Math. Proc. Camb. Phil. Soc. (1983), **94**, 495
Printed in Great Britain

Complex Einstein hypersurfaces of indefinite complex space forms

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(*Received* 11 *January* 1983; *revised* 29 *April* 1983)

1. *Introduction*

In [8], Fialkow classified Einstein hypersurfaces in indefinite space forms, when the Weingarten endomorphism is diagonalizable. Recently, [13]–[15], Magid completed this work when such endomorphism is not diagonalizable. On the other hand, Smyth [17], classified complex Einstein hypersurfaces in complex space forms. Since Barros & Romero, [1], have made a systematic study of indefinite Kählerian manifolds, in particular, indefinite complex space forms, we investigate here complex Einstein non-degenerate hypersurfaces in indefinite complex space forms. In our case, we cannot always diagonalize the Weingarten endomorphism A associated to a unit normal field because it is a self-adjoint endomorphism in an indefinite metric vector space, but we prove that, if A is not diagonalizable, one has either $A^2 = 0$ and $A \neq 0$ or $A^2 = -\gamma^2 I$, $\gamma \in \mathbb{R}$. We obtain in Theorems 4·4 and 5·5 a full classification except in the case that the ambient space is flat and the Weingarten endomorphism A satisfies $A^2 = 0$ and $A \neq 0$. Our classification is based on extending Smyth's results to show that, except in this last case, an Einstein hypersurface is locally symmetric. Using expressions for the curvature tensor to estimate the dimension of the isotropy group, together with Berger's list of symmetric spaces, [3], we then determine the possible Einstein hypersurfaces.

Manifolds, maps, ... will be understood to be of class C^∞.

We denote $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$ an indefinite complex space form of holomorphic sectional curvature $\bar{c}$ and index $2s+2a$, $a = 0, 1$. According as $\bar{c} > 0$, $\bar{c} < 0$, $\bar{c} = 0$, we can identify $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$ with $\mathbb{C}P^{n+1}_{s+a}(\bar{c})$ the indefinite complex projective space of curvature $\bar{c}$, with $\mathbb{C}H^{n+1}_{s+a}(\bar{c})$ the indefinite complex hyperbolic space of curvature $\bar{c}$, or with $\mathbb{C}^{n+1}_{s+a}$ the indefinite complex flat space (see [1], or [18], p. 378).

Let M^{2n}_{2s} be an indefinite complex hypersurface with index $2s$ of $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$. Let g and J be the metric tensors and complex structures on M^{2n}_{2s} and $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$. By $\bar{\nabla}$, ∇, $\bar{R}$, R we denote the metric connections and curvature tensors of $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$ and M^{2n}_{2s} respectively. Let ξ be a unit ($g(\xi, \xi) = \tau = 1, -1$ according as $a = 0, 1$) local normal field on M^{2n}_{2s} and let A be the Weingarten endomorphism associated to ξ, defined by the equation

$$\bar{\nabla}_X \xi = -AX + f(X) J\xi \quad \text{for all } X \text{ tangent to } M^{2n}_{2s}$$

(tangent and normal components to M^{2n}_{2s}), where f is a local 1-forms field on M^{2n}_{2s}. It is easy to see that the Weingarten endomorphism associated to $J\xi$ is JA and $JA = -AJ$.

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We have also the Gauss and Codazzi equations for curvature operators (see [17])

$$R(X,Y)Z = \bar{R}(X,Y)Z + \tau\{g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY\}, \quad (1\cdot1)$$

$$O = (\nabla_X A)Y - (\nabla_Y A)X - f(X)JAY + f(Y)JAX \quad (1\cdot2)$$

for all X, Y, Z tangent to M_{2s}^{2n}.

As $\bar{M}_{2s+2a}^{2n+2}(\bar{c})$ has constant holomorphic sectional curvature $\bar{c}$, we have (see [1], Proposition 2·1):

$$R(X,Y)Z = (\tfrac{1}{4}\bar{c})\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\} + \tau\{g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY\}, \quad (1\cdot3)$$

and so, as $tr\,A = tr\,JA = 0$, we have

$$S(X,Y) = \tfrac{1}{2}(n+1)\bar{c}g(X,Y) - 2\tau g(A^2X,Y) \quad \text{for all} \quad X, Y \text{ tangent to } M_{2s}^{2n}, \quad (1\cdot4)$$

where S is the Ricci tensor on M_{2s}^{2n}.

2. *On the Weingarten endomorphism of an indefinite complex Einstein hypersurface*

If M_{2s}^{2n} is an Einstein manifold one has $S = \rho g$, $\rho \in \mathbb{R}$, and hence, by using (1·4),

$$A^2 + (\tfrac{1}{4}\tau)\{2\rho - \bar{c}(n+1)\}I = 0. \quad (2\cdot1)$$

So A^2 is globally defined on M_{2s}^{2n} and it is independent of the local field ξ. Then the polynomial equation $p(\lambda) = \lambda^2 + \beta = 0$ is satisfied for all Weingarten endomorphisms on M_{2s}^{2n}, where $\beta = (\tfrac{1}{4}\tau)\{2\rho - \bar{c}(n+1)\}$. Hence, if $x \in M_{2s}^{2n}$:

(*a*) The minimal polynomial of A_x is linear, i.e. $A_x = \alpha I$, $\alpha \in \mathbb{R}$. As $\mathrm{tr}\,A_x = 0$, one has $A_x = 0$. From (2·1) it occurs only when $\rho = \frac{1}{2}(n+1)\bar{c}$.

(*b*) The minimal polynomial of A_x is $p(\lambda)$. (i) If one has $\beta < 0$, $p(\lambda)$ has two non zero different roots $\alpha, -\alpha$, and so the elementary divisors of A_x are of the form $\lambda - \alpha$, $\lambda + \alpha$. There exists (see [16], chapter VII, §28·1, where self-adjoint endomorphisms of an indefinite metric vector space are classified from its elementary divisors) an orthonormal basis of $T_xM_{2s}^{2n}$ in which A_x is represented by the matrix

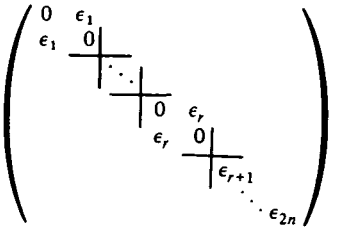
$$\begin{pmatrix} \alpha I_n & \\ & -\alpha I_n \end{pmatrix} \quad (A_x \text{ is diagonalizable}). \quad (2\cdot2)$$

