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Heat kernel-based p-energy norms on metric measure spaces

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Abstract

We investigate heat kernel-based and other p-energy norms $(1 < p < \infty)$ on bounded and unbounded metric measure spaces, in particular, on nested fractals and their blow-ups. With the weak-monotonicity properties for these semi-norms, we generalise the celebrated Bourgain–Brezis-Mironescu (BBM) type characterisation for $p \neq 2$. When the underlying space admits a heat kernel satisfying the sub-Gaussian estimates, we establish the equivalence of various p-energy semi-norms and weak-monotonicity properties, and show that these weak-monotonicity properties hold when $p = 2$ (that is the case of Dirichlet form). Our paper's key results concern the equivalence and verification of various weak-monotonicity properties on fractals. Consequently, many classical results on p-energy norms hold on nested fractals and their blow-ups, including the BBM type characterisation and Gagliardo-Nirenberg inequality.

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1. *Introduction*

1·1. *Historical background and motivation*

Recently, there has been considerable development on the p-energy defined on fractals and general metric measure spaces (initiated by [**31**]). For a smooth Euclidean domain D, its associated p-energy is defined as $\int_D |\nabla u(x)|^p dx$, but for many fractals or metric measure spaces, it is not easy to define proper gradient structures to characterise the smoothness of certain 'core' functions arising naturally from analysis. Previous constructions of p-energy $(1 < p < \infty)$ are based on the graph-approximation to the underlying space, including post-critically finite (p.c.f.) self-similar sets (see definition in Section 4) by Cao, Gu and Qiu [**16**], the Sierpiński carpet by Shimizu [**46**] and more general fractal spaces by Kigami [**37**].

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In this paper, we propose an alternative way to define p-energy norms via equivalent Besov-type norms, which naturally generalise the energy of Dirichlet forms when $p = 2$, and do not make use of the graph-approximation for their definition. Heat kernel-based p-energy norms were introduced by K. Pietruska-Pałuba in [45], and heat semigroup-based norms were later studied by Alonso Ruiz, Baudoin et al. for $1 \leq p < \infty$ in [1–3]. There, the authors focus on generalising classical analysis results including Sobolev embedding, isoperimetric inequalities and bounded variation functions. It is possible to use heat kernel-based p-energy norms to construct equivalent (or exactly the same) p-energy given by the graph-approximation approach, to satisfy further restrictions like convexity or self-similarity for self-similar sets, see for example [16, 23].

Recall the celebrated 'Bourgain–Brezis-Mironescu (BBM) convergence' in [13, corollary 2]:

$$\lim_{\sigma\uparrow 1}(1-\sigma)\int_D\int_D \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\sigma}}dxdy = C_{n,p}\int_D |\nabla u(x)|^p dx \quad (1<p<\infty) \tag{1·1}$$

where D is a smooth domain in $\mathbb{R}^n$. It states that multiplying by a scaling factor $1-\sigma$, the fractional Gagliardo semi-norms of a function converge to the first-order Sobolev semi-norm as $\sigma \to 1$. The BBM type characterisation thus corresponds to the convergence of Besov semi-norms $(\sigma^* - \sigma)B^{\sigma}_{p,p}$ to $B^{\sigma^*}_{p,\infty}$ on metric measure spaces, where σ^* is a certain critical exponent. This characterisation has been studied for instance on fractals with 'property (E)' [23, theorem 1·6], and on spaces supporting a p-Poincaré inequality [20, 29, 41].

In this paper, we establish a BBM type characterisation for both bounded and unbounded metric measure spaces under appropriate weak-monotonicity properties that replace property (E) on fractals from [23]. We utilise the concept of property $(KE)_\sigma$ introduced in [1, definition 6·7] and $(NE)_\sigma$ defined in [11, definition 4·5] (all termed $P(p,\alpha)$ therein), and use $(VE)_\sigma$ for bounded and unbounded fractals based on [23, definition 3·1], where the letters 'E' stands for energy control and 'K','N','V' represent (heat) kernel, (Besov) norm, vertex respectively. Additionally, we explore their slightly weaker variants by replacing 'sup' with 'limsup'.

These weak-monotonicity properties correspond to certain energy-control conditions, which are essentially required and important in all the studies related to p-energy (norms) mentioned above, to guarantee 'some level of L^p infinitesimal regularity and global controlled L^p geometry', as stated in [11]. In analysis on fractals, weak-monotonicity appears in the form of a p-resistance estimate in [16, 37, 46], which is not easy to obtain. Verifying the weak-monotonicity properties $(KE)_\sigma$, $(NE)_\sigma$ and $(VE)_\sigma$ for certain fractal-type spaces, like nested fractals and their blow-ups, is one of the main contributions of this paper.

To do so, we establish the equivalence of $(KE)_\sigma$ and $(NE)_\sigma$ on metric measure spaces, the equivalence of $(NE)_\sigma$ and $(VE)_\sigma$ on certain fractals (including nested fractals), and then verify a relatively easier condition $(VE)_\sigma$ using the arguments in a main theorem of [16]. Such properties are not easy to examine when $p \neq 2$ in general (but property $(KE)_\sigma$ holds automatically when $p = 2$). Alonso Ruiz, Baudoin et al. examined property $(KE)_\sigma$ in the case $p = 1$ using a weak Bakry-Emery type estimate and in the case $p \neq 2$ for spaces with the same critical exponent as Euclidean smooth manifolds (see for example [3, lemma 4·13]). But for fractal-type spaces, different ideas are required, since the critical exponents σ^* from (1·1) are different from manifolds ($\sigma^* = 1$ for smooth manifolds; but even when $p = 2$, $\sigma^* = \log 15/\log 9$ on the Vicsek set [12] and $\sigma^* = \log 5/\log 4$ on the Sierpiński gasket [10]) and (bi-)linearity is missing.

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1·2. *Preliminaries*

1·2·1. *Basic definitions*

Let (M, d, μ) be a metric measure space, that is, (M,d) is a locally compact, separable metric space and μ is a Radon measure with full support on M. Fix a value (which could be infinite)

$$R_0 \in (0, \operatorname{diam}(M)]$$

that will be used for localisation throughout the paper, where

$$\operatorname{diam}(M) := \sup\{d(x, y) : x, y \in M\} \in (0, \infty]$$

is infinite if M is unbounded and is finite if M is bounded.

The measure μ is assumed to be Borel regular with the following positivity and *volume doubling property*: there exists a constant $C_d > 0$ such that for every $x \in M$, $0 < r < \infty$,

$$0 < \mu(B(x, 2r)) \le C_d \mu(B(x, r)) < \infty, \tag{1·2}$$

where $B(x, r) := \{y \in M : d(x, y) < r\}$ denotes the open metric ball centered at $x \in M$ with radius $r > 0$. Denote

$$V(x, r) := \mu(B(x, r)).$$

It is known from [**24**, proposition 5·1] that if (1·2) holds, then there exists $\alpha_1 > 0$ such that

$$\frac{V(x, R)}{V(x, r)} \le C_d \left(\frac{R}{r}\right)^{\alpha_1}, \tag{1·3}$$

for all $x \in M$, $0 < r \le R < \infty$.

For $\alpha > 0$, we say that μ is α*-regular* if there exists a constant $C \ge 1$ such that for all $0 < r < R_0$,

$$C^{-1} r^{\alpha} \le V(x, r) \le C r^{\alpha}. \tag{1·4}$$

We remark that when μ is α-regular, then (1·3) holds with $\alpha_1 = \alpha$.

A family $\{p_t\}_{t>0}$ of non-negative measurable functions on $M \times M$ is a *heat kernel* if it satisfies, for all $x, y \in M$ and $s, t > 0$:

(1) symmetry: $p_t(x, y) = p_t(y, x)$;

(2) Markov property: $\int_M p_t(x, y) d\mu(y) \le 1$;

(3) semigroup property: $p_{s+t}(x, y) = \int_M p_s(x, z) p_t(z, y) d\mu(z)$;

(4) identity approximation: for any $f \in L^2(M, \mu)$, $\int_M p_t(x, y) f(y) d\mu(y) \to f(x)$ as $t \downarrow 0$, strongly in $L^2(M, \mu)$.

For some parameter $\beta^* > 1$, we consider the following two conditions on the heat kernel:

We say that $\{p_t\}_{t>0}$ satisfies the lower heat kernel estimate (LHE), if for all $t \in (0, R_0^{\beta^*})$ and μ-almost all $x, y \in M$,

$$p_t(x, y) \ge \frac{c_1}{V(x, t^{1/\beta^*})} \exp\left(-c_2 \left(\frac{d(x, y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^* - 1}}\right). \tag{1·5}$$

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We say that $\{p_t\}_{t>0}$ satisfies the upper heat kernel estimate (UHE), if for all $t \in (0, R_0^{\beta^*})$ and μ-almost all $x, y \in M$,

$$p_t(x,y) \leq \frac{c_3}{V(x,t^{1/\beta^*})} \exp\left(-c_4\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right). \tag{1·6}$$

Conditions (UHE) and (LHE) hold in many cases. The classical Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \tag{1·7}$$

is a heat kernel for the standard Brownian motion in $\mathbb{R}^n$. Li and Yau have shown in [40] that, for a complete Riemannian manifold with non-negative Ricci curvature, two-sided estimates (UHE) and (LHE) hold with $\beta^* = 2$:

$$p_t(x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{ct}\right), \tag{1·8}$$

where d is the geodesic metric and $V(x,r)$ is the Riemannian volume. The following sub-Gaussian heat kernel estimates

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta^*}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) \tag{1·9}$$

also hold on many fractal spaces with different β^*, such as the Sierpiński carpet and nested fractals [6, 7, 21, 38].

1·2·2. *Besov and Korevaar-Schoen norms*

From now on we fix $p \in (1,\infty)$. As in [11], for $\sigma > 0$, define the semi-norm $[u]_{B^\sigma_{p,\infty}}$ for $u \in L^p(M,\mu)$ by

$$[u]^p_{B^\sigma_{p,\infty}} := \sup_{r\in(0,R_0)} r^{-p\sigma} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x)-u(y)|^p d\mu(y) d\mu(x), \tag{1·10}$$

and define

$$B^\sigma_{p,\infty} := B^\sigma_{p,\infty}(M) = \{u : ||u||_{B^\sigma_{p,\infty}} < \infty\},$$

with the norm $||u||_{B^\sigma_{p,\infty}} := ||u||_p + [u]_{B^\sigma_{p,\infty}}$. Further, define the semi-norm $[u]_{B^\sigma_{p,p}}$ by

$$[u]^p_{B^\sigma_{p,p}} := \int_0^{R_0} r^{-p\sigma} \left(\int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x)-u(y)|^p d\mu(y) d\mu(x)\right) \frac{dr}{r} \tag{1·11}$$

and

$$B^\sigma_{p,p} := B^\sigma_{p,p}(M) = \{u : ||u||_{B^\sigma_{p,p}} < \infty\},$$

with the norm $||u||_{B^\sigma_{p,p}} := ||u||_p + [u]_{B^\sigma_{p,p}}$.

