Math. Proc. Camb. Phil. Soc. (1994), **116**, 527
Printed in Great Britain

The generalized Yosida–Hewitt theorem

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(*Received* 2 *March* 1993; *revised* 16 *March* 1994)

0. *Introduction*

The Yosida–Hewitt (YH, for short) theorem [**YH**] has many versions and generalizations in diverse settings, e.g. functionals on vector lattices and spaces of vector-valued functions, measures with values in Banach spaces, topological groups and vector lattices, etc. In this paper we derive a very general form of the YH theorem dealing with the much more general case of operators acting in vector lattices (VLs, for short) and Banach spaces (BSs, for short). A unified approach to all settings mentioned above may be founded on decompositions for operators in VLs.

Two different standpoints to the YH type theorems are available in the literature. The first goes back to the original publication defining the 'antinormal' part as a certain disjoint complement to the 'normal' one (we speak of weak YH decompositions). The second approach goes deeper and characterizes this disjoint complement intrinsically i.e. as a measure, functional or operator which is equal to zero on a huge set (then we speak of strong YH decompositions). In the case of functionals, strong decompositions have a variety of different and unexpected applications; directly to complex analysis, control theory, subdifferential calculus; indirectly, namely through the theory of [**BL**], to optimization, game theory, geometry of BSs, mathematical economics.

Our main decomposition (Theorem 1) is strong. New information in the case of vector measures is derived in Section 3. Moreover in Section 2 we discuss strict singularity of operators.

1. *The Yosida–Hewitt theorem for operators in vector lattices*

Throughout the paper all VLs will be assumed Archimedean, E and F are VLs and F is order complete. The symbols $L^{\sim}(E,F)$, $L_n^{\sim}(E,F)$ and $L_{n\sigma}^{\sim}(E,F)$ denote the spaces of operators from E to F that are respectively regular, order continuous and σ-order continuous. If there is no reason to emphasize the role played by E and F, we simply

https://doi.org/10.1017/S0305004100072790 Published online by Cambridge University Press

write $L^\sim$, $L_n^\sim$ and $L_{n\sigma}^\sim$. The spaces $L_n^\sim$ and $L_{n\sigma}^\sim$ are bands in $L^\sim$. The null ideal of an operator $T \in L^\sim(E,F)$ is $N(T) = \{e \in E : |T|(|e|) = 0\}$.

Since there is a band projection onto any band in an order complete VL, one derives immediately the following weak forms of the YH-decomposition:

$$L^\sim = L_n^\sim \oplus (L_n^\sim)^d, \quad L^\sim = L_{n\sigma}^\sim \oplus (L_{n\sigma}^\sim)^d.$$

To formulate our strong form of the YH-decomposition, we need some definitions. It is well known that, even for functionals, this form is not valid without certain additional assumptions. For example, in the VL $E = C[0,1]$ there are no non-trivial order continuous functionals (cf. [**V**]) but the Rieman integral gives an example of a non-singular functional.

Definition 1. An operator $T \in L^\sim(E,F)$ is called singular if it vanishes on a suitable o.d.i (= order dense ideal) $G \subseteq E$. The set of singular operators is denoted by $L_s^\sim = L_s^\sim(E,F)$. Analogously, if T vanishes on a σ-o.d.i., then T is called σ-singular and the set of such operators is denoted by $L_{s\sigma}^\sim = L_{s\sigma}^\sim(E,F)$.

It is obvious that $L_s^\sim$ and $L_{s\sigma}^\sim$ are ideals in $L^\sim$. Our goal is to learn when they are bands.

For the definitions of both Egorov and countable sup properties, we refer to [**LZ**]. Such properties allow us to cover the spaces of measurable functions on a σ-finite measure space. To cover also the more general case of semi-finite measure spaces we recall the following Definition 2. We would like to thank the referee who suggested this greater generality.

Definition 2. A VL E is said to be weakly σ-distributive if for every $0 < f \in E$ and for every double sequence $f_{n,k} \in E$, we have that $0 \leqslant f_{n,k} \uparrow_k f$ (for all $n \in \mathbb{N}$) implies the existence of a sequence $j(n)$ such that there is an element $g > 0$ with $g \leqslant f_{n,j(n)}$ for every n.

