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PRICING DERIVATIVES UNDER THE WILKIE MODEL

By M. H. D. Kemp

ABSTRACT

This paper describes how to calculate approximate prices of options if investment markets are being modelled according to the Wilkie investment model.

KEYWORDS

Option Pricing; Asset/Liability Modelling

CONTACT ADDRESS

M. H. D. Kemp, M.A., F.I.A., Scudder Threadneedle Investments Ltd, 60 St Mary Axe, London EC3A 8JQ, U.K. Tel: +44(0)20-7464-5000; Fax: +44(0)20-7464-5273

1. Introduction

1.1 *Rationale for Paper*

1.1.1 Over the last decade or so, asset/liability modelling has become a common tool amongst United Kingdom actuaries. Usually, although not always, the investment strategies included in such exercises have involved static asset mixes (or perhaps ones that are rebalanced at the start of each projection time step back to a fixed position). These strategies can be modelled relatively simply, as long as the underlying economic model is suitably defined.

1.1.2 However, there is an increasing awareness by actuaries of the potential usefulness of derivatives in asset/liability management. These are more difficult to model, because different derivatives need to be modelled in a consistent manner for the analysis to be meaningful. It is necessary to work out how to place values on derivatives which are consistent with the returns being simulated on the assets underlying the derivatives.

1.2 *Contents of Paper*

1.2.1 This paper describes how, approximately, to place values on options under the best known model used in asset/liability modelling exercises by U.K. actuaries, i.e. the Wilkie investment model. See e.g. Wilkie (1995). The Wilkie model was not originally designed to cope with option pricing, as will become obvious below.

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1.2.2 However, that does not make it impossible to come up with methods of valuing options under the Wilkie model. Section 2 describes how this can be done for 'total return' options (in which the underlying assets are expressed in terms of indices which have income reinvested). Section 3 extends the results to include the more usual sorts of options based on price indices (or on the prices of individual securities) which do not have income reinvested.

1.3 *The Applicability of the Wilkie Model to Derivatives*

1.3.1 Another question that asset/liability modellers will be interested in is whether their chosen model is a good representation of reality.

1.3.2 This paper is based on the contents of Appendix C of Kemp (1996), extended to cover more sorts of derivatives and to provide some further discussion of the problems that arise when trying to value options under the Wilkie model. In Kemp (1996) the author criticises the use of the Wilkie model for testing the merits of dynamic investment strategies or investment strategies involving derivatives. Therefore, this paper should *not* be taken as an endorsement of the Wilkie model for these purposes.

1.3.3 Weaknesses of the Wilkie model in this respect include:

(a) The option prices derived by the 'natural' extension of the Wilkie model to a continuous time framework, as described below, generally bear little resemblance to actual market prices (for shorter-dated options), and they do not replicate observed market dynamics. They show little of the variation in implied volatilities actually experienced in the marketplace, and they fail to mimic any term structure of implied volatilities that actually exists. They also fail to replicate the skew towards the downside typically seen in equity option implied volatilities.
(b) For non-total return options, the pricing formulae require the values of zero coupon bonds or estimates of equity forward prices, neither of which are actually available from the 1995 version of the Wilkie model.
(c) The risk neutral probability distributions underlying option values under the 'natural' extension of the Wilkie model to continuous time show unusual characteristics compared to the spread of 'real world' outcomes predicted by the model. See e.g. Kemp (1996; 1997). These characteristics arise because of the mean reverting characteristics of the Wilkie model, which imply the existence of strategies likely to generate implausibly high excess returns.

1.3.4 The author therefore recommends that asset/liability modellers exercise considerable caution when using the Wilkie model to analyse dynamic investment strategies or strategies involving derivatives. Nevertheless, there may be cases where it is inconvenient to use any other model or where the strategies being modelled, whilst still involving derivatives, are sufficiently static for these weaknesses to be tolerable. Examples of the latter might include

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regular purchase of short-dated options over an extended period, the length of which is believed to be sufficiently long to bring the average of the prices actually paid in the market close to those implied by the Wilkie model.