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The space $B^{\sigma}_{p,\infty}$ coincides with the Korevaar-Schoen space $KS^{\sigma}_{p,\infty}$ in [11, section 4·2] where the Korevaar-Schoen semi-norm $[u]_{KS^{\sigma}_{p,\infty}}$ is given by

$$[u]^p_{KS^{\sigma}_{p,\infty}} = \limsup_{r\to 0} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} \frac{|u(x)-u(y)|^p}{r^{p\sigma}} d\mu(y) d\mu(x) < \infty. \tag{1·12}$$

In our paper, we define the *critical exponent of* (M, d, μ) by

$$\sigma^{\#}_p = \sup\{\sigma > 0 : B^{\sigma}_{p,\infty} \text{ contains non-constant functions}\}.$$

In many related studies, the critical exponent is defined by

$$\sigma^{*}_p = \sup\{\sigma > 0 : B^{\sigma}_{p,\infty} \text{ is dense in } L^p(M,\mu)\}.$$

Clearly, $\sigma^*_p \le \sigma^{\#}_p$. Whenever the critical exponents appear in our paper, we assume that they are finite. It is known by [11, theorem 4·2] that when the chain condition holds,

$$\sigma^{\#}_p \le 1 + \frac{\alpha_1}{p} < \infty.$$

1·2·3. *Heat kernel-based p-energy norm*

It is well known that, a heat kernel $\{p_t\}_{t>0}$ can induce a Dirichlet form on $L^2(M,\mu)$ by

$$\mathcal{E}(u,u) = \lim_{t\to 0^+} \mathcal{E}_t(u,u),\ u \in L^2(M,\mu)$$
$$\mathcal{D}(\mathcal{E}) = \{u \in L^2(M,\mu) : \mathcal{E}(u,u) < \infty\}$$

(see for example [22, section 1·3]), where

$$\mathcal{E}_t(u,u) := \frac{1}{2t} \int_M \int_M |u(x)-u(y)|^2 p_t(x,y) d\mu(y) d\mu(x). \tag{1·13}$$

The limit exists, since for any given $u \in L^2(M,\mu)$, $\mathcal{E}_t(u,u)$ decreases in t by using the spectral theorem.

Following [1, 45], given a heat kernel $\{p_t\}_{t>0}$, we denote its p-energy semi-norm on $L^p(M,\mu)$ by $\left(E^{\sigma}_{p,\infty}(\,\cdot\,)\right)^{1/p}$, where

$$E^{\sigma}_{p,\infty}(u) := \sup_{t\in(0,R_0^{\beta^*})} \frac{1}{t^{p\sigma/\beta^*}} \int_M \int_M |u(x)-u(y)|^p p_t(x,y) d\mu(y) d\mu(x),$$

with its domain

$$\mathcal{D}(E^{\sigma}_{p,\infty}) := \{u \in L^p(M,\mu) : E^{\sigma}_{p,\infty}(u) < \infty\}.$$

We call $\left(E^{\sigma}_{p,\infty}(\,\cdot\,)\right)^{1/p} + ||\cdot||_p$ the heat kernel-based p-energy norm. This is a natural way to extend the Dirichlet form case for $p = 2$ to $p \in (1,\infty)$. $E^{\sigma^{\#}_p}_{p,\infty}$ is equivalent to the p-energy on certain fractals (see Section 4).

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To extend the BBM type characterisation from $p=2$ to $1<p<\infty$, we define

$$E^{\sigma}_{p,p}(u):=\int_0^{R_0^{\beta^*}}\frac{1}{t^{p\sigma/\beta^*}}\left(\int_M\int_M|u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t},$$

with domain

$$\mathcal{D}(E^{\sigma}_{p,p}):=\{u\in L^p(M,\mu):E^{\sigma}_{p,p}(u)<\infty\}.$$

It is known that (see for example [44]), when the two-sided heat kernel estimates hold, the proper scaling of $E^{\sigma}_{2,2}$ converges to $E^{\sigma_2^{\#}}_{2,\infty}$ thanks to the fact that $E^{\sigma}_{2,2}$ is equivalent to the Dirichlet form defined by the subordinated heat kernel. In this paper, we establish an analogous convergence of $E^{\sigma}_{p,p}$ to $E^{\sigma_p^{\#}}_{p,\infty}$. However, $(E^{\sigma}_{p,p})^{1/p}$ does not seem to be the p-energy semi-norm of the subordinated heat kernel, so we need weak-monotonicity properties to replace the subordination technique in the case $p=2$ (see Theorem 2·2).

1·3. *Weak-monotonicity properties and main results*

We introduce some notions that will be used for weak-monotonicity properties and rewrite the above notions in the following way. For $u\in L^p(M,\mu)$ and $\sigma,t>0$, denote

$$\Psi^{\sigma}_u(t)=\frac{1}{t^{p\sigma/\beta^*}}\int_M\int_M|u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x),\tag{1·14}$$

then

$$E^{\sigma}_{p,\infty}(u)=\sup_{t\in(0,R_0^{\beta^*})}\Psi^{\sigma}_u(t),\ E^{\sigma}_{p,p}(u)=\int_0^{R_0^{\beta^*}}\Psi^{\sigma}_u(t)\frac{dt}{t}.\tag{1·15}$$

Similarly, denote

$$\Phi^{\sigma}_u(r)=r^{-p\sigma}\int_M\frac{1}{V(x,r)}\int_{B(x,r)}|u(x)-u(y)|^p d\mu(y)d\mu(x),\tag{1·16}$$

then

$$[u]^p_{B^{\sigma}_{p,\infty}}=\sup_{r\in(0,R_0)}\Phi^{\sigma}_u(r),\ [u]^p_{B^{\sigma}_{p,p}}=\int_0^{R_0}\Phi^{\sigma}_u(r)\frac{dr}{r},\ [u]^p_{KS^{\sigma}_{p,\infty}}=\limsup_{r\to0}\Phi^{\sigma}_u(r).$$

Definition 1·1. We say that a metric measure space (M,d,μ) with heat kernel $\{p_t\}_{t>0}$ satisfies property $(KE)_\sigma$ with $\sigma>0$, if there exists a constant $C>0$ such that for all $u\in\mathcal{D}(E^{\sigma}_{p,\infty})$,

$$\sup_{t\in(0,R_0^{\beta^*})}\Psi^{\sigma}_u(t)\le C\liminf_{t\to0}\Psi^{\sigma}_u(t),\tag{KE}_\sigma$$

where $\Psi^{\sigma}_u(t)$ is defined as in (1·14). We further say that property $(\widetilde{KE})_\sigma$ is satisfied with $\sigma>0$, if there exists a constant $C>0$ such that for all $u\in\mathcal{D}(E^{\sigma}_{p,\infty})$,

$$\limsup_{t\to0}\Psi^{\sigma}_u(t)\le C\liminf_{t\to0}\Psi^{\sigma}_u(t),\tag{$\widetilde{KE}$}_\sigma$$

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Remark 1·2. When $p = 2$, property $(KE)_{\beta^*/2}$ automatically holds as $\mathcal{E}_t(u,u)$ decreases in t for any $u \in \mathcal{D}(E_{2,\infty}^{\beta^*/2})$.

Definition 1·3. We say that a metric measure space (M,d,μ) satisfies property $(NE)_\sigma$ with $\sigma > 0$ if there exists $C > 0$ such that for all $u \in B_{p,\infty}^\sigma$,

$$\sup_{r\in(0,R_0)} \Phi_u^\sigma(r) \le C \liminf_{r\to 0} \Phi_u^\sigma(r). \tag{NE}_\sigma$$

We further say that property $(\widetilde{NE})_\sigma$ is satisfied with $\sigma > 0$, if there exists $C > 0$ such that for all $u \in B_{p,\infty}^\sigma$,

$$\limsup_{r\to 0} \Phi_u^\sigma(r) \le C \liminf_{r\to 0} \Phi_u^\sigma(r). \tag{$\widetilde{NE}$}_\sigma$$

Remark 1·4. It is known from [**11**, lemma 4·7] that, property $(\mathrm{NE})_\sigma$ can only hold when $\sigma \ge \sigma_p^\#$. So property $(NE)_\sigma$ is only interested when $\sigma = \sigma_p^\#$, and whenever we say that property $(NE)_\sigma$ is satisfied, we automatically assume $\sigma = \sigma_p^\#$. Recently, property $(NE)_\sigma$ was verified on nested fractals by Chang and the authors [**17**, theorem 2·3] with the help of a lemma in this paper, on Sierpiński carpets for $p > \dim_{ARC}(K,d)$ by Yang [**50**, theorem 2·8] and for all $p > 1$ by Murugan and Shimizu [**43**, theorem 1·4]. Kajino and Shimizu [**34**, section 5] studied property $(NE)_\sigma$ under the 'p-contraction property' based on Kigami's partition of metric spaces [**37**].

Our first result is that, two kinds of BBM type characterisation under $(KE)_\sigma$ and $(NE)_\sigma$ are established in Theorems 2·2 and 2·4, without using heat kernel estimates.

Our second result is that, under the heat kernel estimates (UHE) and (LHE), we show the equivalence of the heat kernel-based p-energy semi-norms and Besov-Korevaar-Schoen semi-norms on bounded and unbounded metric measure spaces in Lemma 3·1, and the following weak-monotonicity equivalence.

THEOREM 1·5. *If a metric measure space (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (UHE) and (LHE), then we have the following equivalences for $\sigma > 0$:*

(i) $(\widetilde{KE})_\sigma \iff (\widetilde{NE})_\sigma$;

(ii) $(KE)_\sigma \iff (NE)_\sigma$.

Now we introduce our results for fractals. In what follows, we assume that K is a connected homogeneous p.c.f. self-similar set. Let $\alpha = \dim_H(K)$ and denote the α-dimensional Hausdorff measure on K by μ. Let $K^F := \bigcup_{f\in F} f(K)$ be a fractal glue-up where F is a proper set of functions (see details in Subsection 4·1) and equip it with a natural glue-up measure μ^F as in (4·4). This notion is used to unify bounded and unbounded fractals. Following [**23**], let

$$E_n^{(p)}(u) := \sum_{x,y\in V_w,|w|=n} |u(x)-u(y)|^p \quad \text{and} \quad \mathcal{E}_n^\sigma(u) := \rho^{-n(p\sigma-\alpha)} E_n^{(p)}(u). \tag{1·17}$$

Similarly, let

$$E_n^{(p),F}(u) := \sum_{f\in F} E_n^{(p)}(u\circ f) \text{ and } \mathcal{E}_n^{\sigma,F}(u) := \rho^{-n(p\sigma-\alpha)} E_n^{(p),F}(u). \tag{1·18}$$

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Definition 1·6. We say that a fractal glue-up K^F satisfies property $(VE)_\sigma$ with $\sigma > 0$, if there exists a constant $C > 0$ such that for all $u \in L^p(K^F, \mu^F)$,

$$\sup_{n\geq 0} \mathcal{E}_n^{\sigma,F}(u) \leq C \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u). \tag{$VE)_\sigma$}$$

We say that a fractal glue-up K^F satisfies property $(\widetilde{VE})_\sigma$ with $\sigma > 0$, if there exists $C > 0$ such that for all $u \in L^p(K^F, \mu^F)$,

$$\limsup_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \leq C \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u). \tag{$\widetilde{VE})_\sigma$}$$

The proof of the following equivalence is more delicate. It involves proper truncation, an improvement of the p-energies equivalence in [**23**] with annulus decomposition, and comparing the vertex-energies of different levels. Also, analysis on unbounded fractals requires more arguments than the bounded cases.

THEOREM 1·7. *Let K^F be a fractal glue-up, where K is a connected homogeneous p.c.f. self-similar set, then we have the following equivalences*:

(i) $(\widetilde{VE})_\sigma \Longleftrightarrow (\widetilde{NE})_\sigma$ *when* $\sigma > \alpha/p$;

(ii) $(VE)_{\sigma_p^\#} \Longleftrightarrow (NE)_{\sigma_p^\#}$ *when* $\sigma_p^\# > \alpha/p$ *and* $R_0 = C_H$ *(a positive number determined by* K^F*).*

Remark 1·8. The equivalence of the integral-type properties $(KE)_\sigma$, $(NE)_\sigma$ and of the discrete-type property $(VE)_\sigma$ does not rely on the p.c.f. property, but requires the continuity of the functions in $B^\sigma_{p,\infty}$. The reason why we consider p.c.f. fractals is to avoid complexity in defining $(VE)_\sigma$ properly. Also verifying $(VE)_\sigma$ is relatively easier under the p.c.f. property (see Proposition 4·13 and Lemma 4·14).