Moreover, if C_α is the α-eigenspace of A_x, then $JC_\alpha = C_{-\alpha}$, and so this orthonormal basis can be taken to be of the form $E_1, \ldots, E_n, JE_1, \ldots, JE_n$. (ii) If $\beta = 0$, i.e. $A_x^2 = 0$ and $A_x \neq 0$, the elementary divisors of A_x are λ, λ^2. From the classification result in [16], there exists a basis $E_1, U_1, \ldots, E_r, U_r, V_{r+1}, \ldots, V_{2n-r}$ of $T_xM_{2s}^{2n}$ with respect to which A_x is represented by

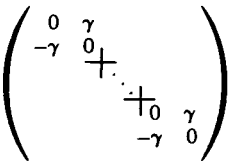
$$\begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots\ 0 \end{pmatrix} \quad (2\cdot3)$$

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and g by

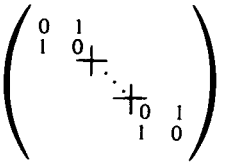


$1 \leqslant r \leqslant n$ and $\epsilon_i = \pm 1$, $1 \leqslant i \leqslant 2n$. Clearly, A_x has maximal rank if and only if $r = n$, and in this case $\operatorname{Ker} A_x = \operatorname{Im} A_x$, and index $M^{2n}_{2s} = 2s = n$. So, $\dim M^{2n}_{2s} = 4s$. (iii) If $\beta = \gamma^2 > 0$, i.e. $A_x^2 = -\gamma^2 I$, $\gamma \neq 0$, then $p(\lambda)$ is an irreducible polynomial. All elementary divisors of A_x are of the form $\lambda^2 + \gamma^2$, and so, from [16], there exists a basis of $T_x M^{2n}_{2s}, V_1, W_1, \ldots, V_n, W_n$, for which A_x is represented by



(2·4)

and g by



So, $2s = \text{index } M^{2n}_{2s} = n$ and $\dim M^{2n}_{2s} = 4s$.

Remark. The authors know of Magid's paper [13] where he obtains these canonical forms for A_x (in the real case) from Petrov's book (A. Z. Petrov, Einstein spaces, Pergamon Press, 1969).

The following result, based on the polynomial equation (2·1) for the Weingarten endomorphism A, shows us that the behaviour of a complex Einstein hypersurface in an indefinite complex space form is quite different from the definite case. In fact, we can prove:

THEOREM 2·1. *Let M^{2n}_{2s} be a complex Einstein hypersurface of an indefinite complex space form. If the Weingarten endomorphism A associated to a unit normal field satisfies $A^2 \neq 0$ or $A = 0$ then M^{2n}_{2s} is locally symmetric.*

Proof. We have from the preceding discussions that A can be diagonalized or $A^2 = -\gamma^2 I$, $\gamma \neq 0$. In the first case, we can prove, in exactly the same way as Smith in [17], from Codazzi's equation (1·2), the following formulae

$$\nabla_X A = f(X) JA \quad \text{and} \quad \nabla_X JA = -f(X) A \tag{2·5}$$

for all X tangent to M.

We suppose now that $A^2 = -\gamma^2 I$ and hence

$$(\nabla_V A) \cdot A + A \cdot (\nabla_V A) = 0 \tag{2·6}$$

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for all V tangent to M, and if X, Y are also tangent to M, then

$$g(AY,(\nabla_V A)X)+g((\nabla_V A)Y,AX)=0 \tag{2·7}$$

because $\nabla_V A$ and A are self-adjoint endomorphisms for g. From (2·7), putting $V=AZ$, Z tangent to M, and by using the Codazzi equation (1·2):

$$g(AY,(\nabla_X A)AZ)-f(X)g(JA^2Z,AY)+f(AZ)g(JAX,AY)$$
$$+g((\nabla_Y A)AZ,AX)+f(AZ)g(JAY,AX)-f(Y)g(JA^2Z,AX)=0.$$

But $A^2=-\gamma^2 I$ and from (2·6):

$$g(Y,(\nabla_X A)Z)+g((\nabla_Y A)Z,X)+f(X)g(JZ,AY)+f(Y)g(JZ,AX)=0.$$

By using the Codazzi equation again:

$$g((\nabla_X A)Y,Z)=f(X)g(JAY,Z)$$

for all X, Y, Z tangent to M. So, in this case, we obtain the same formulae (2·5) as in the first case. It is easy to prove from (2·5) and the Gauss equation (1·1) that $\nabla R=0$.

Remark. This is false if $A\neq 0$ and $A^2=0$ (see § 6).

3. *Some examples*

Example 3·1. As in [1], Example 3·1, we put $\mathbb{C}_s^n$ the complex vector space $\mathbb{C}^n$ endowed with the real part of the Hermitian form $b_{s,n}$ given by

$$b_{s,n}(z,w)=-\sum_{i=1}^{s} z_i\overline{w}_i+\sum_{j=s+1}^{n} z_j\overline{w}_j,$$

where $z=(z_i)$, $w=(w_i)$, $1\leqslant i\leqslant n$. In an obvious way one has a totally geodesic complex immersion of $\mathbb{C}_s^n$ in $\mathbb{C}_{s+a}^{n+1}$, $a=0,1$.

Also in [1], Example 3·2, the 'Hopf fibration' of $\mathbb{C}P_{s+a}^{n+1}(\bar{c})$, $\bar{c}>0$, by the indefinite sphere $S_{2s+2a}^{2(n+1)+1}=\{z\in\mathbb{C}_{s+a}^{n+2}/b_{s+a,n+2}(z,z)=4/\bar{c}\}$ with structural group S^1 and projection π has been studied. A totally geodesic immersion of S_{2s}^{2n+1} in S_{2s+2a}^{2n+3} provides another complex totally geodesic immersion of $\mathbb{C}P_s^n(\bar{c})$ in $\mathbb{C}P_{s+a}^{n+1}(\bar{c})$ with $a=0,1$. Since $\mathbb{C}H_{n-s-a+1}^{n+1}(-\bar{c})$ is obtained from $\mathbb{C}P_{s+a}^{n+1}(\bar{c})$ by changing the metric tensor g to its negative, this immersion provides another complex totally geodesic immersion of $\mathbb{C}H_{n-s}^{n}(-\bar{c})$ in $\mathbb{C}H_{n-s-a+1}^{n+1}(-\bar{c})$.

Example 3·2. Let M_{2s}^{2n} be the complex hypersurface of $\mathbb{C}P_s^{n+1}(\bar{c})$ which is obtained by projecting $N=\{z\in S_{2s}^{2n+3}/-z_1^2-z_2^2-\ldots-z_s^2+z_{s+1}^2+\ldots+z_{n+2}^2=0\}$. We call M_{2s}^{2n} the indefinite complex quadric and we denote it by $\mathbb{C}Q_s^n$. In a similar way to [11], chapter XI, example 10·6, one can see that $\mathbb{C}Q_s^n$ is an indefinite complex Einstein hypersurface of $\mathbb{C}P_s^{n+1}(\bar{c})$ with $\rho=(\frac{1}{2}n)\bar{c}$ and that it is identifiable with the Hermitian (half-Kählerian according to Berger, see [3], p. 158) symmetric non-compact space

$$SO^s(n+2)/SO^s(n)\times SO(2)$$

endowed with the negative of the Killing form of $SO^s(n+2)$. As in Example 3·1, one obtains another indefinite complex Einstein hypersurface M'^{2n}_{2n-2s} in $\mathbb{C}H_{n-s+1}^{n+1}(-\bar{c})$ which is identifiable with $SO^{n-s}(n+2)/SO^{n-s}(n)\times SO(2)$ endowed with the negative of the Killing form of $SO^{n-s}(n+2)$.

