THE REVIEW OF SYMBOLIC LOGIC
Volume 13, Number 4, December 2020

NONREPRESENTABLE RELATION ALGEBRAS FROM GROUPS

HAJNAL ANDRÉKA and ISTVÁN NÉMETI
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
and
STEVEN GIVANT
Department of Mathematics and Computer Science, Mills College

Abstract. A series of nonrepresentable relation algebras is constructed from groups. We use them to prove that there are continuum many subvarieties between the variety of representable relation algebras and the variety of coset relation algebras. We present our main construction in terms of polygroupoids.

§1. Introduction. In 1941, Tarski [39] defined the class RA of relation algebras. This was the culmination of previous extensive work by Peirce and Schröder, who worked towards extending Boole's system to capture more of our reasoning than propositional calculus does. In the early 1940s, Tarski proved that set theory can be built on the equational theory of relation algebras [40]. It may be just an accidental fact that mathematics today is based on first-order logic and not on the equational theory of relation algebras [29,36]. Still today, RA is an active field of research. It is a central topic in algebraic logic [13,14,21,30], but relation algebras also have deep connections with other areas of mathematics, such as geometry, graph theory, combinatorics, group theory, linguistics, arrow logic, and modal logic, to list some [5, 7, 12, 26–28, 32]. Relation algebras are extensively used in computer science [6, 35, 37].

In the 1940s, J. C. C. McKinsey noticed that the complex algebra (the natural algebra of complexes or subsets) of a group is a relation algebra. Jónsson and Tarski proved that the complex algebra of a Brandt groupoid is a relation algebra [25, sec. 5]. Comer [8] defined polygroupoids (these turned out the same as regular, reversible-in-itself multigroups in Dresher-Ore [11]) and proved that relation algebras are exactly subalgebras of complex algebras of polygroupoids. He showed that some theorems of relation algebra theory are easier to prove for polygroupoids first and then transfer the result to relation algebras [9]. He suggested a research project of doing a portion of relation algebra theory in terms of polygroupoids. In this article, we take up his initiative since we find that a large part of our work can be more conveniently formulated in terms of polygroupoids than in terms of relation algebra theory. In particular, we can introduce the class RRA of representable relation algebras by generalizing the notion of the Cayley representation of a group to

Received: October 25, 2018.
2010 *Mathematics Subject Classification*: Primary 03G15, 20Bxx, 03Gxx, Secondary 03E20, 03C05.
Key words and phrases: group, relation algebra, Brandt groupoid, polygroupoid, atom structure, complex algebra, group relation algebra, nonrepresentable relation algebra, coset algebra, measurable relation algebra.

© Association for Symbolic Logic, 2019

doi:10.1017/S1755020319000224

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

polygroupoids. Representable relation algebras are the primary or generic examples of relation algebras.

A generalization of group relation algebras is introduced in [15,17], by using an intricate system of groups and their factor groups (see Figure 1). They are all representable, their representations closely follow the Cayley representations of the groups involved. However, with a slight "shift" we can make some of these group relation algebras nonrepresentable. One such example is presented in [1, sec. 5]. In this article, we present a generalization of that example, and this way we construct an infinite series of nonrepresentable shifted group relation algebras (coset relation algebras, in short). We use these algebras to prove that there are uncountably many varieties of coset relation algebras that all contain the variety of representable relation algebras (Theorem 5.1).

It is known to be difficult to construct nonrepresentable relation algebras because the finitely axiomatized class of relation algebras approximates surprisingly well the non-finitely axiomatizable class of representable relation algebras. Our method for constructing nonrepresentable relation algebras has novel features; it differs in intuition from the ones available in the relation algebra literature. One novel feature is that they are constructed from systems of groups with only slightly distorting the group system structure. Only how the groups are connected is distorted, the groups themselves are kept intact. Another feature is that the cause of nonrepresentability in these algebras is "sheer" structure, not size, not that some parts of the algebras are too big or too small.

Why does one want to construct nonrepresentable relation algebras, when the representable ones are of the primary interest? For example, because they can be used to prove various properties of representable relation algebras. Monk [34] uses a series of nonrepresentable relation algebras to prove that the variety RRA cannot be axiomatized by a finite set of equations. Similarly, Jónsson [24] proves that each equational axiom system of RRA must use infinitely many variables. His proof relies on nonrepresentable relation algebras constructed from projective planes. For a sample of constructions of nonrepresentable algebras, see [4, 20, 22, 31, 33, 38].

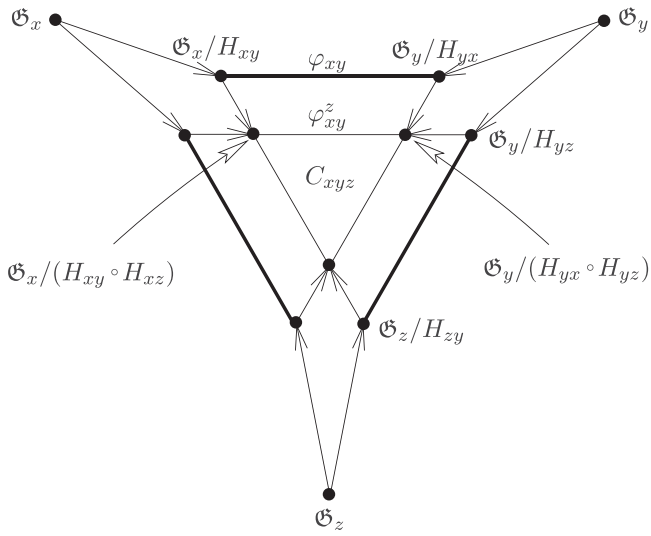


Fig. 1. Group coset frame $(\langle \mathfrak{G}_x : x \in I\rangle, \langle \varphi_{xy} : (x, y) \in \mathcal{E}\rangle, \langle C_{xyz} : (x, y), (y, z) \in \mathcal{E}\rangle)$.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

In §2, we introduce previous work that we need from relation algebra theory, in terms of polygroupoids. We assume basic knowledge of groups, e.g., normal subgroup and factor group. We hope that this section can be read without any background in relation algebra. We also hope that some knowledge acquired in relation algebra theory can be made available this way to a larger community of mathematicians. In §3 and §4 we still work solely with polygroupoids and groups. In §3, we construct a group frame starting from any commutative group. In §4, we add the "shift" and we prove that the algebras obtained are all nonrepresentable. This is one of the two main theorems of the article. Relation algebras are introduced in §5, where we begin to use relation algebra terminology. We show that the examples we constructed in §3 and §4 are diverse enough in that none of them can be embedded into the other. This feature is used in proving our second main theorem, Theorem 5.1.

§2. Groups, groupoids, and group systems. We recall the definition of polygroupoids. The notion of a polygroupoid is a generalization of that of a group. Brandt groupoids are generalizations of groups in that the binary composition operation is partial in them and that they can have more than one identity elements. Composition can be multivalued in polygroupoids, in addition.

Some notation that will be convenient (and is customary) to use in partial multivalued algebras. A partial multivalued binary operation $\circ$ on a set M is a partial function from $M \times M$ to the set of all nonempty subsets of M. We can define an extension $\circ$ of $\circ$ by defining $a \circ b = \emptyset$ when $a, b \in M$ and $a \circ b$ is not defined. Then $\circ$ is a total function from $M \times M$ to the powerset of M. We can recover $\circ$ from $\circ$ by noting that $a \circ b = \emptyset$ if and only if $a \circ b$ is not defined. We will not distinguish between $\circ$ and $\circ$ in the following. When $X, Y \subseteq M$ we write $X \circ Y = \{c : c \in a \circ b \text{ for some } a \in X \text{ and } b \in Y\}$, and $X^{-1} = \{c^{-1} : c \in X\}$. We often write just a in place of the singleton $\{a\}$.

DEFINITION 2.1 (Polygroupoid, [8, Definition 3.1]). *By a polygroupoid we understand a structure $\mathfrak{M} = (M, \circ, I, ^{-1})$, where $\circ$ is a partial multivalued binary operation on M and $I \subseteq M$ and $^{-1}$ is a unary operation on M that satisfy the following three conditions for all $a, b, c \in M$.*

(i) $(a \circ b) \circ c = a \circ (b \circ c)$,

(ii) $a \circ I = a = I \circ a$,

(iii) *$a \in b \circ c$ holds if and only if $b \in a \circ c^{-1}$ and this holds if and only if $c \in b^{-1} \circ a$.*

Polygroupoids appear in many parts of mathematics. In particular, groups and Brandt groupoids are special polygroupoids. The latter can be characterized as those polygroupoids in which the composition operation $\circ$ is not multivalued (but it may be partial).

An element a of a polygroupoid is called a *loop* if there is $x \in I$ such that $x \circ a \circ x = a$ (intuitively, if the domain and range of a coincide). A polygroupoid is said to be *locally functional* if the product of two loops f, g in $\mathfrak{M}$ cannot have more than one value, i.e., if $|f \circ g| \leq 1$. Let PG denote the class of all polygroupoids and let LPG denote the class of all locally functional polygroupoids.

Surprisingly, locally functional polygroupoids have a rather rich structure. They are put together from various groups and their factor groups in an orderly manner. The novelty here is the appearance of factor groups: they appear because of the multivalued nature of the composition operation.

The notion of a group coset frame emerges as the complete description of the structure of locally functional polygroupoids, see Theorem 2.5 below. For introducing group coset

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

frames, we need some notation concerning groups. When $\mathfrak{G}$ is a group, we usually denote its universe by G, its binary operation by $\circ$, its identity element by e and inverse of $g \in G$ by g^{-1}. When H is a normal subgroup of $\mathfrak{G}$, we denote the set of cosets of H in $\mathfrak{G}$ by G/H. Thus $G/H = \{g \circ H : g \in G\} = \{H \circ g : g \in G\}$, this is the universe of the factor group $\mathfrak{G}/H$. More generally, when $X \subseteq G$ we denote $X/H = \{g \circ H : g \in X\}$ and $g/H = g \circ H$. Since groups are special polygroupoids, we use the notation introduced for polygroupoids for groups, too. In particular, when H, K are two normal subgroups of $\mathfrak{G}$, their complex product, $H \circ K$, is the normal subgroup generated by them. Now, $H \circ K$ is both a union of cosets of H and a union of cosets of K. Often, we will identify $H \circ K$ with H/K or K/H while the exact connection is $H \circ K = \bigcup H/K$. We hope this will not lead to confusion.