Theorem 1·5 and Theorem 1·7 are very important in our paper with many corollaries. In Section 4·4, we will apply them to obtain Corollary 4·20, and further deduce Theorem 4·21 focusing on nested fractals and their blow-ups. Moreover, they also deduce Corollary 3·11 and Corollary 4·22 that are important to Dirichlet forms when $p = 2$.

1·4. *Structure of the paper*

The organisation of the paper is as follows. In Section 2, we establish a BBM type characterisation under appropriate weak-monotonicity properties (Theorems 2·2 and 2·4). In Section 3, we demonstrate the equivalence between heat kernel-based and Besov-Korevaar-Schoen p-energy semi-norms under two-sided estimates (UHE) and (LHE) (Lemma 3·1), and the equivalence of the corresponding weak-monotonicity properties (Theorem 1·5). In Section 4, we study fractal glue-ups K^F. In Subsection 4·1, we introduce some geometric properties of K^F and present some useful lemmas. In Subsection 4·2, we verify $(VE)_\sigma$ for nested fractal glue-ups. In Subsection 4·3, we prove Theorem 1·7. In Subsection 4·4, we show that the BBM type characterisation and Gagliardo-Nirenberg inequality hold true for nested fractal blow-ups (Theorem 4·21).

Notation. The letters C, C', C_i,C_i', C_i'', c are universal positive constants depending only on M which may vary at each occurrence. The sign $\asymp$ means that both $\leq$ and $\geq$ are true with uniform values of C depending only on M.

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2. *Consequence of* $(KE)_\sigma$ *and* $(NE)_\sigma$*: BBM type characterisation*

In this section, we present our BBM type characterisation for heat kernel-based and Besov-Korevaar-Schoen p-energy semi-norms. Under the settings in Section 3, these two BBM type characterisation results will coincide.

2·1. *BBM type characterisation of heat kernel-based p-energy semi-norms*

In our proof, we need the following embedding relation.

PROPOSITION 2·1. *For any* $\delta \in (0, \sigma)$*, we have*

$$\mathcal{D}(E^\sigma_{p,\infty}) \subseteq \mathcal{D}(E^{\sigma-\delta}_{p,p}). \tag{2·1}$$

Proof. By the elementary inequality $|a-b|^p \le 2^{p-1}(|a|^p + |b|^p)$, we have

$$\begin{aligned}
&\int_M \int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\
&\le 2^{p-1}\int_M \int_M (|u(x)|^p + |u(y)|^p)p_t(x,y)d\mu(x)d\mu(y)\\
&= 2^p \int_M |u(x)|^p \left(\int_M p_t(x,y)d\mu(y)\right) d\mu(x) \le 2^p ||u||_p^p.
\end{aligned} \tag{2·2}$$

Fix $\epsilon \in (0, R_0^{\beta^*})$. By (2·2),

$$\begin{aligned}
E^{\sigma-\delta}_{p,p}(u) &= \int_0^{R_0^{\beta^*}} \frac{1}{t^{p(\sigma-\delta)/\beta^*}} \left(\int_M \int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t}\\
&\le \int_0^{\epsilon} \frac{1}{t^{p(\sigma-\delta)/\beta^*}} \left(\int_M \int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t}\\
&\quad + \int_\epsilon^{\infty} \frac{1}{t^{p(\sigma-\delta)/\beta^*}} \left(\int_M \int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t}\\
&\le \int_0^{\epsilon} t^{-1+p\delta/\beta^*}dt \sup_{t\in(0,R_0^{\beta^*})} \Psi^\sigma_u(t) + 2^p||u||_p^p \int_\epsilon^\infty \frac{dt}{t^{1+p(\sigma-\delta)/\beta^*}}\\
&= \frac{\beta^*\epsilon^{p\delta/\beta^*}}{p\delta} \sup_{t\in(0,R_0^{\beta^*})} \Psi^\sigma_u(t) + 2^p\|u\|_p^p \frac{\beta^*\epsilon^{-p(\sigma-\delta)/\beta^*}}{p(\sigma-\delta)}\\
&= \frac{\beta^*\epsilon^{p\delta/\beta^*}}{p\delta} E^\sigma_{p,\infty}(u) + \frac{2^p\beta^*||u||_p^p}{p(\sigma-\delta)}\epsilon^{-p(\sigma-\delta)/\beta^*} < \infty,
\end{aligned}$$

thus showing (2·1).

THEOREM 2·2. *Suppose that* (M,d,μ) *admits heat kernel* $\{p_t\}_{t>0}$ *satisfies property* $(KE)_{\tilde\sigma_p}$ *with some* $\tilde\sigma_p > 0$*. Then there exists a constant* $C \ge 1$ *such that for all* $u \in \mathcal{D}(E^{\tilde\sigma_p}_{p,\infty})$*,*

$$C^{-1}E^{\tilde\sigma_p}_{p,\infty}(u) \le \liminf_{\sigma\uparrow\tilde\sigma_p} (\tilde\sigma_p-\sigma)E^\sigma_{p,p}(u) \le \limsup_{\sigma\uparrow\tilde\sigma_p} (\tilde\sigma_p-\sigma)E^\sigma_{p,p}(u) \le CE^{\tilde\sigma_p}_{p,\infty}(u). \tag{2·3}$$

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Proof. Let $u \in \mathcal{D}(E^{\tilde{\sigma}_p}_{p,\infty})$ and $\sigma \in (0, \tilde{\sigma}_p)$. By Proposition 2·1, $u \in \mathcal{D}(E^{\sigma}_{p,p})$. Note that

$$\Psi^{\sigma}_u(t) = t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\Psi^{\tilde{\sigma}_p}_u(t).$$

For the left-hand side, by (1·15), we have

$$\begin{aligned}
\liminf_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)E^{\sigma}_{p,p}(u) &= \liminf_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)\int_0^{R_0^{\beta^*}} t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\Psi^{\tilde{\sigma}_p}_u(t)\frac{dt}{t}\\
&\geq \liminf_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)\int_0^{\tilde{\sigma}_p-\sigma} t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\Psi^{\tilde{\sigma}_p}_u(t)\frac{dt}{t}\\
&\geq \liminf_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)\int_0^{\tilde{\sigma}_p-\sigma} t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\frac{dt}{t}\inf_{t\in(0,\tilde{\sigma}_p-\sigma)}\Psi^{\tilde{\sigma}_p}_u(t)\\
&= \frac{\beta^*}{p}\liminf_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\inf_{t\in(0,\tilde{\sigma}_p-\sigma)}\Psi^{\tilde{\sigma}_p}_u(t)\\
&= \frac{\beta^*}{p}\liminf_{\sigma\uparrow\tilde{\sigma}_p}\inf_{t\in(0,\tilde{\sigma}_p-\sigma)}\Psi^{\tilde{\sigma}_p}_u(t) = \frac{\beta^*}{p}\liminf_{t\to 0}\Psi^{\tilde{\sigma}_p}_u(t).
\end{aligned}$$

It follows from property $(KE)_{\tilde{\sigma}_p}$ that

$$\liminf_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)E^{\sigma}_{p,p}(u) \geq \frac{C\beta^*}{p}E^{\tilde{\sigma}_p}_{p,\infty}(u).$$

For the right-hand side, let $A \in (0, R_0)$ be a finite positive number. By (2·2), for any $\sigma \in (0, \tilde{\sigma}_p)$,

$$\begin{aligned}
E^{\sigma}_{p,p}(u) &= \int_0^{R_0^{\beta^*}} \frac{1}{t^{p\sigma/\beta^*}}\left(\int_M\int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t}\\
&\leq \int_0^{A^{\beta^*}} \frac{1}{t^{p\sigma/\beta^*}}\left(\int_M\int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t} + \int_{A^{\beta^*}}^{\infty}\frac{2^p||u||_p^p}{t^{p\sigma/\beta^*}}\frac{dt}{t}\\
&= \int_0^{A^{\beta^*}} t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\Psi^{\tilde{\sigma}_p}_u(t)\frac{dt}{t} + \frac{2^p\beta^*||u||_p^p}{p\sigma A^{p\sigma}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)E^{\sigma}_{p,p}(u) &\leq \limsup_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)\int_0^{A^{\beta^*}} t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\Psi^{\tilde{\sigma}_p}_u(t)\frac{dt}{t}\\
&\quad + \limsup_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)\frac{2^p\beta^*||u||_p^p}{p\sigma A^{p\sigma}}\\
&\leq \limsup_{\sigma\uparrow\tilde{\sigma}_p}(\tilde{\sigma}_p-\sigma)\int_0^{A^{\beta^*}} t^{p(\tilde{\sigma}_p-\sigma)/\beta^*}\frac{dt}{t}\sup_{t\in(0,A^{\beta^*})}\Psi^{\tilde{\sigma}_p}_u(t) + 0
\end{aligned}$$

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$$= \frac{\beta^*}{p} \limsup_{\sigma \uparrow \tilde{\sigma}_p} A^{p(\tilde{\sigma}_p - \sigma)} \sup_{t \in (0, A^{\beta^*})} \Psi_u^{\tilde{\sigma}_p}(t)$$

$$= \frac{\beta^*}{p} \sup_{t \in (0, A^{\beta^*})} \Psi_u^{\tilde{\sigma}_p}(t) \leq \frac{\beta^*}{p} E_{p,\infty}^{\tilde{\sigma}_p}(u). \tag{2·4}$$

The proof is complete.

2·2. *BBM type characterisation of Besov-Korevaar-Schoen p-energy semi-norms*

The proof of this version is similar to the previous one.

PROPOSITION 2·3. *For any* $\delta \in (0, \sigma)$*, we have*

$$B_{p,\infty}^{\sigma} \subseteq B_{p,p}^{\sigma-\delta}. \tag{2·5}$$

Proof. By the elementary inequality $|a-b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, we have

$$\begin{aligned}
&\int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) d\mu(x) \\
&\leq 2^{p-1} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} (|u(x)|^p + |u(y)|^p) d\mu(y) d\mu(x) \\
&= 2^{p-1} \left(||u||_p^p + \int_M \int_M \frac{\mathbf{1}_{\{d(y,x)<r\}}}{V(x,r)} |u(y)|^p d\mu(y) d\mu(x) \right) \\
&= 2^{p-1} \left(||u||_p^p + \int_M \left(\int_{B(y,r)} \frac{1}{V(x,r)} d\mu(x) \right) |u(y)|^p d\mu(y) \right) \\
&\leq 2^{p-1} \left(||u||_p^p + \sup_{x \in M} \frac{V(x,2r)}{V(x,r)} \int_M |u(y)|^p d\mu(y) \right) \leq C ||u||_p^p.
\end{aligned} \tag{2·6}$$

Fix $\epsilon \in (0, R_0)$. By (2·6),

$$\begin{aligned}
[u]_{B_{p,p}^{\sigma-\delta}}^p &= \int_0^{R_0} r^{-(p\sigma - p\delta)} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) d\mu(x) \frac{dr}{r} \\
&\leq \int_0^{\epsilon} r^{-(p\sigma - p\delta)} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) d\mu(x) \frac{dr}{r} \\
&\quad + \int_{\epsilon}^{\infty} r^{-(p\sigma - p\delta)} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) d\mu(x) \frac{dr}{r} \\
&\leq \int_0^{\epsilon} r^{-1+p\delta} dr \sup_{r \in (0,R_0)} \Phi_u^{\sigma}(r) + C ||u||_p^p \int_{\epsilon}^{\infty} r^{-(p\sigma - p\delta)} \frac{dr}{r} \\
&= \frac{\epsilon^{p\delta}}{p\delta} \sup_{r \in (0,R_0)} \Phi_u^{\sigma}(r) + C ||u||_p^p \frac{\epsilon^{-p(\sigma-\delta)}}{p(\sigma-\delta)} = \frac{\epsilon^{p\delta}}{p\delta} [u]_{B_{p,\infty}^{\sigma}}^p + C' ||u||_p^p < \infty,
\end{aligned}$$

thus showing (2·5).