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Example 3·3. Let M_{2s}^{4s} be the complex hypersurface of $\mathbb{C}P_{s+1}^{2s+1}(\bar{c})$ which is obtained by projecting $N' = \{z \in S_{2s+2}^{4s+3}/z_1 z_{s+2} + z_2 z_{s+3} + \ldots + z_{s+1} z_{2s+2} = 0\}$. It is easy to see that M_{2s}^{4s} is an indefinite complex Einstein hypersurface of $\mathbb{C}P_{s+1}^{2s+1}(\bar{c})$ with $A^2 = -\gamma^2 I$ and $\gamma^2 = \frac{1}{4}\bar{c}$. One sees that the Lie group $SO^*(2s+2)$ ([3], p. 113), considered as a subgroup of $SU^{s+1}(2s+2)$, acts on $\mathbb{C}P_{s+1}^{2s+1}(\bar{c})$ and leaves M_{2s}^{4s} invariant. Moreover this action is transitive and the isotropy group in any point of M_{2s}^{4s} is isomorphic to the subgroup $SO^*(2s) \times SO(2)$ of $SO^*(2s+2)$. Then M_{2s}^{4s} can be identified with the Hermitian non compact symmetric space $SO^*(2s+2)/SO^*(2s) \times SO(2)$ endowed with the negative of the Killing form of $SO^*(2s+2)$. We denote it by $\mathbb{C}Q_s^*$. A reasoning as in Example 3·1 proves that $\mathbb{C}Q_s^*$ is also an indefinite complex Einstein hypersurface of $\mathbb{C}H_s^{2s+1}(-\bar{c})$ with $A^2 = -\gamma^2 I$ and $\gamma^2 = \frac{1}{4}\bar{c}$.

Example 3·4. Let V be a complex vector space with $\dim_{\mathbf{C}} V = s$ and $\mathbb{C}_q^r$ as in Example 3·1. We denote $V^* = Hom_{\mathbf{C}}(V, \mathbb{C})$. We take $e_1, \ldots, e_s$ a basis in V and let $\phi_1, \ldots, \phi_s$ by its dual basis in V^*. If $x = z_1 e_1 + \ldots + z_s e_s$, we put

$$\bar{x} = \bar{z}_1 e_1 + \ldots + \bar{z}_s e_s \quad \text{and} \quad x^* = z_1 \phi_1 + \ldots + z_s \phi_s.$$

If $\phi = w_1 \phi_1 + \ldots + w_s \phi_s$ we also put $\overline{\phi} = \overline{w}_1 \phi_1 + \ldots + \overline{w}_s \phi_s$. It is clear that the complex vector space $\mathbb{C} \times V \times V^* \times \mathbb{C}_q^r$ endowed with the real part of the Hermitian form B given by

$$B((z, x, \phi, v), (t, y, \psi, w)) = z\bar{t} - \tfrac{1}{2}(\phi(\bar{y}) + \overline{\psi}(x)) + b_{q,r}(v, w)$$

can be identified with the indefinite flat complex space form $\mathbb{C}_{s+q}^{2s+r+1}$. Let M_{2s+2q}^{4s+2r} be the complex hypersurface of $\mathbb{C}_{s+q}^{2s+r+1}$ given by

$$M_{2s+2q}^{4s+2r} = \{(z, x, \phi, v) \in \mathbb{C}_{s+q}^{2s+r+1}/4z = x^*(x)\}.$$

One can see that M_{2s+2q}^{4s+2r} is an indefinite complex Einstein hypersurface of $\mathbb{C}_{s+q}^{2s+r+1}$ with $A^2 = 0$ (this space is Ricci-flat but non flat). If $r = q = 0$, then the rank of A is maximal in each point of M_{2s}^{4s}.

4. *The diagonalizable case*

This case includes previous (a) and (i) of (b) in § 2. We use a similar method to Smyth in [17], and we call it 'proper case'. Here, we have that A_x is represented by (2·2) with $\alpha \geqslant 0$.

PROPOSITION 4·1. *Let M_{2s}^{2n} be a proper indefinite complex Einstein hypersurface of $\overline{M}_{2s+2a}^{2n+2}(\bar{c})$ with $n > 2$. Then at each point of M_{2s}^{2n} there exists a non degenerate holomorphic section with sectional curvature $\bar{c}$.*

Proof. If $n > 2$, it occurs $s > 2$ or $n - s > 2$. So, we take vectors E_i, E_j in the basis which diagonalizes A_x in (2·2), with $g(E_i, E_i) = g(E_j, E_j) = 1$ or -1. Then

$$X = (E_i + E_j)/\sqrt{2}$$

is a non null vector in $T_x M$. From the Gauss equation (1·3) one sees that

$$R(X, JX, JX, X) = \bar{c}.$$

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COROLLARY 4·2. *Let M^{2n}_{2s} be a proper indefinite complex Einstein hypersurface of $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$. If $n > 2$, the following assertions are equivalent:* (i) *M^{2n}_{2s} is totally geodesic in $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$.* (ii) *$M^{2n}_{2s}$ has constant holomorphic sectional curvature.*

Proof. If M is totally geodesic in $\bar{M}$, then $A = 0$ and from the Gauss equation (1·3) and [1], proposition 2·1, one finishes. Conversely, from Proposition 4·1, M has constant holomorphic sectional curvature $\bar{c}$. Moreover, if X is a unit vector tangent to M, according to the Gauss equation (1·3), we have

$$\bar{c} = R(X, JX, JX, X) = \bar{c} - 2r\{g(AX, X)^2 + g(JAX, X)^2\}$$

and so $g(AX, X) = 0$ for all X tangent to M and hence $A = 0$.

Remark. This gives a short proof of the equivalence of statements (i) and (ii) of theorem 1 of Smith's work. Note that the statement (iii) of this theorem is now absurd because if at one point of M^{2n}_{2s} all sectional curvatures are bounded then the sectional curvature is constant at this point [12]. On the other hand, if M^{2n}_{2s} is totally geodesic one has by (1·3)

$$R(X, Y, Y, X) = (\tfrac{1}{4}\bar{c})\{\pm 1 + 3g(JX, Y)^2\},$$

where $g(X, Y) = 0, g(X, X) = \pm 1, g(Y, Y) = \pm 1$ and one can obtain different sectional curvatures.

