Readers will have been saddened to read the news of the death of Douglas Quadling in March 2015. Tributes to his immense contribution to the work of The Mathematical Association were featured in the June 2015 issue of *MA News* and the July 2015 issue of the *Gazette*. He was a regular contributor to Problem Corner whose superb expository skills and vast knowledge of mathematics shone through every solution he submitted. His wise, authoritative and enthusiastic support will be greatly missed.

doi:10.1017/mag.2015.107 N.J.L.

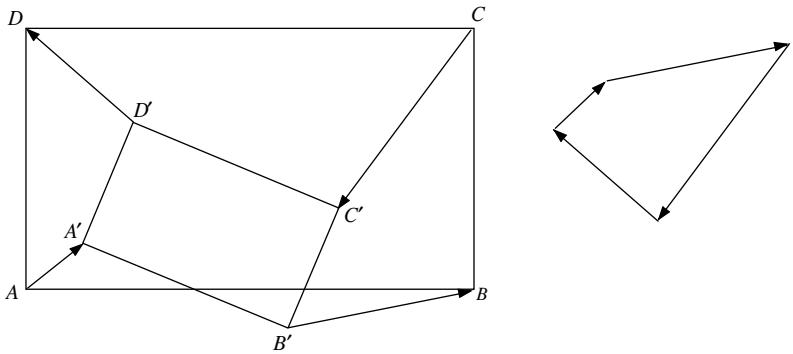
Student Problems

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by January 20th 2016. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. Please print your own name clearly! The names of all successful solvers will be published in the March 2016 edition.

Problem 2015.5 (Dao Thanh Oai)

Draw any two similar rectangles $ABCD$ and $A'B'C'D'$ with vertices corresponding in the natural way.



Prove that, as shown above, the vectors $\overrightarrow{AA'}$, $\overrightarrow{B'B}$, $\overrightarrow{CC'}$ and $\overrightarrow{D'D}$ always form a closed quadrilateral. What can you determine about the diagonals of this quadrilateral?

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Problem 2015.6 (Paul Stephenson)

Consider powers of 10 written as binary numbers, for example:

$$10^1 = 1\mathbf{0}10_2, \qquad 10^2 = 11\mathbf{00}100_2, \qquad 10^3 = 1111\mathbf{0}1000_2.$$

For 10^n, the number of zero digits to the right of the final 1 is just n. Can you find and prove a formula for the number of zeros between the penultimate and final 1s? These zeros are shown emboldened above.

Solutions to 2015.3 and 2015.4

Problem 2015.3 was solved by Tom Syndercombe (Bishop Wordsworth's School, Salisbury). Problem 2015.4 was solved by Xiusi Li (Francis Holland School, Regent's Park).

Problem 2015.3

For any quadruplet (a, b, c, d) of numbers, define a second quadruplet to be

$$\left(\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+a}{2}\right).$$

Repeat this process indefinitely and describe what happens.

Solution

As spotted by Tom, a good start to solving this problem is to note that the mean of the four numbers does not change during this process.

This problem has a further neat connection with statistics. From the second iteration onwards, the process repeatedly halves the variance of the numbers in the quadruplet.

Let $M = \frac{1}{4}(a + b + c + d)$ then the second quadruplet has the form $(M + x, M + y, M - x, M - y)$. The third quadruplet is then

$$\left(M + \frac{x+y}{2}, M + \frac{y-x}{2}, M - \frac{x+y}{2}, M - \frac{y-x}{2}\right).$$

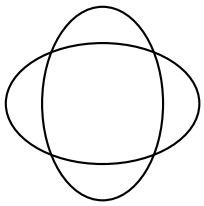
Since $\left(\frac{x+y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = \frac{x^2+y^2}{2}$ the differences from M tend to zero and so the quadruplets tend to

$$\left(\frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}\right).$$

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Problem 2015.4

The overlapping ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (with $a > b > 0$) form five bounded regions as shown in the diagram. What value of $\frac{b}{a}$ makes all five regions have the same area?

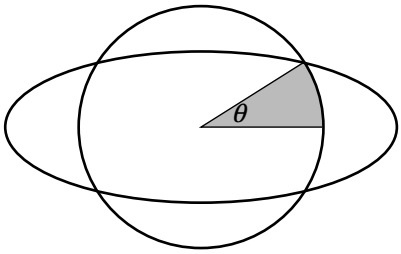


Solution

This problem, suggested by Nick Lord, can be solved in a very direct manner by using integration. Making good use of symmetry, Xiusi managed this with commendable accuracy and efficiency. However, an elementary solution can be found by considering the ellipses to be stretched circles.

The central area must be equal to a third of the area of one of the ellipses, i.e. $\frac{1}{3}\pi ab$.

Apply a 1-way stretch with scale factor $\frac{a}{b}$ parallel to the x-axis. This produces the diagram below, where $\tan\theta = \frac{b}{a}$



The shaded area is

$$\frac{1}{8} \times \frac{\pi ab}{3} \times \frac{a}{b} = \frac{\pi a^2}{24},$$

i.e. $\frac{1}{24}$ th of the area of the circle.

Therefore

$$\frac{b}{a} = \tan^{-1}\frac{\pi}{12} = 2 - \sqrt{3}.$$

The first prize of £25 is awarded to Xiusi Li. The second prize of £20 is awarded to Tom Syndercombe.

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