THEOREM 1(a). *Let E be a VL with the Egorov property and F an order complete VL with the countable sup property. Then*

$$L^\sim(E,F) = L_{n\sigma}^\sim(E,F) \oplus L_{s\sigma}^\sim(E,F). \tag{1}$$

THEOREM 1(b). *Let E be a weakly σ-distributive VL and F an order complete VL with the countable sup property. Then*

$$L^\sim(E,F) = L_n^\sim(E,F) \oplus L_s^\sim(E,F). \tag{2}$$

Proof. We first prove (1). Of course we must prove $L_{s\sigma}^\sim = (L_{n\sigma}^\sim)^d$, that is

$$(L_{n\sigma}^\sim)^d \subseteq L_{s\sigma}^\sim \tag{3}$$

the reverse inclusion being trivial.

To do this, let us consider the band projection $P_1 : L^\sim \to L_{n\sigma}^\sim$. By [**AB**, theorem 4·6], we have, for $T \in L_+^\sim$ and $e \in E_+$,

$$T_1(e) = P_1 T(e) = \inf\{\sup_k T(e_k) : 0 \leqslant e_k \uparrow e\}. \tag{4}$$

We claim that the infimum in (4) is attained, i.e. for a certain sequence $0 \leqslant f_l \uparrow e$ we have $T_1(e) = \sup_l T(f_l)$.

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To prove this claim, note that, because of the countable sup property of F, in (4) it suffices to take a countable set of sequences $0 \leqslant e_k \uparrow e$. So we can write

$$T_1^{\perp}(e) = \inf_{m \in \mathbb{N}} \{\sup_k T(e_{m,k}) : 0 \leqslant e_{m,k} \uparrow_k e\} = \inf_m g_m,$$

where $g_m = \sup_k T(e_{m,k})$. Due to the Egorov property, there exists a sequence f_l such that $0 \leqslant f_l \uparrow e$ and $f_l \leqslant e_{m,j(l,m)}$. Consequently, for any $m \in \mathbb{N}$, $\sup_l T(f_l) \leqslant \sup_k T(e_{m,k}) = g_m$ and $\sup_l T(f_l) \leqslant \inf_m g_m = T_1^{\perp}(e)$ that finally means $T_1^{\perp}(e) = \sup_l T(f_l)$ as we claimed.

If $0 \leqslant T \in (L_{n\sigma}^{\sim})^d$, then $T_1^{\perp} = 0$. So, for $e \in E_+$, we find $0 \leqslant f_l \uparrow e$ with $0 = \sup_l T(f_l)$. This says that $N(T)$ is σ-order dense and proves (3).

Now we turn to the proof of the formula (2). Let us note that in the more general case of σ-weak distributivity the argument above furnishes $(L_{n\sigma}^{\sim})^d \subseteq L_s^{\sim}$. Using this inclusion and taking into account that $L_n^{\sim} \subseteq L_{n\sigma}^{\sim}$, we have $(L_n^{\sim})^d \subseteq [L_{n\sigma}^{\sim} \cap (L_n^{\sim})^d] \oplus L_s^{\sim}$ and, therefore, it will be sufficient to obtain $L_{n\sigma}^{\sim} \cap (L_n^{\sim})^d \subseteq L_s^{\sim}$. For this, let us take $0 \leqslant T \in L_{n\sigma}^{\sim} \cap (L_n^{\sim})^d$ and restrict it to its carrier $C(T) = N(T)^d$ defining the strictly positive operator $T_0 = T\restriction_{C(T)} \in L_{n\sigma}^{\sim}(C(T), F)$.

It is obvious that $C(T)$ has the countable sup property and, hence, $T_0 \in L_n^{\sim}(C(T), F)$. Let $S (\in L_n^{\sim}(E, F))$ be the smallest positive extension of T_0 to E (cf. [**Z**, theorem 83·7]). The inequality $0 \leqslant S \leqslant T \in (L_n^{\sim})^d$ implies $S \in (L_n^{\sim})^d$ and, consequently, $S = 0$. The latter says that $C(T) = \{0\}$, namely that $T \in L_s^{\sim}$ as we had to prove.

Remark. If E is an order complete VL with the Egorov property and the continuum hypothesis (CH) is true, then E has the countable sup property [**KVP**]. This means that there is no difference between formulas (1) and (2) since $L_n^{\sim} = L_{n\sigma}^{\sim}$ and $L_s^{\sim} = L_{s\sigma}^{\sim}$. It is interesting to mention that without the order completeness assumption this is not the case. Let us consider the space $E = C(\beta\mathbb{N} \setminus \mathbb{N})$ of continuous functions on the Stone–Čech remainder of the set of natural numbers. It is proved in [**Ve**] that this space has the Egorov property, hence both formulas (1) and (2) are true. But they appear otherwise. Indeed, this Banach lattice has σ-order continuous norm which implies that $L^{\sim}(E, F) = L_{n\sigma}^{\sim}(E, F)$ for any VL F, i.e. (1) is read as $L^{\sim} = L_{n\sigma}^{\sim} \oplus \{0\}$. On the other hand, there are no non-trivial ideals with order continuous norm in E [**Ve**]. This (in combination with σ-order continuity) implies that $L_n^{\sim}(E, F) = \{0\}$ at least in the case when F has a total set of order continuous functionals. Hence, (2) is read as $L^{\sim} = \{0\} \oplus L_s^{\sim}$, that is $L_{n\sigma}^{\sim} = L_s^{\sim}$ in this case.