Now, we investigate the restricted holonomy group of these manifolds.

THEOREM 4·3. *Let M^{2n}_{2s}, $n > 2$, be a proper indefinite complex Einstein hypersurface of $\bar{M}^{2n+2}_{2s+2a}(\bar{c})$ and let $\mathfrak{h}$ be the Lie algebra of the restricted holonomy group in a point x of M^{2n}_{2s}.* (i) *If $\bar{c} \neq 0$ and $\rho \neq (n/2)\bar{c}$, then $\dim \mathfrak{h} \geqslant \frac{1}{2}n(n+1)$.* (ii) *If $\bar{c} = 0$ and $\rho \neq 0$, then $\dim \mathfrak{h} = n^2 \geqslant \frac{1}{2}n(n+1)$. If $\bar{c} = 0$ and $\rho = 0$, then M^{2n}_{2s} is flat.* (iii) *Moreover, the action of $\mathfrak{h}$ on $T_x M^{2n}_{2s}$ is always weakly irreducible and completely reducible if $\mathfrak{h} \neq 0$ (see* [6], p. 24 *for definitions).*

Proof. Let $E_1, \ldots, E_s, U_{s+1}, \ldots, U_n, JE_1, \ldots, JE_s, JU_{s+1}, \ldots, JU_n$ be an orthonormal basis of $T_x M$ with $g(E_i, E_i) = -1$ and $g(U_j, U_j) = 1$ as in (2·2). Then $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{u}^s(n)$ of the structural group $U^s(n)$ of the principal bundle of orthonormal frames on M. Since the complex structure J and the metric tensor g, referred to our basis, are respectively given by

$$\left(\begin{array}{c|c} & -I_n \\ \hline I_n & \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} G^s_n & \\ \hline & G^s_n \end{array}\right)$$

where $G^s_n = \operatorname{diag}(-I_s, I_{n-s})$, we have that the endomorphisms of $\mathfrak{h}$ are of the form:

$$\left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right)$$

where A, B are $n \times n$-matrices which verify

$$AG^s_n - G^s_n A^t = 0 \quad \text{and} \quad BG^s_n - G^s_n B^t = 0. \tag{4·1}$$

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It is well known that the Lie algebra $\mathfrak{h}$ is spanned for all endomorphisms of T_xM of the form $R(X, Y)$ with $X, Y \in T_xM$. By using the Gauss equation (1·3) and from a single calculation one can obtain the endomorphisms $R(E_r, E_t)$, $R(U_i, U_j)$, $R(E_r, JE_r)$, $R(U_i, JU_i), R(E_r, U_i), R(E_r, JU_i), R(E_r, JE_t), R(U_i, JU_j), r \neq t, i \neq j, 1 \leqslant r, t \leqslant s, s+1 \leqslant i, j \leqslant n$. Also we obtain in a similar way to [17],

$$-\sum_{r=1}^{s} R(E_r, JE_r)E_i + \sum_{r=s+1}^{n} R(U_r, JU_r)E_i = -2\rho JE_i,$$

$$-\sum_{r=1}^{s} R(E_r, JE_r)U_j + \sum_{r=s+1}^{n} R(U_r, JU_r)U_j = -2\rho JU_j.$$

So, if $\rho \neq 0$, then $J \in \mathfrak{h}$. Hence $\psi_i = R(U_i, JU_i) + \frac{1}{2}\bar{c}J$, $\phi_r = R(E_r, JE_r) - \frac{1}{2}\bar{c}J$ are in $\mathfrak{h}$ and

$$\psi_i U_j = 0, \quad j \neq i, \quad \psi_i U_i = \tfrac{1}{2}(\bar{c}n - 2\rho)JU_i,$$
$$\phi_t E_r = 0, \quad t \neq r, \quad \phi_t E_t = \tfrac{1}{2}(2\rho - \bar{c}n)JE_t,$$
$$\psi_i E_r = 0 \quad \text{for all} \quad r, \quad \phi_t U_j = 0 \quad \text{for all} \quad j.$$

(i) Now, we suppose $\bar{c} \neq 0, \rho \neq 0$ and $\rho \neq \frac{1}{2}n\bar{c}$. Then the endomorphisms $R(E_r, JE_t)$, $R(U_i, JU_j)$, $R(E_r, JU_i)$, ϕ_r and ψ_i span the matrices of $\mathfrak{h}$ of the form

$$\left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right) \tag{4·2}$$

where B is a $n \times n$-matrix with $BG_n^s - G_n^s B^t = 0$. If also $\rho \neq \frac{1}{2}(n+2)\bar{c}$, the endomorphisms $R(E_r, E_t)$, $R(U_i, U_j)$, $R(E_r, U_i)$ span the matrices of $\mathfrak{h}$ of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right) \tag{4·3}$$

where A is a $n \times n$-matrix with $AG_n^s + G_n^s A^t = 0$. In this case $\mathfrak{h} = \mathfrak{u}^s(n)$ and so $\dim \mathfrak{h} = n^2 \geqslant \frac{1}{2}n(n+1)$, and the action of $\mathfrak{h}$ on T_xM is irreducible and hence weakly irreducible and completely reducible. If $\rho = \frac{1}{2}(n+2)\bar{c}$, then $\mathfrak{h}$ contains as a subspace all the endomorphisms of the form (4·2). However, non-trivial subspaces of T_xM are not invariant by the action of this subspace. So, this action is also irreducible. Moreover, as the dimension of this subspace is $\frac{1}{2}n(n+1)$ one has $\dim \mathfrak{h} \geqslant \frac{1}{2}n(n+1)$. If $\bar{c} \neq 0$ and $\rho = 0$, then we have $\rho \neq \frac{1}{2}(n+2)\bar{c}$ and $\rho \neq \frac{1}{2}n\bar{c}$. So, the endomorphisms $R(E_r, E_t)$, $R(U_i, U_j)$, $R(E_r, U_i)$ span the matrices of $\mathfrak{h}$ of the form (4·3) and the endomorphisms $R(E_r, JE_t)$, $R(U_i, JU_j)$, $R(E_r, JU_i)$ span the matrices of $\mathfrak{h}$ of the form (4·2) where B is a $n \times n$-matrix with $BG_n^s - G_n^s B^t = 0$ and all its diagonal elements are zero. These matrices form a subspace of $\mathfrak{h}$ of dimension $n(n-1)$. So, $\dim \mathfrak{h} \geqslant n(n-1) \geqslant \frac{1}{2}n(n+1)$ because $n > 2$. Also, non-trivial subspaces of T_xM are not invariant by the action of this subspace.