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THEOREM 2·4. *Suppose that (M, d, μ) satisfies property $(NE)_{\sigma_p^\#}$, then there exists a positive constant C such that for all $u \in B^{\sigma_p^\#}_{p,\infty}$,*

$$C^{-1}[u]^p_{B^{\sigma_p^\#}_{p,\infty}} \le \liminf_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)[u]^p_{B^\sigma_{p,p}} \le \limsup_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)[u]^p_{B^\sigma_{p,p}} \le C[u]^p_{B^{\sigma_p^\#}_{p,\infty}}. \tag{2·7}$$

Suppose that (M, d, μ) satisfies property $(\widetilde{NE})_{\sigma_p}$ with some $\sigma_p > 0$, then there exists a positive constant C such that for all $u \in B^{\sigma_p}_{p,\infty}$,

$$C^{-1}[u]^p_{KS^{\sigma_p}_{p,\infty}} \le \liminf_{\sigma\uparrow\sigma_p}(\sigma_p - \sigma)[u]^p_{B^\sigma_{p,p}} \le \limsup_{\sigma\uparrow\sigma_p}(\sigma_p - \sigma)[u]^p_{B^\sigma_{p,p}} \le C[u]^p_{KS^{\sigma_p}_{p,\infty}}. \tag{2·8}$$

Proof. We show (2·7) first. For the left-hand side, similarly, we have that for any $\sigma_p > 0$,

$$\begin{aligned}
\liminf_{\sigma\uparrow\sigma_p}(\sigma_p - \sigma)[u]^p_{B^\sigma_{p,p}} &= \liminf_{\sigma\uparrow\sigma_p}(\sigma_p - \sigma)\int_0^{R_0} r^{p(\sigma_p-\sigma)}\Phi_u^{\sigma_p}(r)\frac{dr}{r} \\
&\ge \liminf_{\sigma\uparrow\sigma_p}(\sigma_p - \sigma)\int_0^{\sigma_p-\sigma} r^{p(\sigma_p-\sigma)}\Phi_u^{\sigma_p}(r)\frac{dr}{r} \\
&\ge \liminf_{\sigma\uparrow\sigma_p}(\sigma_p - \sigma)\int_0^{\sigma_p-\sigma} r^{p(\sigma_p-\sigma)}\frac{dr}{r}\inf_{r\in(0,\sigma_p-\sigma)}\Phi_u^{\sigma_p}(r) \\
&= \frac{1}{p}\liminf_{\sigma\uparrow\sigma_p}\inf_{r\in(0,\sigma_p-\sigma)}\Phi_u^{\sigma_p}(r) = \frac{1}{p}\liminf_{r\to 0}\Phi_u^{\sigma_p}(r).
\end{aligned} \tag{2·9}$$

Therefore, by property $(NE)_{\sigma_p^\#}$, we have for $\sigma_p = \sigma_p^\#$ in (2·9) that

$$\liminf_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)[u]^p_{B^\sigma_{p,p}} \ge \frac{C^{-1}}{p}[u]^p_{B^{\sigma_p^\#}_{p,\infty}}. \tag{2·10}$$

For the other side, let $A \in (0, R_0)$ be a finite positive number. We have from (2·6) that

$$\begin{aligned}
[u]^p_{B^\sigma_{p,p}} &= \int_0^{R_0} r^{-p\sigma}\left(\int_M \frac{1}{V(x,r)}\int_{B(x,r)}|u(x)-u(y)|^p d\mu(y)d\mu(x)\right)\frac{dr}{r} \\
&\le \int_0^{A} r^{-p\sigma}\left(\int_M \frac{1}{V(x,r)}\int_{B(x,r)}|u(x)-u(y)|^p d\mu(y)d\mu(x)\right)\frac{dr}{r} + C\int_A^\infty r^{-p\sigma}||u||_p^p\frac{dr}{r} \\
&= \int_0^A r^{p(\sigma_p^\#-\sigma)}\Phi_u^{\sigma_p^\#}(r)\frac{dr}{r} + \frac{C||u||_p^p}{p\sigma A^{p\sigma}}.
\end{aligned} \tag{2·11}$$

It follows that

$$\begin{aligned}
\limsup_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)[u]^p_{B^\sigma_{p,p}} &\le \limsup_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)\int_0^A r^{p(\sigma_p^\#-\sigma)}\Phi_u^{\sigma_p^\#}(r)\frac{dr}{r} + \limsup_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)\frac{C||u||_p^p}{p\sigma A^{p\sigma}} \\
&\le \limsup_{\sigma\uparrow\sigma_p^\#}(\sigma_p^\# - \sigma)\int_0^A r^{p(\sigma_p^\#-\sigma)}\frac{dr}{r}\sup_{r\in(0,R_0)}\Phi_u^{\sigma_p^\#}(r) + 0 \\
&= \frac{1}{p}\limsup_{\sigma\uparrow\sigma_p^\#}A^{p(\sigma_p^\#-\sigma)}\sup_{r\in(0,R_0)}\Phi_u^{\sigma_p^\#}(r) = \frac{1}{p}[u]^p_{B^{\sigma_p^\#}_{p,\infty}},
\end{aligned} \tag{2·12}$$

thus showing (2·7).

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Next, we show (2·8). By property $(\widetilde{NE})_{\sigma_p}$ and (2·9), we have

$$\liminf_{\sigma\uparrow\sigma_p}(\sigma_p-\sigma)[u]^p_{B^\sigma_{p,p}}\geq C^{-1}[u]^p_{KS^{\sigma_p}_{p,\infty}}. \tag{2·13}$$

Then for the other side of (2·8), by the definition of lim sup, there exists $\epsilon\in(0,R_0)$ such that

$$\sup_{r\in(0,\epsilon)}\Phi_u^{\sigma_p}(r)\leq 2\limsup_{r\to 0}\Phi_u^{\sigma_p}(r).$$

It follows from replacing $\sigma_p^{\#}$ by σ_p and taking $A=\epsilon$ in (2·11) that,

$$\begin{aligned}\limsup_{\sigma\uparrow\sigma_p}(\sigma_p-\sigma)[u]^p_{B^\sigma_{p,p}}&\leq\limsup_{\sigma\uparrow\sigma_p}(\sigma_p-\sigma)\int_0^\epsilon r^{p(\sigma_p-\sigma)}\Phi_u^{\sigma_p}(r)\frac{dr}{r}+\limsup_{\sigma\uparrow\sigma_p}(\sigma_p-\sigma)\frac{C||u||_p^p}{p\sigma\epsilon^{p\sigma}}\\&\leq\limsup_{\sigma\uparrow\sigma_p}(\sigma_p-\sigma)\int_0^\epsilon r^{p(\sigma_p-\sigma)}\frac{dr}{r}\sup_{r\in(0,\epsilon)}\Phi_u^{\sigma_p}(r)+0\\&=\frac{1}{p}\sup_{r\in(0,\epsilon)}\Phi_u^{\sigma_p}(r)\leq\frac{2}{p}\limsup_{r\to 0}\Phi_u^{\sigma_p}(r)=\frac{2}{p}[u]^p_{KS^{\sigma_p}_{p,\infty}}.\end{aligned} \tag{2·14}$$

The proof is complete.

3. *Equivalences on metric measure spaces*

In this section, we assume that the underlying metric measure space (M,d,μ) admits a heat kernel satisfying (UHE) and (LHE).

3·1. *Equivalence of integral-type semi-norms*

In this subsection, fix $\sigma>0$. The main result of this subsection is the following.

LEMMA 3·1. *If (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (UHE) and (LHE), then $\mathcal{D}(E^\sigma_{p,\infty})=B^\sigma_{p,\infty}$ and for all $u\in B^\sigma_{p,\infty}$,*

$$E^\sigma_{p,\infty}(u)\asymp[u]^p_{B^\sigma_{p,\infty}}\ \text{and}\ [u]^p_{KS^\sigma_{p,\infty}}\asymp\limsup_{t\to 0}\Psi_u^\sigma(t). \tag{3·1}$$

In addition, $\mathcal{D}(E^\sigma_{p,p})=B^\sigma_{p,p}$ and for all $u\in B^\sigma_{p,p}$,

$$E^\sigma_{p,p}(u)\asymp[u]^p_{B^\sigma_{p,p}}. \tag{3·2}$$

We prove Lemma 3·1 by pure elementary analysis (without probability arguments) under slightly milder assumptions, and split the proof into Propositions 3·2-3·6. Part of Lemma 3·1 was also studied by K. Pietruska-Pałuba in [**45**, theorems 3·1 and 3·2] using probability arguments, and by Alonso Ruiz, Baudoin et al. in [**2**, proposition 4·2] under Gaussian heat kernel estimate. Recently, Cao and Qiu in [**15**] proved the equivalence of the Korevaar-Schoen norm and another type of Besov norm.

PROPOSITION 3·2. *If (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (LHE), then there exists a positive constant C such that for all $u\in\mathcal{D}(E^\sigma_{p,p})$,*

$$[u]^p_{B^\sigma_{p,p}}\leq CE^\sigma_{p,p}(u). \tag{3·3}$$

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Proof. By (LHE), we have

$$\begin{aligned}
&\frac{1}{t^{p\sigma/\beta^*}}\int_M\int_{B(x,t^{1/\beta^*})}|u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x)\\
&\geq \frac{c_1}{t^{p\sigma/\beta^*}}\int_M\frac{1}{V(x,t^{1/\beta^*})}\int_{B(x,t^{1/\beta^*})}|u(x)-u(y)|^p\exp\left(-c_2\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right)d\mu(y)d\mu(x)\\
&\geq c_1e^{-c_2}\Phi_u^\sigma(t^{1/\beta^*}). \qquad (3{\cdot}4)
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_{p,p}^\sigma(u)&=\int_0^{R_0^{\beta^*}}\frac{1}{t^{p\sigma/\beta^*}}\left(\int_M\int_M|u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)\right)\frac{dt}{t}\\
&\geq \beta^*c_1e^{-c_2}\int_0^{R_0}\Phi_u^\sigma(r)\frac{dr}{r}=\beta^*c_1e^{-c_2}[u]_{B_{p,p}^\sigma}^p,
\end{aligned}$$

which completes the proof.

