightarrow$ *u.f.p. holds*; *the other implications are strict. Every u.e.i. ceer R is* $\Sigma^0_1$*-universal as an equivalence relation, i.e., every ceer can be reduced to it.*

*Proof.* For a proof see (Andrews *et al.*, 2014). The fact that precomplete ceers are u.e.i. was first noticed in (Visser, 1980). □

Via the equivalence relation induced by a preordering relation we export Definition 2.1 to preorders and prestructures, as follows.

DEFINITION 2.4. *A c.e. preorder* $R = \langle \omega, \leq_R \rangle$ *is called* effectively inseparable*,* (*respectively*: uniformly effectively inseparable*,* precomplete*,* uniformly finitely precomplete)*, if so is the associated ceer* $\equiv_R$*.*

REMARK 2.5. *A warning on the terminology introduced in the previous definition is in order*: *to be meticulous, one could find the definition of an e.i. prelattice not coherent with Definition 2.4, for which an e.i. preorder is a c.e. preorder in which all pairs of distinct equivalence classes are e.i., whereas for e.i. prelattices we only require, following* (*Montagna & Sorbi, 1985*)*, effective inseparability of the equivalence classes of* 0, 1*. It will turn out in any case* (*Theorem 4.1*) *that e.i. prelattices are also e.i. preorders in the sense of Definition 2.4, in fact they are u.e.i.*

The following observation shows a simple application of effective inseparability to c.e. preorders and c.e. prelattices.

OBSERVATION 2.6. *If* $R = \langle \omega, \leq_R \rangle$ *is a c.e. preorder so that* $([u]_R, [v]_R)$*, with* $u <_R v$*, is an e.i. pair of sets then for every nonempty c.e. set X one can find* (*uniformly from u, v and a c.e. index of X*) *an element y such that if* $u <_R X <_R v$ *then* $y|_R X$*. If in addition R is a c.e. prelattice then we can find y as above such that if* $u <_R X <_R v$ *then* $y|_R X$ *and* $u <_R y <_R v$*.*

*Proof.* Assume that $u, v, X$ are as in the statement of the observation. Let $p$ be a productive function for the pair $([u]_R, [v]_R)$ of e.i. sets. Let

$$U = \{y : (\exists x \in X)[y \leq_R x]\} \quad \text{and} \quad V = \{y : (\exists x \in X)[y \geq_R x]\},$$

(thus $u \in U$ and $v \in V$) and by the Reduction Principle (see (Rogers, 1967)) let $U' \subseteq U$, $V' \subseteq V$ be c.e. sets such that $U' \cap V' = \emptyset$ and $U \cup V = U' \cup V'$. If $u <_R X <_R v$ then clearly $[u]_R \subseteq U'$ and $[v]_R \subseteq V'$. If $u', v'$ are c.e. indices of $U', V'$ then $p(u', v') \notin U \cup V$ and thus $p(u', v')|_R x$ with all $x \in X$. To show uniformity, use the fact that $u', v'$ can be uniformly found starting from $u, v$, and a c.e. index of $X$.

If instead of just a c.e. preorder we start with a c.e. prelattice $R = \langle \omega, \wedge, \vee, \leq_R \rangle$ then define

$$U = \{y : (\exists x \in X)[(u \vee y) \wedge v \leq_R x]\} \quad \text{and} \quad V = \{y : (\exists x \in X)[(u \vee y) \wedge v \geq_R x]\}.$$

Arguing as above, by the Reduction Principle get from $U, V$ disjoint c.e. sets $U', V'$ with indices $u', v'$ such that $U \cup V = U' \cup V'$: then it is easy to see that $y = (u \vee p(u', v')) \wedge v$ satisfies the claim. □

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**§3. Examples of e.i. prelattices.** In this section we point out several examples of e.i. prelattices, many of them coming from the already existing literature.

DEFINITION 3.1. *If $S$ is a c.e. set $S$ of formulas of a first-order language $L$, by a* Gödel numbering of $S$ *we will mean* any *computable* 1-1 *function from $S$ to $\omega$.*

***3.1. E.i. Boolean prealgebras.*** E.i. Boolean prealgebras appear very naturally in logic.

EXAMPLE 3.2. *Consider any c.e. consistent extension $T$ of Robinson's system $R$ or $Q$ for first-order arithmetic* (*as presented for instance in Definitions III.6.10 and III.8.2, respectively, of* (*Smoryński, 1991*)), *with the usual first order language $L$ for arithmetic*: *we assume these systems to be consistent. Our choice of $R$ is motivated by the fact that we would like to be as general as possible, and indeed $R$ is often recognized as a prototypical example of a weak theory of arithmetic, and although $Q$ interprets $R$* (*see, e.g.,* (*Smoryński, 1991, Theorem III.8.3*)) *we explicitly mention $Q$ as well as an example of a weak finitely axiomatizable theory.*

*Let us fix a Gödel numbering $\gamma$ of Sent($L$)* (*all sentences of $L$*) onto $\omega$ *so that* $\gamma(\neg 0 = 0) = 0$ *and* $\gamma(0 = 0) = 1$.

DEFINITION 3.3. *The* Boolean prealgebra of $T$ relatively to $\gamma$ *is the Boolean prealgebra*

$$B_T^\gamma = \langle \omega, \wedge^\gamma, \vee^\gamma, \neg^\gamma, 0, 1, \leq_T^\gamma \rangle,$$

*where $\wedge^\gamma, \vee^\gamma, \neg^\gamma$ are* (*via $\gamma$*) *the usual propositional connectives* (*i.e., $x \wedge^\gamma y = \gamma(\gamma^{-1}(x) \wedge \gamma^{-1}(x))$, etc.*), *and $x \leq_T^\gamma y$ if $\vdash_T \gamma^{-1}(x) \to \gamma^{-1}(y)$.*

REMARK 3.4. *It is easy to see that up to computable isomorphisms induced by computable permutations of $\omega$, the Boolean prealgebra $B_T^\gamma$ does not depend on the chosen $\gamma$, so it is fair to refer to $B_T^\gamma$ as just*

$$B_T = \langle \omega, \wedge, \vee, \neg, 0, 1, \leq_T \rangle,$$

*without mentioning, unless when really needed, the chosen Gödel numbering. Notice that $B_T$ is not trivial by consistency of $T$.*

THEOREM 3.5. *$B_T$ is an e.i. Boolean prealgebra.*

*Proof.* This immediately follows from the well known fact (see, e.g., (Smoryński, 1991)) that the sets of theorems and anti-theorems of $T$ (where $\sigma$ is an *anti-theorem* if $\neg\sigma$ is a theorem) form an e.i. pair of c.e. sets. We give however a somewhat more detailed proof which will be exploited again for Theorem 3.10 below. Fix a $\gamma$ as above such that $B_T = B_T^\gamma$, and let $(A_0, A_1)$ be a disjoint pair of c.e. sets which are e.i. Consider (see, for instance (Smoryński, 1991, Corollary III.6.20)) two strict $\Sigma_1$ formulas $H_0(v), H_1(v)$ (i.e., each $H_i(v)$ is of the form $(\exists w)\psi_i(w, v)$ with $\psi_i$ containing no quantifiers or only quantifiers which are bounded by variables) such that $A_i = \{m : \mathbb{N} \models H_i(\overline{m})\}$ (where $\overline{m}$ is the numeral term corresponding to the number $m$) and let $H(v)$ be the formula $(\exists w)(\psi_0(w, v) \wedge (\forall w' \leq w)\neg\psi_1(w', v))$. Using (Smoryński, 1991, Lemma III.6.27) one can show that if $m \in A_0$ then $\vdash_U H(\overline{m})$ and if $m \in A_1$ then $\vdash_U \neg H(\overline{m})$, where $U$ is either $R$ or $Q$. Let now $(X_0, X_1)$ be the disjoint pair of c.e. sets $X_0 = \{m : \vdash_T \neg H(\overline{m})\}$ and $X_1 = \{m : \vdash_T H(\overline{m})\}$: since $(X_0, X_1) \supseteq (A_1, A_0)$ we have that $(X_0, X_1)$ is e.i.: on the other hand, the computable function $m \mapsto \gamma(H(\overline{m}))$ gives a 1-reduction of $(X_0, X_1)$ to the pair $([0]_T, [1]_T)$ (where $[x]_T$ denotes the equivalence class of a number $x$ under the equivalence relation induced by $\leq_T$): therefore the pair $([0]_T, [1]_T)$ is e.i. too, and thus $B_T$ is an e.i. Boolean prealgebra. □

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Moreover, the associated quotient structure of $B_T$ is isomorphic to the Lindenbaum sentence algebra of $T$, which leads us to the following definition.

DEFINITION 3.6. *$B_T$ as above is called the* Boolean prealgebra of the Lindenbaum sentence algebra of $T$.

For introductions to e.i. Boolean prealgebras and related topics, see (listed in chronological order) (Montagna & Sorbi, 1985; Nerode & Remmel, 1985; Nies, 2000; Shavrukov, 2010).

***3.2. C.e. precomplete prelattices.*** If $L$ is a c.e. precomplete bounded prelattice, then $\equiv_L$ is a precomplete ceer and thus by Fact 2.3 yields a partition of $\omega$ such that all pairs of distinct equivalence classes are e.i. Therefore $L$ is an e.i. prelattice.

***3.3. C.e. prelattices of sentences.*** Let us consider a decidable first-order language $L$ and a c.e. consistent theory $T$ on $L$, and let $\mathcal{C} \subseteq \mathrm{Sent}(L)$ be an infinite c.e. set closed under the connectives $\wedge$ and $\vee$, and intersecting both the set of theorems and the set of anti-theorems of $T$. Fix a Gödel numbering $\gamma$ of $\mathcal{C}$ onto $\omega$ for which 0 and 1 are the Gödel numbers of a theorem and an anti-theorem, respectively.

DEFINITION 3.7. *The* prelattice of $\mathcal{C}$-sentences relatively to $\gamma$ *is the c.e. bounded prelattice*

$$L^{\gamma}_{\mathcal{C}/T} = \langle \omega, \wedge^{\gamma}, \vee^{\gamma}, 0, 1, \leq^{\gamma}_{\mathcal{C}/T} \rangle,$$

*where $x \leq^{\gamma}_{\mathcal{C}/T} y$ if and only if $\vdash_T \gamma^{-1}(x) \to \gamma^{-1}(y)$, and the operations are given (via $\gamma$) by the propositional connectives. A* prelattice of sentences *is a prelattice of the form $L^{\gamma}_{\mathcal{C}/T}$, for some choice of $T$, $\mathcal{C}$, and $\gamma$.*

Once again it is immediate to see that up to computable isomorphisms induced by computable permutations of $\omega$ the definition does not depend on the choice of $\gamma$, and thus, unless when strictly necessary for definiteness, reference to the chosen $\gamma$ will always be omitted in the following.

Also, up to a computable isomorphism provided by a computable permutation of $\omega$ it does not matter whether we consider $\mathcal{C}$ or its closure under provable equivalence in $T$: indeed, suppose that $\mathcal{C}$ is a set of sentences providing the prelattice of sentences $L_{\mathcal{C}/T}$ (relatively, say, to $\gamma$), let

$$[\mathcal{C}]_T = \{\sigma \in \mathrm{Sent}(L) : (\exists \tau \in \mathcal{C})[\vdash_T \tau \leftrightarrow \sigma]\},$$

and let us consider the prelattice of sentences $L_{[\mathcal{C}]_T/T}$ (relatively say to $\delta$). It is easy to see that $h = \delta \circ \gamma^{-1}$ is a computable function satisfying for every $x, y$,

$$x \leq_{\mathcal{C}/T} y \Leftrightarrow h(x) \leq_{[\mathcal{C}]_T/T} h(y),$$

and for every $y$ there exists $x$ such that $h(x) \equiv_{[\mathcal{C}]_T/T} y$. By a standard back-and-forth argument, and using that the equivalence classes of both $\equiv_{\mathcal{C}/T}$ and $\equiv_{[\mathcal{C}]_T/T}$ are infinite, it is easy to see that from $h$ one can construct a computable permutation $f$ of $\omega$ inducing an isomorphism of $L_{\mathcal{C}/T}$ with $L_{[\mathcal{C}]_T/T}$: At stage 0 we define $f_0 = \emptyset$. At stage $2s+1$, if $s \in \mathrm{dom}(f_{2s})$ then let $f_{2s+1} = f_{2s}$; otherwise effectively search for the first $y$ for which $h(s) \equiv_{[\mathcal{C}]_T/T} y$ and $y \notin \mathrm{range}(f_{2s})$, and define $f_{2s+1} = f_{2s} \cup \{(s, y)\}$. At stage $2s+2$, if $s \in \mathrm{range}(f_{2s+1})$ then let $f_{2s+2} = f_{2s+1}$; otherwise effectively search for the first $x$ for which $h(x) \equiv_{[\mathcal{C}]_T/T} s$, and $x \notin \mathrm{dom}(f_{2s+1})$, and define $f_{2s+1} = f_{2s} \cup \{(x, s)\}$.