DEFINITION 2.2 (Group coset system, [17], [1, sec. 3]). *Let I be any set and $\mathcal{E} \subseteq I \times I$ be an equivalence relation on I. Assume that $\mathcal{G} = \langle \mathfrak{G}_x : x \in I \rangle$ is a system of groups, $\varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle$ is a system of isomorphisms between their factor groups,*

$$\varphi_{xy} : \mathfrak{G}_x/H_{xy} \to \mathfrak{G}_y/K_{xy} \quad \text{is an isomorphism}$$

and $C = \langle C_{xyz} : (x, y), (y, z) \in \mathcal{E} \rangle$ is a system of cosets of these factors, in more detail,

$$C_{xyz} \quad \text{is a coset of } H_{xy} \circ H_{xz}.$$

Then we call $(\mathcal{G}, \varphi, C)$ a group coset system.

Group coset frames are group coset systems in which there is a concert between the ingredients of the group coset system, see Figure 1. When f is a function with domain A and $X \subseteq A$, we denote the image of X under f by $f[X] = \{f(x) : x \in X\}$. When C is a coset of a normal subgroup H, we denote by $\tau(C)$ the inner automorphism it induces, thus

$$\tau(C)(D) = C^{-1} \circ D \circ C$$

for all cosets D of H. We will denote compositions of functions with a sign different from the usual $\circ$; moreover, we will use relation-type composition as in category theory where the order of applying the functions is different. In more detail, the composition of binary relations R, S is

$$R \mid S = \{(u, v) : (u, w) \in R \text{ and } (w, v) \in S \text{ for some } w\}.$$

When $f : U \to V$ is a function, we consider it as a binary relation $\{(u, f(u)) : u \in U\}$, then $f \mid g : U \to W$ if $g : V \to W$ and

$$(f \mid g)(u) = g(f(u)).$$

DEFINITION 2.3 (Group coset frame, [1, Definitions 3.2, 4.1]). *A group coset system $(\mathcal{G}, \varphi, C)$ is called a* group coset frame *if the following eight conditions are satisfied for all $(x, y), (y, z), (z, w) \in \mathcal{E}$.*

(i) *φ_{xx} is the identity function on $G_x/\{e_x\}$, where e_x is the identity element of G_x. Thus, $H_{xx} = \{e_x\}$.*

(ii) *φ_{yx} is the inverse of φ_{xy}. In particular, $K_{xy} = H_{yx}$.*

(iii) *$\varphi_{xy}[H_{xz}/H_{xy}] = H_{yz}/H_{yx}$.*

Assume that (iii) *holds. Define $\varphi^z_{xy}(g/(H_{xy} \circ H_{xz})) = \varphi_{xy}(g/H_{xy}) \circ H_{yz}$. Now, φ^z_{xy} is well defined, because of* (iii).

(iv) *$\varphi^z_{xy} \mid \varphi^x_{yz} = \tau(C_{xyz}) \mid \varphi^y_{xz}$.*

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

We now list the connections between the cosets C_{xyz} that are required to hold in a group coset frame.

(v) $C_{xyy} = H_{xy}$.
(vi) $\varphi_{xz}[C_{xyz}] = C^{-1}_{zyx}$.
(vii) $\varphi_{xy}[C_{xyz}] = C^{-1}_{yxz}$.
(viii) $C_{xyz} \circ C_{xzw} = \varphi_{yx}[C_{yzw} \circ H_{yx}] \circ C_{xyw}$.

Theorem 2.5 below states that group coset frames completely capture the structure of locally functional polygroupoids, in an analogous manner as a system of groups captures the structure of a Brandt groupoid. Let us call a polygroupoid *connected* when for all identity elements x, y there is an element a such that $x \circ a \circ y = a$. It is known that the structure of each connected Brandt groupoid can be described by a group $\mathfrak{G}$ and a set I, as follows. To each group $\mathfrak{G}$ and set I, a Brandt groupoid $\mathfrak{B}(\mathfrak{G}, I)$ is defined, such that the universe is $\{(x, g, y) : x, y \in I \text{ and } g \in G\}$ and multiplication is $(x, g, y) \circ (y, h, z) = (x, g \circ h, z)$. One can prove that each connected Brandt groupoid is isomorphic to one of these $\mathfrak{B}(\mathfrak{G}, I)$. When $\mathfrak{B}$ is not connected, we have an equivalence relation $\mathcal{E}$ on I and distinct blocks of $\mathcal{E}$ may have distinct groups. Now, group coset frames capture the structures of locally functional polygroupoids, in an analogous way. First we introduce the polygroupoid analogues of $\mathfrak{B}(\mathfrak{G}, I)$.

DEFINITION 2.4 (Structure associated to group coset system). *Assume that $(\mathcal{G}, \varphi, C)$ is a group coset system. We define the structure $\mathfrak{P}(\mathcal{G}, \varphi, C)$ as follows. The universe is*

$$P = \{(x, g, y) : (x, y) \in \mathcal{E} \text{ and } g \in G_x/H_{xy}\}.$$

Multiplication on P is a multivalued binary partial function defined as follows. We will denote the multiple $(x, g, y) \circ (y, h, z)$ simply by juxtaposition $(x, g, y)(y, h, z)$. Assume that $(x, g, y), (w, h, z) \in P$. Their multiple is defined exactly when $y = w$ and

$$(x, g, y)(y, h, z) = \{(x, k, z) \in P : k \subseteq \varphi_{yx}(\varphi_{xy}(g) \circ h) \circ C_{xyz}\}.$$

The set E of identities of $\mathfrak{P}$ is

$$E = \{(x, H_{xx}, x) : x \in I\}$$

and inverse is defined as

$$(x, g, y)^{-1} = (y, \varphi_{xy}(g^{-1}), x).$$

The following theorem states that the concrete polygroupoids defined in the above definition exhaust all locally functional polygroupoids.

THEOREM 2.5 (Group coset frame theorem, [1,18]). LPG *is exactly the class of structures associated to group coset frames, up to isomorphisms. In more detail*:

(i) *$\mathfrak{P}(\mathcal{G}, \varphi, C)$ is a locally functional polygroupoid if and only if $(\mathcal{G}, \varphi, C)$ is a group coset frame.*
(ii) *For each locally functional polygroupoid $\mathfrak{P}$ there is a group coset frame $(\mathcal{G}, \varphi, C)$ such that $\mathfrak{P}$ is isomorphic to $\mathfrak{P}(\mathcal{G}, \varphi, C)$.*

Proof. Proving part (i) of the theorem in relation algebraic context is the main subject of [1] and part (ii) in relation algebraic form is the main representation theorem of [18]. After showing how the present theorem follows from the results in these articles, we sketch a direct proof of (ii) in terms of polygroupoids.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

To be able to apply the above relation algebraic theorems, we begin by showing that $\mathfrak{P} = \mathfrak{P}(\mathcal{G}, \varphi, C)$ is isomorphic to the structure of atoms of the algebra $\mathfrak{C} = \mathfrak{C}(\mathcal{G}, \varphi, C)$ defined in [1], below Definition 3.2.

The definition of $\mathfrak{C}$ is as follows. First, we make the universes of the groups $\mathfrak{G}_x$ disjoint. Let $(x, y) \in \mathcal{E}$ and let H be a coset of H_{xy} in $\mathfrak{G}_x$. Define (see [1, Definition 2.1])

$$R_{xy,H} = \bigcup \{K \times \varphi_{xy}[K \circ H] : K \in G_x/H_{xy}\}.$$

The atoms of $\mathfrak{C}$ are the binary relations $R_{xy,H}$ and multiplication $\otimes$ of $\mathfrak{C}$ between atoms is (see [1, Definition 3.1])

$$R_{xy,H} \otimes R_{yz,K} = \bigcup \{R_{xz,Q} : Q \subseteq \varphi_{yx}[\varphi_{xy}(H) \circ K] \circ C_{xyz}\},$$

and $R_{xy,H} \otimes R_{wz,K} = \emptyset$ when $w \neq y$. The identity atoms are $R_{xx,H}$ where $H = H_{xx}$, by [1, Theorem 2.4]. The inverse of $R_{xy,H}$ is $R_{yx,K}$ where $K = \varphi_{xy}(H^{-1})$, see [1, Theorem 2.5]. Therefore, the function assigning $R_{xy,H}$ to $(x, H, y) \in P$ is the desired isomorphism between $\mathfrak{P}$ and the structure of atoms of $\mathfrak{C}$.

Theorem 4.2 in [1] states that $\mathfrak{C}$ is a complete atomic measurable relation algebra when $(\mathcal{G}, \varphi, C)$ is a group coset frame. Comer [8, Theorem 4.2] states that the structure of atoms of a complete atomic relation algebra is a polygroupoid. Thus $\mathfrak{P} \in \mathsf{PG}$. It is locally functional because $\mathfrak{C}$ is measurable, but this can be seen easier directly, as follows. The loops among the elements of $\mathfrak{P}$ are of form (x, g, x) where $x \in I$ and g is a coset of H_{xx}. When $(\mathcal{G}, \varphi, C)$ is a group coset frame, we have that $H_{xx} = \{e_x\}$, φ_{xx} is the identity map on cosets of H_{xx} and $C_{xxx} = H_{xx}$, so $\varphi_{xx}(\varphi_{xx}(g) \circ h) \circ C_{xxx} = g \circ h$ which is a single coset of H_{xx} and therefore $(x, g, x)(x, h, x) = \{(x, k, x) : k \subseteq \varphi_{xx}(\varphi_{xx}(g) \circ h) \circ C_{xxx}\}$ consists of a single atom. Also, $(x, g, x)(y, h, y) = \emptyset$ when $y \neq x$. Thus $\mathfrak{P}$ is locally functional. This proves one direction of (i). The other direction, namely, that $(\mathcal{G}, \varphi, C)$ is a group coset frame when $\mathfrak{P}$ is a locally functional polygroupoid follows from Theorems 2.4, 2.5, 2.8, 3.4, 3.5, 3.7, and 3.8 in [1].

To prove (ii) of the theorem, let us assume that $\mathfrak{P}$ is a locally functional polygroupoid. Let $\mathfrak{C}$ be its natural algebra of complexes, defined in [8], for example. Then $\mathfrak{C}$ is a complete atomic relation algebra, by [8, Theorem 4.2]. That $\mathfrak{P}$ is locally functional translates immediately to the statement that $\mathfrak{C}$ is measurable (for the definition of a measurable relation algebra see [17]). Now, the main representation theorem of [18], Theorem 7.4, states that there is a group coset frame $(\mathcal{G}, \varphi, C)$ such that the structure of atoms of $\mathfrak{C}$ is $\mathfrak{P}(\mathcal{G}, \varphi, C)$. This proves (ii).

Next we sketch a direct proof of Theorem 2.5(ii), in terms of polygroupoids. This will also show how to extract the group coset frame $(\mathcal{G}, \varphi, C)$ from the polygroupoid.

