Thus $\sum_{\rho\in P}^{*}\{d_\rho\}$ is a direct summand of S containing H. Moreover, since the maps $H \to Z$ defined by $h \to h^\rho$ ($\rho \in P$) are distinct, therefore

$$|P| \leqslant |\mathrm{Hom}\,(H, Z)| \leqslant \aleph_0^{\mathfrak{n}} = 2^{\mathfrak{n}} < \mathfrak{m} = |\Lambda|.$$

It follows that $M = \Lambda - P$ is non-empty; and we conclude that $\sum_{\rho\in P}^{*}\{d_\rho\}$ is a proper direct summand of S containing H.

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A problem on measure

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(*Received* 14 *June* 1962)

Given a set $\mathscr{E}$ in n-space, a δ-covering $U(\delta, \mathscr{E})$, $\delta > 0$, of $\mathscr{E}$ is defined as a set of convex sets, each of diameter $\leqslant \delta$ such that $\mathscr{E} \subset U(\delta, \mathscr{E})$. Given an increasing function $\phi(t)$ defined for $t > 0$ and such that $\phi(+0) = 0$, we define the function $\Lambda_\delta^\phi \mathscr{E}$ of a positive variable δ and of a set $\mathscr{E}$ of n-space by the equation

$$\Lambda_\delta^\phi \mathscr{E} = \text{l.bd.} \sum_{U(\delta,\mathscr{E})} \phi(d),$$

where d is the diameter of a general set of $U(\delta, \mathscr{E})$ and the sum is considered for all $U(\delta, \mathscr{E})$.

The Λ^ϕ outer measure of $\mathscr{E}$ is then defined by the equation

$$\Lambda^\phi \mathscr{E} = \lim_{\delta\to 0} \Lambda_\delta^\phi \mathscr{E}.$$

Carathéodori gives a well-known definition of a Λ^ϕ-measurable set and then $\Lambda^\phi \mathscr{E}$ is called the Λ^ϕ-measure, Let $\psi(t)$ be a function satisfying the same conditions as $\phi(t)$ and the condition $\lim_{t\to 0} \psi(t)/\phi(t) = 0$. Then we say that Λ^ψ is a higher measure than Λ^ϕ. We obviously have $\Lambda^\psi \mathscr{E} = 0$ if $\Lambda^\phi \mathscr{E} < \infty$.

A set $\mathscr{E}$ is said to be of σ-finite Λ^ϕ-measure if $\mathscr{E}$ can be represented as the sum $\mathscr{E} = \sum_{n=1}^{\infty} \mathscr{E}_n$ where every $\mathscr{E}_n$ is of finite Λ^ϕ-measure.

In (2) there was proved

Theorem 7. *If an analytic set $\mathscr{E}$ is not of σ-finite Λ^ϕ-measure then there exists a higher Λ^ψ measure such that $\mathscr{E}$ is also not of σ-finite Λ^ψ-measure.*

The question naturally arises whether the theorem still holds if the condition of $\mathscr{E}$ being analytic is dropped.

https://doi.org/10.1017/S030500410000219X Published online by Cambridge University Press

LEMMA. *If a set $\mathscr{E}$ is the sum of more than $\aleph_0$ disjoined sets, each of positive Λ^ϕ-measure, then $\mathscr{E}$ is not of σ-finite Λ^ϕ-measure.*

For suppose
$$\mathscr{E} = \sum_{x \in X} \mathscr{E}_x = \sum_{i=1}^{\infty} \mathscr{F}_i,$$
where $|X| > \aleph_0$, $\mathscr{E}_x$, $x \in X$ are disjoined and $\Lambda^\phi \mathscr{E}_x > 0$ and $\Lambda^\phi \mathscr{F}_i < \infty$ for every $x \in X$ and every i.

Obviously for any fixed i $\Lambda^\phi \mathscr{E}_x \mathscr{F}_i$ is > 0 only for an enumerable set of values of x. Hence there is only an enumerable set of values of x such that $\Lambda^\phi \mathscr{E}_x \mathscr{F}_i > 0$ for some value of i, and for any other x'
$$0 < \Lambda^\phi \mathscr{E}_{x'} = \Lambda^\phi \left(\mathscr{E}_{x'} \sum_1^\infty \mathscr{F}_i \right)$$
$$= \sum_1^\infty \Lambda^\phi \mathscr{E}_x \mathscr{F}_i = 0.$$
This contradiction proves the lemma.

Let now y_t, $t \in T$, be a set of real numbers of power $> \aleph_0$ concentrated about the rational numbers r_n, $n = 1, 2, \ldots$, that is such that any set of intervals covering all the rational numbers covers all y_t except for an enumerable subset of them. The definition and the proof of existence of concentrated sets is given in (1).