Prelattices of sentences for c.e. arithmetical first-order theories $T$ have been extensively studied by Shavrukov and Visser (see, for instance (Shavrukov, 2010) and (Shavrukov

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& Visser, 2014)), in particular taking $\mathcal{C}$ to be the set of $\Sigma_n$-sentences for some $n \geq 1$. The reader is referred to (Smoryński, 1991) for a clear introduction to the classes $\Sigma_n$: the definition of $\Sigma_n$ is purely syntactic (and in particular each class is decidable), depending only on the language and not on any specific theory $T$, but our previous observation shows that whether we take $\Sigma_n$ or its closure under provable equivalence in a chosen $T$ we get prelattices that are isomorphic via an isomorphism provided by a computable permutation of $\omega$ as long as in either prelattice the preordering relation is provided by provable implication in $T$.

It is easy to see that the quotient structure associated with $L_{\Sigma_n/T}$ is isomorphic to the Lindenbaum lattice of $\Sigma_n$-sentences of $T$, i.e., the bounded sublattice of the Lindenbaum sentence algebra of $T$, restricted to the equivalence classes of $\Sigma_n$-sentences. This motivates the following definition.

DEFINITION 3.8. *$L_{\Sigma_n/T}$ will be called the* prelattice of the Lindenbaum lattice of $\Sigma_n$-sentences of $T$.

For the following theorem (where *EA*, called Elementary Arithmetic, is the theory $I\Delta_0 +$ Exp) see also (Shavrukov & Visser, 2014, Example 4.1).

THEOREM 3.9 (Visser, 1980). *If $T$ is a c.e. consistent extension of elementary arithmetic EA then, for $n \geq 1$, $L_{\Sigma_n/T}$ is precomplete.*

*Proof.* If $\ulcorner\ \urcorner$ is a conventional Gödel numbering encoding the syntactic objects and formulas of $L$ as normally used in the arithmetization of metamathematics (representability of partial computable functions, c.e. sets, etc.: see e.g., (Smoryński, 1991) or (Hajek & Pudlak, 1998) for details), and $\varphi$ is a partial computable function then from (Visser, 1980) we can see that there exists a computable function $f_\varphi$ which always land at $\ulcorner\ \urcorner$-numbers of $\Sigma_n$ sentences, and if $\varphi(x) \downarrow = \ulcorner\sigma\urcorner$, with $\sigma \in \Sigma_n$, then $f_\varphi(x) = \ulcorner\tau\urcorner$ with $\vdash_T \sigma \leftrightarrow \tau$. Let $\gamma$ be a Gödel numbering of the $\Sigma_n$-sentences onto $\omega$ so that $L_{\Sigma_n/T} = L^{\gamma}_{\Sigma_n/T}$. Now, given a partial computable function $\varphi$, consider the partial computable function $\psi$ with the same domain as $\varphi$ and $\psi(x) = \ulcorner\sigma\urcorner$ if $\sigma = \gamma^{-1}(\varphi(x))$, and let $g = \gamma(\tau)$ if $f_\psi(x) = \ulcorner\tau\urcorner$: it is immediate to see that $g$ makes $\varphi$ total modulo $\equiv^{\gamma}_T$, showing that $\equiv^{\gamma}_T$ is precomplete. □

As every c.e. bounded precomplete lattice is e.i., it follows that if $T$ is a c.e. consistent extension of elementary arithmetic *EA* then $L_{\Sigma_n/T}$ is an e.i. prelattice. For Robinson's weaker systems $Q$ and $R$, it is not known if $L_{\Sigma_n/T}$ is precomplete. The following theorem shows however that we still have an e.i. prelattice.

THEOREM 3.10. *If $T$ is a c.e. consistent extension of $Q$ or $R$ then $L_{\Sigma_n/T}$ is an e.i. prelattice.*

*Proof.* Let $n \geq 1$ be given, and fix a suitable Gödel numbering $\gamma$ of the $\Sigma_n$-sentences onto $\omega$ so that $L_{\Sigma_n/T} = L^{\gamma}_{\Sigma_n/T}$. By consistency of $T$, $L_{\Sigma_n/T}$ is not trivial. Notice that the formula $H(v)$ used in the proof of Theorem 3.5 is $\Sigma_1$: therefore, if $(X_0, X_1)$ is the e.i. pair of c.e. sets as in that proof then the computable function $m \mapsto \gamma(H(\overline{m}))$ gives a 1-reduction of $(X_0, X_1)$ to the pair $([0]_{\Sigma_n/T}, [1]_{\Sigma_n/T})$ which is therefore e.i. □

***3.4. E.i. Heyting prealgebras.*** Examples of e.i. Heyting prealgebras which are not Boolean prealgebras come from intuitionistic logic. For instance, Let $iT$ be any c.e. consistent intuitionistic extension of Heyting Arithmetic *HA* in the usual arithmetical first order language $L$, and, having fixed a Gödel numbering $\gamma$ from Sent($L$) onto $\omega$ with $\gamma(\neg 0 = 0) = 0$ and $\gamma(0 = 0) = 1$, let $\mathrm{iT}^{\gamma} = \langle \omega, \wedge^{\gamma}, \vee^{\gamma}, \rightarrow^{\gamma}, \neg^{\gamma}, 0, 1, \leq^{\gamma}_{iT} \rangle$ be the c.e. Heyting prealgebra where the operations are the connectives (via Gödel numbers given

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by $\gamma$) and $x \leq_{iT}^{\gamma} y$ if and only if $\vdash_{iT} \gamma^{-1}(x) \rightarrow \gamma^{-1}(y)$. Once again, up to computable isomorphisms induced by computable permutations of $\omega$, the definition is independent of $\gamma$, so the superscript $\gamma$ will be omitted unless when strictly necessary for a clearer understanding of the text. As the Lindenbaum sentence algebra of $iT$ is isomorphic with the quotient structure of iT it is fair to give the following definition.

DEFINITION 3.11. *The Heyting prealgebra* iT *is called the* Heyting prealgebra of the Lindenbaum sentence algebra of $iT$.

By an argument similar to the proof of Theorem 3.5, using similar arithmetization tools (available in Heyting Arithmetic), one can show effective inseparability of iT. We choose however a different argument, based on the Gödel–Gentzen double-negation translation, which we particularly like for its simplicity and clarity.

LEMMA 3.12. iT *is an e.i. Heyting prealgebra.*

*Proof.* Let $PA$ denote Peano Arithmetic, and let $B_{PA}$ be the Boolean prealgebra of the Lindenbaum sentence algebra of $PA$. The Gödel–Gentzen double-negation translation (see e.g., (Avigad & Feferman, 1998) for mathematical and historical details) ensures that there is a computable mapping $G$ transforming each arithmetical sentence $\sigma$ into an arithmetical sentence $G(\sigma)$ such that $\vdash_{PA} \sigma$ if and only if $\vdash_{HA} G(\sigma)$. Moreover $G(\neg\sigma) = \neg G(\sigma)$, thus $\vdash_{PA} \neg\sigma$ if and only if $\vdash_{HA} \neg G(\sigma)$ as well. From this, one easily extracts a 1-reduction $([0]_{PA}, [1]_{PA}) \leq_1 ([0]_{HA}, [1]_{HA})$ of disjoint pairs of sets, implying that $([0]_{HA}, [1]_{HA})$ is e.i. as so is $([0]_{PA}, [1]_{PA})$: therefore the pair $([0]_{iT}, [1]_{iT})$ is e.i. too as it contains the pair $([0]_{HA}, [1]_{HA})$. □

REMARK 3.13. *There is of course a much wider range of examples of e.i. Heyting prealgebras than the ones covered by Lemma 3.12*: *we expect that forthcoming work will address the issue of examples of e.i. Heyting prealgebras arising from weak intuitionistic fragments of arithmetic, for which one cannot use the Gödel–Gentzen translation.*

***3.5. E.i. free distributive prelattices.*** Useful examples (and also counterexamples) of e.i. prestructures come from free prestructures.

DEFINITION 3.14. *Define a prestructure to be* free on a countably infinite set of generators *if the corresponding associated quotient structure is free on a countably infinite set of generators.*

Consider a computable presentation of the form $F^d_{0,1}(X) = \langle \omega, \wedge, \vee, 0, 1\rangle$ of the free bounded distributive lattice on a decidable infinite set $X$ (with $0, 1 \notin X$)) which is computably listed without repetitions by $\{x_i : i \in \omega\}$. Fix a pair $(U, V)$ of e.i. c.e. sets, and let $\alpha$ be the c.e. congruence on $F^d_{0,1}(X)$ generated by the set of pairs $\{(x_i, 0) : i \in U\} \cup \{(x_i, 1) : i \in V\}$. (This argument to encode effective inseparability in a c.e. quotient of $F^d_{0,1}(X)$ has been suggested in private communication by Shavrukov.) Define $L^d_{01} = \langle \omega, \wedge, \vee, 0, 1, \leq_{L^d_{01}}\rangle$ where $\wedge, \vee$ are the same operations as in $F^d_{0,1}(X)$, and $x \leq_{L^d_{01}} y$ if $[x]_\alpha = [x \wedge y]_\alpha$. Throughout the remainder of this section leading to Example 3.15, for simplicity denote $L = L^d_{01}$. It is easy to see that $L$ is a c.e. prelattice whose associated quotient structure is $F^d_{0,1}(X)_{/\alpha}$. On the other hand, we can argue that $F^d_{0,1}(X)_{/\alpha}$ is isomorphic with $F^d_{0,1}(\{x_i : i \notin U \cup V\})$: if $f : X \longrightarrow F^d_{0,1}(\{x_i : i \notin U \cup V\})$ is the function mapping $x_i \mapsto 0$ if $i \in U$, $x_i \mapsto 1$ if $i \in V$, and $x_i \mapsto x_i$ if $i \notin U \cup V$, then by freeness $f$ extends to a (unique) homomorphism $g : F^d_{0,1}(X) \longrightarrow F^d_{0,1}(\{x_i : i \notin U \cup V\})$ which is onto as

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$f$ is onto a generating set, and it is not difficult to see that $\ker(g) = \alpha$, i.e., $g(x) = g(y)$ if and only if $x \ \alpha \ y$. Thus by the First Isomorphism Theorem of universal algebra we see that $F^d_{0,1}(X)_{/\alpha} \simeq F^d_{0,1}(\{x_i : i \notin U \cup V\})$. In conclusion, $L$ is free bounded distributive on a countably infinite set of generators; moreover the function $f(i) = x_i$ provides a 1-reduction of the e.i. pair $(U, V)$ to the pair $([0]_L, [1]_L)$, giving that $L$ is e.i.

We have shown:

EXAMPLE 3.15. *$L^d_{01}$ is an e.i. distributive prelattice, which is a free bounded distributive prelattice on countably many generators.*

***3.6. E.i. nondistributive prelattices.*** The same construction as in §3.5 but starting with the free bounded nondistributive lattice $F^{01}_{nd}(X)$ on $X$ (with $X$ decidable, infinite, and $0, 1 \notin X$), leads to a c.e. prelattice $L^{nd}_{01}$.