2. Total Return Options

2.1 *Total Return Options*

2.1.1 In this section we will concentrate on 'total return' derivatives. An example would be a European-style put option, which gives the holder the right to sell at time T a portfolio invested in an equity index, with gross income reinvested, represented by S_t, for a price set by reference to an initial exercise price E, rolled up in line with the total return on cash.

2.1.2 Suppose that F_t is the cash index with gross income reinvested. Then the pay-off of this option at time T is:

$$\max\left(E.\frac{F_T}{F_0} - S_T, 0\right).$$

2.1.3 The cash and equity indices can be replaced by any other asset categories being modelled. Thus, this pay off covers both put and call options, as can be seen by switching around S and F.

2.2 *Pricing in the Absence of Transaction Costs*

2.2.1 The main advantage of considering total return options before other sorts of options is that their no arbitrage pricing formula is particularly simple, if we assume markets are arbitrage free and are frictionless (i.e. no transaction costs, no limits on short selling, borrowing, etc.). The price of such a put option at time t, in the absence of transaction costs, if the cumulative quadratic variation of the option $C(t)$ is continuous, and $C(T) - C(t)$ is fixed whatever the path taken by the underlying stochastic process, is given by the following formula (see Neuberger (1990) or Kemp (1997)):

$$P(S, t) = E.N(-d_2) - S.N(-d_1)$$

where:

$$d_1 = \frac{\log(S/E) + (C(T) - C(t))/2}{\sqrt{C(T) - C(t)}}$$

$$d_2 = d_1 - \sqrt{C(T) - C(t)}$$

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and

$$N(x) = \text{cumulative normal distribution function.}$$

2.2.2 The cumulative quadratic variation is here defined as the limit as the partitioning into separate time steps tends to infinity of:

$$C(T) = \sum_{t=0}^{T} x_t = \sum_{t=0}^{T} \left(\log\left(\frac{S(t+1)}{S(t)}\right) - \log\left(\frac{E(t+1)}{E(t)}\right)\right)^2.$$

2.2.3 The option pricing formula quoted above can be shown to be valid using the following sort of reasoning. Suppose we adopt the hedging strategy that, in a Black-Scholes world, perfectly replicates a put option for a given maturity *except in one respect*, namely we replace t by:

$$\bar{t} = \frac{C(t)}{C(T)} T.$$

In effect we use a time clock that runs at a special rate different to normal elapsed time, running more quickly when the cumulative quadratic variation is changing more rapidly. There will be no discontinuities, because we have assumed that the cumulative quadratic variation is continuous. When the cumulative quadratic variation reaches $C(T) - C(t)$, then the replicating portfolio that such a strategy will generate will exactly match that required to satisfy a put option pay off, as long as the put option has the same maturity date as the replicating strategy. However, we have assumed that we know the value of $C(T) - C(t)$ at outset, and therefore we know that this option matures at time T (in normal elapsed time), just like the contract that we are seeking to value.

2.2.4 As a result, the formula in ¶2.2.1 is, not surprisingly, the same as the Black-Scholes formula, as long as we equate σ, the implied volatility of the option, with $\sqrt{(C(T) - C(t))/(T - t)}$.

2.2.5 The Wilkie model is a no arbitrage model (see Kemp, 1996), and, as generally formulated, ignores transaction costs. Therefore, to value total return options we merely need to analyse the behaviour of the cumulative quadratic variation in the Wilkie model framework.

2.3 *The Discrete Nature of the Wilkie Model*

2.3.1 This immediately raises a significant problem. The Wilkie model is a discrete time (yearly) model, rather than a continuous time model. Thus, the cumulative quadratic variation is not well defined. In technical terms, the Wilkie model in its discrete form describes an *incomplete* market.

2.3.2 We can therefore, if we want, create continuous time extensions to the Wilkie model in all sorts of different ways.

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2.3.3 For example, we could, in theory, assume that there was a very small quadratic variation for a small, but non-zero, period at the start of the year, and from then on to the end of the year the possible return paths spread out smoothly (e.g. linearly) to their final values, choosing the movements so that the spread of returns at the end of the year match those arising from the discrete Wilkie model. Smooth movements do not add to cumulative *quadratic* variation, so the cumulative quadratic variation for the entire year could, in theory, be as small as we liked, provided that it was still greater than zero.