(ii) If $\bar{c} = 0$ and $\rho \neq 0$, one has also $\rho \neq \frac{1}{2}n\bar{c}$ and $\rho \neq \frac{1}{2}(n+2)\bar{c}$. So, $\mathfrak{h} = \mathfrak{u}^s(n)$ and one finishes in this case. If $\bar{c} = 0$ and $\rho = 0$, it is $\alpha = 0$ in (2·2) and M is totally geodesic in $\bar{M}$. Then M has constant holomorphic sectional curvature 0.

Now, we can state

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THEOREM 4·4. *If $n > 2$, we have:* (i) *The proper indefinite complex Einstein hypersurfaces of $\mathbb{C}P^{n+1}_{s+a}(\bar{c})$, $\bar{c} > 0$ which are complete and 1-connected are only $\mathbb{C}P^n_s(\bar{c})$ $(a = 0, 1)$ and $\mathbb{C}Q^n_s$ with $a = 0$.* (ii) *In $\mathbb{C}H^{n+1}_{s+a}(-\bar{c})$ are only $\mathbb{C}H^n_s(-\bar{c})$ $(a=0, 1)$ and $\mathbb{C}Q^n_s$ with $a=1$.* (iii) *In $\mathbb{C}^{n+1}_{s+a}$ are only $\mathbb{C}^n_s$ $(a = 0, 1)$.*

Proof. From Theorem 2·1, as we suppose that M is complete and 1-connected, M is an indefinite Hermitian symmetric space (half-Kählerian according to Berger). Moreover, from Theorem 4·3 (iii) and by using [6], proposition 5·4, p. 24, M is identifiable with a coset space G/H where G is a semisimple Lie group. So, it must appear in the Berger table ([3], tableau II, p. 157). As G is semisimple (see § 3, [4] p. 92) $\dim \mathfrak{h}$ coincides with the dimension of the linear isotropy group of G/H.

(i) If $\rho \neq \frac{1}{2}n\bar{c}$ we have $\dim \mathfrak{h} \geqslant \frac{1}{2}n(n+1)$ according to Theorem 4·3. Half-Kählerian symmetric spaces in the Berger table with real dimension $2n$ and $\dim \mathfrak{h} \geqslant \frac{1}{2}n(n+1)$ are only

$$\left.\begin{aligned} &SU^i(m)/SU^k(k+h)\times SU^{i-k}(m-k-h)\times SO(2) \quad \text{with} \quad k+h=1 \\ &\qquad\qquad \text{or} \quad m-k-h=1 \quad (m=n+1) \\ &SO^*(6)/SU^i(3)\times SO(2) \\ &SO^{2i}(6)/SU^i(3)\times SO(2) \end{aligned}\right\}. \tag{4·4}$$

We have local isomorphisms $SO(6) = SO^6(6) = SU(4)$, $SO^2(6) = SO^4(6) = SU^2(4)$ and $SO^*(6) = SU^1(4)$ (see [9], p. 352) and so we can consider only the first coset space in (4·4). Hence M is isometric, up to a constant factor, to $SU^s(n+1)/SU^s(n)\times SO(2)$ or $SU^{s+1}(n+1)/SU^s(n)\times SO(2)$. Therefore (see [1], theorem 3·4) to an indefinite complex projective or hyperbolic space $\mathbb{C}P^n_s(c)$, $c > 0$, or $\mathbb{C}H^n_s(d)$, $d < 0$. It is easy to see that a similar reasoning as in [10], corollary, p. 120, proves that a complex structure in $\mathbb{C}P^n_s(c)$ or $\mathbb{C}H^n_s(d)$ endowed with the metric tensors as in [1], which makes it indefinite Kählerian is unique up to conjugation. So, M is holomorphically isometric to $\mathbb{C}P^n_s(c)$ or $\mathbb{C}H^n_s(d)$. From Proposition 4·1 the last case is impossible and the first case occurs with $c = \bar{c}$. In view of Corollary 4·2 M is totally geodesic in $\mathbb{C}P^{n+1}_{s+a}(\bar{c})$ and hence M is immersed as a complex projective hyperplane.

If $\rho = \frac{1}{2}n\bar{c}$, one has $\alpha^2 = (\frac{1}{2}\tau)\{\bar{c}(n+1)-2\rho\} = \tau\bar{c} > 0$ and so if $\bar{c} > 0$ then $\tau = 1$. From Example 3·2 and an analogue to [11], chapter VI, theorem 7·8 (see also [17], proposition 11), one sees that in this case M is holomorphically isometric to the indefinite complex quadric $\mathbb{C}Q^n_s$ immersed in $\mathbb{C}P^{n+1}_s(\bar{c})$. We now show that two different complex imbeddings ω_1 and ω_2 of $\mathbb{C}Q^n_s$ in $\mathbb{C}P^{n+1}_s(\bar{c})$ differ only by a rigid motion ϕ of $\mathbb{C}P^{n+1}_s(\bar{c})$. It is hard to see that a reasoning as in [17] (based on Calabi, 'Isometric imbedding of complex manifolds', *Ann. of Math.*, **58** (1953), 1–23) remains valid here. We utilize a different argument in which it is essential that $\mathbb{C}Q^n_s$ is a complex hypersurface. Firstly, we prove that, for such two imbeddings, we can choose local unit normal fields ξ_1 and ξ_2 such that the respective Weingarten endomorphisms A_1 and A_2 coincide. In fact, let ξ_1 be arbitrary for ω_1 and let $\bar{\xi}$ be arbitrary for ω_2. Its Weingarten endomorphisms A_1 and $\bar{A}$ satisfy (from the Gauss equation (1·1)):

$$\begin{aligned} &g(A_1Y,Z)A_1X - g(A_1X,Z)A_1Y + g(JA_1Y,Z)JA_1X - g(JA_1X,Z)JA_1Y \\ &\qquad = g(\bar{A}Y,Z)\bar{A}X - g(\bar{A}X,Z)\bar{A}Y + g(J\bar{A}Y,Z)J\bar{A}X - g(J\bar{A}X,Z)J\bar{A}Y \end{aligned}$$

and $A_1^2 = \bar{A}^2 = \bar{c}I$ (from (2·1)), for all X, Y, Z tangent to $\mathbb{C}Q^n_s$. Putting $Y = JX$ and $Z = A_1W$ where $g(X, W) = g(JX, W) = 0$, one obtains

$$g(A_1\bar{A}JX, W)\bar{A}X - g(A_1\bar{A}X, W)\bar{A}JX = 0$$

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and so $g(A_1\bar{A}JX, W) = g(A_1\bar{A}X, W) = 0$ for all X tangent to $\mathbb{C}Q_s^n$ and W orthogonal to X and JX because $\bar{A}$ is non singular. Hence the holomorphic section span $\{X, JX\}$ is invariant by $A_1\bar{A}$ when X is a non null vector. A short reasoning of continuity proves that this occurs for all holomorphic sections. Then, if X is tangent to $\mathbb{C}Q_s^n$, we have $A_1\bar{A}X = aX + bJX$ with $a, b \in \mathbb{R}$, and, if Y is also tangent to $\mathbb{C}Q_s^n$, then