EXAMPLE 3.16. *$L^{nd}_{01}$ is an e.i. nondistributive prelattice which is free bounded on countably many generators.*

**§4. Effectively inseparable prelattices and universality.** Montagna and Sorbi (Montagna & Sorbi, 1985, Theorem 2.1) proved that if $L$ is an e.i. prelattice and $R = \langle \omega, \leq_R \rangle$ is a decidable prepartial order then $\leq_R \ \preceq \ \leq_L$, but left open (Montagna & Sorbi, 1985, Remark 2.1) whether such a prepartial order $\leq_L$ is universal. In the next section, we show that this is so.

***4.1. E.i. prelattices vs. uniform finite precompleteness.*** We first show that effective inseparability of the pair $([0]_L, [1]_L)$ makes the ceer $\equiv_L$ (and thus $\leq_L$, according to Definition 2.4) u.f.p.

THEOREM 4.1. *Let $L$ be a c.e. bounded prelattice. Then $L$ is e.i. if and only if the ceer $\equiv_L$ is u.f.p.*

*Proof.* If $\equiv_L$ is u.f.p. then by Fact 2.3 all pairs of distinct equivalence classes are e.i.

Suppose now that $L$ is e.i. We need to define a computable function $k(D, e, x)$ such that if $\varphi_e(x) \downarrow \equiv_L d$ for some $d \in D$ then $k(D, e, x) \equiv_L \varphi_e(x)$.

Let $p$ be a productive function for the pair $([0]_L, [1]_L)$. Let

$$\{u_{d,D,e,x}, v_{d,D,e,x} : \ D \text{ finite subset of } \omega, d \in D, e, x \in \omega\},$$

be a computable set of indices we control by the Recursion Theorem. For a pair $(u_{d,D,e,x}, v_{d,D,e,x})$ in this set let $c_{d,D,e,x} = p(u_{d,D,e,x}, v_{d,D,e,x})$ and $a_{d,D,e,x} = d \wedge c_{d,D,e,x}$. Define

$$k(D, e, x) = \bigvee_{d \in D} a_{d,D,e,x}.$$

We now specify how to computably enumerate $W_{u_{d,D,e,x}}$ and $W_{v_{d,D,e,x}}$ for $d \in D$. We wait for $\varphi_e(x)$ to converge to some $y$ which is $\equiv_L$ to some element in $D$, and while waiting, we let $W_{u_{d,D,e,x}}$ and $W_{v_{d,D,e,x}}$ enumerate $[0]_L$ and $[1]_L$, respectively. If we wait forever then for all $d \in D$ we end up with $W_{u_{d,D,e,x}} = [0]_L$ and $W_{v_{d,D,e,x}} = [1]_L$. If the wait terminates, let $d_0 \in D$ be the first seen so that $\varphi_e(x) \equiv_L d_0$, and then enumerate $c_{d_0,D,e,x}$ into $W_{u_{d_0,D,e,x}}$, while keeping on with $W_{u_{d_0,D,e,x}}$ enumerating $[0]_L$ and with $W_{v_{d_0,D,e,x}}$ enumerating $[1]_L$: this ends up with $W_{u_{d_0,D,e,x}} = [0]_L \cup \{c_{d_0,D,e,x}\}$ and $W_{v_{d_0,D,e,x}} = [1]_L$, thus forcing $c_{d_0,D,e,x} \in [1]_L$ and thus $a_{d_0,D,e,x} \equiv_L d_0$. For all $d \in D$ with $d \neq d_0$, we let $W_{u_{d,D,e,x}} = [0]_L$ and $W_{v_{d,D,e,x}} = [1]_L \cup \{c_{d,D,e,x}\}$: this forces $c_{d,D,e,x} \in [0]_L$ and thus $a_{d,D,e,x} \equiv_L 0$ for each such $d$.

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In conclusion, if $\varphi_e(x)$ converges to a number that is $\equiv_L$-equivalent to some $d \in D$ then, for the number $d_0$ for which we see this for the first time, we have

$$k(D, e, x) = \bigvee_{d \in D} a_{d,D,e,x} \equiv_L d_0,$$

as desired. □

***4.2. E.i. prelattices are universal.*** We now show that effective inseparability for a c.e. prelattice implies universality with respect to the class of all c.e. preorders.

THEOREM 4.2. *If $L = \langle \omega, \wedge, \vee, 0, 1, \leq_L \rangle$ is an e.i. prelattice then $\leq_L$ is universal.*

*Proof.* By Theorem 4.1 let $j(D, e, x)$ be a function witnessing that $\equiv_L$ is u.f.p.

CLAIM 4.3. *There is a computable function $k(a, b, D, e, x)$ so that if $a \leq_L b$, then $k(a, b, D, e, x) \in [a, b]_{\leq_L}$, and if $\varphi_e(x) \equiv_L d \in D$ for some $d \in D$ then $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$. In particular, if $\varphi_e(x) \equiv_L d \in D$ for some $d \in D$ with $d \in [a, b]_{\leq_L}$, then $k(a, b, D, e, x) \equiv_L d$.*

*Proof.* Let $k(a, b, D, e, x) = (j(D, e, x) \wedge b) \vee a$. Then if $a \leq_L b$, then $k(a, b, D, e, x) \in [a, b]_{\leq_L}$. If $\varphi_e(x) \equiv_L d \in D$, then $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$. Thus, if $d \in [a, b]_{\leq_L}$, then $k(a, b, D, e, x) \equiv_L d$. □

Since we know that $\equiv_L$ has this behavior with regard to $k$, we use a "speedup" of the enumeration of $\equiv_L$ so that this effect happens immediately. That is, whenever at a stage $s$ we need to define, given $a, b, D$, some computation $\varphi_e(x)$ to converge to some $d \in D$ relying on the fact that this entails $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$, we momentarily stop the construction until we in fact see $\equiv_L$ to give this equivalence, and only after that we proceed with the construction: we may therefore pretend that when we make $\varphi_e(x)$ converge then at the same stage we also see $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$.

Let $R = \langle \omega, \leq_R \rangle$ be any c.e. preorder on $\omega$. We must construct a computable function $f : \omega \to \omega$ so that $n \leq_R m$ if and only if $f(n) \leq_L f(m)$. We fix an infinite decidable set $E$ of indices that we control via the Recursion Theorem. The idea is then to define the map $f$ reducing $\leq_R$ to $\leq_L$ inductively. Having defined $f(j)$ for each $j < n$, we define $f(n)$ as follows. We consider first the partial order of the set $T^0_n$ of all possible prepartial orders on the set $\{j \mid j < n\}$, ordered by $O_1 \preceq^0_n O_2$ if $O_2$ extends $O_1$, that is, if $i \leq_{O_1} j$ implies $i \leq_{O_2} j$ for each $i, j < n$. This defines a finite partial order $(T^0_n, \preceq^0_n)$. Next we consider the partial order $(T_n, \preceq_n)$ where

$$\begin{aligned} T_n = \{(O, X, Y) \mid O \in T^0_n, X, Y \subseteq \{j \mid j < n\} \\ \text{and } a \leq_O b \text{ for every } a \in X \text{ and } b \in Y \\ \text{and if } i \leq_O a \in X, \text{ then } i \in X, \text{ and if } j \geq_O b \in Y \text{ then } j \in Y\}. \end{aligned}$$

In other words, $T_n$ is comprised of prepartial orders of the set $\{j \mid j < n\}$ with a chosen "interval." We say $(O_1, X_1, Y_1) \preceq_n (O_2, X_2, Y_2)$ if $O_1 \preceq^0_n O_2$ and $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. In other words, we have refined both the order and the interval. Once again, $(T_n, \preceq_n)$ is a finite partial order.

For each element $\tau = (O, X, Y) \in T_n$, we define an $x_\tau \in \omega$. We begin by defining $x_\tau$ for the leaf $v$ in $T_n$. Note that the leaf corresponds to a prepartial order where all elements are $O$-equivalent, $X = Y = \{j \mid j < n\}$. We define $x_v$ to be

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$$x_v = k(\bigvee_{i\in X} f(i), \bigwedge_{j\in Y} f(j), \{f(j) \mid j < n\} \cup \{0, 1\}, e_v, 0),$$

where $e_v \in E$ is distinct from any previously mentioned element of $E$.

For every node $\tau \in T_n$, we define

$$x_\tau = k(\bigvee_{i\in X} f(i), \bigwedge_{j\in Y} f(j), \{x_\sigma \mid \tau \preceq_n \sigma\} \cup \{0, 1\}, e_\tau, 0),$$

where $e_\tau \in E$ is distinct from any previously mentioned element of $E$.

We fix notation that $\tau = (O_\tau, X_\tau, Y_\tau)$ and $x_\tau = k(a_\tau, b_\tau, S_\tau, e_\tau, 0)$. Let $\lambda$ be the root in $T_n$, i.e., the prepartial ordering given by the identity (providing an antichain), and $X = Y = \emptyset$.

We define $f(n)$ to be the element $x_\lambda$, where $\lambda$ is the root in $T_n$ ($\lambda = \lambda(n)$ depends in fact on $n$). The idea is to use in a construction in stages the indices $e_\tau$ for various $\tau$ to ensure that: If the elements $\{j \mid j < n\}$ in $\leq_R$ have order type $O$, and $X = \{j \mid j < n \wedge j \leq_R n\}$ and $Y = \{j \mid j < n \wedge j \geq_R n\}$ at stage $s$, then $x_\lambda \equiv_L x_{(O,X,Y)}$. Since there are only finitely many elements of $T_n$, we will only need to act at finitely many stages to ensure this is true ($(O, X, Y) = \tau(n, s)$ depends in fact on $n$ and $s$).

We work with computable approximations $\{\leq_{R,s}: s \in \omega\}$, $\{\leq_{L,s}: s \in \omega\}$ to $\leq_R$, $\leq_L$, respectively (and consequent approximations $\{\equiv_{R,s}: s \in \omega\}$, $\{\equiv_{L,s}: s \in \omega\}$ to $\equiv_R$, $\equiv_L$, respectively): we may assume that these sequences are increasing sequences of preorders $\leq_{R,s}$, $\leq_{L,s}$ with unions $R, S$, respectively, such that the predicates (in $x, y, s$) $x \leq_{R,s} y$, $x \leq_{L,s} y$ are decidable and $\leq_{R,0}$, $\leq_{L,0}$ are the identity preorders on $\omega$. In the construction below it is understood that at each stage $s$, when we monitor the c.e. progress of the two c.e preorderings and of the corresponding equivalence relations, we work with the corresponding approximations at that stage.

Recall that $f(n) = x_{\lambda(n)}$ where $\lambda(n)$ is the root in $T_n$.

*Stage* 0. Do nothing.

*Stage* $s + 1$. Let $n$ be given, and suppose we have dealt already with all $j < n$. We now deal with $n$. Recall that $f(n) = x_\lambda$. Let $O_1$ represent the order type of $R$ restricted to $\{j \mid j < n\}$ at stage $s$, and $O_2$ be the order type of $R$ restricted to $\{j \mid j < n\}$ at stage $s + 1$. Let $X_1$ be the set of elements in $\{j \mid j < n\}$ which are $\leq_R n$ at stage $s$ and let $Y_1$ be the set of elements in $\{j \mid j < n\}$ which are $\geq_R n$ at stage $s$. Let $X_2$, $Y_2$ be defined similarly at stage $s + 1$. Then $(O_1, X_1, Y_1) \preceq_n (O_2, X_2, Y_2)$.

If $(O_1, X_1, Y_1) \neq (O_2, X_2, Y_2)$ then make $\varphi_{e_{(O_1,X_1,Y_1)}}(0)$ converge to equal the value $x_{(O_2,X_2,Y_2)}$. Note that this causes (by our speedup) that $x_{(O_1,X_1,Y_1)} \equiv_L x_{(O_2,X_2,Y_2)}$ also at stage $s + 1$. (We will show by induction that then $f(n) = x_\lambda$ will become $\equiv_L$-equivalent to $x_{(O_2,X_2,Y_2)}$, which will ensure that $f(n) \leq_L f(j)$ for every $j \in Y_2$ and $f(n) \geq_L f(j)$ for every $j \in X_2$.)