2.3.4 Conversely, we could, in theory, assume that there was a huge degree of randomness throughout almost all of the year, but just before the end of the year the spread of return paths collapsed back inwards, so that at the year end the spread matched that arising from the discrete Wilkie model. The cumulative quadratic variation could then be as large as we liked.

2.3.5 Thus, in theory the Wilkie model can be made continuous, and produce option prices with implied volatilities anywhere in the range $0 < \sigma < \infty$, i.e. with no practical constraints whatsoever!

2.3.6 However, the total return components of the Wilkie model generally develop in a log-normal, or close to log-normal, fashion. Log-normal returns are typically associated with random walk type behaviour. The most *natural* way of extending the Wilkie model to continuous time is, therefore, to assume that intra-year the returns are Brownian motions which cumulatively generate by the end of the year the same variation as the Wilkie model assumes appears over the year as a whole.

2.3.7 We shall concentrate on this 'natural' extension to the Wilkie model in the remainder of this paper. To be precise, we will assume that the random innovations accrue with uniform quadratic variation through the year, i.e. behave like Brownian motions. We will also assume that intra-year returns, etc. for economic factors exactly match what would have been produced by the Wilkie model (based on parameter values ruling at the outset of the given year), but based on the cumulative innovations to that point in time rather than to the year end.

2.3.8 The innovations are Weiner processes, and therefore, over the year as a whole, the returns generated in this fashion will exactly match those generated by the Wilkie model. Intra-year, the returns, etc. are generalised Brownian motions (because of the characteristics of the mean and variance of returns over all short time periods of length Δh, including those over each integer time step). The non-linearities of the Wilkie model mean that the formulae for instantaneous variances (and means) will be difficult to describe in analytical terms, and will involve autoregressive characteristics. In practice, however, the non-linearities within a year are (nearly always) small, and therefore option prices can (nearly always) be approximated by a suitable linear approximation.

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2.4 *Total Return Components for Various Asset Categories under the Wilkie Model*

2.4.1 The total return components of the Wilkie model develop according to the following formulae, where the total return indices at time t are: $PR(t)$ for equities, $CR(t)$ for consols (fixed-interest gilts), $BR(t)$ for cash, $RR(t)$ for index-linked gilts and $AR(t)$ for property (see e.g. Wilkie, 1995):

$$\begin{aligned}\log(PR(t)/PR(t-1)) &= DW \times DM(t) + (1-DW) \times I(t) + DMU \\ &\quad + DSD \times DZ(t) + \log(1 + \exp\{-YW \times I(t) - YN(t)\}) \\ &\quad + YW \times I(t-1) + YN(t-1) \\ &\quad + DY \times YSD \times YZ(t-1) + DB \times DSD \times DZ(t-1)\end{aligned}$$

$$\begin{aligned}\log(CR(t)/CR(t-1)) &= \log\left(1 + \frac{1}{CM(t) + CMU \times \exp(CN(t))}\right) \\ &\quad + \log(CM(t-1) + CMU \times \exp(CN(t-1)))\end{aligned}$$

$$\begin{aligned}\log(BR(t)/BR(t-1)) &= \log(1 + \{CM(t-1) \\ &\quad + CMU \times \exp(CN(t-1))\} \times \exp(-BD(t-1)))\end{aligned}$$

$$\log(RR(t)/RR(t-1)) = I(t) + \log(1 + \exp(-\ln R(t)) + \ln R(t-1))$$

$$\begin{aligned}\log(AR(t)/AR(t-1)) &= \log(1 + \exp(-\ln Z(t))) + EW \times EM(t) \\ &\quad + (1-EW) \times I(t) + EMU + EBZ \times ZSD \times ZZ(t) \\ &\quad + ESD \times EZ(t) + \ln Z(t-1).\end{aligned}$$