$$A_1\bar{A}Y = cY + dJY$$

with $c, d \in \mathbb{R}$. As $A_1^2 = \bar{A}^2 = \bar{c}I$, we have

$$\bar{A}X = pA_1X + qJA_1X \quad \text{and} \quad \bar{A}Y = rA_1Y + sJA_1Y$$

with $p, q, r, s \in \mathbb{R}$. On the other hand one has $\bar{A}(X+Y) = uA_1(X+Y) + vJA_1(X+Y)$ and, so, if X, Y are linearly independent, we have $p = r = u$ and $q = s = v$ because A_1 is non singular. Finally, we conclude that $\bar{A} = pA_1 + qJA_1$. Moreover, as $A_1^2 = \bar{A}^2 = \bar{c}I$ and $A_1JA_1 = -JA_1^2$, $p^2 + q^2 = 1$. We put $\xi_2 = p\xi + qJ\xi$ and we obtain that the Weingarten endomorphisms A_1 and A_2 associated to ξ_1 and ξ_2 respectively coincide. Now, let x any point of $\mathbb{C}Q_s^n$ and U a neighbourhood of x on which ξ_1 and ξ_2 are defined as before. We consider the holomorphic linear isometry $\theta: T_{\omega_1(x)}\mathbb{C}P_s^{n+1} \to T_{\omega_2(x)}\mathbb{C}P_s^{n+1}$ given by

$$\theta(\xi_1) = \xi_2, \quad \theta(J\xi_1) = J\xi_2, \quad \theta(d\omega_1(X)) = d\omega_2(X) \quad \text{for all} \quad X \text{ in } T_x\mathbb{C}Q_s^n.$$

As in [11], chapter VI, theorem 7·8 we have a holomorphic motion ϕ of $\mathbb{C}P_s^{n+1}(\bar{c})$ with $\phi\omega_1(x) = \omega_2(x)$, $d\phi \cdot d\omega_1 = d\omega_2$ and $d\phi(\xi_1) = \xi_2$ at the point x. Moreover, as $A_1 = A_2$ on U, we have $\phi \cdot \omega_1 = \omega_2$ on U by a reasoning as in [4], p. 207–208. From considerations of analyticity it follows that $\phi \cdot \omega_1 = \omega_2$ on $\mathbb{C}Q_s^n$ ([11], chapter VI, theorem 6·1).

(ii) All arguments here are the same as in (i).

(iii) Now, if $\rho \neq 0$, then M is holomorphically isometric to one of the symmetric spaces given in (4·4), because $\dim \mathcal{H} \geqslant \frac{1}{2}n(n+1)$, according to Theorem 4·3. So, M has constant holomorphic sectional curvature c and $c = 0$ from Proposition 4·1. But this is not possible because the symmetric spaces in (4·4) are indefinite complex projective or hyperbolic spaces. Hence $\rho = 0$ and by using Theorem 4·3 M is flat. One finishes now from [1].

5. *The case* $A^2 = -\gamma^2 I$

We need to know the behaviour of the complex structure of M_{2s}^{4s} with respect to the basis of $T_xM_{2s}^{4s}$, $x \in M_{2s}^{4s}$, where A is of the form as in (2·4).

PROPOSITION 5·1. *Let M_{2s}^{4s} be an indefinite complex Einstein hypersurface of $\bar{M}_{2s+2a}^{4s+2}(\bar{c})$ with $A^2 = -\gamma^2 I$, $\gamma \neq 0$. If $x \in M_{2s}^{4s}$, there exists a basis in $T_xM_{2s}^{4s}$: $E_1, U_1, \ldots, E_s, U_s$, $E_{1^*}, U_{1^*}, \ldots, E_{s^*}, U_{s^*}$, where $E_{i^*} = JU_i$, $U_{i^*} = JE_i$, $1 \leqslant i \leqslant s$, in which A and g are of the form* (2·4).

Proof. We can take a null vector X in T_xM, $X \neq 0$, with $g(AX, X) \neq 0$ from (2·4). Let $Y = \gamma X + JAX$, then span $\{Y, AY\}$ is a Lorentz A-invariant plane. We call it π. Moreover it is orthogonal to $J\pi$. So, A is a self-adjoint endomorphism on π, and, from the Maltsev result, there exists a basis E_1, U_1 in π in which A and g are respectively represented by

$$\begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Then $E_{1^*} = JU_1$, $U_{1^*} = JE_1$ is a basis of $J\pi$ in the same conditions. As $(\pi \oplus J\pi)^\perp$ (orthogonal complement in T_xM) is J and A-invariant one proves the Proposition following in the same way.

Firstly, we shall prove that a complex Einstein hypersurface with $A^2 = -\gamma^2 I, \gamma \neq 0$ cannot be immersed in any complex space form. For this, we prove

LEMMA 5·2. *Under the hypothesis of Proposition* 5·1 *one has*

$$R(X, Y)AZ = AR(X, Y)Z - \lambda g(X, JY)JAZ$$

for all X, Y, Z *tangent to* M_{2s}^{4s} *and where* $\lambda = 2\tau\gamma^2 - \frac{1}{2}\bar{c}$.

Proof. The equalities (2·5) give

$$\nabla_X AY = A\nabla_X Y + f(X)JAY \quad \text{and} \quad \nabla_X JAY = JA\nabla_X Y - f(X)AY. \tag{5·1}$$

Now, as $R(X, Y)AZ = \nabla_X \nabla_Y AZ - \nabla_Y \nabla_X AZ - \nabla_{[X,Y]}AZ$, from (5·1)

$$R(X, Y)AZ = AR(X, Y)Z + 2df(X, Y)JAZ \tag{5·2}$$

for all X, Y, Z tangent to M. But, as in [17], proposition 4, we have

$$\bar{S}(X, JY) = S(X, JY) + 2df(X, Y),$$

where $\bar{S}$ is the Ricci tensor of $\bar{M}$. So, $2df(X, Y) = \frac{1}{2}\{(n+2)\bar{c} - 2\rho\}g(X, JY)$ and from (2·1) we have $\frac{1}{2}(n+2)\bar{c} - \rho = -2\tau\gamma^2 + \frac{1}{2}\bar{c}$ and we finish.

A consequence is

THEOREM 5·3. *Let* M_{2s}^{4s} *be an indefinite complex Einstein hypersurface of* $\bar{M}_{2s+2a}^{4s+2}(\bar{c})$ *with* $A^2 = -\gamma^2 I$. *Then* $\bar{c} = -4\tau\gamma^2$.