If $(O_1, X_1, Y_1) = (O_2, X_2, Y_2)$ then go to $n + 1$.

If ever we see, at any stage, $f(n) \leq_L f(m)$ when we have not yet seen $n \leq_R m$, then we halt the entire construction and call action $\diamondsuit$. Action $\diamondsuit$, if ever called, will cause a contradiction, which will guarantee that this will never happen and action $\diamondsuit$ will never in fact be called.

***Action*** $\diamondsuit$**.** We now describe action $\diamondsuit$. Suppose that at stage $s$ of the construction, we see $f(n) \leq_L f(m)$ and we have not yet seen $n \leq_R m$. In particular, at this stage $f(n) \equiv_L x_\tau$ and $f(m) \equiv_L x_\sigma$ where $\tau = \tau(n, s)$ and $\sigma = \tau(m, s)$ are as given for $n$ and $m$ by Claim 4.4, so

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that we have not as yet caused $\varphi_{e_\tau}(0)$ or $\varphi_{e_\sigma}(0)$ to converge. Recall that $a_\tau = \bigvee_{i\in X_\tau} f(i)$ and $b_\tau = \bigwedge_{j\in Y_\tau} f(j)$, and similarly for $\sigma$.

Now, for every $z \in \{n\} \cup Y_\tau$, we perform the following action, which we call One($z$): take $\rho = \tau(z, s)$ as given for $z$ again by Claim 4.4 (in particular $x_\rho \equiv_L f(z)$ and $\varphi_{e_\rho}(0)\uparrow$ at this stage). Now, make $\varphi_{e_\rho}(0)$ converge to equal 1. This forces $f(z) \equiv_L b_\rho \vee a_\rho$ (by our speedup this happens at $s$). In other words, we cause at $s$ $f(z)$ to become as large as possible.

Also, for every $z \in \{m\} \cup X_\sigma$, we perform the following action, which we call Zero($z$): take $\rho$ as given for $z$ again by Claim 4.4 (in particular $x_\rho \equiv_L f(z)$ and $\varphi_{e_\rho}(0)\uparrow$ at this stage). Now, make $\varphi_{e_\rho}(0)$ converge to equal 0, which causes $f(z) \equiv_L a_\rho$ (by our speedup this happens at $s$). In other words, we cause at $s$ $f(z)$ to become as small as possible.

***Verification.*** We now verify that the above construction yields a function $f : \omega \to \omega$ so that $n \leq_R m$ if and only if $f(n) \leq_L f(m)$.

CLAIM 4.4. *Suppose that we have not called action $\diamondsuit$ at any stage $<s$, and let $m$ be given. Let $\tau(m, s) = (O, X, Y) \in T_m$ be the triple defined by: $O$ is the order type of $\{j \mid j < m\}$ in $R$ at stage $s$, $X = \{j \mid j < m \text{ and } j \leq_R m\}$ at stage $s$, and $Y = \{j \mid j < m \text{ and } j \geq_R m\}$ at stage $s$. Then (letting for simplicity $\tau = \tau(m, s)$) $f(m) \equiv_L x_\tau$ at stage $s$ and $\varphi_{e_\tau}(0)\uparrow$ at stage $s$, and $f$ preserves the prepartial order given by $R$ at stage $s$ on the interval $\{j : j \leq m\}$.*

*Proof.* At stage 0, we have $f(m) = x_\lambda$, so the claim holds for every $m$ at stage 0.

Assume by induction that the claim holds for every $m$ at stage $s$ and for every $j < m$ at stage $s + 1$.

Notice that we can act at stage $s + 1$, as the construction has not been stopped due to $\diamondsuit$. Let $\tau(m, s) = (O, X, Y)$ be the triple defined as in the statement of the Claim, i.e., $O$ is the order type of $\{j \mid j < m\}$ in $R$ at stage $s$, $X = \{j \mid j < m \text{ and } j \leq_R m\}$ at stage $s$, and $Y = \{j \mid j < m \text{ and } j \geq_R m\}$ at stage $s$. By induction, if $\tau = \tau(m, s)$ then, at $s$, $f(m) \equiv_L x_\tau$. Let $\sigma = \tau(m, s + 1)$ be the triple defined analogously at stage $s + 1$, so we make $\varphi_{e_\tau}(0)\downarrow = x_\sigma$. But then, by our speedup, at $s + 1$

$$x_\tau = k(a_\tau, b_\tau, S_\tau, e_\tau, 0) \equiv_L (x_\sigma \wedge b_\tau) \vee a_\tau.$$

Moreover, by inductive hypothesis at $s + 1$ (on $f$ preserving at $s + 1$ the preorder of $R$ on numbers smaller than $m$), $a_\tau \leq_L b_\tau$ and $a_\sigma \leq_L b_\sigma$: this implies by property of function $k$ that $a_\tau \leq_L x_\tau \leq_L b_\tau$ and $a_\sigma \leq_L x_\sigma \leq_L b_\sigma$. Now, since $X_\tau \subseteq X_\sigma$ and $Y_\tau \subseteq Y_\sigma$, we get

$$a_\tau \leq_L a_\sigma \leq_L b_\sigma \leq_L b_\tau,$$

and so at stage $s+1$ (all the following $\equiv_L$-equivalences by our speedup may be assumed to be seen at $s+1$) we have that $f(m) \equiv_L x_\tau \equiv_L x_\sigma$: indeed $x_\sigma \wedge b_\tau \equiv_L x_\sigma$ and $x_\sigma \vee a_\tau \equiv_L x_\sigma$. Further, there is no condition which could have caused $\varphi_{e_\sigma}(0)$ to have converged before this stage, so we still have $\varphi_{e_\sigma}(0)\uparrow$ at stage $s + 1$.

We finally show that $f$ preserves the preorder given at stage $s + 1$ by $R$ on $\{j : j \leq m\}$. By inductive hypothesis, the preorder given at stage $s + 1$ by $R$ on $\{j : j < m\}$ is preserved. Also, at $s + 1$, since $f(m) \equiv_L x_\sigma$ we have that $a_\sigma \leq_L f(m) \leq_L b_\sigma$, so all the inequalities which occur between $m$ and any $j < m$ in $R$ are preserved in $\leq_L$ at $s + 1$. □

It now remains to show that we never call action $\diamondsuit$. Then, Claim 4.4 shows that $f$ preserves the preorder from $R$ and the fact that we do not call action $\diamondsuit$ shows that we never have any more inequalities holding in $\leq_L$ than in $\leq_R$. Thus $f$ is a reduction of $\leq_R$ to $\leq_L$.

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CLAIM 4.5. *We never call action* $\diamondsuit$.

*Proof.* Suppose we call action $\diamondsuit$ at stage $s$ because we see $f(n) \leq_L f(m)$ and we have not yet seen that $n \leq_R m$. By calling action One($z$) for every $z \in \{n\} \cup Y_\tau$, we arrange that for each of these elements $z, f(z)$ become (by our speedup this happens at $s$) $\equiv_L$-equivalent to $b_\rho \vee a_\rho$, as in the construction (with $\rho, \tau, \sigma$ given for $m, n, z$ at $s$ by Claim 4.4), $f(m) \equiv_L x_\tau$, $f(n) \equiv_L x_\sigma, f(z) \equiv_L x_\rho$ and we have not as yet caused either of $\varphi_{x_\tau}(0)$, $\varphi_{x_\sigma}(0)$, $\varphi_{x_\rho}(0)$ to converge. But since at $s$ the order in $L$ is the image of the order in $R$ (since we have not as yet called action $\diamondsuit$), we get that $f(z) \equiv_L b_\rho$, since $b_\rho \geq_L a_\rho$. We now show that for every $z \in \{n\} \cup Y_\tau, f(z)$ becomes $\equiv_L$-equivalent to 1 (in the following, $\equiv_L$ and $\leq_R$ are understood to be approximated at stage $s$). Suppose towards a contradiction that for some element in this set, the image under $f$ is not $\equiv_L$-equivalent to 1, and let $z$ be $\leq_R$-maximal among those such that $f(z)$ is not $L$-equivalent to 1. Subject to this restraint, we further choose $z$ to be minimal (as a number in $\omega$). Now, if $y \in Y_\rho$ (so that $y < z$ and thus $y < n$ as $z \leq n$ being $z \in Y_\tau \cup \{n\}$) then $y \geq_R z \geq_R n$ (if $z = n$ then at $s$ $z \geq_R n$ by our assumptions on the approximations to $\leq_R$) and thus $y \in Y_\tau$: by $R$-maximality of $z$, we see that either $y \equiv_R z$ or $f(y) \equiv_L 1$. In the former case, we have $f(y) \equiv_L f(z)$ (since at $s$ $f$ preserves the preorder) and that $z$ was not minimal in $\{n\} \cup Y_\tau$ with $f(z) \not\equiv_L 1$. Thus, we always have the latter case, and we see that for every $y \in Y_\rho$ we have $f(y) \equiv_L 1$. But $f(z) \equiv_L b_\rho$ which is the meet of the images of the members of $Y_\rho$, and thus $\equiv_L 1$. Thus, $f(z) \equiv_L 1$.

The same proof shows that for every $z \in \{m\} \cup X_\sigma, f(z)$ becomes $\equiv_L$-equivalent to 0. Thus we have that $1 \equiv_L f(n) \leq_L f(m) \equiv_L 0$ (taking in turn $z = n$ and $z = m$), which is a contradiction. Thus, the action $\diamondsuit$ is never called. □

This ends the proof. □

REMARK 4.6. *It should be noted that the proofs of Theorems 4.1 and 4.2 do not require distributivity in their assumptions. On the other hand, for a c.e. prelattice distributivity is not a consequence of being e.i.: Example 3.16 shows in fact that there are e.i. nondistributive prelattices.*

*Notice however that the proofs of Theorems 4.1 and 4.2 make use of the existence of both operations of* $\wedge$ *and* $\vee$. *The following observation shows that this is indeed necessary. We say that a bounded c.e. upper pre-semi-lattice* $U = \langle \vee, 0, 1, \leq_U \rangle$ *is* effectively inseparable (e.i) *if, again, the pair* $([0]_U, [1]_U)$ *is effectively inseparable.*

OBSERVATION 4.7. *There exists an e.i. upper pre-semi-lattice whose preordering relation is not universal.*

*Proof.* The same construction as in §3.5 but starting with the free bounded upper pre-semi-lattice on $X$ (a decidable, infinite set with $0, 1 \notin X$) leads to a c.e. bounded upper pre-semi-lattice $U_{01}$ which is e.i. and as a bounded upper pre-semi-lattice is free on a countably infinite set of generators.

We claim that the c.e. order $\langle \omega, \leq^* \rangle$ where $x \leq^* y$ if $x \geq y$ can not be reduced to $\leq_{U_{01}}$, thus showing that $\leq_{U_{01}}$ is not universal. This is so because there is no infinite descending chain in $U_{01}$, as every element other than 1 is the join of a unique finite set of generators, which are atoms in $U_{01}$. □

***4.3. Doing without boundedness.*** We have shown (Theorem 4.1) that every e.i. prelattice is u.f.p. and thus u.e.i. by Fact 2.3. In this section we reverse the last implication by showing that for c.e. prelattices u.e.i.-ness, even without boundedness, is enough to have u.f.p.-ness.