Assume that $\mathfrak{P} = (M, \circ, I, ^{-1})$ is a locally functional polygroupoid. For any $x, y \in I$ define $M_{xy} = \{a \in M : x \circ a \circ y = a\}$. Let $\mathcal{E} = \{(x, y) \in I \times I : M_{xy} \neq \emptyset\}$. It can be shown that $\mathcal{E}$ is an equivalence relation on I.

Let $G_x = M_{xx}$. It can be shown that $\mathfrak{G}_x = (G_x, \circ, \{x\}, ^{-1})$ is a group for each $x \in I$, because $\mathfrak{P}$ is locally functional. This gives our system $\mathcal{G} = \langle \mathfrak{G}_x : x \in I \rangle$ of groups.

Next we construct the isomorphism φ_{xy} between quotients of $\mathfrak{G}_x$ and $\mathfrak{G}_y$ for $(x, y) \in \mathcal{E}$. For any $a \in M_{xy}$ let

$$H_{xy}(a) = \{g \in G_x : g \circ a = a\} \quad \text{and} \quad K_{xy}(a) = \{g \in G_y : a \circ g = a\}$$

and let $\varphi_{xy}(a)$ take a coset H of $H_{xy}(a)$ to a coset K of $K_{xy}(a)$ just in case $H \circ a = a \circ K$. It can be shown that $H_{xy}(a)$, $K_{xy}(a)$ are normal subgroups of $\mathfrak{G}_x$ and $\mathfrak{G}_y$, respectively, and they do not depend on the choice of a. Moreover, $\varphi_{xy}(a) : \mathfrak{G}_x/H_{xy}(a) \to \mathfrak{G}_y/K_{xy}(a)$ is

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

an isomorphism, but this isomorphism may be different for different elements a. To select a coherent system of isomorphisms, let us select a system $a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ of elements such that $a_{xy} \in M_{xy}$, $a_{xx} = x$ and $a_{xy}^{-1} = a_{yx}$ for all $(x, y) \in \mathcal{E}$. Such a system of elements can be chosen, it is called a semi-scaffold. Let us define $\varphi_{xy} = \varphi_{xy}(a_{xy})$, and let $\varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle$. It can be proved that φ satisfies frame conditions (i), (ii), and (iii).

Finally, we define C_{xyz} so that frame condition (iv) holds, as follows. Ideally, we would want that $a_{xz} \in a_{xy} \circ a_{yz}$ also hold in the semi-scaffold (then a is called a scaffold), but this cannot always be achieved. We define C_{xyz} to make up for this, by letting $g \in G_x$ be such that $g \circ a_{xz} \in a_{xy} \circ a_{yz}$ (one can prove that there is such a g), and then $C_{xyz} = g \circ H_{xy} \circ H_{xz}$. Finally, one can show that coset conditions (v)–(viii) hold because $\mathfrak{P}$ is a polygroupoid. □

The notion of the Cayley representation of a group can be generalized to polygroupoids. Since we have multivalued composition in polygroupoids, we will represent the elements by binary relations in place of bijective functions (permutations) as in a Cayley representation. When R is a binary relation, its inverse is $R^{-1} = \{(v, u) : (u, v) \in R\}$.

DEFINITION 2.6 (Representation of polygroupoid). *A* representation *of a polygroupoid* $\mathfrak{M} = (M, \circ, I, ^{-1})$ *on a set* U *is a function* rep *that assigns disjoint nonempty binary relations on* U *to elements of* M*, that is*

$\mathsf{rep}(a) \subseteq U \times U$ *is nonempty and* $\mathsf{rep}(a)$ *is disjoint from* $\mathsf{rep}(b)$ *whenever* $a \neq b$*,*

that satisfies the following three conditions for all $a, b \in M$*.*

$$\mathsf{rep}(a) \mid \mathsf{rep}(b) = \bigcup \mathsf{rep}[a \circ b],$$
$$\bigcup \{\mathsf{rep}(x) : x \in I\} = \{(u, u) : u \in U\}, \quad \text{and}$$
$$\mathsf{rep}(a)^{-1} = \mathsf{rep}(a^{-1}).$$

Let RPG *denote the class of representable polygroupoids.*

What we call representations are called *color schemes* in [10, p. 31] and [9, Example 3, p. 79], and their connections with Cayley-representations of groups are elaborated in [10, Theorem 5]. (More precisely, the above articles deal with polygroups only, i.e., polygroupoids where I is a singleton, but everything can be generalized to polygroupoids, see [9, Remark 1, p. 85].)

Brandt groupoids are all representable, and in fact, their representations can be put together from the Cayley representations of the groups involved. Our original hope, when we started to work with measurable relation algebras, was to prove the same for all locally functional polygroupoids. However, after proving several representation theorems for special classes (see [2, 15, 17]), we found a nonrepresentable locally functional polygroupoid (see [17, p. 56]). This construction is presented in [1, sec. 5]). A generalization of this example is outlined in [17, p. 59], but this more involved construction has never been published. In this article, we present concrete definitions for the algebras and we give a detailed proof for their nonrepresentability (Theorem 4.2). The construction and proof we present here are simpler than the original ones in [16].

To prove nonrepresentability of our polygroupoids, we will use Theorem 2.8 below which characterizes the representable locally functional polygroupoids.

DEFINITION 2.7 (Scaffold, [17, Definition 2]). *Let* $\mathfrak{M} = (M, \circ, I, ^{-1})$ *be a polygroupoid. A scaffold in* $\mathfrak{M}$ *is a system* $a = \langle a_{xy} : x, y \in I, x \circ M \circ y \neq \emptyset \rangle$ *of elements of* M *that satisfies the following, for all* $x, y, z \in I$ *such that* $x \circ M \circ y \neq \emptyset$*,* $y \circ M \circ z \neq \emptyset$*.*

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

$a_{xy} = x \circ a_{xy} \circ y, \quad a_{xx} \in I, \quad a_{xy} = a_{yx}^{-1} \quad \text{and} \quad a_{xz} \in a_{xy} \circ a_{yz}.$

Let us call $(\mathcal{G}, \varphi)$ a *group frame* if $(\mathcal{G}, \varphi, C)$ is a group coset frame where all the C_{xyz}s are the identity cosets $H_{xy} \circ H_{xz}$. The following theorem says that choosing the coset system C in a nontrivial way is essential in constructing nonrepresentable LPGs.

THEOREM 2.8 (Group frame theorem, [15, 18]). *Let* $\mathfrak{M} \in \mathsf{LPG}$. *The following are equivalent.*

(i) $\mathfrak{M}$ *is representable.*

(ii) $\mathfrak{M}$ *is isomorphic to* $\mathfrak{P}(\mathcal{G}, \varphi, C)$ *for some group coset frame* $(\mathcal{G}, \varphi, C)$ *where all the* C_{xyz}*s are the identity cosets* $H_{xy} \circ H_{xz}$.

(iii) *There is a scaffold in* $\mathfrak{M}$.

Proof. Assume that $\mathfrak{M} \in \mathsf{LPG}$ and let $\mathfrak{C}$ be its algebra of complexes. Then $\mathfrak{C}$ is a complete atomic measurable relation algebra, by the proof of Theorem 2.5. Any representation of $\mathfrak{M}$ can be extended to a function defined on the complexes, simply by defining $\mathsf{rep}(X) = \bigcup\{\mathsf{rep}(x) : x \in X\}$. It can be proved that rep so defined is a complete representation of $\mathfrak{C}$ as defined in the relation algebra literature, and any complete representation of $\mathfrak{C}$ is a representation of $\mathfrak{M}$ when restricted to the atoms, that is, to say to the elements of M. Thus, $\mathfrak{C}$ is completely representable if and only if $\mathfrak{M}$ is representable. By the proof of Theorem 2.5, we have that $\mathfrak{C}$ is a complete and atomic measurable relation algebra. Thus, $\mathfrak{C}$ is completely representable if and only if it has a scaffold, by Corollary 7.7 and Theorem 7.8 of [18]. Finally, $\mathfrak{C}$ is completely representable if and only if its structure of atoms is determined by a group frame, by Theorem 7.6 of [18] and Theorem 4.2 of [15]. □

§3. Construction of group frames. In this section, we construct a series of group frames, one for each commutative group $\mathfrak{F}$. A coset system C will be added in the next section. Our goal is to add a coset system C such that the so obtained group coset frame will determine a nonrepresentable polygroupoid. This places restrictions on how we can define our group frame. For example, any group coset frame $(\langle \mathfrak{G}_x : x \in I\rangle, \varphi, C)$ where I has less than five elements determines a representable polygroupoid, so we want to have at least five nodes x in the group system. When all the groups $\mathfrak{G}_x$ are the product of at most two finite cyclic groups, we cannot achieve nonrepresentability, either. Therefore we will have $\mathfrak{F} \times \mathfrak{F} \times \mathfrak{F}$ at the nodes of our group frames. Finally, we have to have "three distinct levels" in the construction, because if at each node x we have $H_{xy} = \{e_x\}$, or we have $H_{xy} = H_{xz}$ for all distinct $y, z, x \in I$, then nonrepresentability cannot be achieved. Therefore we want to have that $H_{xy} \neq \{e_x\}$ and $H_{xy} \circ H_{xz} \neq H_{xy}$ for all distinct x, y, z. These "guiding" theorems are mentioned in [17], and their proofs are in [16].

In this section, we use additive notation for groups instead of multiplicative notation as in the rest of the article. Let $\mathfrak{F} = \langle F, +\rangle$ be a commutative group with zero element 0. We denote the inverse of $g \in F$ by $-g$. We shall define a particular group frame (G, φ) with all the $\mathfrak{G}_x$ being $\mathfrak{F}^3 = \mathfrak{F} \times \mathfrak{F} \times \mathfrak{F}$.

3.1. The normal subgroups. Define the following subsets of F^3:

$L_0 = \{(a, 0, 0) : a \in F\},$
$L_1 = \{(0, a, 0) : a \in F\},$
$L_2 = \{(0, 0, a) : a \in F\},$
$L_3 = \{(a, a, a) : a \in F\}.$

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

The properties of these subsets we will use in the article are gathered in the following two lemmas. Let $\mathcal{L} = \{L_i : i < 4\}$.

LEMMA 3.1. *Let $\mathfrak{S}_{\mathcal{L}}$ denote the group of all permutations of $\mathcal{L}$ and let $\mathsf{Aut}(\mathfrak{F}^3)$ denote the group of all automorphisms of $\mathfrak{F}^3$. There is a group homomorphism $\alpha : \mathfrak{S}_{\mathcal{L}} \to \mathsf{Aut}(\mathfrak{F}^3)$ which satisfies the following two properties for all $\pi \in S_{\mathcal{L}}$ and and $L \in \mathcal{L}$.*

(i) $\alpha(\pi)[L] = \pi(L)$.