PROPOSITION 3·3. *If (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (UHE), then there exists a positive constant C such that for all $u\in B_{p,p}^\sigma$,*

$$E_{p,p}^\sigma(u)\leq C[u]_{B_{p,p}^\sigma}^p.$$

Proof. We decompose $B(x,t^{1/\beta^*})^c$ as the union of annuli $B(x,2^nt^{1/\beta^*})\setminus B(x,2^{n-1}t^{1/\beta^*})$ for $x\in M$, where the integers $1\leq n\leq\log_2(R_0t^{-1/\beta^*})$, then by (1·6),

$$\begin{aligned}
&\frac{1}{t^{p\sigma/\beta^*}}\int_M\int_{B(x,t^{1/\beta^*})^c}|u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x)\\
&=\frac{1}{t^{p\sigma/\beta^*}}\int_M\sum_{n=1}^{[\log_2(R_0t^{-1/\beta^*})]}\int_{B(x,2^nt^{1/\beta^*})\setminus B(x,2^{n-1}t^{1/\beta^*})}|u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x)\\
&\leq\frac{1}{t^{p\sigma/\beta^*}}\int_M\frac{c_3}{V(x,t^{1/\beta^*})}\sum_{n=1}^{[\log_2(R_0t^{-1/\beta^*})]}\int_{B(x,2^nt^{1/\beta^*})\setminus B(x,2^{n-1}t^{1/\beta^*})}|u(x)-u(y)|^p\\
&\times\exp\left(-c_42^{\frac{(n-1)\beta^*}{\beta^*-1}}\right)d\mu(y)d\mu(x)\\
&\leq\sum_{n=1}^{[\log_2(R_0t^{-1/\beta^*})]}\frac{1}{t^{p\sigma/\beta^*}}\int_M\frac{c_3}{V(x,t^{1/\beta^*})}\int_{B(x,2^nt^{1/\beta^*})}|u(x)-u(y)|^p\\
&\times\exp\left(-c_42^{\frac{(n-1)\beta^*}{\beta^*-1}}\right)d\mu(y)d\mu(x)\\
&\leq\sum_{n=1}^{[\log_2(R_0t^{-1/\beta^*})]}\exp\left(-c_42^{\frac{(n-1)\beta^*}{\beta^*-1}}\right)\frac{2^{np\sigma}}{(2^nt^{1/\beta^*})^{p\sigma}}\int_M\frac{c_3C_d2^{n\alpha_1}}{V(x,2^nt^{1/\beta^*})} \qquad (3{\cdot}5)
\end{aligned}$$

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$$\times \int_{B(x,2^n t^{1/\beta^*})} |u(x)-u(y)|^p d\mu(y)d\mu(x)$$
$$= c_3 C_d \sum_{n=1}^{\infty} 1_{\{n\le \log_2 (R_0 t^{-1/\beta^*})\}} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} \Phi_u^\sigma(2^n t^{1/\beta^*}),$$

where we use the volume doubling property (1·3) in the last inequality above

$$\frac{V(x,2^n t^{1/\beta^*})}{V(x,t^{1/\beta^*})} \le C_d 2^{n\alpha_1}. \tag{3·6}$$

Using (1·6) again, we have

$$\frac{1}{t^{p\sigma/\beta^*}} \int_M \int_{B(x,t^{1/\beta^*})} |u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x)$$
$$\le \frac{1}{t^{p\sigma/\beta^*}} \int_M \int_{B(x,t^{1/\beta^*})} |u(x)-u(y)|^p \frac{c_3}{V(x,t^{1/\beta^*})} \exp\left(-c_4\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) d\mu(y)d\mu(x)$$
$$\le c_3 \Phi_u^\sigma(t^{1/\beta^*}). \tag{3·7}$$

Therefore, by (3·5) and (3·7),

$$\Psi_u^\sigma(t) \le c_3 C_d \sum_{n=1}^{\infty} 1_{\{n\le \log_2 (R_0 t^{-1/\beta^*})\}} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} \Phi_u^\sigma(2^n t^{1/\beta^*}) + c_3 \Phi_u^\sigma(t^{1/\beta^*})$$
$$\le c_3 C_d \sum_{n=0}^{\infty} 1_{\{n\le \log_2 (R_0 t^{-1/\beta^*})\}} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} \Phi_u^\sigma(2^n t^{1/\beta^*}), \tag{3·8}$$

it follows that

$$E_{p,p}^\sigma(u) = \int_0^{R_0^{\beta^*}} \Psi_u^\sigma(t)\frac{dt}{t} \le c_3 C_d \sum_{n=0}^{\infty} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) \int_0^{2^{-n\beta^*} R_0^{\beta^*}} 2^{(p\sigma+\alpha_1)n} \Phi_u^\sigma(2^n t^{1/\beta^*})\frac{dt}{t}$$
$$= c_3 \beta^* C_d \sum_{n=0}^{\infty} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} \int_0^{R_0} \Phi_u^\sigma(r)\frac{dr}{r} = c_3 \beta^* C_d C_0 [u]_{B_{p,p}^\sigma}^p,$$

where

$$C_0 = \sum_{n=0}^{\infty} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} < \infty. \tag{3·9}$$

The proof is complete.

The following two propositions deal with the 'local' semi-norms.

PROPOSITION 3·4. *If (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (LHE), then there exists a positive constant C such that for all $u \in \mathcal{D}(E_{p,\infty}^\sigma)$,*

$$[u]_{B_{p,\infty}^\sigma}^p \le C E_{p,\infty}^\sigma(u).$$

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Proof. By (3·4), we have

$$E^{\sigma}_{p,\infty}(u) = \sup_{t\in(0,R_0^{\beta^*})} \frac{1}{t^{p\sigma/\beta^*}} \int_M \int_M |u(x)-u(y)|^p p_t(x,y)d\mu(x)d\mu(y)$$

$$\geq c_1 e^{-c_2} \sup_{t\in(0,R_0^{\beta^*})} \Phi^{\sigma}_u(t^{1/\beta^*}) = c_1 e^{c_2} \sup_{r\in(0,R_0)} \Phi^{\sigma}_u(r) = c_1 e^{c_2} [u]^p_{B^{\sigma}_{p,\infty}},$$

which ends the proof.

PROPOSITION 3·5. *If (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (UHE), then there exists a positive constant C such that for all $u \in B^{\sigma}_{p,\infty}$,*

$$E^{\sigma}_{p,\infty}(u) \leq C[u]^p_{B^{\sigma}_{p,\infty}}.$$

Proof. By (3·8) and (3·6), we have

$$\Psi^{\sigma}_u(t) \leq c_3 C_d \sum_{n=0}^{\infty} 1_{\{n\leq \log_2 (R_0 t^{-1/\beta^*})\}} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} \Phi^{\sigma}_u(2^n t^{1/\beta^*})$$

$$= c_3 C_d \sum_{n=0}^{\infty} \exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right) 2^{(p\sigma+\alpha_1)n} [u]^p_{B^{\sigma}_{p,\infty}} = c_3 C_d C_0 [u]^p_{B^{\sigma}_{p,\infty}}, \tag{3·10}$$

where C_0 is finite by (3·9). The proof is complete by taking the supremum in $(0, R_0^{\beta^*})$ on both sides of (3·10).

The following proposition concerns the Korevaar-Schoen semi-norm.

PROPOSITION 3·6. *If (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (UHE) and (LHE), then for all $u \in B^{\sigma}_{p,\infty}$,*

$$\limsup_{t\to 0} \Psi^{\sigma}_u(t) \asymp \limsup_{r\to 0} \Phi^{\sigma}_u(r). \tag{3·11}$$

Proof. By Proposition 3·4, we have

$$\sup_{r\in(0,R_0)} \Phi^{\sigma}_u(r) \leq C \sup_{t\in(0,R_0^{\beta^*})} \Psi^{\sigma}_u(t), \tag{3·12}$$

for any $R_0 \in (0,\mathrm{diam}(M)]$. Therefore, if we choose $R_0 = \epsilon$ in (3·12), then

$$\limsup_{t\to 0} \Psi^{\sigma}_u(t) = \lim_{\epsilon\to 0} \sup_{t\in(0,\epsilon)} \Psi^{\sigma}_u(t)$$

$$\geq C_1 \lim_{\epsilon\to 0} \sup_{r\in(0,\epsilon^{1/\beta^*})} \Phi^{\sigma}_u(r) = C_1 \limsup_{r\to 0} \Phi^{\sigma}_u(r).$$

Next, we consider the left-hand side of (3·11). By (3·8) and the dominated convergence theorem, we have that

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$$\begin{aligned}
&\limsup_{t\to 0}\frac{1}{t^{p\sigma/\beta^*}}\int_M\int_M |u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x)\\
&\le C_1\sum_{n=0}^{\infty}\exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right)\limsup_{t\to 0}\frac{1_{\{n\le \log_2(R_0 t^{-1/\beta^*})\}}}{(2^n t^{1/\beta^*})^{p\sigma}}\int_M\frac{2^{(p\sigma+\alpha_1)n}}{V(x,2^n t^{1/\beta^*})}\\
&\quad\times\int_{B(x,2^n t^{1/\beta^*})}|u(x)-u(y)|^p d\mu(y)d\mu(x)\\
&\le C_1\sum_{n=0}^{\infty}\exp\left(-c_4 2^{\frac{(n-1)\beta^*}{\beta^*-1}}\right)2^{(p\sigma+\alpha_1)n}\limsup_{r\to 0}\Phi_u^\sigma(r)\le C_1C_0\limsup_{r\to 0}\Phi_u^\sigma(r),
\end{aligned}$$

where $C_1=c_3C_d$ and the constant C_0 is defined in (3·9). The proof is complete.

Proof of Lemma 3·1. By Propositions 3·2 and 3·3, we obtain $\mathcal{D}(E^\sigma_{p,p})=B^\sigma_{p,p}$ and (3·2). By Propositions 3·4 and 3·5, we obtain $\mathcal{D}(E^\sigma_{p,\infty})=B^\sigma_{p,\infty}$ and $E^\sigma_{p,\infty}(u)\asymp[u]^p_{B^\sigma_{p,\infty}}$. By Proposition 3·6, we finally obtain (3·1). The proof is complete.

3·2. *Equivalence between* $(KE)_\sigma$ *and* $(NE)_\sigma$

We start with an easy side $(NE)_\sigma\Rightarrow(KE)_\sigma$.

PROPOSITION 3·7. *Suppose that* (M,d,μ) *admits a heat kernel* $\{p_t\}_{t>0}$ *satisfying (LHE). Then there exists* $C>0$ *such that for all* $u\in\mathcal{D}(E^\sigma_{p,\infty})$,

$$\Psi_u^\sigma(t)\ge C\Phi_u^\sigma(t^{1/\beta^*}).$$

Consequently,

$$\liminf_{t\to 0}\Psi_u^\sigma(t)\ge C\liminf_{r\to 0}\Phi_u^\sigma(r), \tag{3·13}$$

where the constant C *depends only on* c_1 *in (LHE).*

Proof. By (3·4), we have

$$\begin{aligned}
\Psi_u^\sigma(t)&\ge\frac{1}{t^{p\sigma/\beta^*}}\int_M\int_{B(x,\delta t^{1/\beta^*})}|u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x)\\
&\ge\frac{c_1C_d^{-1}\delta^{\alpha_1}}{t^{p\sigma/\beta^*}}\int_M\frac{1}{V(x,\delta t^{1/\beta^*})}\int_{B(x,\delta t^{1/\beta^*})}|u(x)-u(y)|^p d\mu(y)d\mu(x)\\
&=c_1C_d^{-1}\delta^{p\sigma+\alpha_1}\Phi_u(\delta t^{1/\beta^*}).
\end{aligned}$$

Taking the lower limit on both sides of the above inequality, we have

$$\liminf_{t\to 0}\Psi_u^\sigma(t)\ge c_1C_d^{-1}\delta^{p\sigma+\alpha_1}\liminf_{t\to 0}\Phi_u^\sigma(t^{1/\beta^*})=c_1C_d^{-1}\delta^{p\sigma+\alpha_1}\liminf_{r\to 0}\Phi_u^\sigma(r),$$

which implies (3·13).