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LEMMA 4.8. *Let $L = \langle \omega, \wedge, \vee, \leq_L \rangle$ be a c.e. prelattice, with $a, b$ such that $a <_L b$ and the pair $([a]_L, [b]_L)$ is e.i. Then there exists a productive function $p$ for this pair such that if $[a]_L \subseteq W_u$, $[b]_L \subseteq W_v$ and $W_u \cap W_v = \emptyset$ then $p(u, v) \in [a, b]_{\leq_L}$.*

*Proof.* Let $q$ be a productive function for the pair $([a]_L, [b]_L)$. We use two indices $u', v'$ (uniformly depending on $u, v$, respectively) which we control by the Recursion Theorem. We program enumerations of $W_{u'}$ and $W_{v'}$ as follows. Let $j = (a \vee q(u', v')) \wedge b$. Then $W_{u'}$ and $W_{v'}$ enumerate $[a]_L$ and $[b]_L$, respectively: if we see $j \in W_u$ then we also enumerate $q(u', v')$ into $W_{u'}$; if we see $j \in W_v$ then we also enumerate $q(u', v')$ into $W_{v'}$. In other words,

$$W_{u'} = \begin{cases} [a]_L, & \text{if } j \notin W_u, \\ [a]_L \cup \{q(u'v')\}, & \text{otherwise,} \end{cases}$$

$$W_{v'} = \begin{cases} [b]_L, & \text{if } j \notin W_v, \\ [b]_L \cup \{q(u'v')\}, & \text{otherwise.} \end{cases}$$

Assume now that $[a]_L \subseteq W_u$ and $[b]_L \subseteq W_v$ and $W_u \cap W_v = \emptyset$. Then $j \notin W_u \cup W_v$: if for instance $j \in W_u$ then we have forced $q(u', v') \in [b]_L$ giving $j \in [b]_L \subseteq W_v$, thus $W_u \cap W_v \neq \emptyset$, contradiction. Thus, $p(u, v) = j$ is our desired productive function. □

LEMMA 4.9. *Let $L = \langle \omega, \wedge, \vee, \leq_L \rangle$ be a c.e. prelattice, with $a, b$ such that $a <_L b$ and $([a]_L, [b]_L)$ is e.i. Then the interval $[a, b]_{\leq_L}$ is computably isomorphic to an e.i. prelattice.*

*Proof.* We first observe that the interval $[a, b]_{\leq_L}$ is infinite, as the equivalence classes of $a$ and $b$ are infinite. We argue as in Remark 1.5. Let $h : \omega \longrightarrow [a, b]_{\leq_L}$ be a computable bijection with $h(0) = a$ and $h(1) = b$. This induces a c.e. prelattice $L' = \langle \omega, \wedge, \vee, 0, 1, \leq_{L'} \rangle$ where $x \leq_{L'} y$ if and only if $h(x) \leq_L h(y)$. By Lemma 4.8 the equivalence classes $[0]_{L'}, [1]_{L'}$ form an e.i. pair: if $[0]_{L'} \subseteq W_u$ and $[1]_{L'} \subseteq W_v$, and $W_u \cap W_v$ are disjoint, then $h[W_u]$ and $h[W_v]$ are disjoint supersets of $[a]_L, [b]_L$, respectively, so that by the previous lemma (which provides a productive function which always lands in $[a, b]_{\leq_L}$ when applied to indices of disjoint c.e. supersets of $[a]_L, [b]_L$) we can effectively find $y \in [a, b]_{\leq_L}$ so that $h^{-1}(y)$ is defined but $y \notin h[W_u] \cup h[W_v]$, and thus $h^{-1}(y) \notin W_u \cup W_v$. □

Theorem 4.10 and Observation 4.11 below answer questions raised in private communication by Shavrukov.

THEOREM 4.10. *A c.e. prelattice is u.e.i. if and only if it is u.f.p.*

*Proof.* The implication $\Leftarrow$ follows from Fact 2.3. For the other implication, let $L = \langle \omega, \wedge, \vee, \leq_L \rangle$ be a c.e. prelattice which is u.e.i. By Remark 2.2 and Lemma 4.8, uniformly in $a, b$ we can find a binary computable function $q_{a,b}$ so that if $a <_L b$ then $q_{a,b}$ has the properties stated for $p$ in Lemma 4.8. Thus, by Lemma 4.9 from any pair $a, b$ we can uniformly go to an index of a total computable binary function $p_{a,b}$, and indices of a partial computable function $h_{a,b}$, two computable binary functions $\wedge_{a,b}, \vee_{a,b}$, and a c.e. relation $\leq_{a,b}$, such that the following properties hold: (1) if $a \not\leq_L b$ then $h_{a,b}$ never converges; (2) if $a <_L b$ then $L_{a,b} = \langle \omega, \wedge_{a,b}, \vee_{a,b}, 0, 1, \leq_{a,b} \rangle$ is an e.i. prelattice, $h_{a,b} : \omega \longrightarrow [a, b]_{\leq_L}$ is a bijection giving an order-theoretic isomorphism of $L_{a,b}$ with $[a, b]_{\leq_L}$ (with $h_{a,b}(0) = a$ and $h_{a,b}(1) = b$), and $p_{a,b}$ is a productive function of the pair $([0]_{L_{a,b}}, [1]_{L_{a,b}})$; (3) if $a \equiv_L b$, then $h_{a,b}$ is a bijection between $[a]_L$ and $\omega$. It follows that if $a \leq_L b$ then $h_{a,b}$ is a bijection between $\omega$ and $[a, b]_{\leq_L}$, and thus from the canonical index of a finite set $D$ we can effectively find the canonical index of the image $h_{a,b}[D]$. Finally, by the proof of

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Theorem 4.1 (which is uniform) from $a, b$, using totality of $p_{a,b}$, we can uniformly find an index of a total computable function $k_{a,b}(D, e, x)$ which witnesses that $L_{a,b}$ is u.f.p. if $a <_L b$.

Given now any finite set $D$ let $a_D = \bigwedge D$ and $b_D = \bigvee D$ where the meet and the join are taken in $L$: notice that $a_D \leq_L b_D$, thus $h_{a_D,b_D}$ is a bijection between $\omega$ and $[a_D, b_D]_{\leq_L}$; and let $f$ be a computable function such that $\varphi_{f(e,D)} = h^{-1}_{a_D,b_D} \circ \varphi_e$. But then it is easy to see that the computable function

$$k(D, e, x) = h_{a_D,b_D}(k_{a_D,b_D}(h^{-1}_{a_D,b_D}[D], f(e, D), x))$$

witnesses that $L$ is u.f.p.: if $a_D <_L b_D$ this follows from the fact that $k_{a_D,b_D}$ witnesses that the e.i. prelattice $L_{a_D,b_D}$ is u.f.p.; if $a_D \equiv_L b_D$ (when $L_{a_D,b_D}$ collapses to one point) this follows from the fact that $[D]_L = [a_D]_L$, and in this case $h_{a_D,b_D}$ always lands in $[a_D]_L$. □

All the examples seen so far of u.e.i. (or, equivalently, u.f.p.) c.e. prelattices are in fact e.i. prelattices, which by the very definition, are bounded. The following observation answers the natural question of whether u.e.i.-ness entails boundedness.

OBSERVATION 4.11. *There exist c.e. prelattices which are u.e.i. but not bounded.*

*Proof.* Fix $L$ to be any e.i. prelattice. Let $M$ be $\oplus_{i\in\omega} L$, that is the direct sum of $\omega$ copies of $L$: $M$ has universe the set of elements in $L^\omega$ with only finitely many non-0 entries, and the operations and preordering relation of $M$ are component-wise; clearly this universe can be coded as $\omega$. $M$ is not bounded from above, though it is not hard to show that $M$ is u.e.i. In fact, we claim that for any pair of u.e.i. c.e. prelattices $A$ and $B$, the direct sum $A \oplus B$ is a u.e.i. c.e. prelattice, and an index for a uniform productive function for $A \oplus B$ can be found uniformly from indices for uniform productive functions for $A$ and for $B$. Before proving the claim, let us conclude the argument to show that $M$ is u.e.i.: since every pair of elements from $M$ appears in a finite sum of copies of $L$, their $\equiv_M$-equivalence classes, if distinct, are e.i. by the claim, and it is uniform to find an index witnessing this, so $M$ is u.e.i. If we want an example which is neither bounded above nor below, consider $M \oplus M^*$, where $M^*$ is the same as $M$ but with the order reversed.

We are only left to show the claim, that if $A$ and $B$ are u.e.i. c.e. lattices then the c.e. prelattice $A \oplus B$, which equals $A \times B$, is u.e.i. as well. We code $A \oplus B$ as $\omega$ by using the Cantor pairing function, and let $\pi_0, \pi_1$ be the computable projections associated with the Cantor pairing function. Since u.e.i. implies u.f.p. (Fact 2.3) we know that $A$ and $B$ are u.f.p.: pick two totalizers $k^A, k^B$ for the equivalence relations $\equiv_A$ and $\equiv_A$ of $A$ and $B$, respectively. In order to show that $A \oplus B$ is u.f.p., for every $D, e, x$ let $u^A_{D,e}, u^B_{D,e}$ be indices of partial computable functions as follows: to compute $\varphi_{u^A_{D,e}}(x)$ wait until $\varphi_e(x) \downarrow \equiv_{A\oplus B} d$ for some $d \in D$: if and when this happens then make $\varphi_{u^A_{D,e,x}}(0)$ converge to $\pi_0(d)$; otherwise $\varphi_{u^A_{D,e,x}}(0)$ diverges. The computation $\varphi_{u^B_{D,e,x}}(0)$ is defined similarly, outputting $\pi_1(d)$ when it converges. Define now

$$k(D, e, x) = \langle k^A(\pi_0[D], u^A_{D,e}, x), k^B(\pi_1[D], u^B_{D,e}, x)\rangle.$$

Using that $k^A, k^B$ are totalizers for the equivalence relations $\equiv_A$ and $\equiv_B$ of $A, B$, respectively, it is straightforward to see that $k$ is a totalizer for $\equiv_{A\oplus B}$. Since (Fact 2.3) u.f.p. implies u.e.i. we have that $A \oplus B$ is u.e.i. Uniformity from indices of uniform productive functions for $A$ and $B$, respectively, to indices of a productive function for $A \oplus B$ follows from the following facts: the implications stated in Fact 2.3 are uniform, i.e., one can uniformly go from an index of a uniform productive function for a u.e.i. ceer to an index

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of a totalizer of that ceer, and vice versa; and the proof of the above claim shows that one can uniformly go from indices of $k^A, k^B$ to indices of $k$. □

REMARK 4.12. *Notice that $M$ and $M \oplus M^*$ are distributive if and only if $L$ is distributive.*

**§5. Local universality and uniform density.** When dealing with ordered or preordered structures, one of the most immediate and important tasks is to study density questions, in particular whether a given structure is dense or where density locally holds or fails. For prelattices, effective inseparability turns out to be, as far as density goes, a powerful tool of investigation.

***5.1. Density.*** We first look at the case in which a c.e. prelattice $L$ need not be bounded but it has a nontrivial interval $[a, b]_{\leq L}$ such that the pair $([a]_L, [b]_L)$ is e.i. Before proving in Corollary 5.3 that the bounded interval $[a, b]_{\leq L}$ enjoys the universality property already shown of any e.i. prelattice, namely, every c.e. preorder can be computably embedded into the interval $[a, b]_{\leq L}$, we first notice, by the following example, that such a c.e. prelattice, even if bounded, need not be u.e.i.