(ii) *$\alpha(\pi)$ is the identity on L whenever $\pi(L) = L$.*

Proof. In this proof, instead of permutations of $\mathcal{L} = \{L_i : i < 4\}$ it will be more convenient to work with permutations of $4 = \{i : i < 4\}$, by identifying a permutation Π of $\mathcal{L}$ with the permutation π of 4 that takes i to j just in case Π takes L_i to L_j. Let $\mathfrak{S}_4$ denote the group of all permutations of 4.

For $i < 4$ we define $g_i : F \to F^3$ as follows. For all $a \in F$ we let

$$g_0(a) = (a, 0, 0),$$
$$g_1(a) = (0, a, 0),$$
$$g_2(a) = (0, 0, a),$$
$$g_3(a) = (-a, -a, -a).$$

Then it is straightforward to check that for all $i < 4$ and $a \in F$

$$g_i : \mathfrak{F} \to \mathfrak{F}^3 \text{ is a homomorphism and } \textstyle\sum_{i<4} g_i(a) = 0. \tag{1}$$

For each $\pi \in S_4$ we define a function $\alpha_\pi : F^3 \to F^3$ by

$$\alpha_\pi(a) = \textstyle\sum_{i<3} g_{\pi(i)}(a_i), \quad \text{where } a = (a_0, a_1, a_2) \in F^3. \tag{2}$$

We are going to prove that $\alpha_\pi \in \mathsf{Aut}(\mathfrak{F}^3)$. First we prove

$$\alpha_\pi : \mathfrak{F}^3 \to \mathfrak{F}^3 \text{ is a homomorphism.} \tag{3}$$

Indeed, α_π respects $+$ because

$$\alpha_\pi(a) + \alpha_\pi(b) = \textstyle\sum_{i<3} g_{\pi(i)}(a_i) + \sum_{i<3} g_{\pi(i)}(b_i) =$$
$$\textstyle\sum_{i<3}(g_{\pi(i)}(a_i) + g_{\pi(i)}(b_i)) = \sum_{i<3} g_{\pi(i)}(a_i + b_i) = \alpha_\pi(a + b).$$

Above, the first and fourth equalities hold by the definition of α_π in (2), the second holds by commutativity of $\mathfrak{F}^3$, and the third by homomorphism of the g_is stated in (1).
Similarly, inverse and zero are respected:

$$\alpha_\pi(-a) = \textstyle\sum_{i<3} g_{\pi(i)}(-a_i) = \sum_{i<3} -g_{\pi(i)}(a_i) = -\sum_{i<3} g_{\pi(i)}(a_i) = -\alpha_\pi(a),$$
$$\alpha_\pi(0) = \alpha_\pi(0, 0, 0) = \textstyle\sum_{i<3} g_{\pi(i)}(0) = \sum_{i<3} 0 = 0.$$

So α_π is indeed a homomorphism, and this proves (3).

Next we prove

$$\alpha_\pi(g_i(a)) = g_{\pi(i)}(a), \quad \text{for each } \pi \in S_4, a \in F \text{ and } i < 4\,. \tag{4}$$

Indeed, if $i < 3$, this is clear by definition of α_π. For example,

$$\alpha_\pi(g_0(a)) = \alpha_\pi(a, 0, 0) = g_{\pi(0)}(a) + g_{\pi(1)}(0) + g_{\pi(2)}(0) = g_{\pi(0)}(a) + 0 + 0 =$$
$$g_{\pi(0)}(a).$$

To prove (4) for $i = 3$, let $a \in F$. Then

$$\alpha_\pi(g_3(a)) = \alpha_\pi(-[g_0(a) + g_1(a) + g_2(a)]) =$$

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

$-[\alpha_\pi(g_0(a)) + \alpha_\pi(g_1(a)) + \alpha_\pi(g_2(a))] = -[g_{\pi(0)}(a) + g_{\pi(1)}(a) + g_{\pi(2)}(a)] = g_{\pi(3)}(a)$.

Above, the first equality holds by the second part of (1), the second equality holds as α_π is a homomorphism, the third equality holds by the cases of (4) that we have already proven and the last equality holds by the second part of (1) and $\pi \in S_4$. This proves (4).

Let us define the function α with domain S_4 such that α assigns α_π to $\pi \in S_4$. We are ready to prove that α is the desired homomorphism.

$$\alpha : \mathfrak{S}_4 \to \mathsf{Aut}(\mathfrak{F}^3) \text{ is a homomorphism.} \tag{5}$$

To prove (5), first we prove that α preserves composition of functions and it sends the identity map on 4 to the identity map on F^3. From this, we will infer that α_π has an inverse, so it is indeed in $\mathsf{Aut}(\mathfrak{F}^3)$.

Let $\circ$ denote the composition operation between functions, that is

$$(\alpha_\pi \circ \alpha_\sigma)(a) = \alpha_\pi(\alpha_\sigma(a)) \text{ and } (\pi \circ \sigma)(i) = \pi(\sigma(i)). \tag{6}$$

We now prove preservation of composition by α.

$$\alpha_\pi \circ \alpha_\sigma = \alpha_{\pi \circ \sigma}, \quad \text{for all } \pi, \sigma \in S_4. \tag{7}$$

Indeed, for given $a = (a_0, a_1, a_2) \in F^3$,

$(\alpha_\pi \circ \alpha_\sigma)(a) = \alpha_\pi(\alpha_\sigma(a)) = \alpha_\pi(\sum_{i<3} g_{\sigma(i)}(a_i)) = \sum_{i<3} \alpha_\pi(g_{\sigma(i)}(a_i)) = \sum_{i<3} g_{\pi(\sigma(i))}(a_i) = \sum_{i<3} g_{(\pi \circ \sigma)(i)}(a_i) = \alpha_{\pi \circ \sigma}(a)$.

The first equality holds by the definition of $\circ$ in (7), the second equality holds by the definition of α_σ in (2), the third equality holds because α_π is a homomorphism as proved in (3), the fourth holds by (4), the fifth by the definition of $\pi \circ \sigma$ in (6), and the last one by the definition of $\alpha_{\pi \circ \sigma}$ in (2).

Let ι denote the identity map on 4, i.e., $\iota(i) = i$ for all $i < 4$.

$\alpha_\iota(a) = g_0(a_0) + g_1(a_1) + g_2(a_2) = (a_0, 0, 0) + (0, a_1, 0) + (0, 0, a_2) = (a_0, a_1, a_2) = a$.

Thus, α_ι is indeed the identity map on F^3. Let σ denote the inverse π^{-1} of π. Now, $\alpha_\pi \circ \alpha_\sigma = \alpha_{\pi \circ \sigma} = \alpha_\iota$ by (7), and similarly $\alpha_\sigma \circ \alpha_\pi = \alpha_\iota$. Since α_ι is the identity map, this shows that α_σ is the inverse of α_π. Hence, $\alpha_\pi \in Aut(\mathfrak{F}^3)$ for all $\pi \in S_4$ and, moreover, α preserves inverse. This completes the proof of (5).

Finally, we show that α satisfies the two extra conditions required in the statement of Lemma 3.1. Indeed, they follow from the observation that $L_i = \{g_i(a) : a \in F\}$ and (4):

$\alpha(\pi)[L_i] = \alpha_\pi[\{g_i(a) : a \in F\}] = \{g_{\pi(i)}(a) : a \in F\} = L_{\pi(i)}$, and
$\alpha(\pi)(x) = \alpha_\pi g_i(a) = g_{\pi(i)}(a) = g_i(a) = x$, whenever $\pi(i) = i$ and $x \in L_i, x = g_i(a)$.

Lemma 3.1 has been proved. □

LEMMA 3.2. *Assume $i, j, k < 4$ are distinct.*

(i) *L_i is a normal subgroup of $\mathfrak{F}^3$.*
(ii) *$L_i + L_j$ intersected with $L_i + L_k$ is L_i.*
(iii) *$L_i + L_j + L_k = F^3$.*
(iv) *Let C, D be cosets of $L_i + L_j$ and $L_i + L_k$, respectively. Then $C \cap D$ is a coset of L_i.*
(v) *$\mathfrak{F}^3/L_i \cong \mathfrak{F} \times \mathfrak{F}$, and $\mathfrak{F}^3/(L_i + L_j) \cong \mathfrak{F}$.*

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

Proof. It is enough to check Lemma 3.2 for $(i, j, k) = (0, 1, 2)$, because after that we can use Lemma 3.1 and the fact that all the statements in the lemma are preserved by automorphisms, to infer that Lemma 3.2 holds for all distinct $i, j, k < 4$.

Clearly, L_0 is closed under addition, so it is a subgroup of $\mathfrak{F}^3$, and it is normal because $\mathfrak{F}$ is commutative and so $\mathfrak{F}^3$ is also commutative.

We have $L_0 + L_1 = \{(a, b, 0) : a, b \in F\}$ and $L_0 + L_2 = \{(a, 0, c) : a, c \in F\}$, their intersection is $L_0 = \{(a, 0, 0) : a \in F\}$.

Clearly, $L_0 + L_1 + L_2 = \{(a, b, c) : a, b, c \in F\} = F^3$.

Let C, D be cosets of $L_0 + L_1$ and $L_0 + L_2$, respectively. Then $C = \{(a, b, c) : a, b \in F\}$ for some $c \in F$ and $D = \{(a, b, c) : a, c \in F\}$ for some $b \in F$. Their intersection is $C \cap D = \{(a, b, c) : a \in F\}$ which is the coset $L_0 + \{(0, b, c)\}$ of L_0.

Finally, the elements of F^3/L_0 are $\{(a, b, c) : a \in F\}$ for some $b, c \in F$, and it is easy to see that the function assigning (b, c) to $\{(a, b, c) : a \in F\}$ is an isomorphism from $\mathfrak{F}^3/L_0$ onto $\mathfrak{F} \times \mathfrak{F}$. The proof for $\mathfrak{F}^3/(L_0 + L_1) \cong \mathfrak{F}$ is similar.

This proves Lemma 3.2. □

3.2. The system of quotient isomorphisms. Let I be any five-element set. Define

$$\mathfrak{G}_x = \mathfrak{F}^3, \qquad \text{for all } x \in I,$$

and $\mathcal{G} = \langle \mathfrak{G}_x : x \in I \rangle$. Let us specify H_{xy} for distinct $x, y \in I$ such that

$$\{H_{xy} : y \in I, y \neq x\} = \{L_i : i < 4\} \qquad \text{for any } x \in I. \tag{1}$$

This can be done because I has at least five elements. Since I has five elements, we have that H_{xy} and H_{xz} are distinct for distinct x, y, z.