2.4.2 In these formulae, we have used the convention $\ln YMU = \ln(YMU)$, etc. The 'state' vectors are: $I(t)$ the rate of RPI inflation at time t; $YN(t)$ the element of the dividend yield not explained by prices; $CN(t)$ the element of long-term interest rates not explained by retail prices; $BD(t)$ the relationship between long and short-term interest rates; $\ln Z(t)$ the log of the property yield; $\ln R(t)$ the log of the real yield on index-linked stocks; and three 'quasi-state' vectors $DM(t)$, $CM(t)$ and $EM(t)$, which are moving averages of past RPI inflation. They develop as follows:

$$I(t) = QMU + QA \times (I(t-1) - QMU) + QSD \times QZ(t)$$

$$YN(t) = \ln YMU + YA \times (YN(t-1) - \ln YMU) + YSD \times YZ(t)$$

$$CN(t) = CA1 \times CN(t-1) + CY \times YSD \times YZ(t) + CSD \times CZ(t)$$

$$BD(t) = BMU + BA \times (BD(t-1) - BMU) + BSD \times BZ(t)$$

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$$\ln Z(t) = \ln ZMU + ZA \times (\ln Z(t-1) - \ln ZMU) + ZSD \times ZZ(t)$$

$$\ln R(t) = \ln RMU + RA \times (\ln R(t-1) - \ln RMU) + RBC \times CSD \times CZ(t) + RSD \times RZ(t)$$

$$DM(t) = DD \times I(t) + (1 - DD) \times DM(t-1)$$

$$CM(t) = CD \times I(t) + (1 - CD) \times CM(t-1)$$

$$EM(t) = ED \times I(t) + (1 - ED) \times EM(t-1).$$

2.4.3 The eight random components $QZ(t)$, $DZ(t)$, $YZ(t)$, $CZ(t)$, $BZ(t)$, $RZ(t)$, $ZZ(t)$ and $EZ(t)$ are independent identically distributed unit normal random variables, and are the source of the variability of the model (the full Wilkie model also has an earnings index which involves a ninth random component $WZ(t)$, but we do not consider this part of the model further in this paper). The remaining terms are constants chosen as parameters when the Wilkie model is first established by the user.

2.4.4 To first order, and for one period (year) ahead, we can expand the right-hand sides of the formulae in ¶2.4.1 in the following form, for each asset category i:

$$y_i(t) = c_i + a_{i,YZ} YZ(t) + a_{i,DZ} DZ(t) + \ldots$$

where c_i and all the $a_{i,Z}$ are constants dependent only on the state variables at time $t-1$ or before. The relevant values of the $a_{i,Z}$ are then as follows.

2.4.4.1 *Equities* ($i = 1$). We note that:

$$DM(t) = DD \times I(t) + (1 - DD) \times DM(t-1)$$

$$I(t) = QMU + QA \times (I(t-1) - QMU) + QSD \times QZ(t)$$

$$\log(1 + e^{a+x}) \approx \log(1 + e^a) + \left(\frac{e^a}{1 + e^a}\right) x.$$

Hence we have, approximately:

$$\log(PR(t)/PR(t-1)) \approx c + (DW \times DD + 1 - DW - k \times YW) \times QSD \times QZ(t) + DSD \times DZ(t) - k \times YSD \times YZ(t)$$

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where:

$$a = -YW \times (QMU + QA \times (I(t-1) - QMU)) - \ln YMU - YA \times (YN(t-1) - \ln YMU)$$

and

$$k = \frac{e^a}{1+e^a}.$$

If the model is in its 'neutral' state, according to the terminology in Wilkie (1995), then $I(t-1) = QMU$ and $YN(t-1) = YMU$, so the expression for a simplifies to $-YW.QMU - \ln YMU$. Then, using the parameters suggested by Wilkie (1995), viz: $DW = 0.58$, $DD = 0.13$, $QSD = 0.0425$, $YW = 1.8$, $DSD = 0.07$, $YSD = 0.155$, $YMU = 3.75\%$, $QMU = 0.047$, we get $\ln YMU = -3.283414$, $a = 3.199$ and $k = 0.96$.

