Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 104 NOVEMBER 1988 PART 3

Math. Proc. Camb. Phil. Soc. (1988), **104**, 417
Printed in Great Britain

Some groups which classify knots

BY B. ZIMMERMANN

Università degli Studi di Trieste, Dipartimento di Scienze Matematiche, 34100 *Trieste, Italy*

(*Received* 19 *November* 1987)

Groups associated to a knot in S^3 which classify the knot have been constructed by Conway and Gordon [**4**], Simon [**12**] and Whitten [**16**]; see also [**1**] and [**19**] for certain special classes of knots. In this note we show that for a knot K in S^3 many of the groups $\pi_1(S^3\backslash K)/\langle m^a l^b\rangle$ classify the knot; here m resp. l denote a meridian resp. longitude of the knot and $\langle\ \rangle$ denotes normal closure. This is based on Thurston's orbifold geometrization theorem [**13–15**] and the following result (see also [**8**, **19**]):

THEOREM 1. *Let O_1 and O_2 be good closed irreducible 3-orbifolds which possess a decomposition into geometric pieces (along Euclidean 2-suborbifolds, see* [**2**, **15**]) *and have infinite (orbifold-) fundamental group. Then O_1 and O_2 are diffeomorphic (as orbifolds) if and only if $\pi_1 O_1$ and $\pi_1 O_2$ are isomorphic.*

Remark. An orbifold is said to be good if it admits a finite covering which is a manifold (see [**2**, **6**] for basic definitions concerning orbifolds). It has been shown by McCullough and Miller [**8**] that an irreducible 3-orbifold with a geometric decomposition is good.

Proof of Theorem 1. If O_1 and O_2 are hyperbolic, resp. Seifert fibred, resp. Euclidean, Theorem 1 follows from Mostow's rigidity theorem [**10**], resp. theorem 1·1 of [**18**], resp. from a Bieberbach theorem for crystallographic groups (see e.g. [**17**]). For 3-orbifolds with a non-trivial geometric decomposition and solvable 3-orbifolds it follows from theorems 0·1 and 3·1 of [**18**]. ∎

Now let K be a knot in S^3. Let $S^3\backslash\mathring{N}(K)$ be the knot exterior and m resp. l a meridian resp. longitude of K on $\partial N(K)$. Let p, q be integers with $(p, q) = 1$, and let $M_{p,q} = M_{p,q}(K)$ be the closed 3-manifold obtained by p/q-surgery on K, i.e. by attaching a solid torus T to $S^3\backslash\mathring{N}(K)$ along $\partial N(K)$ such that a meridian of T is attached to the curve $m^p l^q$ on $\partial N(K)$ (see [**3**, **11**]). The central curve in T gives a knot $K_{p,q}$ in $M_{p,q}$. Let $O^n_{p,q} = O^n_{p,q}(K)$ be the 3-orbifold with underlying space $M_{p,q}$ and branching-order n associated to its singular set $K_{p,q}$, and $M^n_{p,q} = M^n_{p,q}(K)$ the n-fold

https://doi.org/10.1017/S0305004100065609 Published online by Cambridge University Press

cyclic branched covering of $M_{p,q}$ along $K_{p,q}$, with covering transformation h, where $h^n = \mathrm{id}$ and $M^n_{p,q}/h = O^n_{p,q}$. We have a short exact sequence

$$1 \to \pi_1 M^n_{p,q} \hookrightarrow \pi_1 O^n_{p,q} \to \mathbb{Z}_n \cong \langle h \rangle \to 1,$$

i.e. $\pi_1 M^n_{p,q}$ is a normal subgroup of index n in the orbifold-fundamental group $\pi_1 O^n_{p,q}$. Also $\pi_1 M_{p,q}$ is isomorphic to $\pi_1(S^3\backslash K)/\langle m^p l^q \rangle$ and $\pi_1 O^n_{p,q}$ is isomorphic to $\pi_1(S^3\backslash K)/\langle (m^p l^q)^n \rangle$, with $(m^p l^q)^n = m^{np} l^{nq}$.

THEOREM 2. *Let K_1 and K_2 be knots in S^3. Let p, q and n be integers such that $(p, q) = 1$, $0 < |p| < q$ and $n > 2$. Then the pairs (S^3, K_1) and (S^3, K_2) are diffeomorphic if and only if the groups $\pi_1 O^n_{p,q}(K_1)$ and $\pi_1 O^n_{p,q}(K_2)$ are isomorphic.*

Proof. One implication in Theorem 2 is clear. For the other, suppose that $\pi_1 O^n_{p,q}(K_1)$ and $\pi_1 O^n_{p,q}(K_2)$ are isomorphic and that neither K_1 nor K_2 is the trivial knot. Let K be K_1 or K_2. Because $q \geqslant 2$, by results of Gordon and Luecke [7] $M_{p,q}$ is irreducible and moreover $K_{p,q}$ is a prime knot in $M_{p,q}$, so $O^n_{p,q}$ is an irreducible 3-orbifold. By Thurston's orbifold geometrization theorem $O^n_{p,q}$ possesses a geometric decomposition.

Claim. $\pi_1 O^n_{p,q}$ is infinite.