We give a proof for the following claim which appeared in [**3**, The proof of Lemma 4·13].

PROPOSITION 3·8. *Suppose that* (M,d,μ) *admits a heat kernel* $\{p_t\}_{t>0}$ *satisfying (UHE) and (LHE). Then there exist* $C,c>1$ *and* $c'>0$ *such that for any* $\delta>0$ *and for* μ*-almost all*

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$x, y \in M$ *with* $d(x,y) > \delta t^{1/\beta^*}$,

$$p_t(x,y) \le C \exp\left(-c'\delta^{\frac{\beta^*}{\beta^*-1}}\right) p_{ct}(x,y).$$

Proof. By (UHE) and the volume doubling property (1·3), we have

$$\begin{aligned} p_t(x,y) &\le \frac{c_3}{V(x,t^{1/\beta^*})} \exp\left(-c_4\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) \\ &\le \frac{c_3 C_d c^{\alpha_1/\beta^*}}{V(x,(ct)^{1/\beta^*})} \exp\left(-c_4\left(\frac{d(x,y)}{(ct)^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}} \cdot c^{\frac{1}{\beta^*-1}}\right) \\ &= \frac{c_3 C_d c^{\alpha_1/\beta^*}}{c_1} \frac{c_1}{V(x,(ct)^{1/\beta^*})} \exp\left(-c_4\left(\frac{d(x,y)}{(ct)^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}} \cdot c^{\frac{1}{\beta^*-1}}\right), \end{aligned}$$

for μ-almost all $x, y \in M$. Taking $\tilde{c} = c_4 c^{\frac{1}{\beta^*-1}} - c_2$, we use (LHE) to obtain

$$p_t(x,y) \le \frac{c_3 C_d c^{\alpha_1/\beta^*}}{c_1} \exp\left(-\tilde{c}\left(\frac{d(x,y)}{(ct)^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) p_{ct}(x,y).$$

We choose $c > 1$ to ensure $c^{\frac{1}{\beta^*-1}} c_4 > c_2$ so that $\tilde{c} > 0$. Let $c' = \tilde{c}/c^{\frac{1}{\beta^*-1}}$, then for $d(x,y) > \delta t^{1/\beta^*}$,

$$p_t(x,y) \le \frac{c_3 C_d c^{\alpha_1/\beta^*}}{c_1} \exp\left(-c'\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) p_{ct}(x,y) \le C \exp\left(-c'\delta^{\frac{\beta^*}{\beta^*-1}}\right) p_{ct}(x,y),$$

where $C = c_3 C_d c^{\alpha_1/\beta^*}/c_1$. The proof is complete.

The proof of the implication $(KE)_\sigma \Rightarrow (NE)_\sigma$ is inspired by [**3**, lemma 4·13] where $p = 1$ there.

LEMMA 3·9. *Suppose that* (M,d,μ) *admits a heat kernel* $\{p_t\}_{t>0}$ *satisfying (UHE) and (LHE). Assuming the property* $(\widetilde{KE})_\sigma$ *with* $\sigma > 0$, *we have*

$$\liminf_{t\to 0} \Psi_u^\sigma(t) \le C \liminf_{r\to 0} \Phi_u^\sigma(r), \tag{3·14}$$

for all $u \in \mathcal{D}(E^\sigma_{p,\infty})$.

Proof. Temporally fix $u \in \mathcal{D}(E^\sigma_{p,\infty})$. Without loss of generality, assume that $\liminf_{t\to 0} \Psi_u^\sigma(t) > 0$. Fix $\delta > 1$ which will be determined later in (3·17). For $t \in (0, R_0^{\beta^*})$, decompose

$$\Psi_u^\sigma(t) = \Psi_1(t) + \Psi_2(t),$$

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where

$$\Psi_1(t) = \frac{1}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)\le \delta t^{1/\beta^*}\}} |u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x),$$
$$\Psi_2(t) = \frac{1}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)> \delta t^{1/\beta^*}\}} |u(x)-u(y)|^p p_t(x,y)d\mu(y)d\mu(x).$$

For $\Psi_1(t)$, when $d(x,y)\le \delta t^{1/\beta^*}$, we have

$$\begin{aligned}\frac{c_1}{V(x,t^{1/\beta^*})}\exp\left(-c_2\delta^{\frac{\beta^*}{\beta^*-1}}\right) &\le \frac{c_1}{V(x,t^{1/\beta^*})}\exp\left(-c_2\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right)\\ &\le p_t(x,y) \le \frac{c_3}{V(x,t^{1/\beta^*})}\exp\left(-c_4\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right)\\ &\le \frac{c_5}{V(x,t^{1/\beta^*})}.\end{aligned}$$

Therefore, noting that $\delta > 1$, we have from (1·3) that

$$\begin{aligned}\Psi_1(t) &\le \frac{c_3}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)\le\delta t^{1/\beta^*}\}} \frac{|u(x)-u(y)|^p}{V(x,t^{1/\beta^*})}d\mu(y)d\mu(x)\\ &\le \frac{c_3C_d\delta^{\alpha_1}}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)\le\delta t^{1/\beta^*}\}} \frac{|u(x)-u(y)|^p}{V(x,\delta t^{1/\beta^*})}d\mu(y)d\mu(x) = C_1\Phi_u^\sigma(\delta t^{1/\beta^*}),\end{aligned} \tag{3·15}$$

where $C_1 = c_3C_d\delta^{p\sigma+\alpha_1}$.

For $\Psi_2(t)$, by Proposition 3·8, we have that

$$\Psi_2(t) \le A\Psi_u^\sigma(ct), \tag{3·16}$$

where $A = Cc^{p\sigma/\beta^*}\exp(-c'\delta^{\frac{\beta^*}{\beta^*-1}})$ can be sufficiently small by choosing sufficiently large δ (since C, c, c' are independent constants). So, from now on, we choose sufficiently large δ such that

$$A = Cc^{p\sigma/\beta^*}\exp\left(-c'\delta^{\frac{\beta^*}{\beta^*-1}}\right) < \frac{1}{2}. \tag{3·17}$$

Combining (3·15) and (3·16), we have

$$\Psi_u^\sigma(t) \le C_1\Phi_u^\sigma(\delta t^{1/\beta^*}) + A\Psi_u^\sigma(ct). \tag{3·18}$$

By property $(\widetilde{KE})_\sigma$, there exists $t_0 > 0$ such that for all $t\in(0,t_0)$,

$$\Psi_u^\sigma(t) \le \sup_{0<t<t_0}\Psi_u^\sigma(t) \le 2\limsup_{t\to0}\Psi_u^\sigma(t) \le C_2\liminf_{t\to0}\Psi_u^\sigma(t),$$

where C_2 comes from property $(\widetilde{KE})_\sigma$. Therefore, we have from (3·18) that, for all $t\in(0,\frac{t_0}{c})$,

$$\Psi_u^\sigma(t) \le C_1\Phi_u^\sigma(\delta t^{1/\beta^*}) + AC_2\liminf_{t\to0}\Psi_u^\sigma(t).$$

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Fix $\delta > 0$ such that $AC_2 < 1$ with the aforementioned requirement $A < 1/2$. It follows that

$$\liminf_{t\to 0} \Psi_u^\sigma(t) \le C_1 \liminf_{t\to 0} \Phi_u^\sigma(t) + AC_2 \liminf_{t\to 0} \Psi_u^\sigma(t),$$

which implies

$$\liminf_{t\to 0} \Psi_u^\sigma(t) \le C_3 \liminf_{t\to 0} \Phi_u^\sigma(t),$$

where $C_3 = \frac{C_1}{1-AC_2}$. The proof is complete.

Proof of Theorem 1·5. Since the two-sided estimates (UHE) and (LHE) hold, we have that $\mathcal{D}(E^\sigma_{p,\infty}) = B^\sigma_{p,\infty}$ for any $\sigma > 0$ by Lemma 3·1.

By (3·11) and (3·14), $(\widetilde{KE})_\sigma$ implies $(\widetilde{NE})_\sigma$ since for all $u \in B^\sigma_{p,\infty} = \mathcal{D}(E^\sigma_{p,\infty})$,

$$\limsup_{r\to 0} \Phi_u^\sigma(r) \le C \limsup_{t\to 0} \Psi_u^\sigma(t) \le C' \liminf_{t\to 0} \Psi_u^\sigma(t) \le C'' \liminf_{r\to 0} \Phi_u^\sigma(r). \tag{3·19}$$

By (3·11) and (3·13), $(\widetilde{NE})_\sigma$ implies $(\widetilde{KE})_\sigma$ since

$$\limsup_{t\to 0} \Psi_u^\sigma(t) \le C_1 \limsup_{r\to 0} \Phi_u^\sigma(r) \le C_1' \liminf_{r\to 0} \Phi_u^\sigma(r) \le C_1'' \liminf_{t\to 0} \Psi_u^\sigma(t). \tag{3·20}$$

Therefore, by (3·19) and (3·20), we show Theorem 1·5 (i).

By (3·14) and Proposition 3·4, $(KE)_\sigma$ implies $(NE)_\sigma$ since

$$\sup_{r\in(0,R_0)} \Phi_u^\sigma(r) \le C_2 \sup_{t\in(0,R_0^{\beta^*})} \Psi_u^\sigma(t) \le C_2' \liminf_{t\to 0} \Psi_u^\sigma(t) \le C_2'' \liminf_{r\to 0} \Phi_u^\sigma(r), \tag{3·21}$$

where we use the simple fact $(KE)_\sigma \Rightarrow (\widetilde{KE})_\sigma$. Similarly, by (3·13) and Proposition 3·5 adjoint with the fact $(NE)_\sigma \Rightarrow (\widetilde{NE})_\sigma$, we immediately derive $(KE)_\sigma$ since

$$\sup_{t\in(0,R_0^{\beta^*})} \Psi_u^\sigma(t) \le C_3 \sup_{r\in(0,R_0)} \Phi_u^\sigma(r) \le C_3' \liminf_{r\to 0} \Phi_u^\sigma(r) \le C_3'' \liminf_{t\to 0} \Psi_u^\sigma(t). \tag{3·22}$$

Thus, Theorem 1·5 (ii) holds by (3·21) and (3·22).

Remark 3·10. We say that a heat kernel $\{p_t\}_{t>0}$ satisfies the near diagonal lower estimate (NLE), if there exists $\delta \in (0,\infty)$ such that, for all $t \in (0, R_0^{\beta^*})$ and μ-almost all $x, y \in M$:

$$p_t(x,y) \ge \frac{c_1}{V(x, t^{1/\beta^*})} \quad \text{whenever } d(x,y) \le \delta t^{1/\beta^*}. \tag{3·23}$$

We conjecture that Lemma 3·1 and Lemma 3·9 cannot be derived from the weaker heat kernel estimates (UHE) and (NLE) without further assumptions (like the chain condition).

By Theorem 1·5 and Remark 1·2, we have the following interesting corollary for $p = 2$, which is important for the construction of local Dirichlet forms based on Besov norms.

COROLLARY 3·11. *Suppose that (M,d,μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (UHE) and (LHE), then $(NE)_{\beta^*/2}$ holds when $p = 2$.*

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4. *Connected homogeneous p.c.f self-similar sets and their glue-ups*

In this section, we analyse bounded and unbounded 'fractal glue-ups' by using connected homogeneous p.c.f self-similar sets as 'tiles'. We first obtain equivalent discrete semi-norms (vertex energies) in Subsection 4·1. Then in Subsection 4·2, we introduce weak-monotonicity properties for vertex energies, including property (E) in [**23**, definition 3·1], properties $(VE)_\sigma$ and $(\widetilde{VE})_\sigma$. We will verify property (E) on nested fractals. In Subsection 4·3, we show the equivalence between $(KE)_\sigma$ and $(VE)_\sigma$. Lastly, we show that aforementioned consequences hold for certain 'fractal glue-ups', for example, nested fractals and their fractal blow-ups in Subsection 4·4.