Hence we can approximate the random components in the log price movement as:

$$a_{1,QZ} \approx (DW \times DD + 1 - DW - 0.96 \times YW) \times QSD = -0.052$$

$$a_{1,DZ} = DSD = 0.070$$

$$a_{1,YZ} \approx -0.96 \times YSD = -0.149.$$

2.4.4.2 *Fixed-interest gilts* ($i = 2$). The only contribution to the $a_{2,Z}$ will be from the following term, which can be expanded approximately as:

$$\log\left(1 + \frac{1}{CM(t) + CMU \times \exp(CN(t))}\right) = \log\left(1 + \frac{1}{a+x}\right) \approx \log\left(1 + \frac{1}{a}\right) - \frac{x}{a(a+1)}$$

where:

$$a = CD \times (QMU + QA \times (I(t-1) - QMU)) + (1 - CD) \times CM(t-1) + CMU \times (1 + CA1 \times CN(t-1))$$

and

$$x \approx CD \times QSD \times QZ(t) + CMU \times CY \times YSD \times YZ(t) + CMU \times CSD \times CZ(t).$$

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If the model is in its 'neutral' state, then $I(t-1) = QMU$, $CM(t-1) = CMU$, $CN(t) = 0$. Then, using the parameters suggested by Wilkie (1995), viz: $CD = 0.045$, $QSD = 0.0425$, $QMU = 0.047$, $CSD = 0.185$, $YSD = 0.155$, $CY = 0.34$, $CMU = 0.0305$, we get $a = QMU + CMU = 0.0775$, and we can approximate the random components in the log price movement as:

$$a_{2,QZ} = -\frac{CD \times QSD}{(QMU + CMU)(1 + QMU + CMU)} = -0.023$$

$$a_{2,YZ} = -\frac{CMU \times CY \times YSD}{(QMU + CMU)(1 + QMU + CMU)} = -0.019$$

$$a_{2,CZ} = -\frac{CMU \times CSD}{(QMU + CMU)(1 + QMU + CMU)} = -0.068.$$

The calculations are less accurate for this asset category than for the others (see Section 2.7).

2.4.4.3 *Cash* ($i = 3$). The formula for $\log(BR(t)/BR(t-1))$ has no dependency on random components other than via state variables at time $t-1$. Hence all $a_{3,Z}$ are zero.

2.4.4.4 *Index-linked gilts* ($i = 4$). We can expand the formula for $\log(RR(t)/RR(t-1))$ as follows:

$$\begin{aligned}\log(RR(t)/RR(t-1)) &= I(t) + \log(1 + \exp(-\ln R(t)) + \ln R(t-1) \\ &\approx \ln R(t-1) + QMU + QA \times (I(t-1) - QMU) \\ &\quad + QSD \times QZ(t) + \log(1 + e^a) \\ &\quad - k(RBC \times CSD \times CZ(t) + RSD \times RZ(t))\end{aligned}$$

where:

$$a = -\ln RMU - RA \times (\ln R(t-1) - \ln RMU) \text{ and } k = \frac{e^a}{1 + e^a}.$$

If the model is in its 'neutral' state, then $R(t-1) = RMU$, so a simplifies to $-\ln RMU$. Then, using the parameters suggested by Wilkie (1995), viz: $QMU = 0.047$, $QSD = 0.0425$, $RBC = 0.22$, $CSD = 0.185$, $RSD = 0.05$, $RMU = 4.0\%$, we get $a = -\ln RMU = 3.218875$ and $k = 0.96$.

Hence we can approximate the random components in the log price movement as:

$$a_{4,QZ} = QSD = 0.0425$$

$$a_{4,CZ} = -0.96 \times RBC \times CSD = -0.96 \times 0.22 \times 0.185 = -0.039$$

$$a_{4,RZ} = -0.96 \times RSD = -0.96 \times 0.05 = -0.048.$$

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2.4.4.5 *Property* ($i = 5$). The log price movement can be expanded approximately as:

$$\begin{aligned}\log(AR(t)/AR(t-1)) \approx c - k \times ZSD \times ZZ(t) + EW \times ED \times QSD \times QZ(t) \\ + (1 - EW) \times QSD \times QZ(t) + EBZ \times ZSD \times ZZ(t) \\ + ESD \times EZ(t)\end{aligned}$$

where:

$$a = -\ln ZMU - ZA \times (\ln Z(t-1) - \ln ZMU) \text{ and } k = \frac{e^a}{1+e^a}.$$