The countable sup property of F is essential for the validity of Theorem 1. This is illustrated by the following

Example. Take $E = L^{\infty}(0, 1)$ and, by the Kreins–Kakutani representation theorem, realize it as a space $C(Q)$ for an extremally disconnected compact Hausdorff space Q. Let T be the identical embedding of $E = C(Q)$ into $F = l^{\infty}(Q)$. We claim that $T \in (L_n^{\sim})^d \setminus L_s^{\sim}$.

It is clear that T does not belong to $L_s^{\sim}$. To show that $T \in (L_n^{\sim})^d$, let T_n be the order continuous component of T and f_0 the constant function 1 on Q. Proving $T_n(f_0) = 0$ we get that $T_n = 0$. Let $\{U_\alpha\}$ be a base of neighbourhoods of an arbitrary point $q \in Q$. Set $F_\alpha = Q \setminus U_\alpha$ and $\alpha_1 \geqslant \alpha_2$ if $U_{\alpha_1} \subseteq U_{\alpha_2}$. By complete regularity, for any α we get $f_\alpha \in C(Q)$ with $f_\alpha(q) = 0$ and $f_\alpha = 1$ over F_α. As a consequence, we can affirm that $f_\alpha \uparrow f_0$ in $C(Q)$ (Q has no isolated points since $C(Q)$ is order isomorphic to $L^{\infty}(0, 1)$)

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and by [**AB**, theorem 4·6] $T_n(f_0) \leqslant \sup_\alpha T(f_\alpha)$ (in $l^\infty(Q)$), hence $T_n(f_0)(q) = 0$. Since q is arbitrary we have finished.

2. *Strictly singular operators*

The results by Lozanovskiĭ [**Lo1**] will be now extended to the case of operators.

Definition 3. An operator $T \in L^\sim(E,F)$ is called strictly singular if for any non-trivial band B in E, there exists a non-trivial band $K \subseteq B$ such that $T\restriction_K = 0$. The set of all strictly singular operators is denoted by $L^\sim_{ss} = L^\sim_{ss}(E,F)$.

Obviously $L^\sim_{ss} \subseteq L^\sim_s$. In [**Lo1**, **Lo2**] examples of singular but non strictly singular functionals are given (the case of Marcinkiewicz spaces is considered in detail). It is easy to see that $L^\sim_{ss}(L^\infty,F) = L^\sim_s(L^\infty,F)$.

PROPOSITION 1. *The ideal $L^\sim_{ss}$ is order dense in $L^\sim_s$.*

Proof. For a given $0 < T \in L^\sim_s$ we must show the existence of an $S \in L^\sim_{ss}$ with $0 < S \leqslant T$. Assume first that E is order complete. Let G be an o.d.i in E such that $T\restriction_G = 0$. Take $e \in E$ satisfying $T(e) > 0$. We set

$$M = \{h \in E : h \wedge (e-h) = 0 \text{ and } e-h \in G\}.$$

Denote by P_h the order projection onto the band generated by h. For $f \in E_+$ define: $S(f) = \inf_{h\in M} T(P_h(f))$. As usual we can expand S to a positive linear operator defined on the entire space E. Note that for $h \in M$ we have both $T(h) = T(e)$ and $P_h(e) = h$. Hence, $S(e) = T(e) > 0$; this says $S > 0$ while it is clear that $S \leqslant T$. It remains to prove that $S \in L^\sim_{ss}$. For this, take a non-trivial band B in E and $0 < g \in B \cap G$ such that $P_g e \in G$. Clearly $e - P_g e \in M$ and, therefore, for the band K of Definition 2 we can choose the band generated by g.

For an E which is not order complete we should pass to its Dedekind completion E^δ observing that, for $0 < S \in L^\sim(E^\delta,F), S \in L^\sim_{ss}(E^\delta,F) \Rightarrow S\restriction_E \in L^\sim_{ss}(E,F)$.