4·1. *Equivalent discrete semi-norms of fractal glue-ups*

We first introduce connected homogeneous p.c.f. self-similar sets in $\mathbb{R}^d$ $(d \geq 2)$. Let $\{\phi_i\}_{i=1}^N$ $(N \geq 2)$ be an IFS where each $\phi_i : \mathbb{R}^d \to \mathbb{R}^d$ is of the form

$$\phi_i(x) = \rho(x - b_i) + b_i, \tag{4·1}$$

with $\rho \in (0, 1)$, $b_i \in \mathbb{R}^d$ for $i = 1, ..., N$. Let K be the attractor of the iterated function system (IFS) $\{\phi_i\}_{i=1}^N$, namely, K is the unique non-empty compact set in $\mathbb{R}^d$ satisfying

$$K = \bigcup_{i=1}^{N} \phi_i(K).$$

To explain what 'p.c.f' is, we introduce the natural symbolic space associated with the IFS $\{\phi_i\}_{i=1}^N$. Let $W = \{1, 2, ..., N\}$, W_n be the set of words with length n over W, and $W^{\mathbb{N}}$ be the set of all infinite words over W. For $w = \mathrm{w}_1\mathrm{w}_2 \cdots \in W^{\mathbb{N}}$, the canonical projection $\pi : W^{\mathbb{N}} \to K$ is defined by $\pi(w) := \bigcap_{n \in \mathbb{N}^*} F_{\mathrm{w}_1 \ldots \mathrm{w}_n}(K)$, where $F_{\mathrm{w}_1 \ldots \mathrm{w}_n} := F_{\mathrm{w}_1} \circ \cdots \circ F_{\mathrm{w}_n}$. Then the critical set Γ and the post-critical set $\mathcal{P}$ is defined by

$$\Gamma = \pi^{-1}\left(\bigcup_{1 \leq i < j \leq N} \left(\phi_i(K) \cap \phi_j(K)\right)\right), \quad \mathcal{P} = \bigcup_{m \geq 1} \tau^m(\Gamma),$$

where τ is the left shift by one index on $W^{\mathbb{N}}$ (see for example [**36**, definition 1·3·13]). We say that the IFS $\{\phi_i\}_{i=1}^N$ is *post-critically finite* (p.c.f.) if $\mathcal{P}$ is finite. Define

$$V_0 = \pi(\mathcal{P}), \quad V_w := F_w(V_0), \quad V_n = \bigcup_{w \in W_n} V_w, \quad V_* = \bigcup_{n \geq 1} V_n, \tag{4·2}$$

then K is the closure of V_* with respect to the Euclidean metric. For $w \in W_n$,

$$K_w := F_w(K)$$

is called an n-cell of K. Without loss of generality, we always assume that

$$\operatorname{diam}(K) = 1 \tag{4·3}$$

by an affine transformation on $\{b_i\}_{i=1}^N$. From now on, denote by K a homogeneous p.c.f. self-similar set that is connected. It is known that a homogeneous p.c.f. IFS in the form of (4·1) satisfies the *open set condition* (OSC) (see for example [**19**, theorem 1·1]). Hence the

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Hausdorff dimension of K is

$$\alpha = \dim_H(K) = -\log N/\log \rho,$$

and the α-dimensional Hausdorff measure on K, denote by μ, is α-*regular*.

It is known from [27, proposition 2·5] that *Condition (H)* holds for connected homogeneous p.c.f. self-similar sets. That is, there exists $c > 0$ depending only on K such that, for any two words w and w' with the same length $m \geq 1$, $K_w \cap K_{w'} = \emptyset$ implies that dist $(K_w, K_{w'}) \geq c\rho^m$.

In order to unify the notation for bounded and unbounded fractals, we introduce the concept of *fractal glue-ups*. The definition is adapted from the idea of fractafold in [48] although it is not the same. Given a compact set $K \subset \mathbb{R}^d$ and a set F of similitudes on $\mathbb{R}^d$, define

$$K^F := \bigcup_{f \in F} f(K) \subset \mathbb{R}^d.$$

In this paper, we need the following requirements:

(1) isometries: all similitudes in F are isometries;

(2) just-touching property: for any $f \neq g \in F$,

$$f(K) \bigcap g(K) = f(V_0) \bigcap g(V_0);$$

(3) condition (H): There exists a constant $C_H \in (0, 1)$ such that, if $|x - y| < C_H\rho^m$ and $x \in f_1(K_w)$ and $y \in f_2(K_{\tilde{w}})$ for two words w and $\tilde{w}$ with the same length $m \geq 0$ and $f_1, f_2 \in F$, then $f_1(K_w)$ intersects $f_2(K_{\tilde{w}})$;

(4) connectedness: K^F is connected.

From now on, whenever we mention a *fractal glue-up* K^F, we automatically assume that these four conditions are satisfied. We call each $f(K)$ a *tile* ($f \in F$). The above conditions (2) and (3) imply the following *uniform finitely-joint* property.

PROPOSITION 4·1. *For any metric ball B(x,R) of radius $R > 0$ in K^F, the number of tiles that intersect it is bounded by a constant $C_K(R) < \infty$, which depends only on C_H and the IFS. Thus for any tile f(K), the number of tiles $\tilde{f}(K)$ that intersect it is bounded by $C_K(1)$.*

Proof. Denote all the tiles that intersect $B(x,R)$ by $f_n(K)$, $1 \leq n \leq N_0$ (here N_0 might be ∞). Denote by t the smallest integer satisfying

$$N^t > |V_0|.$$

Choose a level-t cell $\phi_w(K)$ ($|w| = t$) that does not intersect V_0. Such a cell exists by the pigeonhole principle. Condition (H) implies that all $f_n(\phi_w(K))$ are separated by distance $C_H\rho^t$ for $1 \leq n \leq N_0$, but they all belong to the ball $B(x, R+1)$ by (4·3). Let $C_K(R)$ be the maximal cardinality of disjoint balls of radii $C_H\rho^t/2$ inside a ball with diameter $R+1$ in $\mathbb{R}^d$. Since there are N_0 disjoint balls $B(f_n(\phi_w(v)), C_H\rho^t/2)$ inside $B(x, R+1)$, we obtain $N_0 \leq C_K(R)$. The second inclusion follows from choosing $R = 1$ and $x \in f(K)$.

Remark 4·2. The concept of the fractal glue-up unifies the notions of bounded and unbounded fractals. For example, $K = K^F$ when $F = \{Id\}$. We are mainly interested in the

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following unbounded fractal by blowing up K in [47, section 3]. However, fractal glue-ups may lack self-similarity on large scales.

EXAMPLE 4.3. (Fractal blow-up). *Let* $\{\phi_i\}_{i=1}^N$ *be defined as in (4·1). Define*

$$K_l := \underbrace{\phi_1^{-1} \circ \phi_1^{-1} \circ \cdots \circ \phi_1^{-1}}_{l} K = \rho^{-l} K = \bigcup_{f \in F_l} f(K),$$

where $F_l = \left\{ x + \sum_{j=1}^{l} \rho^{-j} c_j : c_j \in \{(1-\rho) b_i\}_{i=1}^N \right\}$.

Clearly, F_l and K_l are increasing sequences, so that $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$.

The final fractal blow-up of K given by

$$K_\infty = \bigcup_{l=1}^{\infty} K_l = \lim_{l \to \infty} K_l,$$

is in the form K^F with $F = \bigcup_{l=1}^{\infty} F_l = \lim_{l \to \infty} F_l$.

From now on, we assume that K is a connected homogeneous p.c.f. self-similar set when we say that K^F is a fractal glue-up. In what follows, we use the superscript F to distinguish the underlying space K and K^F for the energies and norms etc.

Let

$$\mu^F := \sum_{f \in F} \mu \circ f^{-1}, \tag{4·4}$$

be the glue-up measure on K^F. Then μ^F is α-regular by our assumptions on F. Let

$$[u]^p_{B^{\sigma,F}_{p,\infty}} := \sup_{n \geq 0} \Phi_u^{\sigma,F}(C_H \rho^n), \tag{4·5}$$

where $\Phi_u^{\sigma,F}(r) := r^{-(p\sigma+\alpha)} \int_{K^F} \int_{B(x,r)} |u(x) - u(y)|^p d\mu^F(y) d\mu^F(x)$, and

$$B^{\sigma,F}_{p,\infty} := \{ u \in L^p(K^F, \mu^F) : [u]_{B^{\sigma,F}_{p,\infty}} < \infty \}.$$

It is easy to see that

$$\sup_{n \geq 0} \Phi_u^{\sigma,F}(C_H \rho^n) \asymp \sup_{r \in (0, C_H)} \Phi_u^{\sigma,F}(r), \tag{4·6}$$

which implies that

$$B^{\sigma,F}_{p,\infty} = B^{\sigma}_{p,\infty}(K^F) \quad \text{and} \quad [u]_{B^{\sigma,F}_{p,\infty}} \asymp [u]_{B^{\sigma}_{p,\infty}(K^F)} (\text{with } R_0 = C_H \text{ in } (1·10)). \tag{4·7}$$

Also, we have

$$\liminf_{n \to \infty} \Phi_u^{\sigma,F}(\rho^n) \asymp \liminf_{r \to 0} \Phi_u^{\sigma,F}(r) \quad \text{and} \quad \limsup_{n \to \infty} \Phi_u^{\sigma,F}(\rho^n) \asymp \limsup_{r \to 0} \Phi_u^{\sigma,F}(r). \tag{4·8}$$

We need the following Morrey-Sobolev inequality to show that $B^{\sigma}_{p,\infty}(K^F)$ essentially embeds into $C(K^F)$ when $\sigma > \alpha/p$.

LEMMA 4·4 ([11, theorem 3·2]). *Let* (M, d, μ) *be a metric measure space with* μ *satisfying (1·4). When* $\sigma > \alpha/p$, *for any* $u \in B^{\sigma}_{p,\infty}(M)$, *there exists a continuous version*

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$\tilde{u} \in C^{(p\sigma-\alpha)/p}(M)$ satisfying $\tilde{u} = u$ μ-almost everywhere in M and

$$|\tilde{u}(x) - \tilde{u}(y)| \le C|x-y|^{(p\sigma-\alpha)/p} \sup_{r\in(0,3d(x,y)]} \Phi_u^{\sigma}(r) \le C|x-y|^{(p\sigma-\alpha)/p}[u]_{B^{\sigma}_{p,\infty}(M)}, \tag{4·9}$$

for all $x, y \in M$ with $d(x, y) < R_0/3$, where C is a positive constant. Here $C^{\beta}(M)$ denotes the class of Hölder continuous functions of order β on M.

Next, we estimate

$$I^F_{\infty,n}(u) := \int_{K^F}\int_{B(x,C_H\rho^n)} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x).$$

By (4·5), for all $u \in B^{\sigma,F}_{p,\infty}$,

$$\rho^{-n(p\sigma+\alpha)} I^F_{\infty,n}(u) = C_H^{p\sigma+\alpha} \Phi_u^{\sigma,F}(C_H\rho^n). \tag{4·10}$$

Define

$$\mu_m := \frac{1}{|V_m|}\sum_{a\in V_m} \delta_a$$

and

$$\mu^F_m := \sum_{f\in F} \mu_m \circ f^{-1} = \sum_{f\in F} \frac{1}{|V_m|}\sum_{a\in V_m} \delta_{f(a)}, \tag{4·11}$$

where δ_a is the Dirac measure at point a and $|A|$ denotes the cardinality of a finite set A. Let

$$I^F_{m,n}(u) := \int_{K^F}\int_{B(x,C_H\rho^n)} |u(x)-u(y)|^p d\mu^F_m(y) d\mu^F_m(x). \tag{4·12}$$

We will use these discrete sums to estimate the integral $I^F_{\infty,n}(u)$.