If the model is in its 'neutral' state, then $Z(t-1) = ZMU$, so a simplifies to $-\ln ZMU$. Then, using the parameters suggested by Wilkie (1995), viz: $EW = 1$, $ED = 0.13$, $QSD = 0.0425$, $ESD = 0.06$, $ZSD = 0.12$, $ZMU = 7.4\%$, $EBZ = 0.24$, we get $a = -\ln ZMU = -2.6037$ and $k = 0.93$.

Hence, we can approximate the random components in the log price movement as:

$$a_{5,QZ} = (EW \times ED + 1 - EW) \times QSD = 0.0047$$

$$a_{5,EZ} = ESD = 0.060$$

$$a_{5,ZZ} = (-0.93 + EBZ) \times ZSD = -0.083.$$

2.5 *The 'Natural' Way to make the Wilkie Model Continuous*

2.5.1 Using the approach described in Section 2.3, a 'natural' way to make the Wilkie model a continuous time model is to assume that, over the period $t-1$ to t, the random components $YZ(t)$, . . . behave like unit Brownian motions, with standard deviation over time h of $\sqrt{h}$. The cumulative quadratic variation accruing over an infinitesimal period h, for a total return option linking asset category i, with asset category j, is then a stochastic variable equal to:

$$C_{i,j}(t+h) - C_{i,j}(t) = ((Z_i - E(Z_i)) - (Z_j - E(Z_j)))^2$$

where:

$$Z_i = \log\left(\frac{R_i(t+h)}{R_i(t)}\right) \text{ and } Z_j = \log\left(\frac{R_j(t+h)}{R_j(t)}\right).$$

2.5.2 If all the $a_{i,Z}$ were actually constants (as calculated above), then we would have (since the normal error terms $YZ(t)$, . . . are all independent of each other and have unit variance):

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$$C_{i,j}(t+h) - C_{i,j}(t) = ((a_{i,QZ} - a_{j,QZ})^2 + (a_{i,DZ} - a_{j,DZ})^2 + \ldots)h = \sigma_{i,j}^2 h, \text{ say,}$$

and hence:

$$C_{i,j}(T) - C_{i,j}(t) = \sigma_{i,j}^2 (T-t)$$

where i and j are from the set $\{PR, CR, BR, RR, AR\}$ and $\sigma_{i,j}^2$ are:

	Cash	Fixed-interest	ILGs	Property	Equities
Cash		0.005514	0.005631	0.010511	0.029805
Fixed-interest	0.005514		0.007796	0.016241	0.027265
ILGs	0.005631	0.007796		0.015743	0.039856
Property	0.010511	0.016241	0.015743		0.040805
Equities	0.029805	0.027265	0.039856	0.040805	

Of course the $a_{i,Z}$ are not, in fact, fixed (they will vary, depending on how far away from the 'neutral' state the model is), and we consider whether this is a reasonable assumption in Section 2.7.

2.5.3 The different markets can, therefore, be treated as if they had the following standard deviations and correlations:

Sector	Standard deviation relative to cash (% p.a.)	Correlations				
		Cash	Fixed-interest	ILGs	Property	Equities
Cash	0.0	-				
Fixed-interest	7.4	-	1.00			
ILGs	7.5	-	0.30	1.00		
Property	10.3	-	−0.01	0.03	1.00	
Equities	17.3	-	0.31	−0.17	−0.21	1.00

For example, the correlation $c_{i,j}$ between fixed-interest and index-linked gilts satisfies: $\text{var}(i-j) = \text{var}(i) + \text{var}(j) + 2.c_{i,j}\sqrt{\text{var}(i).\text{var}(j)}$, where $\text{var}(i) = 0.005514$, $\text{var}(j) = 0.005631$ and $\text{var}(i-j) = 0.007796$, as per the table in ¶2.5.2.

2.5.4 The risk neutral probability distribution for options (of any term) would then be a multi-variate log-normal distribution, with the above annualised standard deviations and correlation matrix (to the accuracy of the assumption that the $a_{i,Z}$ are constants).