Let $\mathscr{E}$ be the set of intervals $0 < x < 1, y = y_t$ in the coordinate-plane. By the lemma $\mathscr{E}$ is not of σ-finite linear measure. Take now any Λ^ϕ-measure higher than the linear measure. We have $\lim_{l \to 0} \phi(l)/l = 0$. Take arbitrary $\delta > 0$, $\epsilon > 0$ and define a sequence $\{l_n\}$ of reciprocals of integers such that $0 < l_n < \frac{1}{2}\delta$ and
$$\sum_{n=1}^{\infty} \frac{\phi(l_n \sqrt{2})}{l_n} < \frac{\epsilon}{2}.$$

For every n define $y_n' < r_n < y_n''$ such that $y_n'' - y_n' = l_n$ and define a covering set $U(\delta, \mathscr{E})$ in the following way. By the definition of concentrated sets there is only an enumerable set of y_t outside the intervals (y_n', y_n''). Write $\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2$, where $\mathscr{E}_1$ is the part of $\mathscr{E}$ included in the strips $y_n' \leqslant y \leqslant y_n''$ and $\mathscr{E}_2$ is the rest, so that $\mathscr{E}_2$ has only an enumerable set of unit intervals. Hence $\Lambda^\phi \mathscr{E}_2 = 0$ and there exists a covering $U(\delta, \mathscr{E}_2)$ such that
$$\sum_{U(\delta, \mathscr{E}_2)} \phi(d) > \frac{\epsilon}{2}.$$

Now $\mathscr{E}_1$ is contained in the rectangles R_n with sides $x = 0$, $x = 1$, $y = y_n'$, $y = y_n''$. Divide each R_n into squares of side l_n, there will be exactly $1/l_n$ such squares. The set of all these squares will form a covering $U(\delta, \mathscr{E}_1)$. We shall have
$$\sum_{U(\delta, \mathscr{E})} \phi(d) = \sum_{n=1}^{\infty} \frac{1}{l_n} \phi(l_n \sqrt{2}) < \tfrac{1}{2}\epsilon.$$
Writing $U(\delta, \mathscr{E}_1) + U(\delta, \mathscr{E}_2) = U(\delta, \mathscr{E})$ we have
$$\sum_{U(\delta, \mathscr{E})} \phi(d) < \epsilon$$
which proves that for any Λ^ϕ-measure higher than the linear measure $\Lambda^\phi \mathscr{E} = 0$, and thus gives an answer in the negative to our problem.

https://doi.org/10.1017/S030500410000219X Published online by Cambridge University Press

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Two problems on convergence

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(*Received* 3 *July* 1962)

PROBLEM I. *Does there exist a sequence* $\{u_k(x)\}$ *of positive functions such that for any sequence* $\{a_k\}$ *of positive coefficients the series* $\sum_1^\infty a_k u_k(x)$ *converges for all values of* x, *except at most* $\aleph_0$ *of them*?

PROBLEM II. *Does there exist a sequence* $\{u_k(x)\}$ *of positive functions such that for any sequence* $\{a_k\}$ *of positive coefficients the series* $\sum_1^\infty a_k u_k(x)$ *diverges for all values of* x, *except at most* $\aleph_0$ *of them*?

If the functions $u_k(x)$ are subject to the additional condition of being measurable then the problems are answered in the negative. For, define the sets

$$\mathscr{E}_1 = \{x/-\infty < x < +\infty,\ u_1(x) > b_1\},$$
$$\mathscr{E}_2 = \{x/x \in \mathscr{E}_1,\ u_2(x) > b_2\},$$
$$\mathscr{E}_3 = \{x/x \in \mathscr{E}_2,\ u_3(x) > b_3\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and so on, where $b_1, b_2, b_3, \ldots$ are positive numbers. We can choose b_1 so that $m\mathscr{E}_1 > 1$, then b_2 so that $m\mathscr{E}_2 > 1$, b_3 so that $m\mathscr{E}_3 > 1$ and so on. The sequence of sets $\{\mathscr{E}_n\}$ is decreasing. Writing $\lim \mathscr{E}_k = \mathscr{E}$ we have $m\mathscr{E} \geqslant 1$ and $\mathscr{E} \subset \mathscr{E}_k$. Hence for every $x \in \mathscr{E}$, and every k, $u_k(x) > b_k$. Putting $a_k = b_k^{-1}$ the series

$$\Sigma a_k u_k(x)$$

diverges at all points on $\mathscr{E}$, that is at $\aleph_c$ points.

Similarly we can see that the Problem II is answered in the negative.

The solution of the problems in the general case is arrived at by means of 'fundamental sequences' of decreasing functions. Given two real-valued functions $f(n)$ and $g(n)$ of a positive integer n, we say that $f(n)$ is ultimately greater than $g(n)$, and we write $f(n) \succ g(n)$, if there exists n_0 such that for $n > n_0$, $f(n) > g(n)$.

DEFINITION. *A transfinite sequence* $\{\phi_i(n)\}$, $i < \omega_1$, *of positive decreasing functions, such that every positive function* $f(n)$ *is ultimately greater than* $\phi_i(n)$ *for all* i *with the exception of at most* $\aleph_0$ *of them, is called a fundamental sequence.*

https://doi.org/10.1017/S030500410000219X Published online by Cambridge University Press