EXAMPLE 5.1. *Let $L$ be a u.e.i. prelattice. Form $L'$ by placing a new element called* 0 *below every element of $L$ and a new element called* 1 *above every element of $L$. Formally, let $h$ be a computable bijection of $\omega$ with $\omega \setminus \{0, 1\}$, and define $\leq_{L'}$ be the c.e. preordering given by $x \leq_{L'} y$ if and only if*

$$x = 0 \vee y = 1 \vee [\{x, y\} \cap \{0, 1\} = \emptyset \,\&\, h^{-1}(x) \leq_L h^{-1}(y)].$$

*It is now easy to see that there is a c.e. bounded prelattice $L' = \langle \omega, \wedge, \vee, 0, 1, \leq_{L'} \rangle$ such that $[0]_{L'} = \{0\}$, $[1]_{L'} = \{1\}$, and thus the pair $([0]_{L'}, [1]_{L'})$ is not e.i. On the other hand, as is readily seen, all disjoint pairs $([a]_{L'}, [b]_{L'})$ with $\{a, b\} \cap \{0, 1\} = \emptyset$ are e.i.*

DEFINITION 5.2. *A c.e. preorder $R$ is* locally universal *if for every pair $a, b$ such that $a <_R b$ one can computably embed any c.e. preorder in the interval $[a, b]_R$. A c.e. prelattice (Boolean prealgebra, Heyting prealgebra, etc.) is* locally universal *if so is its preordering relation.*

COROLLARY 5.3. *If $L = \langle \omega, \wedge, \vee, 0, 1, \leq_L \rangle$ is a c.e. prelattice and $([a]_L, [b]_L)$ are e.i. with $a <_L b$ then one can reduce any c.e. preorder to the interval $[a, b]_{\leq_L}$.*

*Proof.* Trivial, since by Lemma 4.9 $[a, b]_{\leq_L}$ is computably isomorphic to an e.i. prelattice. □

COROLLARY 5.4. *Every u.e.i. c.e. prelattice is locally universal.*

*Proof.* This is immediate by Corollary 5.3 since in a u.e.i. c.e. prelattice, all pairs of distinct equivalence classes are e.i. □

REMARK 5.5. *If $L$ is a u.e.i. c.e. prelattice then local universality is uniform: from $a, b$ and a c.e. index for a c.e. preorder $R$ one can uniformly find an index of a computable function $f$ which is a computable embedding of $R$ into $[a, b]_L$ if $a <_L b$. This follows from the fact that the proof of Theorem 4.2 is uniform in an index of $\leq_R$, and from the observations on uniformity of Lemmatas 4.8 and 4.9 made for the proof of Theorem 4.10.*

***5.2. Uniform density.*** Uniform density for c.e. prelattices appearing in logic and for c.e. precomplete prelattices is treated in detail in (Shavrukov & Visser, 2014), to which

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the reader is referred also for motivations and historical remarks. We recall the definition.

DEFINITION 5.6. *A c.e. preordering $\leq_R$ is* uniformly dense *if there exists a computable function $f(a, b)$ so that for all $a, b$ if $a <_R b$ then $a <_R f(a, b) <_R b$, and for all pairs $a', b'$ such that $a \equiv_R a'$ and $b \equiv_R b'$ we have that $f(a, b) \equiv_R f(a', b')$. A c.e. prelattice (Boolean prealgebra, Heyting prealgebra, etc.) is* uniformly dense *if so is its preordering relation.*

The proof of the next theorem follows, via a natural generalization, along the lines of the proof of uniform density for c.e. precomplete prelattices in (Shavrukov & Visser, 2014).

THEOREM 5.7. *Every u.e.i. c.e. prelattice is uniformly dense.*

*Proof.* Let $L$ be a u.e.i. prelattice, and let $k(D, e, x)$ be a computable function witnessing that $\equiv_L$ is u.f.p.: see Theorem 4.10.

Let $\{e_{a,b} : a, b \in \omega\}$ be a computable list of indices we control by the Recursion Theorem. Denote by $\equiv_L^2$ the equivalence relation

$$(a, b) \equiv_L^2 (a', b') \Leftrightarrow \left[a \equiv_L a' \,\&\, b \equiv_L b'\right],$$

and define

$$f(a, b) = (a \vee j(a, b)) \wedge b,$$

where $j(a, b) = k(D_{a,b}, e_{a,b}, 0)$ and $D_{a,b} = \{a, b, j(a', b') : \langle a', b'\rangle < \langle a, b\rangle\}$. (Here and in the following, the reader is invited to make sure to distinguish between the natural orderings of $\omega$ for which we use the symbols $\leq$, $<$, and the preordering or the strict preordering of $L$ for which we use the symbols $\leq_L$, $<_L$.)

Now we specify how to compute the various $\varphi_{e_{a,b}}(0)$.

At step 0 all computations $\varphi_{e_{a,b}}(x)$ are undefined.

At step $s + 1$ we consider all pairs $(a, b)$ with $\langle a, b\rangle \leq s$, for which $\varphi_{e_{a,b}}(0)$ is still undefined:

1. for each such pair $(a, b)$ let $(a_m, b_m) \equiv_L^2 (a, b)$ (at $s$) and $\langle a_m, b_m\rangle < \langle a, b\rangle$ is least with this property, if there is any such pair. If so, define $\varphi_{e_{a,b}}(0) = j(a_m, b_m)$. By the properties of the totalizer $k$, this makes $j(a_m, b_m) \equiv_L j(a, b)$ and thus $f(a_m, b_m) \equiv_L f(a, b)$;
2. if after this, $(a, b)$ is still a pair such that $\varphi_{e_{a,b}}(0)$ is undefined and $a \leq_L b$ at this stage then
   (a) if $(a \vee j(a, b)) \wedge b) \equiv_L a$ (at $s$) then define $\varphi_{e_{a,b}}(0) = b$: this forces $j(a, b) \equiv_L b$ and thus $a \equiv_L b$;
   (b) if $(a \vee j(a, b)) \wedge b) \equiv_L b$ (at $s$) then define $\varphi_{e_{a,b}}(0) = a$: this forces $j(a, b) \equiv_L a$ and thus, again, $a \equiv_L b$.

This ends the construction. Notice that $\varphi_{e_{a,b}}(0)$ is eventually defined if there is $(a', b')$ in the same $\equiv_L^2$-class as $(a, b)$ and $\langle a', b'\rangle < \langle a, b\rangle$.

We now show that $f$ is extensional. As $f(a, b) \equiv_L a$ if $a \equiv_L b$, then extensionality trivially holds when $a \equiv_L b$. Suppose that $a \not\equiv_L b$. Then $\varphi_{e_{a,b}}(0)$ has not been defined through either subclause of clause (2), as this would give $a \equiv_L b$. We claim that in this case $f(a, b) \equiv f(a_m, b_m)$ where $(a_m, b_m)$ is in the same $\equiv_L^2$-class as $(a, b)$ and $\langle a_m, b_m\rangle$ is least with this property. We prove by induction on $\langle a, b\rangle$ that this is true for all $(a, b)$ which is in the same $\equiv_L^2$-class as $(a_m, b_m)$. The claim is trivial when $(a, b) = (a_m, b_m)$. Otherwise $\varphi_{e_{a,b}}(0)$ is defined at some stage (not through (2) as we have already seen). Thus we have

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forced $j(a, b) \equiv_L j(a', b')$ with some $(a', b')$ in the same $\equiv_L^2$-class as $(a, b)$ but $\langle a', b' \rangle < \langle a, b \rangle$. This makes $f(a, b) \equiv_L f(a', b')$. By inductive assumption, $f(a', b') \equiv_L f(a_m, b_m)$: hence $f(a, b) \equiv_L f(a_m, b_m)$.

Finally we show that if $a <_L b$ then $a <_L f(a, b) <_L b$: notice that we have $a \leq_L f(a, b) \leq_L b$ for free as $a \leq_L b$. Indeed by extensionality we may assume that $\langle a, b \rangle$ is least among the pairs in the $\equiv_L^2$-class of $(a, b)$. If $a <_L f(a, b) <_L b$ does not hold then sooner or later clause (2) would force $a \equiv_L b$, a contradiction. □

**§6. Reducibilities inducing monomorphisms.** If $\mathcal{A}$ is a class of preordered structures of the same type, it is natural to study reductions (called $\mathcal{A}$-*reductions*) between the structures in $\mathcal{A}$ which *respect* the operations. For instance:

DEFINITION 6.1. *Let $\mathcal{L}$ be the class of c.e. prelattices, and assume that $L_1 = \langle \omega, \vee_1, \wedge_1, \leq_1 \rangle$ and $L_2 = \langle \omega, \vee_2, \wedge_2, \leq_2 \rangle$ lie in $\mathcal{L}$. We say that a computable functions $f$* $\mathcal{L}$-reduces *$L_1$ to $L_2$ (notation: $L_1 \preceq_{\mathcal{L}} L_2$) if $f$ reduces $\leq_{L_1}$ to $\leq_{L_2}$ and for all $x, y \in L_1$,*

$$f(x \vee_1 y) \equiv_{L_2} f(x) \vee_2 f(y),$$
$$f(x \wedge_1 y) \equiv_{L_2} f(x) \wedge_2 f(y).$$

*Clearly $f$ induces a monomorphism between the associated quotient structures. Given a class $\mathcal{A}$ of c.e. prestructures of the same type, we say that a c.e. prestructure $L$ is* $\mathcal{A}$-universal (*or* universal with respect to $\mathcal{A}$) *if $L$ has a type containing the type of the prestructures in $\mathcal{A}$ and $A \preceq_{\mathcal{A}} L^{red}$ for every $A \in \mathcal{A}$ (where $L^{red}$ is the prestructure in $\mathcal{A}$ obtained from $L$ by restricting its type to the type of $\mathcal{A}$).*

In addition to using the symbol $\mathcal{L}$ to denote the class of c.e. prelattices, in the following we will use the following notations: $\mathcal{B}$ is the class of c.e. Boolean prealgebras; $\mathcal{L}_{0,1}^d$ is comprised of the c.e. distributive bounded prelattices; $\mathcal{L}_0^d$ is comprised of the c.e. distributive prelattices with a least element; $\mathcal{L}^d$ is the class of c.e. distributive prelattices. We will keep the term "universal" without specifications, to mean universality with respect to c.e. preorders: thus a c.e. preordered structure $L$ with c.e. preordering relation $\leq_L$ is *universal* if $\leq_R \preceq \leq_L$ for every c.e. preorder $R$.

LEMMA 6.2 (Montagna & Sorbi, 1985). *$\mathcal{L}^d$-universality implies universality; $\mathcal{L}_{0,1}^d$-universality implies $\mathcal{L}^d$-universality; $\mathcal{B}$-universality implies $\mathcal{L}_{0,1}^d$-universality.*

*Proof.* The first claim follows from the fact that every c.e. preorder can be reduced to the preordering relation of some c.e. distributive prelattice, see for instance, the proof of (Montagna & Sorbi, 1985, Theorem 3.1). The second claim is trivial. For the third claim see (Montagna & Sorbi, 1985, Theorem 3.2). □

THEOREM 6.3 (Montagna & Sorbi, 1985). *Every e.i. Boolean prealgebra is $\mathcal{B}$-universal (by Lemma 6.2, this implies universality and $\mathcal{A}$-universality for the classes $\mathcal{A}$ specified in the Lemma).*

Visser (1982) introduced a c.e. extension of Heyting Arithmetic, called $HA^*$ (which is obtained, as described in (de Jongh & Visser, 1996), as $HA$ plus the Completeness Principle for $HA^*$ by a fixed point construction). Let $\mathrm{HA}^*$ be the Heyting prealgebra of the Lindenbaum sentence algebra of $HA^*$ (i.e., $\mathrm{HA}^* = \mathrm{i}(HA^*)$ in the notation of §3.4)): clearly $\mathrm{HA}^*$ is an e.i. Heyting prealgebra as $HA^*$ extends $HA$.

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Building on some proof techniques appearing in (Shavrukov, 1993; Visser, 1985; Zambella, 1993), De Jongh and Visser (de Jongh & Visser, 1996) prove the following $\mathcal{A}$-universality result, where in this case $\mathcal{A}$ is the class of all c.e. prime Heyting prealgebras (a Heyting algebra is *prime* if its greatest element is join-irreducible).

THEOREM 6.4 (de Jongh & Visser, 1996). HA$^*$ *is* $\mathcal{A}$*-universal.*

***6.1. An e.i. prelattice which not*** $\mathcal{L}^d_{0,1}$***-universal.*** Although every e.i. prelattice is universal, and despite the fact that the e.i. Boolean prealgebras are $\mathcal{B}$-universal by Theorem 6.3, and also the fact there exist e.i. distributive prelattices which are $\mathcal{L}^d_{0,1}$-universal as we will observe in §7, the following theorem shows that one should not be led to conclude that all e.i. prelattices are $\mathcal{L}$-universal, or at least that all e.i. distributive prelattices are $\mathcal{L}^d$-universal.