2.6 *Total Return Option Pricing*

2.6.1 If all the $a_{i,Z}$ were actually constants, then C(t) would also be continuous.

2.6.2 Using the result in Section 2.2, the value of any total return put option would then be given by the following formula (using the $\sigma_{i,j}$ shown in the table in Section 2.5):

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$$P(S, t) = E.N(-d_2) - S.N(-d_1)$$

where:

$$d_1 = \frac{\log(S/E) + \sigma_{i,j}^2(T-t)/2}{\sigma_{i,j}\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma_{i,j}\sqrt{T-t}.$$

2.7 *How Good is the Linear Approximation?*

2.7.1 The $a_{i,Z}$ are not actually constant, but vary depending on the state of the Wilkie model at the particular time in question and on the values taken by the random innovations.

2.7.2 However, the cumulative quadratic variation is still continuous. The logic of Neuberger (1990) implies that, if σ_a and σ_b were such that we always have $\sigma_a^2(T-t) < C(T-t) < \sigma_b^2(T-t)$, then σ_a and σ_b would be lower and upper limits on the implied volatilities to use in the formula in Section 2.6.

2.7.3 Deviations from the above calculation of $\sigma_{i,j}$ arise in the following places:

(a) in equities, index-linked and property: in the expansion of terms of the form $\log(1 + e^{a+x})$;
(b) in fixed-interest: in the expansion of $\log(1 + 1/(a + x))$; and
(c) the model does not remain in its neutral position.

2.7.4 If we expand the relevant term for $a_{i,Z}$ for equities, index-linked and property to higher order, we have:

$$\log(1 + e^{a+x}) = \log(1 + e^a) + \log\left(1 + kx + k\frac{x^2}{2!} + k\frac{x^3}{3!} + \ldots\right)$$

$$= \log(1 + e^a) + kx + \left(\frac{k}{2} - \frac{k^2}{2}\right)x^2 + \left(\frac{k}{6} - \frac{k^2}{2} + \frac{k^3}{3}\right)x^3 + O(x^4).$$

2.7.5 For all three sectors, the second and higher order terms in x are negligible in relation to the first order term (the kx), as long as k is greater than, say, 0.85 (k has to be less than 1) and as long as the $QZ(t)$, etc. do not take highly extreme values. This is because, for all three sectors, the values of x are linear multiples, with relatively small coefficients, of the $QZ(t)$, etc. For example, for equities:

$$x = -YW \times QSD \times QZ(t) + YSD \times YZ(t)$$

$$= -1.8 \times 0.0425 \times QZ(t) + 0.155 \times YZ(t).$$

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2.7.6 An analysis of the factors influencing a also shows that, for these three sectors, a is almost always constrained to stay reasonably close to $-\ln YMU$, $-\ln RMU$ and $-\ln ZMU$. As a rule of thumb, we can assume effective lower limits on a of around 2.5, 2.5 and 1.8 respectively, except in very rare circumstances. These correspond to lower limits on k of 0.92, 0.92 and 0.86 respectively (and upper limits of unity).

2.7.7 'Very rare' here loosely equates to, say, circumstances that should occur with frequencies less than, say, approximately 1 in 1000 (using general reasoning concerning the nature of the stochastic process driving the relevant as). However, these rules of thumb should be taken as indicative, rather than precise.

2.7.8 We can therefore, without material loss of accuracy, place practical lower and upper limits on the implied volatilities to use when valuing options linked to equities, index-linked gilts, property and/or cash, using the ranges for the ks as in ¶2.7.6.

2.7.9 For options linked to U.K. equities (versus cash), this means that, in almost all circumstances, the implied volatilities should be within the rather narrow range 16.6% to 17.9%. This range bears little resemblance to the much wider spread of implied volatilities actually seen in the marketplace. For example, in the two years to end June 1998, the 1-month implied volatilities for equity options ranged from about 10% to over 30%. Implied volatilities for longer-dated equity options show smaller variation, but still much greater than the 16.6% to 17.9% determined above.