Let $x, y \in I$ be distinct. We define a function π_{xy} on $\mathcal{L} = \{L_i : i < 4\}$ as follows.

$$\pi_{xy}(H_{xy}) = H_{yx} \quad \text{and} \quad \pi_{xy}(H_{xu}) = H_{yu} \quad \text{for any } u \in I \setminus \{x, y\}. \tag{2}$$

Below, $[u, v] = \{(u, v), (v, u)\} \cup \{(w, w) : w \in \mathcal{L} \setminus \{u, v\}\}$ denotes the *transposition* of u, v in $\mathcal{L}$.

LEMMA 3.3. *Let $x, y, z \in I$ be distinct.*

(i) *π_{xy} is a permutation of $\mathcal{L}$.*
(ii) $\pi_{yx} = \pi_{xy}^{-1}$.
(iii) $\pi_{xy} \mid \pi_{yz} = \pi_{xz} \mid [H_{zx}, H_{zy}]$.

Proof. Let x, y, u, v, w be a repetition-free listing of I. Then π_{xy} takes

$$H_{xy}, H_{xu}, H_{xv}, H_{xw} \quad \text{to} \quad H_{yx}, H_{yu}, H_{yv}, H_{yw},$$

respectively, by (2). Both lists are repetition-free listings of $\mathcal{L}$, by (1). This shows that π_{xy} is a permutation of $\mathcal{L}$.

To prove (ii), let $p \in I$ be distinct from x, y. Then

$$\pi_{yx}\pi_{xy}(H_{xy}) = \pi_{yx}(H_{yx}) = H_{xy} \text{ and}$$
$$\pi_{yx}\pi_{xy}(H_{xp}) = \pi_{yx}(H_{yp}) = H_{xp}.$$

Thus $\pi_{xy} \mid \pi_{yx}$ is the identity mapping on $\mathcal{L}$. Together with (i), this proves (ii).

To prove (iii), let $p \in I$ be distinct from x, y, z and let π_1 denote $[H_{zx}, H_{zy}]$. Now

$$\pi_{yz}\pi_{xy}(H_{xy}) = \pi_{yz}(H_{yx}) = H_{zx} \quad \text{and} \quad \pi_1\pi_{xz}(H_{xy}) = \pi_1(H_{zy}) = H_{zx}.$$
$$\pi_{yz}\pi_{xy}(H_{xz}) = \pi_{yz}(H_{yz}) = H_{zy} \quad \text{and} \quad \pi_1\pi_{xz}(H_{xz}) = \pi_1(H_{zx}) = H_{zy}.$$

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

$$\pi_{yz}\pi_{xy}(H_{xp}) = \pi_{yz}(H_{yp}) = H_{zp} \quad \text{and} \quad \pi_1\pi_{xz}(H_{xp}) = \pi_1(H_{zp}) = H_{zp}.$$

This shows that (iii) holds. □

Let $\langle \alpha(\pi) : \pi \in S_{\mathcal{L}} \rangle$ be a system of automorphisms of $\mathfrak{F}^3$ that satisfies the properties stated in Lemma 3.1. For distinct $x, y \in I$ define

$$\alpha_{xy} = \alpha(\pi_{xy}). \tag{3}$$

Then α_{xy} is an automorphism of $\mathfrak{F}^3$ that takes H_{xy} to H_{yx}. Hence, α_{xy} induces an isomorphism φ_{xy} between $\mathfrak{F}^3/H_{xy}$ and $\mathfrak{F}^3/H_{yx}$ by setting

$$\varphi_{xy}(g/H_{xy}) = \alpha_{xy}(g)/H_{yx}. \tag{4}$$

For any $x \in I$, let us define φ_{xx} to be the identity mapping on $\mathfrak{G}_x/\{e_x\}$. By this, we have defined the system $\varphi = \langle \varphi_{xy} : x, y \in I \rangle$. Figure 2 illustrates the definition.

THEOREM 3.4. $(\mathcal{G}, \varphi)$ *is a group frame.*

Proof. Throughout the proof, let $x, y, z \in I$ be distinct and $g \in F^3$ be arbitrary. φ_{xx} is the identity by definition. We have $\pi_{yx} = \pi_{xy}^{-1}$ by Lemma 3.3(ii), so $\alpha_{yx} = \alpha_{xy}^{-1}$ by (3) and Lemma 3.1, so $\varphi_{yx} = \varphi_{xy}^{-1}$ by (4). Thus, conditions (i) and (ii) of Definition 2.3 hold.

To check conditions (iii) and (iv), first we prove

$$\varphi_{xy}[g/H_{xz}/H_{xy}] = \alpha_{xy}(g)/H_{yz}/H_{yx}. \tag{5}$$

Indeed,

$$\varphi_{xy}[g/H_{xz}/H_{xy}] = \alpha_{xy}[g/H_{xz}]/H_{yx} = \alpha_{xy}[\{g\} + H_{xz}]/H_{xy} = \\ (\{\alpha_{xy}(g)\} + \alpha_{xy}[H_{xz}])/H_{yx} = (\{\alpha_{xy}(g)\} + H_{yz})/H_{yx} = \alpha_{xy}(g)/H_{yz}/H_{yx}.$$

Above, the first three equalities hold by (4), $g/H = \{g\} + H$, and α_{xy} being a homomorphism, respectively. The fourth equality holds because $\alpha_{xy}[H_{xz}] = H_{yz}$ by (3), (2), and Lemma 3.1(i). The last equality holds by $g/H = \{g\} + H$.

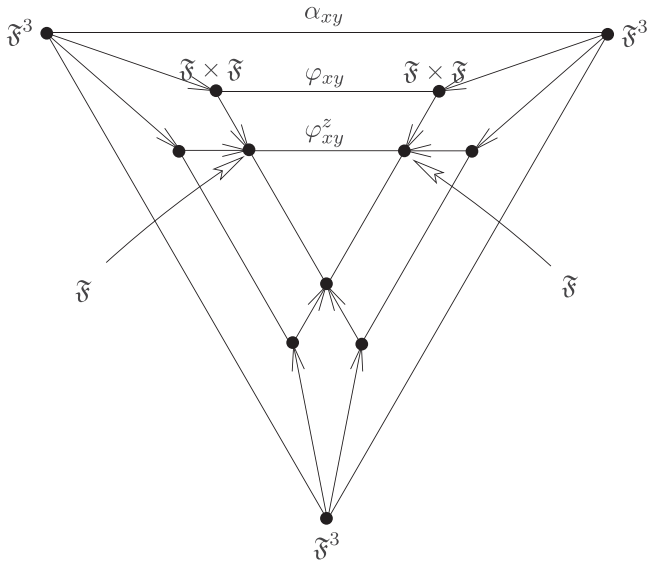


Fig. 2. The group frame $(\mathcal{G}, \varphi)$.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

Condition (iii) in Definition 2.3 is a special case of (5) when we take g to be 0, because $0/H = H$ for any subgroup H. Thus, condition (iii) holds.

To check condition (iv), first we prove

$$\varphi^z_{xy}(g/(H_{xy} + H_{xz})) = \alpha_{xy}(g)/(H_{yx} + H_{yz}). \tag{6}$$

Indeed,

$$\varphi^z_{xy}(g/(H_{xy} + H_{xz})) = \varphi_{xy}(g/H_{xz}) + H_{yz} = \alpha_{xy}(g)/H_{yx} + H_{yz} = \alpha_{xy}(g)/(H_{yx} + H_{yz}).$$

The first equality holds by the definition of φ^z_{xy} in Definition 2.3, the second equality holds by (4), and the last equality holds by $\bigcup g/H/H' = g/(H + H')$. This proves (6).

We will need that the automorphism associated to the transposition of L_i and L_j for distinct $i, j < 4$ is the identity modulo $L_i + L_j$.

$$\alpha([L_i, L_j])(g)/(L_i + L_j) = g/(L_i + L_j). \tag{7}$$

Indeed, let α_1 denote the isomorphism associated to the transposition of L_i and L_j in $\{L_i : i < 4\}$, that is to say, $\alpha_1 = \alpha([L_i, L_j])$ and let Q denote $L_i + L_j$. Then $\alpha_1[L_i] = L_j$ and $\alpha_1[L_j] = L_i$ by Lemma 3.1(i) and the definitions of α_1 and the transposition $[L_i, L_j]$. Therefore, $\alpha_1[Q] = Q$. Let L_p be distinct from L_i and L_j. Then $g = n + f$ for some $n \in Q$ and $f \in L_p$, by Lemma 3.2(iii) since L_i, L_j and L_p are distinct. Also, $\alpha_1(f) = f$ by Lemma 3.1(ii). Hence

$$\alpha_1(g)/Q = \alpha_1(n + f)/Q = \alpha_1(n)/Q + \alpha_1(f)/Q = n/Q + f/Q = (n + f)/Q = g/Q.$$

The first and last equalities hold by $g = n + f$, the second and fourth ones hold since α_1 is a homomorphism and the third equality holds because we have $\alpha_1(n)/Q = n/Q$ by $\alpha_1[Q] = Q$ and $n \in Q$. This proves (7).

We are ready to check condition (iv) in Definition 2.3.

$$\varphi^x_{yz}\varphi^z_{xy}(g/(H_{xy} + H_{xz})) = \varphi^x_{yz}(\alpha_{xy}(g)/(H_{yx} + H_{yz})) = \alpha_{yz}\alpha_{xy}(g)/(H_{zx} + H_{zy}) = \alpha([H_{zx}, H_{zy}])\alpha_{xz}(g)/(H_{zx} + H_{zy}) = \alpha_{xz}(g)/(H_{zx} + H_{zy}) = \varphi^y_{xz}(g/H_{xy} + H_{xz}).$$

The first two and the last equalities hold by (6). The third equality holds by Lemma 3.1 and Lemma 3.3(iii). The fourth equality holds by (7). This finishes checking condition (iv), and the proof of Theorem 3.4 is complete. □

§4. Shifting, proof of nonrepresentability. In this section, we return to multiplicative as opposed to additive notation for groups: we denote the binary operation of a group by $\circ$, the neutral or identity element is denoted by e and the inverse of an element g in a group is denoted by g^{-1}.