Proof. From the Gauss equation (1·3) where we put $X = E_1$, $Y = U_1$ and $Z \in J$ span $\{E_1, U_1\}$ (E_1, U_1 as in Proposition 5·1) we have

$$\begin{aligned} R(E_1, U_1)AZ &= (\tfrac{1}{4}\bar{c})\{g(JU_1, AZ)JE_1 - g(JE_1, AZ)JU_1\} \\ &\quad + \tau\gamma^2\{g(JE_1, Z)JAU_1 + g(JU_1, Z)JAE_1\} \\ AR(E_1, U_1)Z &= (\tfrac{1}{4}\bar{c})\{g(JU_1, Z)AJE_1 - g(JE_1, Z)AJU_1\} \\ &\quad + \tau\gamma^2\{g(JAE_1, Z)JU_1 + g(JAU_1, Z)JE_1\}. \end{aligned}$$

Here, we put $Z = JE_1$ and we obtain, according to Lemma 5·2,

$$(\tfrac{1}{4}\bar{c})\gamma JU_1 + \tau\gamma^3 JU_1 = -(\tfrac{1}{4}\bar{c})\gamma JU_1 - \tau\gamma^3 JU_1$$

and, as $\gamma \neq 0$, one concludes.

COROLLARY 5·4. *Let* M_{2s}^{4s} *be an indefinite complex Einstein hypersurface of* $\bar{M}_{2s+2a}^{4s+2}(\bar{c})$ *with* $A^2 = -\gamma^2 I$. *Then* $\bar{c} \neq 0$ *and* $\bar{c} > 0$ (*resp.* $\bar{c} < 0$) *if and only if* $\tau = -1$ (*resp.* $\tau = 1$).

Since with these conditions M_{2s}^{4s} is locally symmetric, and moreover γ is completely determined from the holomorphic sectional curvature $\bar{c}$ of the ambient space, one has the next classification result:

THEOREM 5·5. $\mathbb{C}Q_s^*$ *is the unique indefinite complex Einstein hypersurface of* $\mathbb{C}P_{s+1}^{2s+1}(\bar{c})$, $\bar{c} > 0$, *and of* $\mathbb{C}H_s^{2s+1}(-\bar{c})$ *which is complete and* 1*-connected and which satisfies* $A^2 = -\gamma^2 I$. *If* $n \neq 2s$ *or* $a \neq 1$ (*resp.* $n \neq 2s$ *or* $a \neq 0$) *there do not exist hypersurfaces satisfying these conditions in* $\mathbb{C}P_{s+a}^{n+1}(\bar{c})$ (*resp.* $\mathbb{C}H_{s+a}^{n+1}(-\bar{c})$).

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Proof. It is immediate from Example 3·3, a similar argument as in [11], chapter VI, theorem 7·8 and the reasoning for rigidity in Theorem 4·4 above.

Remark 1. In this case, the Weingarten endomorphism A on M^{4s}_{2s} provides a complex structure $K = A/\gamma$, such that $KJ = -JK$ and $g(KX, KY) = -g(X, Y)$ for all X, Y tangent to M^{4s}_{2s}. Such a manifold can be called 'anti-Kähler quaternion manifold' (see [14], where anti-Kähler manifolds are defined).

Remark 2. The indefinite complex Einstein hypersurfaces M^{4s}_{2s} of an indefinite complex space form with $A^2 = -\gamma^2 I$, which are complete and 1-connected have the same homotopy type as a definite complex projective space $\mathbb{C}P^s$. It is easy to see from [3], corollaire, p. 166, or from [2], theorem 1, since the maximal compact subgroup of $SO^*(2n)$ is $SU(n) \times SO(2)$.

6. *The case* $A^2 = 0$

Let M^{4r+2p}_{2r+2h} be an indefinite complex Einstein hypersurface of $\bar{M}^{4r+2p+2}_{2r+2h+2a}(\bar{c})$ with $A^2 = 0$ and $A \neq 0$ in each point of M^{4r+2p}_{2r+2h}. From the Codazzi equation (1·2) and by using that $A^2 = 0$ and the fact that since here (2·6) and (2·7) also occur, one has

$$A \cdot (\nabla_X A) = (\nabla_X A) \cdot A = 0 \tag{6·1}$$

for all X tangent to M^{4r+2p}_{2r+2h}. Therefore

$$A \nabla_X AY = 0 \quad \text{and} \quad \nabla_{AX} AY = A \nabla_{AX} Y \tag{6.2}$$

for all X, Y tangent to M^{4r+2p}_{2r+2h}. Now, we suppose that rank A is maximal in each point of M^{4r+2p}_{2r+2h}. Then (2·3), $p = h = 0$ and $\operatorname{Im} A_x = \operatorname{Ker} A_x$ for all $x \in M^{4r}_{2r}$. Let $E_1, U_1, \ldots, E_{2r}, U_{2r}$ be a basis of $T_x M^{4r}_{2r}$ as in (2·3). We extend $U_1, \ldots, U_{2r}$ near x to differentiable fields $Y_1, \ldots, Y_{2r}$. Now, we put $X_1 = AY_1, \ldots, X_{2r} = AY_{2r}$. Since

$$(X_1)_x = E_1, \ldots, (X_{2r})_x = E_{2r},$$

the fields $X_1, \ldots, X_{2r}$ are linearly independent near x. Then $\operatorname{Ker} A$ is a differentiable distribution on M^{4r}_{2r} of dimension $2r$ because $\operatorname{Ker} A_y = \operatorname{span}\{(X_1)_y, \ldots, (X_{2r})_y\}$ for y in a certain neighbourhood of x. Also it is a complex distribution because $AJ = -JA$. From (6·2) and, as in this case, $\operatorname{Ker} A = \operatorname{Im} A$, we have (see [13], proposition, p. 242):

THEOREM 6·1. *Let M^{4r}_{2r} be an indefinite complex Einstein hypersurface of $\bar{M}^{4r+2}_{2r+2a}(\bar{c})$ with $A^2 = 0$ and maximal rank in each point. Then Ker A is a differentiable complex parallel totally degenerate distribution on M^{4r}_{2r}.*

We have also:

THEOREM 6·2. *Under the hypothesis of Theorem* 6·1, *if $r > 1$, then $\bar{c} = 0$.*

Proof. From the Gauss equation (1·3) and as $A^2 = 0$, we have

$$R(U_i, U_j) AZ = (\tfrac{1}{4}\bar{c}) \{g(U_j, AZ) U_i - g(U_i, AZ) U_j + g(JU_j, AZ) JU_i - g(JU_i, AZ) JU_j + 2g(U_i, JU_j) JAZ\}$$

for all Z in $T_x M$, $i \neq j$. From (6·2) $R(U_i, U_j) AZ \in \operatorname{Ker} A_x$. Hence

$$0 = AR(U_i, U_j) AZ = (\tfrac{1}{4}\bar{c}) \{g(E_j, Z) E_i - g(E_i, Z) E_j + g(JE_j, Z) JE_i - g(JE_i, Z) JE_j\}$$

and so $\bar{c} = 0$.