2.7.10 The implied volatilities arising from the 'natural' extension to the Wilkie model are also virtually independent of strike price, in contrast to actual equity option prices.

2.7.11 It is more difficult to calculate equivalent ranges for options linked to fixed-interest securities, and the author believes that they are likely to be wider. This is because the formulae in the Wilkie model for this sector are more complicated than for other sectors, and the non-linearities involved are more pronounced.

2.7.12 Such results are perhaps not surprising, since the Wilkie model, as described above, has constant values for the standard deviations, QSD, YSD, DSD, etc. These assumptions are the main reasons why the resulting implied volatilities remain within such narrow bands. Wilkie (1995) does suggest a variant to his main model in which ARCH (autoregressive conditional heteroscedasticity) components are used within the inflation model. It would be possible to achieve wider spreads of observed implied volatilities by such an approach. However, given the relative sizes of $a_{1,QZ}$, $a_{1,DZ}$ and $a_{1,YZ}$, it would almost certainly be necessary for the ARCH component to apply to $YN(t)$, the element of the dividend yield not explained by prices, instead of, or in addition to, the inflation model. A strike dependent behaviour for implied volatility could be achieved by making the parameters within such an ARCH model dependent on equity market levels.

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Alternatively, such behaviour can be introduced into the model by assuming that markets can jump, e.g. as in the 'cost of capital' approach described in Kemp (1997).

3. Other Sorts Of Put And Call Options

3.1 *Vanilla Options*

3.1.1 Using similar arguments, the value of a vanilla non-total return put option is as follows, see e.g. Kemp (1997), where $V_E(t)$ is the price at time t of a contract delivering E at time T and $V_S(t)$ is the price at time t of a contract delivering 1 share at time T having foregone the dividend income in the meantime:

$$P(S, t) = V_E(t).N(-d_2) - V_S(t).N(-d_1)$$

where:

$$d_1 = \frac{\log(V_S(t)/V_E(t)) + \sigma_{i,j}^2(T-t)/2}{\sigma_{i,j}\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma_{i,j}\sqrt{T-t}.$$

Strictly speaking, in this formula σ needs to be defined as the volatility of $\log(V_S(t)/V_E(t))$.

3.1.2 There is, however, a problem regarding the determination of $V_E(t)$ and $V_S(t)$. Neither are, in general, produced by the 1995 Wilkie model (as $V_S(t)$ is not the same as the share price itself). Suitable approximations might be available, based on assumptions concerning yield curves (for both fixed-interest rates and dividend yields). Alternatively, the strike prices chosen might mean that we know $V_S(t)/V_E(t)$, which would avoid the need to identify either $V_E(t)$ or $V_S(t)$ in isolation.

3.2 *Exotic Options*

More complicated options, e.g. barrier options or other path dependent options, can often be valued using similar parameters to those used in Section 3.

4. Conclusions

4.1 *Can Options be Priced in the Wilkie Model?*

4.1.1 The Wilkie model, being discrete, can be extended to a continuous time model in many different ways. This means that, in theory, options can be priced with essentially any (positive) implied volatility that the user chooses, without breaching no arbitrage constraints.

4.1.2 However, the random innovations in the Wilkie model are

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normally distributed. This means that the most 'natural' way to extend the Wilkie model to continuous time is to assume that these random innovations accrue within a given year like Brownian motions.

4.1.3 Making this assumption, it becomes possible to value options in a meaningful way, using the approach described above.

4.2 *Are the resulting Option Prices Plausible Representations of Reality?*

4.2.1 The resulting prices are not consistent with actually observed market prices, unless the Wilkie model is modified to incorporate autoregressive conditional heteroscedasticity (ARCH) and/or to include the possibility of market jumps. In particular, the above analysis shows that the implied volatilities of options in the Wilkie model world would otherwise be almost constant (for equity options) and be independent of the option strike price. Neither is the case in reality.

4.2.2 Users should, therefore, be cautious about modelling derivatives (or, indeed, any sort of dynamic investment strategy) using the basic Wilkie model without amendment, if the answers are likely to depend on how cheap or expensive the derivatives are assumed to be by the model.

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