Assume $\pi_1 O^n_{p,q}$ is finite. Then $O^n_{p,q}$ belongs to the spherical geometry: it is a spherical Seifert-3-orbifold. In particular, the covering transformation h on $M^n_{p,q}$ is fibre-preserving for a Seifert fibration of $M^n_{p,q}$; because $n > 2$, the fixed-point set of h is a fibre of this fibration. It follows that the knot $K_{p,q}$ is a circle-fibre of the orbifold-fibration of $O^n_{p,q}$, so $M_{p,q}\backslash K_{p,q} \cong S^3\backslash K$ is also Seifert-fibred. But then K is a torus knot. The manifolds obtained by surgery on torus knots have been classified by Moser [9]: if K is a torus knot of type (r, s), they are Seifert fibre spaces with 3 exceptional fibres of orders r, s and $t := |rsq - p|$, if $t \neq 0$. Because of our assumption that $0 < |p| < q$ and because K is not the trivial knot it follows that $t \geqslant 7$, so $M_{p,q}$ is a Seifert fibre space with infinite fundamental group. But then $\pi_1 O^n_{p,q}$ is also infinite, in contradiction to our assumption. This proves the claim.

Now we can assume that $O^n_{p,q}(K_i)$ is a good irreducible 3-orbifold with a geometric decomposition and infinite fundamental group, for $i = 1, 2$. By Theorem 1 there exists a diffeomorphism $\bar{f}$ between $O^n_{p,q}(K_1)$ and $O^n_{p,q}(K_2)$, with $\bar{f}((K_1)_{p,q}) = (K_2)_{p,q}$. Then $\bar{f}$ induces a diffeomorphism $f: S^3\backslash K_1 \to S^3\backslash K_2$, $S^3\backslash K_i \cong M_{p,q}(K_i)\backslash (K_i)_{p,q}$. Let m_i resp. l_i be a meridian resp. longitude of K_i for $i = 1, 2$. Because f maps a meridian of $(K_1)_{p,q}$ to a meridian of $(K_2)_{p,q}$, we can assume that $f(m_1^p l_1^q) = m_2^p l_2^q$, $f(l_1) = l_2^{\pm 1}$.

Suppose first that $f(l_1) = l_2$. Then $f(m_1^p) = m_2^p$, $f(m_1) = m_2$, so f extends to a diffeomorphism between (S^3, K_1) and (S^3, K_2) and the theorem follows. On the other hand, if $f(l_1) = l_2^{-1}$, then $f(m_1^p) = m_2^p l_2^{2q}$. But this is impossible: for $p > 2$ because $(p, q) = 1$, for $|p| = 1$ or 2 by [5], corollary 2 because $q > 1$.

This proves Theorem 2 in the case where K_1, K_2 are non-trivial.

If K_1 is trivial then $\pi_1 O^n_{p,q}(K_i)$ is a finite cyclic group, so the result also holds in this case. This finishes the proof of Theorem 2. ∎

Remark. Theorem 2 can be generalized for links in S^3, but as it does not seem to be known which specific surgeries on a link give irreducible manifolds, at moment it is not possible to specify the surgery-values (p_j, q_j) for the components of the link. It is only known that 'most' surgeries give irreducible manifolds.

https://doi.org/10.1017/S0305004100065609 Published online by Cambridge University Press

REFERENCES

[1] M. Boileau and B. Zimmermann. The π-orbifold group of a link. *Math. Z.* (to appear).
[2] F. Bonahon and L. C. Siebenmann. The characteristic toric splitting of irreducible compact 3-orbifolds. *Math. Ann.* **278** (1987), 441–480.
[3] G. Burde and H. Zieschang. *Knots* (De Gruyter, 1985).
[4] J. H. Conway and C. McA. Gordon. A group to classify knots. *Bull. London Math. Soc.* **7** (1975), 84–86.
[5] M. Culler, C. McA. Gordon, J. Luecke and P. Shalen. Dehn surgery on knots. *Ann. of Math.* (2) **125** (1987), 237–300.
[6] M. Davies and J. Morgan. Finite group actions on homotopy-3-spheres. In *The Smith Conjecture* (Academic Press, 1984).
[7] C. McA. Gordon and J. Luecke. Only integral Dehn surgeries can yield reducible manifolds. *Math. Proc. Cambridge Philos. Soc.* **102** (1987), 97–101.
[8] D. McCullough and A. Miller. Manifold covers of 3-orbifolds with geometric pieces. (Preprint.)
[9] L. Moser. Elementary surgery along a torus knot. *Pacific J. Math.* **38** (1971), 737–745.
[10] G. Mostow. Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms. *Inst. Hautes Études Sci. Publ. Math.* **34** (1968), 53–104.
[11] D. Rolfson. *Knots and Links* (Publish or Perish, 1976).
[12] J. Simon. An algebraic classification of knots in S^3. *Ann. of Math.* (2) **97** (1973), 1–13.
[13] W. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.* **6** (1982), 357–381.
[14] W. Thurston. Hyperbolic structures on 3-manifolds I: deformations of acylindrical manifolds. *Ann. of Math.* (2) **124** (1986), 203–246.
[15] W. Thurston. Three manifolds with symmetry. (Preprint, 1982.)
[16] W. Whitten. Algebraic and geometric characterizations of knots. *Invent. Math.* **26** (1974), 259–270.
[17] J. A. Wolf. *Spaces of Constant Curvature* (McGraw-Hill, 1967).
[18] B. Zimmermann. Finite group actions on Haken-3-manifolds. *Quart. J. Math. Oxford Ser.* 2, **37** (1986), 499–511.
[19] B. Zimmermann. 3-manifolds and orbifold groups of links. In *Proc. Conf. on Differential Geometry and Topology* (Torino, 1987). (To appear.)

https://doi.org/10.1017/S0305004100065609 Published online by Cambridge University Press