Throughout this section, let $\mathfrak{F}$ be any *nontrivial* commutative group, assume that we are given a particular choice for $\langle H_{xy} : x, y \in I\rangle$ that satisfies (1) stated in the previous section, and let $\mathcal{F} = (\mathcal{G}, \varphi)$ denote the group frame we defined from $\mathfrak{F}$ in the previous section. Let us give names for the elements of the index set I, i.e., let

$$I = \{p, q, r, s, t\}.$$

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

We will define the coset system C such that it will be nontrivial only in the triangle pqr.

Let $f \in F^3$ be such that $f \notin H_{pq} \circ H_{pr}$. There is such an element because $\mathfrak{F}^3/H_{pq} \circ H_{pr} \cong \mathfrak{F}$ by Lemma 3.2(v), (1) and because we assume that $\mathfrak{F}$ has more than one element. We now define three elements associated to f (we could say, three "versions" of f):

$$f_p = f, \quad f_q = \alpha_{pq}(f), \quad \text{and} \quad f_r = \alpha_{pr}(f). \tag{8}$$

We call (x, y, z) a *positive* permutation of (p, q, r) if (x, y, z) is one of (p, q, r), (q, r, p), (r, p, q), otherwise we call (x, y, z) a *negative* permutation of (p, q, r).

Define $C = \langle C_{xyz} : x, y, z \in I \rangle$ as follows.

$C_{xyz} = f_x/(H_{xy} \circ H_{xz})$, when (x, y, z) is a positive permutation of (p, q, r),
$C_{xyz} = f_x^{-1}/(H_{xy} \circ H_{xz})$, when (x, y, z) is a negative permutation of (p, q, r),
$C_{xyz} = H_{xy} \circ H_{xz}$, when (x, y, z) is not a permutation of (p, q, r).

THEOREM 4.1. *$(\mathcal{G}, \varphi, C)$ is a group coset frame.*

Proof. We have to show that $(\mathcal{G}, \varphi, C)$ satisfies the conditions listed in Definition 2.3. Since $(\mathcal{G}, \varphi)$ is a group frame by Theorem 3.4, frame conditions (i)–(iii) hold. Also (iv) holds with $\tau(C_{xyz})$ being the identity function on the cosets of $H_{xy} \circ H_{xz}$, by Theorem 3.4. Since $\mathfrak{F}$ is commutative by assumption, we have that $\mathfrak{F}^3$ is also commutative, and thus all inner automorphisms are the identity function in it. Thus, frame condition (iv) holds for our particular choice of C, too.

We now check that C satisfies the four coset conditions (v)–(viii) listed in Definition 2.3. Our system of cosets has the special property that only cosets C_{xyz} with $\{x, y, z\} = \{p, q, r\}$ are not the identity cosets. This special type of coset system is dealt with in [1, Corollary 4.8], we will check the four conditions listed in that corollary.

Let us order the elements of I as $p < q < r < s < t$. The first condition, condition (i) of [1, Corollary 4.8] asks that $C_{pqr}^{-1} = C_{prq}$ and the analogues for qrp, rpq hold. Condition (i) of [1] is satisfied because

$$C_{pqr}^{-1} = (f_p/(H_{pq} \circ H_{pr}))^{-1} = f_p^{-1}/(H_{pq} \circ H_{pr}) = C_{prq}.$$

The cases qrp, rpq are completely analogous. This proves (i) of [1]. Condition (ii) of [1] asks that $\varphi_{pq}[C_{pqr}] = C_{qrp}$. Indeed,

$$\varphi_{pq}[C_{pqr}] = \varphi_{pq}[f_p/(H_{pq} \circ H_{pr})] = \alpha_{pq}(f_p)/(H_{qp} \circ H_{qr}) = f_q/(H_{qp} \circ H_{qr}) = C_{qrp}.$$

The first and last equalities hold by the definitions of C_{pqr}, C_{rpq}, respectively. The second equality holds by (5) and commutativity of $\mathfrak{F}^3$, and the third equality holds by (8). This shows that coset condition (ii) of [1] holds. Checking condition (iii) of [1] is similar:

$$\varphi_{pr}[C_{pqr}] = \varphi_{pr}[f_p/(H_{pq} \circ H_{pr})] = \alpha_{pr}(f_p)/(H_{rp} \circ H_{rq}) = f_r/(H_{rp} \circ H_{rq}) = C_{rpq}.$$

Finally, condition (iv) of [1] asks that

$$C_{pqr} \subseteq H_{pq} \circ H_{pr} \circ H_{ps} \cap H_{pq} \circ H_{pr} \circ H_{pt}.$$

This holds because $H_{pq} \circ H_{pr} \circ H_{px} = G_x = F^3$ for $x = s, t$ by Lemma 3.2(iii) and (1). Thus, C is indeed a system of cosets that satisfies the four coset conditions of [1, Corollary 4.8]. This completes the proof of Theorem 4.1. □

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

THEOREM 4.2. *The polygroupoid* $\mathfrak{P}(\mathcal{G}, \varphi, C)$ *is not representable.*

Proof. Theorem 3.4 states that $(\mathcal{G}, \varphi)$ is a group frame, let $\mathfrak{M} = \mathfrak{P}(\mathcal{G}, \varphi)$ be the structure associated to $(\mathcal{G}, \varphi, H)$ with $H = \langle H_{xy} \circ H_{xz} : x, y, z \in I \rangle$ the "trivial" coset system, and let $\mathfrak{P} = \mathfrak{P}(\mathcal{G}, \varphi, C)$. Both $\mathfrak{P}$ and $\mathfrak{M}$ are locally functional polygroupoids, by Theorems 2.5, 3.4, and 4.1. They have the same universe

$$P = \{(x, g, y) : x, y \in I \text{ and } g \in G_x/H_{xy}\},$$

they have the same identity atoms and inverse operation, by Definition 2.1, only their multiplication operation differ, and that differs only for elements $(x, g, y), (y, h, z) \in P$ with $\{x, y, z\} = \{p, q, r\}$. Let $\otimes$ denote multiplication of $\mathfrak{P}$, and let $*$ denote multiplication of $\mathfrak{M}$.

Let us inspect these two multiplications. Let $(x, g, y), (y, h, z) \in P$ and let us define

$$g(xy)h = \varphi_{yx}(\varphi_{xy}(g) \circ h).$$

Note that $g(xy)h$ is a coset of $H_{xy} \circ H_{xz}$, because g is a coset of H_{xy}, so $\varphi_{xy}(g)$ is a coset of H_{yx}, thus $\varphi_{xy}(g) \circ h$ is a coset of $H_{yx} \circ H_{yz}$ since h is a coset of H_{yz}. Therefore $\varphi_{yx}(\varphi_{xy}(g) \circ h)$ is a coset of $H_{xy} \circ H_{xz}$ by condition (iii) in Definition 2.3. Thus, $g(xy)h$ is a coset of $H_{xy} \circ H_{xz}$, and therefore so is $g(xy)h \circ C_{xyz}$. For convenience, let us define

$$(x, X, y) = \{(x, g, y) : g \subseteq X\}$$

when X is a coset of $H_{xy} \circ H_{xz}$ for some $z \in I$. With this notation we have

$$(x, g, y) * (y, h, z) = (x, g(xy)h, z), \quad \text{and}$$
$$(x, g, y) \otimes (y, h, z) = (x, g(xy)h \circ C_{xyz}, z)\ .$$

By Theorem 2.8 we have that $\mathfrak{M}$ is representable. We call $\mathfrak{M}$ the *representable, or unshifted, pair* of $\mathfrak{P}$.

We prove that $\mathfrak{P}$ is not representable by deriving a contradiction from assuming that it is. Assume that $\mathfrak{P}$ is representable. By Theorem 2.8 then it has a scaffold, let us fix such a scaffold a for $\mathfrak{P}$. Note that the set of identity elements of $\mathfrak{P}$ is $\{(x, e, x) : x \in I\}$, and $\mathfrak{P}$ is connected. Thus, $a = \langle a_{xy} : x, y \in I \rangle$ is a system of elements of P that satisfies for all $x, y, z \in I$

$$a_{xy} = (x, g_{xy}, y) \text{ for some } g_{xy},$$
$$a_{xx} = (x, e, x),$$
$$a_{xy} = a_{yx}^{-1}, \quad \text{and}$$
$$a_{xz} \in a_{xy} \otimes a_{yz}.$$

We now check what this scaffold "does" in the representable pair $\mathfrak{M}$ of $\mathfrak{P}$. Since $\mathfrak{M}$ and $\mathfrak{P}$ have the same universe,

$$a_{xy} = (x, g_{xy}, y) \quad \text{is an element of } \mathfrak{M}, \text{ too.} \tag{9}$$

Since composition $*$ of $\mathfrak{M}$ coincides with composition $\otimes$ of $\mathfrak{P}$ on elements not in the "triangle pqr," we have

$$a_{st} \in a_{sx} * a_{xt} \quad \text{for } x \in \{p, q, r\}, \tag{10}$$

and also

$$a_{pq} \in a_{px} * a_{xq} \quad \text{for } x \in \{s, t\}, \tag{11}$$

and similarly for any yz in place of pq if $y, z \in \{p, q, r\}$. Since converse in $\mathfrak{M}$ is the same as in $\mathfrak{P}$ we continue to have

$$a_{xy} = a_{yx}^{-1} \quad \text{in } \mathfrak{M}. \tag{12}$$

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

From this point on, we will use that $\mathfrak{M}$ is representable. Let rep be any representation of $\mathfrak{M}$. Now, $\mathsf{rep}(a_{st}) \neq \emptyset$ by the definition of a representation, so there are u_s, u_t such that

$$(u_s, u_t) \in \mathsf{rep}(a_{st}). \tag{13}$$

By (10) we have $\mathsf{rep}(a_{st}) \subseteq \mathsf{rep}(a_{sx}) \mid \mathsf{rep}(a_{xt})$; thus, by the definition of relation composition $\mid$ there are u_x for $x \in \{p, q, r\}$ such that

$$(u_s, u_x) \in \mathsf{rep}(a_{sx}) \quad \text{and} \quad (u_x, u_t) \in \mathsf{rep}(a_{xt}), \tag{14}$$

for all $x \in \{p, q, r\}$. See Figure 3.

We now show that the structure of $\mathfrak{M}$, that is the structure of the group frame $(\mathcal{G}, \varphi)$, forces that $(u_x, u_y) \in \mathsf{rep}(a_{xy})$ for $x, y \in \{p, q, r\}$, and this will lead to a contradiction.