PROPOSITION 2. *For a VL E which is decomposable into the direct sum of the bands with both the countable sup and the Egorov property, $L^\sim_{ss}(E,F)$ is σ-order closed, namely*

$$T_k \in L^\sim_{ss}\ (k \in \mathbb{N});\quad 0 \leqslant T_k \uparrow T \in L^\sim \Rightarrow T \in L^\sim_{ss}.$$

Proof. Let B be a band in E. Without any loss of generality we may assume that B is a principle band with the countable sup property and the Egorov property. Using the countable sup property, we can construct inductively a family $\{B_{i,k} : i,k \in \mathbb{N}\}$ of bands with the following properties:

(*a*) $B_{i,k} \subseteq B_{i+1,k} \subseteq B$;

(*b*) $T_k\restriction_{B_{i,k}} = 0\ \forall i$;

(*c*) $\bigcup_i B_{i,k}$ is order dense in B for all k.

Let $0 < e \in B$. Then there exists a double sequence $e_{i,k} \in B_{i,k}$ such that $0 \leqslant e_{i,k} \uparrow_i e$. By the Egorov property we find a sequence f_m with $0 \leqslant f_m \uparrow e$ such that for any m,k there exists $j(m,k)$ for which $f_m \leqslant e_{j(m,k),k}$. It follows that $f_m \in K_m = \bigcap_k B_{j(m,k),k}$ for any $m = 1,2,\ldots$. For any m such that $f_m > 0$ the band K_m is non-zero and any $T_k\restriction_{K_m} = 0$. Hence $T \in L^\sim_{ss}$.

We recall that an element e of a VL is said to be of countable type if the principal ideal generated by e has the countable sup property. Now Propositions 1 and 2 give

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THEOREM 2. *Let E be a VL satisfying the conditions of Proposition 2 and F an order complete VL. Then, if $T \in L_s^{\sim}(E,F)$ is of countable type, it follows that $T \in L_{ss}^{\sim}(E,F)$.*

It is shown in [**Lo1**] that in the space $L^{\infty}[L^1]$ every singular functional is strictly singular despite the fact that its dual does not possess the countable sup property.

3. *o-Weakly compact operators*

In this section we shall give decompositions of operators acting from VLs to BSs. It is then possible to obtain, in certain cases, new information about the purely finitely additive part of a vector measure. To derive decompositions for s-bounded vector measures (both weak, like in [**T**], and strong, as in our final Corollary), we deal with the so called o-weakly compact operators.

Let E be a VL and Y a BS. We recall from [**D**] that a linear operator $T: E \to Y$ is o-weakly compact if, for any $e \in E_+$, the set $\{Tf: |f| \leqslant e\}$ is relatively weakly compact.

To fix notation for the sequel, let us assume that $m: \mathscr{F} \to Y$ is a vector measure and denote by T_m the associated operator acting on the space $\mathscr{S}(\mathscr{F})$ of $\mathscr{F}$-simple real functions in the usual way: $T_m(\sum_i \alpha_i \chi_{A_i}) = \sum_i \alpha_i m(A_i)$. Write $\mathscr{N}(m) = \{A \in \mathscr{F}: B \subseteq A, B \in \mathscr{F} \Rightarrow m(B) = 0\}$. It is relevant for our purposes that a vector measure m is s-bounded if and only if the associated operator T_m is o-weakly compact, see [**D**, proposition 4·3].

For an o-bounded operator $T: E \to Y$ (i.e. such that $T[0,e]$ is norm bounded in Y for every $e \in E_+$) we have a 'natural' monotone (= Riesz) seminorm ρ_T on E:

$$\rho_T(f) = \sup\{\|Te\|_Y: |e| \leqslant |f|\}.$$

In the present context we need to define the concepts corresponding, for example, to those of σ-order continuous operator, singular operator and a concept that corresponds to the disjointness from any σ-order continuous operator.

Definition 4. An operator $T: E \to Y$ is called σ-smooth (respectively smooth) if $f_n \downarrow 0$ (respectively $f_\alpha \downarrow 0$) in E implies $\|Tf_n\|_Y \to 0$ (respectively $\|Tf_\alpha\|_Y \to 0$). An operator T is called σ-singular (respectively singular) if the ideal $\mathscr{N}(T) = \{e \in E: 0 \leqslant f \leqslant |e| \Rightarrow Tf = 0\}$ is σ-order dense (respectively order dense) in E. Finally, T is called purely non-σ-smooth (respectively purely non-smooth) if, for every $y^* \in Y^*$, we have that $y^* \circ T \in (E_{n\sigma}^{\sim})^d$ (respectively $\in (E_n^{\sim})^d$).

Our decomposition results are based on the following refinement of [**AB**, theorem 18·7].