THEOREM 6.5. *There exist e.i. distributive prelattices that are not* $\mathcal{L}^d$*-universal.*

*Proof.* Since we know from Example 3.15 that there is an e.i. distributive prelattice $L^d_{01}$ which is a free bounded distributive prelattice on a countably infinite set of generators, it is enough to show that for every $X$ not all countable distributive lattices which are isomorphic to quotient structures of c.e. prelattices (see Definition 1.1) can be lattice-embedded into $F^d_{0,1}(X)$: the claim then follows from the fact that by Example 3.15 the quotient structure corresponding to $L^d_{01}$ is isomorphic to some such $F^d_{0,1}(X)$. To see this, take $B$ to be any infinite Boolean algebra. Suppose that we have a lattice-theoretic embedding of $B$ into $F^d_{0,1}(X)$. Clearly $0 \in B$ can not be mapped to $0 \in F^d_{0,1}(X)$, as the former 0 is meet-reducible whereas the latter one is not. So the least element $0 \in B$ must be mapped to an element $a \in F^d(X)$ (recall that $F^d_{0,1}(X) = 0 \oplus F^d(X) \oplus 1$ where $F^d(X)$ is the free distributive lattice on $X$, see (Balbes & Dwinger, 1974)). Likewise, the greatest element $1 \in B$ must be mapped to an element $b \in F^d(X)$. Now each element $x \in F^d(X)$ can be written (in a unique way, called *normal form* of $x$) as $\sum_{i\in I_x} U^x_i$ where $I_x$ is a nonempty finite set, $\sum$ denotes join, each $U^x_i$ is a nonempty finite set of distinct generators, and for simplicity we identify (as in clause notation) such a finite set with the meet of its elements (i.e., to be explicit, $x = \sum_{i\in I_x} \prod U^x_i$, with $\prod$ denoting meet); it can be also assumed that the $U^x_i$ are all $\subseteq$-incomparable (if $U^x_i \subseteq U^x_j$, just delete $j$ from $I_x$); moreover, given $x = \sum_{i\in I_x} U^x_i$ and $y = \sum_{i\in I_y} U^y_i$, we have

$$x \leq y \Leftrightarrow (\forall i \in I_x)(\exists j \in I_y)[U^y_j \subseteq U^x_i].$$

Now suppose that $a = x \wedge y$, and $b = x \vee y$: thus (representing $a, b, x, y$ in normal form) we have

$$\sum_{i\in I_a} U^a_i = \sum_{(u,v)\in I_x\times I_y} (U^x_u \cup U^y_v),$$
$$\sum_{i\in I_b} U^b_i = \sum_{u\in I_x} U^x_u + \sum_{v\in I_y} U^y_v.$$

By selecting the pairs $(u, v)$ for which $U^x_u \cup U^y_v$ is minimal under inclusion (so that its meet is maximal) we can pick a subset $J \subseteq I_x \times I_y$ such that

$$\sum_{i\in I_a} U^a_i = \sum_{(u,v)\in J} (U^x_u \cup U^y_v), \tag{1}$$

and all the $U^x_u \cup U^y_v$, with $(u, v) \in J$, are $\subseteq$-incomparable. Using $\geq$ of (1) it follows that

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$$(\forall (u, v) \in J)(\exists i \in I_a)[U_i^a \subseteq U_u^x \cup U_v^y];$$

on the other hand, using $\leq$ of (1) we have

$$(\forall i \in I_a)(\exists (u, v) \in J)[U_u^x \cup U_v^y \subseteq U_i^a].$$

It follows that

$$(\forall (u, v) \in J)(\exists i \in I_a)(\exists (u', v') \in J)[U_{u'}^x \cup U_{v'}^y \subseteq U_i^a \subseteq U_u^x \cup U_v^y] :$$

by incomparability, for all such pairs $(u, v), (u', v') \in J$ it follows that $U_{u'}^x \cup U_{v'}^y = U_u^x \cup U_v^y$, and thus $U_u^x$ and $U_v^y$ are subsets of $U = \bigcup_{i \in I_a} U_i^a$, for all $(u, v) \in J$.

Consider now the equality $b = \sum_{u \in I_x} U_u^x + \sum_{v \in I_y} U_v^y$ and let us pick subsets $H \subseteq I_x$, $K \subseteq I_y$ which select sets $U_u^x$ and $V_v^y$ which are $\subseteq$-minimal and $b = \sum_{u \in H} U_u^x + \sum_{v \in K} U_v^y$. By arguing as before we can show that for every $u \in H$ and $k \in K$ we have that $U_h^x, U_k^y \subseteq V = \bigcup_{i \in I_b} U_i^b$. Finally, suppose that some $U_i^x$ or some $U_j^y$ has been left out of the reduced family that joins to $b$ (i.e., in the former case $i \notin H$ and the latter case $j \notin K$): assume for instance that $i_0 \in I_x \setminus H$, for some $i_0$. Then (by $\subseteq$-incomparability the sets $U_i^x$) there is $j_0 \in I_y$ such that $U_{j_0}^y \subseteq U_{i_0}^x$; but then for the pair $(i_0, j_0)$ we have $U_{i_0}^x \cup U_{j_0}^y = U_{i_0}^x$. If $(i_0, j_0) \in J$ then we know that $U_{i_0}^x \subseteq U$; on the other hand, if $(i_0, j_0) \notin J$ then there exists a pair $(i_1, j_1) \in J$ such that $U_{i_1}^x \cup U_{j_1}^y \subseteq U_{i_0}^x$, which by incomparability implies $i_0 = i_1$ and thus we have that $U_{i_0}^x \subseteq U$, as $U_{i_1}^x \subseteq U$. If instead for some $j_0$ we have $j_0 \in I_y \setminus K$ so that $U_{j_0}^y$ has been left out of the reduced family that joins to $b$ then by a similar argument we conclude that $U_{j_0}^y \subseteq V$. In conclusion, for all $(u, v) \in I_x \times I_y$ we have that $U_u^x \cup U_v^y \subseteq U \cup V$. Thus there can be finitely only many pairs $x, y$ that meet to $a$ and join to $b$, since the normal forms of $x, y$ may involve only generators in $U \cup V$. But being infinite, the Boolean algebra $B$ has infinitely many distinct pairs of elements that meet to 0 and join to 1. Thus the images under the embedding of these pairs would provide an infinite set of pairs of elements that in $F_{0,1}^d(X)$ meet to $a$ and join to $b$, contradiction. □

REMARK 6.6. *Things are even more dramatic if we consider nondistributive prelattices. Galvin and Jònsson* (*Galvin & Jónsson, 1961*) *prove that a distributive lattice L can be embedded in a free lattice if and only if L is a countable linear sum of lattices where*: *each lattice in the linear sum is either a one element lattice, or an eight element Boolean algebra* $2 \times 2 \times 2$, *or a direct product of the two element chain with a countable chain.*

*Therefore no c.e. distributive prelattice with quotient structure isomorphic to* $0 \oplus L \oplus 1$, *where L is not isomorphic with any of the lattices characterized by Galvin and Jònsson as above, can be reduced to* $L_{01}^{nd}$ (*whose quotient structure is isomorphic to a lattice of the form* $0 \oplus F^{nd}(X) \oplus 1$ *where* $F^{nd}(X)$ *is free on X*) *of Example 3.16, preserving* $\wedge$ *and* $\vee$.

**§7. Some applications.** In this section we give some applications of Theorems 4.2 and 5.7. Our results on universality, local universality, and uniform density hold of *all* e.i. prelattices, which form a large enough class to include some c.e. prestructures of particular interest which have been widely studied and for which some or all of these results have been proved already, and also some interesting cases where these were not known. In §7.1 we list some of these new cases. In §7.2 we will briefly review the literature on what was already known about e.i. prestructures, and universality with respect to c.e. preorders, or even $\mathcal{A}$-universality for various classes $\mathcal{A}$.

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***7.1. New applications.*** In the next three corollaries we point out three applications of our theorems which, in their generality, seem to have gone unnoticed so far.

COROLLARY 7.1. *If $L$ is a c.e. precomplete prelattice then $L$ is locally universal, i.e., in any nontrivial interval $[a, b]_L$ one can embed every c.e. preorder.*

*Proof.* By Corollary 5.3. □

COROLLARY 7.2. *If $T$ is a c.e. consistent extension of $R$ or $Q$ then for every $n \geq 1$ the prelattice of sentences $L_{\Sigma_n/T}$ satisfies*:

1. *$L_{\Sigma_n/T}$ is locally universal*;
2. *$L_{\Sigma_n/T}$ is uniformly dense.*

*Proof.* Use Theorem 3.10: then Claim (1) comes from Corollary 5.3. Claim (2) comes from Theorem 5.7. □

COROLLARY 7.3. *If iT is a c.e. consistent intuitionistic theory extending HA then the Heyting prealgebra* iT *of the Lindenbaum sentence algebra of iT satisfies*:

1. iT *is locally universal*;
2. iT *is uniformly dense.*

*Proof.* Claim (1) comes Corollary 5.3. Claim (2) comes from Theorem 5.7. □

***7.2. Reviewing (some of) the existing literature.*** For e.i. Boolean prealgebras, embedding results via computable functions yielding universality were already known in the literature. We have already seen (Theorem 6.3) that each e.i. Boolean prealgebra is $\mathcal{B}$-universal. This theorem alone is enough to establish local universality of the e.i. Boolean prealgebras: in fact there is no need to directly use Corollary 5.3 as any nontrivial interval of an e.i. Boolean prealgebra is itself computably isomorphic (use Lemma 4.9 and the fact that any such interval is a c.e. Boolean prealgebra) to an e.i. Boolean prealgebra, and thus any such interval is $\mathcal{B}$-universal. Moreover, uniform density holds in any e.i. Boolean prealgebra, as all e.i. Boolean prealgebras are computably isomorphic (see Theorem 8.1) and Shavrukov and Visser (Shavrukov & Visser, 2014) show that uniform density holds in any $B_T$ of Example 3.2.

Let us now move to c.e. precomplete prelattices and prelattices of sentences. Uniform density for c.e. precomplete prelattices was proved by Shavrukov and Visser in (Shavrukov & Visser, 2014). Now, if $T$ is a c.e. consistent extension of elementary arithmetic $EA$ then by Theorem 3.9 $L_{\Sigma_n/T}$ is precomplete, and thus uniformly dense. Moreover, it was already known that for such a $T$, and for $n \geq 1$, $L_{\Sigma_n/T}$ is locally $\mathcal{L}^d_0$-universal, and locally $\mathcal{L}^d_{0,1}$-universal if $n \geq 2$. Indeed $\mathcal{L}^d_0$-universality of any $L_{\Sigma_n/T}$ for $n \geq 1$, and $\mathcal{L}^d_{01}$-universality of any $L_{\Sigma_n/T}$ for $n \geq 2$ was first proved in (Montagna & Sorbi, 1985) where it is shown that the reducing functions can in fact be taken of the form $x \mapsto \gamma(\psi(\overline{x}))$ with $\psi(v)$ a $\Sigma_n$ formula, $\overline{x}$ the numeral term corresponding to $x$, and $\gamma(\psi(\overline{x}))$ the Gödel number of $\psi(\overline{x})$ (under the $\gamma$ exploited in building the prelattice of sentences).

Shavrukov (2010) proved that for $n \geq 1$ the c.e. Boolean prealgebra $B_{\Delta_n/T}$ (having as domain the $\Delta_n$-sentences modulo provable equivalence in $T$) is an e.i. Boolean prealgebra unless $n = 1$ and $T$ is $\Sigma_1$-sound; and if $n \geq 1$, $\alpha, \beta$ are $\Sigma_n$ sentences, $\vdash_T \alpha \to \beta$, and $\nvdash_T \beta \to \alpha$, then $[\alpha, \beta]_{\Delta_n/T}$ (i.e., the $\Delta^0_n$ sentences in the interval) is an e.i. Boolean prealgebra unless $n = 1$, $T$ is $\Sigma_1$-sound, and $T \vdash \beta$. $\mathcal{A}$-universality of $L_{\Sigma_n/T}$ (for the relevant class $\mathcal{A}$) finally follows by $\mathcal{B}$-universality of the e.i. Boolean prealgebras (Theorem 6.3) and Lemma 6.2.

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**§8. Computable isomorphism types of universal c.e. preorders.** It would be interesting to characterize "natural" classes of universal prestructures falling in a single computable isomorphism type, as happens for instance for the e.i. Boolean prealgebras. The following result is due essentially to (Montagna & Sorbi, 1985) and (Nerode & Remmel, 1985), based on (Pour-El & Kripke, 1967).