We have $(u_p, u_s) \in \mathsf{rep}(a_{sp})^{-1}$ by $(u_s, u_p) \in \mathsf{rep}(a_{sp})$. Also, $\mathsf{rep}(a_{sp})^{-1} = \mathsf{rep}(a_{sp}^{-1}) = \mathsf{rep}(a_{ps})$ by the properties of rep and by (12), respectively. Now $(u_s, u_q) \in \mathsf{rep}(a_{sq})$ implies that $(u_p, u_q) \in \mathsf{rep}(a_{ps}) \mid \mathsf{rep}(a_{sq}) = \bigcup \mathsf{rep}[a_{ps} * a_{sq}]$. Thus we got

$$(u_p, u_q) \in \mathsf{rep}(b) \qquad \text{for some } b \in a_{ps} * a_{sq}.$$

We get in a completely analogous way that

$$(u_p, u_q) \in \mathsf{rep}(d) \qquad \text{for some } d \in a_{pt} * a_{tq}.$$

Since representations of distinct elements are disjoint, we get that $b = d$, and so

$$(u_p, u_q) \in \mathsf{rep}(b) \qquad \text{for some } b \in (a_{ps} * a_{sq}) \cap (a_{pt} * a_{tq}). \tag{15}$$

From (12) and (11) we also have that

$$a_{pq} \in (a_{ps} * a_{sq}) \cap (a_{pt} * a_{tq}). \tag{16}$$

We are going to show that $(a_{ps} * a_{sq}) \cap (a_{pt} * a_{tq})$ is a one-element set, this will imply $(u_p, u_q) \in \mathsf{rep}(a_{pq})$ by (15), (16).

For all $x, y \in I$ we have that $a_{xy} = (x, g_{xy}, y)$ for some coset g_{xy} of H_{xy}. Thus $a_{ps} * a_{sq} = (p, D, q)$ for some coset D of $H_{pq} \circ H_{ps}$ and $a_{pt} * a_{tq} = (p, E, q)$ for some coset E of $H_{pq} \circ H_{pt}$. Since H_{pq}, H_{ps}, H_{pt} are distinct elements of $\{L_i : i < 4\}$ by the construction of the group frame $(\mathcal{G}, \varphi)$ and D, E are cosets of $H_{pq} \circ H_{ps}$ and $H_{pq} \circ H_{pt}$, respectively, Lemma 3.2(iv) implies that $D \cap E$ is a single coset of H_{pq}. Therefore, $b = (p, D \cap E, q)$ for any

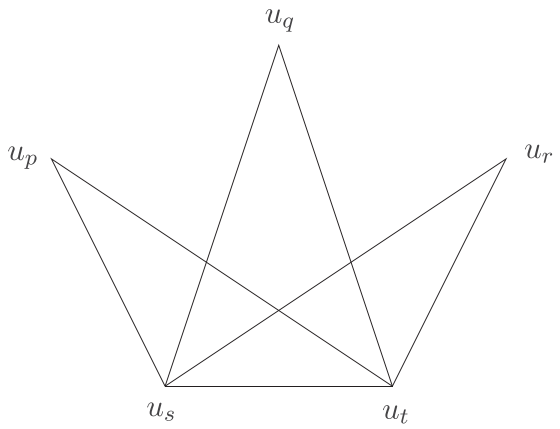


Fig. 3. Selection of the points in the representation of $\mathfrak{M}$.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

element b of $(a_{ps} * a_{sq}) \cap (a_{pt} * a_{tq})$. Thus indeed this set has a single element, and so $(u_p, u_q) \in \mathsf{rep}(a_{pq})$.

In a completely analogous way we get that $(u_p, u_r) \in a_{pr}$ and $(u_r, u_q) \in a_{rq}$. Thus

$$(u_p, u_q) \in \mathsf{rep}(a_{pr}) \mid \mathsf{rep}(a_{rq}) = \bigcup \mathsf{rep}[a_{pr} * a_{rq}].$$

Also $(u_p, u_q) \in \mathsf{rep}(a_{pq})$. By disjointness of the representing relations we get that

$$a_{pq} \in a_{pr} * a_{rq}. \tag{17}$$

By the defining conditions of our scaffold a we also have

$$a_{pq} \in a_{pr} \otimes a_{rq}. \tag{18}$$

However, we got $\otimes$ from $*$ by shifting with a nonidentity coset $K = f_p/(H_{pr} \circ H_{pq})$. Thus we have that $a_{pr} * a_{pq} = (p, D, q)$ for some coset D of $H_{pr} \circ H_{pq}$, by the definition of $*$, and we have that $a_{pr} \otimes a_{pq} = (p, D \circ K, q)$. Since K is a nonidentity coset, we have that D and $D \circ K$ are distinct cosets, so they are disjoint. This in turn shows that

$$a_{pr} * a_{pq} \text{ is disjoint from } a_{pr} \otimes a_{pq}$$

,which contradicts (17), (18). We got a contradiction, so $\mathfrak{P}$ cannot be representable. □

§5. Continuum many subvarieties of coset relation algebras. In this section, we use the series of nonrepresentable locally functional polygroupoids we constructed in §3 and §4 to prove a theorem about relation algebras.

Relation algebras are the subalgebras of complex algebras of polygroupoids, [8, Theorem 4.2]. The definition of the *complex algebra*, or algebra of complexes, of a polygroupoid is rather natural. The elements of the complex algebra are the complexes, that is, to say the subsets, of the structure, the operations and relations of the structure extend naturally to operations on these complexes, and additionally we take the Boolean algebra structure of subsets of a set. Thus the complex algebra of a polygroupoid-type algebra is a "real" algebra, with totally defined operations, see [8, sec. 2]. Complex algebras are one of the key players of algebraic logic.

A relation algebra is *representable* if it is embeddable into the complex algebra of a representable polygroupoid. The standard definition of a representable relation algebra is slightly different, but by using [8, Theorem 4.2], it is not difficult to show that the two definitions are equivalent. (For some detail, see Corollary 3.4(3) and the following Remark 1 in [9], and Remark (2) on p. 30 of [10].) Similarly, a relation algebra is called a *coset relation algebra* if it is embeddable into the complex algebra of a locally functional polygroupoid. Again, it is not difficult to show that the original definition given in [17] is equivalent to this one.

Both the class RRA of representable relation algebras and the class CRA of coset relation algebras are varieties, a technical term for being axiomatized by sets of equations, see [13, Theorem 16.27], [19, Theorem 3.1]. We have $\mathsf{RRA} \subseteq \mathsf{CRA}$ because the natural relation algebra of all subsets of an equivalence relation is also a coset relation algebra ([17, p. 51] and [19, Introduction]).

It is proved in [19, Theorem 4.10] that no variety between RRA and CRA can be finitely axiomatized, and moreover, no such variety can be axiomatized by a set of universal formulas containing only finitely many variables [19, Theorem 4.11]. No such variety has decidable equational theory, either, by [40, Theorem 8.5.(viii)], or [3, Corollary 2.9].

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

The following theorem states that there are many varieties to which the above theorems apply.

THEOREM 5.1. *There are continuum many subvarieties of* CRA *that contain* RRA.

Proof. We follow the proof that Jónsson gave for [23, Theorem 7.8]. An algebra is called *simple* if it has no nontrivial congruences. Jónsson shows that if $\langle \mathfrak{A}_n : n \in J \rangle$ is a series of simple finite relation algebras such that no $\mathfrak{A}_n$ can be embedded into $\mathfrak{A}_m$ for distinct n, m, then the varieties generated by $\{\mathfrak{A}_n : n \in S\}$ are all distinct for the subsets S of J. We will use this idea. However, since we also need that our varieties contain RRA, we will use a bigger variety than what $\{\mathfrak{A}_n : n \in S\}$ generates.

Let $\langle \mathfrak{F}_n : n \in J \rangle$ be an infinite system of finite commutative nontrivial groups of distinct size, e.g., we can take $\mathfrak{F}_n$ to be the n-element cyclic group, and take J to be the set of natural numbers $n \geq 2$. Let us fix now $n \geq 2$. Let $\mathcal{F}_n$ be a group frame that we constructed from $\mathfrak{F}_n \times \mathfrak{F}_n \times \mathfrak{F}_n$ in §3, let $\mathfrak{P}_n$ denote the nonrepresentable polygroupoid that we constructed in §4, and let $\mathfrak{C}_n$ denote the complex algebra of $\mathfrak{P}_n$. We are going to show that $\langle \mathfrak{C}_n : n \in J \rangle$ is a system of simple finite nonrepresentable coset relation algebras.

Since $\mathfrak{F}_n$ is finite, by the construction we have that $\mathfrak{P}_n$ is also finite, so $\mathfrak{C}_n$ is also finite. Since $\mathfrak{P}_n$ is a locally finite polygroupoid, we have that $\mathfrak{C}_n$ is a coset relation algebra. Next we show that $\mathfrak{C}_n$ is nonrepresentable because $\mathfrak{P}_n$ is not representable and finite: Assume in contrary that $\mathfrak{C}_n$ is representable, say rep assigns binary relations to elements of $\mathfrak{C}_n$. Take the restriction of this representation to the atoms of $\mathfrak{C}_n$. Then we get a representation for the polygroupoid $\mathfrak{P}_n$ because $\mathsf{rep}(a) \mid \mathsf{rep}(b) = \bigcup \mathsf{rep}[a \circ b]$ by $\mathfrak{C}_n$ being finite. Since $\mathfrak{P}_n$ is not representable by Theorem 4.2, we get that $\mathfrak{C}_n$ is not representable. Finally, $\mathfrak{C}_n$ is simple because $\mathfrak{P}_n$ is connected, see [8, Theorem 4.5].

$$\mathfrak{C}_n \text{ is a finite, simple, nonrepresentable coset relation algebra.} \tag{1}$$

For any set $S \subseteq J$, let $\mathsf{V}(S)$ denote the class of algebras in CRA into which none of $\mathfrak{C}_n$ with $n \in S$ can be embedded:

$$\mathsf{V}(S) = \{\mathfrak{A} \in \mathsf{CRA} : \mathfrak{C}_n \text{ cannot be embedded into } \mathfrak{A}, \text{ for all } n \in S\}. \tag{2}$$

By [23, Theorem 7.1], every finite simple relation algebra is splitting in the class RA of all relation algebras, thus the biggest variety of RA not containing $\mathfrak{C}_n$ is the class of all relation algebras into which $\mathfrak{C}_n$ cannot be embedded. This is called the conjugate variety of $\mathfrak{C}_n$, let us denote it by $\mathsf{V}^-(\mathfrak{C}_n)$. By $\mathsf{CRA} \subseteq \mathsf{RA}$ then we have that

$$\mathsf{V}(S) = \bigcap \{\mathsf{V}^-(\mathfrak{C}_n) : n \in S\} \cap \mathsf{CRA}.$$

This shows that $\mathsf{V}(S)$ is a variety, since it is the intersection of varieties.