THEOREM 3. *Let E be a VL and $T: E \to Y$ an o-weakly compact operator. Then it admits a factorization $T = W \circ V$ through a Banach lattice F with order continuous norm such that $V: E \to F$ is a lattice homomorphism (both operators W and V are automatically o-weakly compact because of the weak compactness of the intervals in F).*

Proof. The main obstacle to the mere repetition of the argument of [**AB**, theorem 18·7] is the lack of the total set of order bounded functionals on E in the general case. Set $N = \{e \in E: \rho_T(e) = 0\}$ and let $\pi: e \in E \to \dot{e} \in E/N$ denote the canonical projection. Then E/N under the quotient norm is a normed lattice. The operator $\hat{T}(\dot{e}) = T(e)$, $e \in \dot{e}$ is also o-weakly compact since $\hat{T}([\hat{0}, \dot{e}]) \subset T([0,e])$. Now we proceed exactly in the same way as in the proof in [**AB**] (namely showing that the norm completion F of

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E/N has order continuous norm) starting with the operator $\hat{T}: E/N \to Y$. It is essential to note that in Dodds' theorem [**AB**, theorem 18·6] the implication '(1) implies (3)' is valid in the case of the normed lattices (since in the original work [**D**] the results are given for general VL with the total set of order bounded functionals and since Fremlin's theorem [**AB**, Theorem 13·9] is true for any normed lattices).

Now the YH decompositions (weak or strong) are straightforward consequences of YH decompositions for operators between VLs.

In fact, let us assume here that an o-weakly compact operator $T: E \to Y$ is defined on a VL E. By means of Theorem 3, we write $T = W \circ V$ where, in particular, $V: E \to F$ is a lattice homomorphism and F has order continuous norm. Now decompose the positive operator $V \in L^{\sim}(E,F)$ as $V = V_1 + V_2$ where $V_1 \in L_n^{\sim}$ (respectively $\in L_{n\sigma}^{\sim}$) and $V_2 \in (L_n^{\sim})^d$ (respectively $\in (L_{n\sigma}^{\sim})^d$) and set $T_i = W \circ V_i$. It is clear that in this way we get the following weak decomposition corresponding to Traynor's weak form of the Yosida–Hewitt theorem for s-bounded measures.

THEOREM 4. *Any o-weakly compact operator $T: E \to Y$ can be uniquely decomposed as a sum of two o-weakly compact operators T_1, T_2 from E to Y such that T_1 is σ-smooth and T_2 is purely non-σ-smooth.*

A decomposition into a smooth part and a purely non-smooth part also follows.

If, in addition, E has the Egorov property, by using Theorem 1, the above decomposition of T is also strong since $V_2 \in L_s^{\sim}$ (respectively $\in L_{s\sigma}^{\sim}$). The formal statements follow.

THEOREM 5(a). *Let E be a VL with the Egorov property, Y a BS and $T: E \to Y$ an o-weakly compact operator. Then T can be uniquely decomposed as $T = T_1 + T_2$ where T_1 is σ-smooth and T_2 is σ-singular.*

THEOREM 5(b). *Let E be a weakly σ-distributive VL, Y a BS and $T: E \to Y$ an o-weakly compact operator. Then T can be uniquely decomposed as $T = T_1 + T_2$ where T_1 is smooth and T_2 is singular.*

To finish let us consider a σ-finite measure space $(\Omega, \mathscr{F}, \lambda)$. If m is an s-bounded vector measure vanishing on $\mathscr{N}(\lambda)$, the operator $T_m: L^{\infty}(\lambda) \to Y$ is not only continuous but also o-weakly compact. Therefore we have the

COROLLARY. *Let $(\Omega, \mathscr{F}, \lambda)$ be a σ-finite measure space, Y a BS and $m: \mathscr{F} \to Y$ an s-bounded vector measure with $m\restriction_{\mathscr{N}(\lambda)} = 0$. Then m can be uniquely decomposed as $m = m_1 + m_2$, where m_1 and m_2 are s-bounded and null on $\mathscr{N}(\lambda)$, m_1 is countably additive and $\mathscr{N}(m_2)$ is an o.d.i. in $\mathscr{F}$*, i.e.

$$\forall A \in \mathscr{F}: \lambda(A) > 0 \; \exists B \in \mathscr{F} \quad \text{such that} \quad \lambda(B) > 0, B \subseteq A \quad \text{and} \quad m(B') = 0 \; \forall B': B' \subseteq B.$$

A. Bukhvalov thanks Prof V. Aversa and Prof P. de Lucia for hospitality during his visit to the University of Naples. Their kindness was a valuable contribution in the preparation of this paper. The authors wish to acknowledge the many helpful remarks made by the referee.

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