THEOREM 8.1. *All e.i. Boolean prealgebras are computably isomorphic.*

Some natural computable isomorphism types have been pointed out for universal ceers. We recall that a *diagonal* function for an equivalence relation $R$ is a computable function $\delta$ such that $x \not R \delta(x)$ for all $x$. The following are known: All u.f.p. ceers endowed with a diagonal function are computably isomorphic, see (Montagna, 1982), and independently (Lachlan, 1987); all precomplete ceers are computably isomorphic, see (Lachlan, 1987).

Nonetheless, there are several interesting different computable isomorphism types of universal c.e. preorderings, even if we restrict attention to the preorderings relative to e.i. (hence c.e. and bounded by definition) prelattices.

THEOREM 8.2. *The following list provides classes of e.i. prelattices such that for each pair* $(\mathcal{A}, \mathcal{B})$ *of distinct classes* (*i.e.,* (*1*), (*2a*), (*2b*) (*2c*), (*3*))) *in the list there is a pair of objects* $(A, B) \in (\mathcal{A} \times \mathcal{B})$*, with* $A, B$ *not computably isomorphic.*

1. *e.i. Boolean prealgebras*;
2. *e.i. distributive prelattices*; *inside this class, one can further list the following subclasses*:
   - (*a*) *e.i. Heyting prealgebras*;
   - (*b*) *e.i. free distributive prelattices on a countably infinite set of generators*;
   - (*c*) *c.e. precomplete distributive prelattices, including prelattices of sentences of c.e. consistent extensions of Robinson's system R, or Q*;
3. *e.i. nondistributive prelattices.*

*Proof.* Of course distinctions between some of the classes derive immediately from evident structural differences (for instance, a free distributive lattice can not be isomorphic with a Boolean algebra, and so on) which immediately imply that for a given pair $(A, B) \in (\mathcal{A} \times \mathcal{B})$ of prestructures no isomorphism may exist between the corresponding quotient structures, and thus no computable isomorphism may exist between the prestructures. Apart from these obvious considerations, the proof follows from previous results proved in this article or recalled in this article from the literature. We list here some of the distinguishing properties. No precomplete prelattice has a diagonal function (this is a consequence of the celebrated Ershov Fixed Point Theorem for precomplete equivalence relations). If $L$ is the Heyting prealgebra of the Lindenbaum sentence algebra of $HA$ then $L$ is e.i. and has a diagonal function, so no computably isomorphic copy of it can lie in 2c, and can not lie in 2b either since in every free bounded distributive lattice the least element is meet-irreducible, but this does not hold in the Lindenbaum sentence algebra of $HA$. For the same reason, if $n \geq 1$ and $T$ is a c.e. consistent extension of $EA$ then no isomorphic copy of $L_{\Sigma_n/T}$ can lie in (2b). □

One must not think that each of the classes displayed in Theorem 8.2 contains only one computable isomorphism class. For instance if $T$ is $\Sigma_1$-sound then the precomplete distributive prelattice $L_{\Sigma_1/T}$ is not isomorphic to any $L_{\Sigma_n/T}$, for $n \geq 2$: indeed, by $\Sigma_1$-soundness the greatest element of $L_{\Sigma_1/T}$ is join-irreducible, whereas in each $L_{\Sigma_n/T}$ the

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greatest element is join-reducible, being equal to the join of the equivalence classes of $\sigma$ and $\neg\sigma$, for any sentence $\sigma \in \Delta_n$ which is not a theorem.

E.i. Boolean prealgebras are free on infinitely many generators, since so is $B_T$ of Example 3.2. However, even adding freeness, the analogue of Theorem 8.1 for e.i. distributive prelattices fails, as shown in next theorem.

THEOREM 8.3. *There are e.i. distributive prelattices that are free on countably infinitely many generators but are not computably isomorphic.*

*Proof.* We construct two e.i. bounded distributive prelattices $L_1, L_2$, each of which is free on a countably infinite sets of generators, and such that for every $i$, *requirement* $R_i$ is satisfied, requesting that the partial computable function $\varphi_i$ do not induce an isomorphism between the quotient structures associated with $L_1$ and $L_2$. We take $L_2$ to be the prelattice $L^d_{01}$ of Example 3.15, but this time we show how to build this prelattice and the congruence $\alpha$ via a construction in stages, which we then synchronize with (part of) the construction of $L_1$.

As in the discussion in §3.5 consider a computable presentation $F^d_{0,1}(X) = \langle \omega, \wedge, \vee, 0, 1\rangle$ of the free bounded distributive lattice on a decidable infinite set $X$ (with $0, 1 \notin X$) which is computably listed without repetitions by $\{x_i : i \in \omega\}$, and let $(U, V)$ an e.i. pair of c.e. sets.

We start with specifying a construction of $L_2 = L^d_{01}$ in stages. Assume that $\{U_s : s \in \omega\}, \{V_s : s \in \omega\}$ are computable approximations to $U, V$, respectively, i.e., increasing computable sequences of finite sets giving $U = \bigcup_s U_s$ and $V = \bigcup_s V_s$. Let $X_s = \{x_i : i < s\}$, and let $\alpha_s$ be the congruence on $F^d_{01}(X_s)$ generated by the pairs $\{(x_i, 0) : x_i \in X_s \,\&\, i \in U_s\} \cup \{(x_i, 1) : x_i \in X_s \,\&\, i \in V_s\}$. It is easy to see (First Isomorphism Theorem) that $F^d_{01}(X_s)_{/\alpha_s} \simeq F^d_{01}(X_s \smallsetminus \{x_i : i \in U_s \cup V_s\})$, and if $\alpha$ is the congruence on $F^d_{01}(X)$ exhibited in §3.5 then $\alpha = \bigcup_s \alpha_s$. Taking $L_2 = L^d_{01}$ of Example 3.15 we know that $L_2$ is e.i., $L_{2/\alpha} = F^d_{01}(X)_{/\alpha} \simeq F^d_{01}(X \smallsetminus \{x_i : i \in U \cup V\})$, and $L_2$ is free on a countably infinite set of generators.

We use the sequence $\{F^d_{01}(X_s)_{/\alpha_s} : s \in \omega\}$ to approximate $F^d_{01}(X)_{/\alpha}$ in the construction of $L_1$, which we are now going to describe. Split $X$ as $X = Y \cup Z$, with $Y, Z$ decidable and infinite. Split also $Z$ into two infinite decidable sets $Z_1, Z_2$ so that $Z_1$ is computably enumerated without repetitions by $\{z^1_i : i \in \omega\}$. We define in stages sequences $\{X_{1,s} : s \in \omega\}$, $\{H_s : s \in \omega\}$, $\{\beta_s : s \in \omega\}$, where $H_s \subseteq X_{1,s}$, $X_{1,s}$ is a finite subset of $X$, $\beta_s$ is a congruence on $F^d_{0,1}(X_{1,s})$. Moreover let $G_s = X_{1,s} \smallsetminus H_s$. Recall that in a bounded lattice an element $x$ is meet-irreducible if $x \neq 0, 1$ and is not the meet of any two strictly bigger elements.

*Stage* 0. Let $X_{1,0} = H_0 = \emptyset$ so that $F^d_{0,1}(X_{1,0})$ is the two-element bounded lattice, and let $\beta_0$ be the identity equivalence relation on $\{0, 1\}$.

*Stage* $s + 1$, say $s = \langle i, t\rangle$. If $R_i$ has already acted (as described below), or $z^1_i \notin X_{1,s}$, or $\varphi_i(z^1_i)$ has not as yet converged in fewer than $s$ steps to an $x \in F^d_{0,1}(X_s)$, or $\varphi_i(z^1_i)$ has already converged to an $x \in F^d_{0,1}(X_s)$, such that $[x]_{\alpha_s}$ is meet-reducible in $F^d_{0,1}(X_s)_{/\alpha_s}$, then let $X_{1,s+1} = X_{1,s} \cup \{x_s\}$, $H_{s+1} = H_s$, and let $\beta_{s+1}$ be the congruence on $F^d_{0,1}(X_{1,s+1})$ generated by the pairs in $\beta_s$. Otherwise, $R_i$ *acts* by choosing the least two distinct elements $z^2_{k_1}, z^2_{k_2} \in Z_2$ not as yet used at any stage of the construction; in this case, we let $X_{1,s+1} = X_{1,s} \cup \{x_s, z^2_{k_1}, z^2_{k_2}\}$, we let $\beta_{s+1}$ be the congruence on $F^d_{0,1}(X_{1,s+1})$ generated by $\beta_s$ plus the pair $(z^1_i, z^2_{k_1} \wedge z^2_{k_2})$; and, finally, we let $H_{s+1} = H_s \cup \{z^1_j\}$.

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The following can be easily checked. If $H = \bigcup_s H_s$ and $G = X \setminus H$ then $G$ is comprised of all elements $g \in X$ such that $g$ is not collapsed at some stage to the meet of two elements of $Z_2$ by the c.e. congruence $\beta$ on $F^d_{0,1}(X)$ where $\beta = \bigcup_s \beta_s$; moreover, $Y \subseteq G$ as $Y \cap H = \emptyset$. Let now $\alpha_1$ be the c.e. congruence on $F^d_{0,1}(X)$ generated by the pairs in $\beta \cup \{(y_i, 0) : i \in U\} \cup \{(y_i, 1) : i \in V\}$. On numbers $x, y$ define $x \leq_{L_1} y$ if $[x]_{\alpha_1} = [x \wedge y]_{\alpha_1}$. Then $L_1 = \langle \omega, \wedge, \vee, 0, 1, \leq_{L_1} \rangle$, where $\wedge, \vee$ are the same operations as in $F^d_{0,1}(X)$, is a c.e. prelattice whose associated quotient structure is $F^d_{0,1}(X)_{/\alpha_1}$. By an argument similar to the one used for $\alpha$ and $L^d_{01}$ in §3.5, one can show that $L_1$ is e.i. and $F^d_{0,1}(X)_{/\alpha_1} \simeq F^d_{0,1}(G \setminus \{y_i : i \in U \cup V\})$ so that $L_1$ is free on a countably infinite set of generators.

It remains to show that for every $i$, $\varphi_i$ does not induce an isomorphism from $L_{1/\alpha_1} (= F^d_{0,1}(X)_{/\alpha_1})$ to $L_{2/\alpha} (= F^d_{0,1}(X)_{/\alpha})$. Assume $\varphi_i(z^1_i)$ converges and $\varphi_i(z^1_i) = x$: if $[x]_\alpha \in \{[0]_\alpha, [1]_\alpha\}$ or $[x]_\alpha$ is meet-irreducible then at some stage in the construction of $\beta$ we introduce a pair $z^2_{k_1}, z^2_{k_2}$, and make $[z^1_i]_{\alpha_1}$ meet-reducible since we make $[z^1_i]_{\alpha_1} = [z^2_{k_1}]_{\alpha_1} \wedge [z^2_{k_2}]_{\alpha_1}$, and on the other hand $[z^2_{k_1}]_{\alpha_1}, [z^2_{k_2}]_{\alpha_1}$ are incomparable as under the isomorphism $F^d_{0,1}(X)_{/\alpha_1} \simeq F^d_{0,1}(G \setminus \{y_i : i \in U \cup V\})$ they correspond to two distinct generators of $F^d_{0,1}(G \setminus \{y_i : i \in U \cup V\})$. If $[x]_\alpha$ is meet-reducible then for every $s$ with $x \in F^d_{01}(X_s)$ we have that $[x]_{\alpha_s}$ is meet-reducible, thus $[z^1_i]_{\alpha_1}$ is meet-irreducible in $L_{1/\alpha_1}$ as $[z^1_i]_{\alpha_1}$ corresponds (under the isomorphism ) to a generator of $F^d_{0,1}(G \setminus \{y_i : i \in U \cup V\})$ since $z^1_i$ is never extracted from $G$. This shows that $\varphi_i$ does not induce an isomorphism, since meet-reducibility is a property that is invariant under isomorphisms. □

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