All the $\mathfrak{C}_n$ are nonrepresentable by (1), therefore none of them can be embedded into a representable relation algebra (since RRA is closed under taking subalgebras). Also, $\mathsf{RRA} \subseteq \mathsf{CRA}$, hence we get

$$\mathsf{RRA} \subseteq \mathsf{V}(S) \subseteq \mathsf{CRA}.$$

It remains to show that all the $\mathsf{V}(S)$ are distinct. For this it is enough to show, that

$$\mathfrak{C}_n \text{ cannot be embedded into } \mathfrak{C}_m \text{ if } n, m \text{ are distinct.} \tag{3}$$

Indeed, assume that (3) holds and S, Z are distinct subsets of J, say $n \in S$ but $n \notin Z$. Then $\mathfrak{C}_n \notin \mathsf{V}(S)$ since it can be embedded into itself and $n \in S$. Also, $\mathfrak{C}_n \in \mathsf{V}(Z)$ if (3) holds because $n \notin Z$ so no $\mathfrak{C}_m$ with $m \in Z$ can be embedded into $\mathfrak{C}_n$. This shows that $\mathsf{V}(S) \neq \mathsf{V}(Z)$.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

Next we show that (3) holds. Indeed, assume that $\mathfrak{C}_n$ is embedded into $\mathfrak{C}_m$, say by a one-to-one homomorphism $f : \mathfrak{C}_n \to \mathfrak{C}_m$. We show that $n = m$. There are five subidentity atoms both in $\mathfrak{C}_n$ and in $\mathfrak{C}_m$. Consequently, the monomorphism f has to take subidentity atoms to subidentity atoms, because of the following. Since f is a homomorphism, it has to take the five subidentity atoms of $\mathfrak{C}_n$ to five disjoint subidentity elements of $\mathfrak{C}_m$, and since f is a monomorphism, it has to take an atom to a nonzero element. Since $\mathfrak{C}_m$ has only five subidentity atoms, this can be only if f takes subidentity atoms to subidentity atoms.

Assume that $f(x) = y$ for a subidentity atom x of $\mathfrak{C}_n$. Then y is a subidentity atom of $\mathfrak{C}_m$, we have just showed this, and $f(x \circ 1 \circ x) = y \circ 1 \circ y$. Now, the group $\mathfrak{F}_n \times \mathfrak{F}_n \times \mathfrak{F}_n$ can be recovered as the set of atoms below $x \circ 1 \circ x$. In particular, there are n^3 atoms below $x \circ 1 \circ x$ such that their sum is $x \circ 1 \circ x$. Similarly, there are m^3 atoms below $y \circ 1 \circ y$ in $\mathfrak{C}_m$.

Now, the atoms g below $x \circ 1 \circ x$, are special in the sense that $g^{-1} \circ g = x$ is true for all of them (they are functional in relation algebraic terminology). Thus, the same has to be true for their images $f(g)$. It can be checked, that the functional elements below $y \circ 1 \circ y$ in $\mathfrak{C}_m$ are exactly the atoms below $y \circ 1 \circ y$ (they are functional as in $\mathfrak{C}_n$, but the sum of more than one such atoms is never functional). Thus, f has to take the n^3 atoms below $x \circ 1 \circ x$ to m^3 atoms below $y \circ 1 \circ y$ such that the sum of the images is $y \circ 1 \circ y$. This is possible only if in $\mathfrak{C}_m$ also $y \circ 1 \circ y$ is the sum of n^3 atoms. Since $\mathfrak{F}_n$ and $\mathfrak{F}_m$ have different finite cardinalities for $n \neq m$, this is possible only if $n = m$. This proves (3), and with this the proof of the theorem is also complete. □

§6. Acknowledgments. We thank Robin Hirsch for calling our attention to the connection between Brandt groupoids and our group relation algebras. We are indebted to the referee for a thorough reading and many useful comments.

BIBLIOGRAPHY

[1] Andréka, H. & Givant, S. (2018). Coset relation algebras. *Algebra Universalis*, **79**(2), paper 28, 53 pp. Sharit link: https://rdcu.be/L94h.

[2] Andréka, H. & Givant, S. (2019). A representation theorem for measurable relation algebras with cyclic groups. *Transactions of the American Mathematical Society*, **371**(10), 7175–7198.

[3] Andréka, H., Givant, S., & Németi, I. (1997). Decision problems for equational theories of relation algebras. *Memoirs of the American Mathematical Society*, **126**(604), xiv+126pp.

[4] Andréka, H., Maddux, R. D., & Németi, I. (1991). Splitting in relation algebras. *Proceedings of the American Mathematical Society*, **111**(4), 1085–1093.

[5] van Benthem, J. A. F. K. (1991). *Language in Action*. Lambdas and Dynamic Logic, Amsterdam: North-Holland. x+349 pp.

[6] Brink, C., Kahl, W., & Schmidt, G. (editors) (1997). *Relational Methods in Computer Science*. Vienna: Springer Verlag.

[7] Comer, S. D. (1976). Multi-valued loops, geometries, and algebraic logic. *Houston Journal of Mathematics*, **2**, 373–380.

[8] Comer, S. D. (1983). A new foundation for the theory of relations. *Notre Dame Journal of Formal Logic*, **24**(2), 181–187.

[9] Comer, S. D. (1984), Combinatorial aspects of relations. *Algebra Universalis*, **18**(1), 77–94.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

[10] Comer, S. D. (1985). The Cayley representation of polygroups. In Corsini, P., editor. *Convegno su*: *Ipergruppi, altre strutture multivoche e loro applicazioni*. Udine: Università degli Studi di Udine, pp. 27–34.

[11] Dresher, M. & Ore, O. (1938). Theory of multigroups. *American Journal of Mathematics*, **60**, 705–733.

[12] Düntsch, I. (2005). Relation algebras and their application in temporal and spatial reasoning. *Artificial Intelligence Review*, **23**, 315–357.

[13] Givant, S. (2017). *Introduction to Relation Algebras*. Cham: Springer International Publishing AG. xxxii+572 pp.

[14] Givant, S. (2017). *Advanced Topics in Relation Algebras*. Cham: Springer International Publishing AG. xix+605 pp.

[15] Givant, S. (2018). Relation algebras and groups. *Algebra Universalis*, **79**(2), paper 16, 38pp. Sharit link: https://rdcu.be/LJ2Y.

[16] Givant, S. & Andréka, H. Notes on measurable relation algebras. Manuscript, 1996–2001. Cca. 500 pages. Part of these notes is published as [1, 2, 15, 18, 19]. Part of the present paper is also based on these notes.

[17] Givant, S. & Andréka, H. (2002). Groups and algebras of relations. *The Bulletin of Symbolic Logic*, **8**, 38–64.

[18] Givant, S. & Andréka, H. (2018). A representation theorem for measurable relation algebras. *Annals of Pure and Applied Logic*, **169**(11), 1117–1189.

[19] Givant, S. & Andréka, H. (2018). The variety of coset relation algebras. *The Journal of Symbolic Logic*, **83**(4), 1595–1609.

[20] Hirsch, R. & Hodkinson, I. (2001). Representability is not decidable for finite relation algebras. *Transactions of the American Mathematical Society*, **353**, 1403–1425.

[21] Hirsch, R. & Hodkinson, I. (2002). *Relation Algebras by Games*. Amsterdam: North-Holland. 712 pp.

[22] Jónsson, B. (1959). Representation of modular lattices and of relation algebras. *Transactions of the American Mathematical Society*, **92**, 449–464.

[23] Jónsson, B. (1982). Varieties of relation algebras. *Algebra Universalis*, **15**, 273–298.

[24] Jónsson, B. (1991). The theory of binary relations. In Andréka, H., Monk, J. D., and Németi, I., editors. *Algebraic Logic*. Amsterdam: North-Holland, pp. 245–292.

[25] Jónsson, B. & Tarski, A. (1952). Boolean algebras with operators. Part II. *American Journal of Mathematics*, **74**, 127–162.

[26] Khaled, M. (2019). The finitely axomatizable complete theories of nonassociative arrow frames. *Advances in Mathematics*, **346**(13), 194–218.

[27] Lyndon, R. C. (1961). Relation algebras and projective geometries. *Michigan Mathematical Journal*, **8**, 21–28.

[28] Maddux, R. D. (1981). Embedding modular lattices into relation algebras. *Algebra Universalis*, **12**, 242–246.

[29] Maddux, R. D. (1991). The origin of relation algebras in the development and axiomatization of the calculus of relations. *Studia Logica*, **50**, 421–455.

[30] Maddux, R. D. (2006). *Relation Algebras*. Amsterdam: North-Holland. xxvi+731 pp.

[31] Maddux, R. D. (2018). Subcompletions of representable relation algebras. *Algebra Universalis*, **79**(2), Paper 20, 32 pp.

[32] Marx, M. & Venema, Y. (1997). *Multi-Dimensional Modal Logic*. Dordrecht: Kluwer Academic Publishers. xiii+239 pp.

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press

[33] McKenzie, R. (1970). Representations of integral relation algebras. *Michigan Mathematical Journal*, **17**, 279–287.

[34] Monk, J. D. (1964). On representable relation algebras. *Michigan Mathematical Journal*, **11**, 207–210.

[35] Pratt, V. R. (1990). Dynamic algebras as a well-behaved fragment of relation algebras. In Bergman, J., Maddux, R. D., and Pigozzi, D., editors. *Algebraic Logic and Universal Algebra in Computer Science*. Berlin: Springer-Verlag, pp. 77–110.

[36] Pratt, V. R. (1992). Origins of the calculus of binary relations. *Logic in Computer Science, Proceedings of the 7th Annual IEEE Symposion, IEEE*. pp. 248–254.

[37] Schmidt, G. (2011). *Relational Mathematics*. Cambridge: Cambridge University Press. xiii+566 pp.

[38] Simon, A. (1997). Nonrepresentable Algebras of Relations. Ph.D. Dissertation, Budapest: Hungarian Academy of Sciences. iii+86 pp.

[39] Tarski, A. (1941). On the calculus of relations. *The Journal of Symbolic Logic*, **6**, 73–89.

[40] Tarski, A. & Givant, S. (1987). *A Formalization of Set Theory without Variables*. Providence, Rhode Island: American Mathematical Society. xxi+318 pp.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA ST. 13-15, H-1053, HUNGARY
E-mail: andreka.hajnal@renyi.mta.hu
E-mail: nemeti.istvan@renyi.mta.hu

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
MILLS COLLEGE
500 MACARTHUR BOULEVARD
OAKLAND, CA 94613 USA
E-mail: wang@mills.edu

https://doi.org/10.1017/S1755020319000224 Published online by Cambridge University Press