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Remark. In [15], Magid has observed that the Einstein hypersurfaces with $A^2 = 0$ and maximal rank are not necessarily symmetric. One can see that the indefinite complex Einstein hypersurfaces with $A^2 = 0$ and which are symmetric spaces have been studied in [5], § 4, p. 346. In fact, from (2·3), we take $V_1, \ldots, V_{2r}, W_1, \ldots, W_{2r}$ an orthonormal basis of $T_x M^{4r}_{2r}$, $x \in M^{4r}_{2r}$, where A_x is represented by the matrix

$$\begin{pmatrix} \epsilon_1 & & & & -\epsilon_1 & & & \\ & \ddots & & & & \ddots & & \\ & & \epsilon_{2r} & & & & -\epsilon_{2r} & \\ \hline & & & \epsilon_1 & & & & -\epsilon_1 \\ & & & & \ddots & & & \\ & & & & & \epsilon_{2r} & & \\ & & & & & & -\epsilon_{2r} & \end{pmatrix}$$

where $\epsilon_i = \pm 1$, $1 \leqslant i \leqslant 2r$. Let (G, K, P, B) be the symmetric pseudo-Riemannian quadruple associated to M^{4r}_{2r} (see [5], § 1). As in this case $\bar{c} = 0$ (we suppose $r > 1$), one sees from the Gauss equation (1·3) that $AR(X, Y) = R(X, Y)A = 0$ for all X, Y tangent to M^{4r}_{2r}. Therefore, if $H \in K$, H is represented in our basis by a matrix of the form

$$\begin{pmatrix} B & & -B & \\ & & & \\ \hline B & & -B & \\ & & & \end{pmatrix}$$

where B is a skew-symmetric $2r \times 2r$-matrix. It is now easy to see that the quadruple (G, K, P, B) has maximal centre (see [5] § 2). But these quadruples are classified in [5], § 4.

Finally, we give a characterization of these manifolds with respect to its normal connection. Let M^{2n}_{2s} be a non degenerate complex submanifold of $\bar{M}^{2n+2m}_{2s+2t}(\bar{c})$. We have

$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi$$

tangent and normal components with respect to M^{2n}_{2s}. It is well known that D is a connection on the normal bundle to M^{2n}_{2s} in $\bar{M}^{2n+2m}_{2s+2t}(\bar{c})$. We denote by $R^\perp$ its curvature tensor. Then we have the Ricci equation:

$$\bar{R}(X, Y, \xi, \eta) = R^\perp(X, Y, \xi, \eta) - g([A_\xi, A_\eta]X, Y) \tag{6·3}$$

for all X, Y, ξ, η tangent and normal to M^{2n}_{2s} respectively.

We have (see [7], theorem 3·1):

THEOREM 6·3. *Let M^{2n}_{2s}, $n > 2$, be an indefinite complex submanifold which is complete and 1-connected with $\bar{c} \neq 0$. Then there exist no unit normal fields to M^{2n}_{2s} which are parallel with respect to the normal connection.*

Proof. We suppose that there exists such a field ξ with $g(\xi, \xi) = \tau = \pm 1$. So, $Q = \operatorname{span}\{\xi, J\xi\}$ is a nondegenerate parallel holomorphic 2-dimensional subbundle of the normal bundle to M in $\bar{M}$. Moreover, from the Ricci equation (6·3) and as $R^\perp(X, Y, \xi, J\xi) = 0$ for all X, Y tangent to M, we have

$$A^2_\xi = -(\tfrac{1}{4}\bar{c})\tau I, \tag{6·4}$$

where I is the identity transformation. So,

$$\operatorname{Ker} A_\xi = 0 \quad \text{and} \quad \nu_Q(x) = \dim\{X \in T_x M / A_\eta X = 0, \eta \in Q\} = 0.$$

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As in [7], theorem 1, we know that there exists a complex submanifold $M'^{2n+2}_{2s+2a}(\bar{c})$ (where $a = 0, 1$ according as $\tau = 1, -1$) embedded in a totally geodesic way in $\bar{M}$ where M is immersed as an indefinite complex hypersurface. From (6·4) and the Gauss equation (1·3), one sees that M is Einstein for this last immersion. Now, if $\bar{c}\tau > 0$, it is $A^2_\xi = -\gamma^2 I$ with $\gamma^2 = \frac{1}{4}\bar{c}\tau$. But, by using Theorem 5·3, one has $\bar{c} = -4\tau\gamma^2$. So, this case is impossible. If $\bar{c}\tau < 0$, from (6·4), A_ξ is a diagonalizable endomorphism. Hence, from (2·1), we have

$$-(\tfrac{1}{4}\tau)\{2\rho - \bar{c}(n+1)\} = -\tfrac{1}{4}\tau\bar{c},$$

i.e.: $\rho = \frac{1}{2}(n+2)\bar{c}$. As $\bar{c} \neq 0$ and $\rho \neq \frac{1}{2}n\bar{c}$, from Theorem 4·4 M is totally geodesic in M'. But it is not possible according to (6·4).

As a consequence we obtain the mentioned characterization.

THEOREM 6·4. *Let $M^{2n}_{2s}, n > 2$, be an indefinite complex submanifold of $\bar{M}^{2n+2m}_{2s+2t}(\bar{c})$ which is complete and 1-connected. Then $R^\perp = 0$ if and only if $\bar{c} = 0$ and $A^2_\xi = 0$ for all ξ normal to M^{2n}_{2s}.*

Proof. If $R^\perp = 0$, then the normal frames bundle on M is locally trivial. Hence there exists normal unit parallel fields on M. From Theorem 6·3 one has $\bar{c} = 0$. By using the Ricci equation (6·3), one obtains

$$g([A_\xi, A_\eta]X, Y) = 0 \tag{6·5}$$

for all X, Y, ξ, η tangent and normal to M respectively. We put in (6·5) $\eta = J\xi$ and so $g(JA^2_\xi X, Y) = 0$ for all X, Y tangent to M. Then $A^2_\xi = 0$. Conversely, if $\bar{c} = 0$ and $A^2_\xi = 0$ for all ξ normal to M, by using the Ricci equation again, we have

$$R^\perp(X, Y, \xi, J\xi) = -g([A_\xi, A_{J\xi}]X, Y) = 2g(JA^2_\xi X, Y) = 0. \tag{6·6}$$

We now put $\xi + J\eta$ in the place of ξ in (6·6) and we finish.

We would like to express our thanks to the referee for valuable suggestions.

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