LEMMA 4·5. *Let K^F be a fractal glue-up. Then for all $u \in C(K^F)$,*

$$\liminf_{m\to\infty} I^F_{m,n}(u) \ge I^F_{\infty,n}(u). \tag{4·13}$$

The proof of this lemma is based on the proof of [27, lemma 3·2]. The key is to establish the following Proposition 4·6, which is not obvious since K is uncountable. We may assume that K is not contained in any hyperplane $H \subset \mathbb{R}^d$. Otherwise, we could reduce the IFS to H by restricting each similitude ϕ_i to H and applying a reversible affine transformation from H to $\mathbb{R}^{d'}$, where d' is the Hausdorff dimension of H. The attractor of this reduced IFS is a scaled copy of K. Thus we can apply [27, lemma 3·2].

PROPOSITION 4·6. *Assume that K is not contained in any hyperplane $H \subset \mathbb{R}^d$. For any $r > 0$, let $E(r) := \bigcup_{x\in K}\{x\} \times \partial B(x, r)$. Then $\mu \times \mu(E(r)) = 0$.*

Proof. Let

$$R_x(a, b) = \overline{B(x, a)} \setminus B(x, b).$$

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Fix $x_w \in K_w$ for each $w \in W_n$. Then

$$\bigcup_{x\in K_w} \partial B(x,r) \subset R_{x_w}(r-\rho^n, r+\rho^n),$$

since for any $y \in \partial B(x,r)$, $d(x,y) = r$ and $d(x,x_w) \le \rho^n$, thus $d(y,x_w) \in [r-\rho^n, r+\rho^n]$. Note that

$$E(r) := \bigcup_{x\in K}\{x\} \times \partial B(x,r) \subset \bigcup_{w\in W_n} K_w \times R_{x_w}(r-\rho^n, r+\rho^n),$$

thus

$$\begin{aligned}\mu\times\mu(E(r)) &\le \sum_{w\in W_n} \mu(K_w)\mu(R_{x_w}(r-\rho^n, r+\rho^n))\\ &\le \mu(K)\sup_{w\in W_n}\mu(R_{x_w}(r-\rho^n, r+\rho^n))\\ &\le \mu(K)\sup_{x\in K}\mu(R_x(r-\rho^n, r+\rho^n)).\end{aligned} \tag{4·14}$$

We claim that

$$\sup_{x\in K}\mu(R_x(r-\rho^n, r+\rho^n)) \to 0 \text{ as } n\to\infty.$$

Otherwise, if $\mu(R_{x_n}(r-\rho^n, r+\rho^n)) \ge \delta > 0$ for a subsequence $\{x_n\}_n \subset K$, we may choose a subsequence $\{x_{n_i}\}_i$ that converge to $x_\infty \in K$ due to the compactness of K, and further require that $d(x_{n_i}, x_\infty)$ decrease in i. Then

$$A_i := R_{x_\infty}(r-\rho^{n_i} - d(x_{n_i},x_\infty), r+\rho^{n_i}+d(x_{n_i},x_\infty)) \supset R_{x_{n_i}}(r-\rho^{n_i}, r+\rho^{n_i})$$

and $A_i \downarrow \partial B(x_\infty, r)$, thus

$$\mu(\partial B(x_\infty, r)) = \lim_{i\to\infty}\mu(A_i) \ge \delta,$$

which contradicts [**27**, proposition 2·4] that $\mu(\partial B(x,r)) = 0$ for any $x \in K$.

The proof is complete by letting $n\to\infty$ in (4·14).

Proof of Lemma 4·5. Let $F = \{f_i\}_{i\ge 1}$, and let $F_k = \{f_i\}_{1\le i\le k}$. As μ_m weak $*$-converges to μ, we know by definition that the finite glue-up measure $\mu_m^{F_k}$ weak $*$-converges to μ^{F_k} on K^{F_k} (the pieces $f_i(K)$ are mutually measure-disjoint), and that μ^{F_k} is exactly the restriction of μ^F to K^{F_k}. Using the same argument as in that of [**27**, lemma 3·2], for $u \in C(K^F)$, we have

$$\begin{aligned}\lim_{m\to\infty} I_{m,n}^{F_k}(u) &= \lim_{m\to\infty}\int_{K^{F_k}}\int_{B(x,C_H\rho^n)} |u(a)-u(b)|^p d\mu_m^{F_k}(b) d\mu_m^{F_k}(a)\\ &= \int_{K^{F_k}}\int_{B(x,C_H\rho^n)} |u(a)-u(b)|^p d\mu^{F_k}(b) d\mu^{F_k}(a) := I_{\infty,n}^{F_k}(u).\end{aligned}$$

Note that the nonnegative sequence $I_{m,n}^{F_k}(u) \uparrow I_{m,n}^F(u)$ when $k\to\infty$. Denote $I_{m,n}^{F_0}(u) = 0$, then

$$I_{m,n}^F(u) = \sum_{k=0}^{\infty}(I_{m,n}^{F_{k+1}}(u) - I_{m,n}^{F_k}(u)) \tag{4·15}$$

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and, by noting that

$$\liminf_{m\to\infty}(I^{F_{k+1}}_{m,n}(u)-I^{F_k}_{m,n}(u))=I^{F_{k+1}}_{\infty,n}(u)-I^{F_k}_{\infty,n}(u),$$

our conclusion directly follows from Fatou's Lemma.

We are now in the position to prove the first estimate lemma.

LEMMA 4·7. *Let K^F be a fractal glue-up. If $\sigma>\alpha/p$, then for all $u\in B^{\sigma,F}_{p,\infty}$,*

$$I^F_{\infty,n}(u)\le C\rho^{2n\alpha}\sum_{k=n}^{\infty}E^{(p),F}_k(u)\le C'\rho^{n(p\sigma+\alpha)}\sup_{k\ge n}\mathcal{E}^{\sigma,F}_k(u). \tag{4·16}$$

Proof. Let $I^F_{m,n}(u)$ be defined as in (4·12) ($m>n$). Since K^F is connected and satisfies the condition (H), $|x-y|\le C_H\rho^n$ implies that x,y lie in the same or the neighbouring n-cells. For every $w\in W_n$ and $x\in f(K_w)$, if $y\in B(x,C_H\rho^n)$, then there exists $(g,\tilde{w})\in F\times W_n$ such that $y\in g(K_{\tilde{w}})$ and

$$g(K_{\tilde{w}})\cap f(K_w)\ne\emptyset.$$

Therefore,

$$\begin{aligned}I^F_{m,n}(u)&\le\sum_{|w|=n}\sum_{f\in F}\int_{f(K_w)}\left(\sum_{|\tilde{w}|=n}\sum_{g\in F}\int_{g(K_{\tilde{w}})}|u(x)-u(y)|^p1_{\{g(K_{\tilde{w}})\cap f(K_w)\ne\emptyset\}}d\mu^F_m(y)\right)d\mu^F_m(x)\\&=\sum_{|w|=n}\sum_{f\in F}\sum_{|\tilde{w}|=n}\sum_{g\in F}\sum_{x\in f(K_w\cap V_m)}\sum_{y\in g(K_{\tilde{w}}\cap V_m)}\frac{1_{\{g(K_{\tilde{w}})\cap f(K_w)\ne\emptyset\}}}{|V_m|^2}|u(x)-u(y)|^p.\end{aligned}$$

When $g(K_{\tilde{w}})\cap f(K_w)\ne\emptyset$, we can find a common vertex of these two cells and denote it by $z\in f(V_w)\cap g(V_{\tilde{w}})$, due to the just-touching property when $f\ne g$ and the p.c.f. structure when $f=g$. Furthermore, by the inequality $|u(x)-u(y)|^p\le 2^{p-1}\,(|u(x)-u(z)|^p+|u(z)-u(y)|^p)$,

$$1_{\{g(K_{\tilde{w}})\cap f(K_w)\ne\emptyset\}}|u(x)-u(y)|^p\le 2^{p-1}\sum_{z\in f(V_w)\cap g(V_{\tilde{w}})}\left(|u(x)-u(z)|^p+|u(z)-u(y)|^p\right).$$

It follows that

$$\begin{aligned}I^F_{m,n}(u)&\le 2^{p-1}\sum_{|w|=n}\sum_{f\in F}\sum_{|\tilde{w}|=n}\sum_{g\in F}\sum_{x\in f(K_w\cap V_m)}\sum_{y\in g(K_{\tilde{w}}\cap V_m)}\sum_{z\in f(V_w)\cap g(V_{\tilde{w}})}\frac{1}{|V_m|^2}\\&\quad\times\left(|u(x)-u(z)|^p+|u(z)-u(y)|^p\right)\\&\le C_1\sum_{|w|=n}\sum_{f\in F}\sum_{|\tilde{w}|=n}\sum_{g\in F}\sum_{z\in f(V_w)\cap g(V_{\tilde{w}})}\sum_{x\in f(K_w\cap V_m)}\sum_{y\in g(K_{\tilde{w}}\cap V_m)}N^{-2m}\\&\quad\times\left(|u(x)-u(z)|^p+|u(z)-u(y)|^p\right)\\&\le C_1N^{-2m}\sum_{|w|=n}\sum_{f\in F}\sum_{z\in f(V_w)}\sum_{x\in f(K_w\cap V_m)}|u(x)-u(z)|^p\\&\quad\times\left(\sum_{|\tilde{w}|=n}\sum_{g\in F}1_{\{z\in f(V_w)\cap g(V_{\tilde{w}})\}}|g(K_{\tilde{w}}\cap V_m)|\right)\end{aligned}$$

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$$+ C_1 N^{-2m} \sum_{|\tilde{w}|=n} \sum_{g\in F} \sum_{z\in g(V_{\tilde{w}})} \sum_{y\in g(K_{\tilde{w}}\cap V_m)} |u(z)-u(y)|^p$$

$$\times \left(\sum_{|w|=n} \sum_{f\in F} 1_{\{z\in f(V_w)\cap g(V_{\tilde{w}})\}} |f(K_w \cap V_m)| \right)$$

$$\le C_2 N^{-(m+n)} \left(\sum_{|w|=n} \sum_{f\in F} \sum_{z\in f(V_w)} \sum_{x\in f(K_w\cap V_m)} |u(x)-u(z)|^p \right.$$

$$\left. + \sum_{|\tilde{w}|=n} \sum_{g\in F} \sum_{z\in g(V_{\tilde{w}})} \sum_{y\in g(K_{\tilde{w}}\cap V_m)} |u(z)-u(y)|^p \right)$$

$$= 2C_2 N^{-(m+n)} \sum_{|w|=n} \sum_{f\in F} \sum_{x\in f(K_w\cap V_m)} \sum_{z\in f(V_w)} |u(x)-u(z)|^p, \tag{4·17}$$

where we use Proposition 4·1 for each common vertex $z \in f(V_w) \cap g(V_{\tilde{w}})$ in the last inequality, so that

$$\sum_{|\tilde{w}|=n} \sum_{g\in F} 1_{\{z\in f(V_w)\cap g(V_{\tilde{w}})\}}$$

is uniformly bounded for any $(f, w) \in F \times W_n$, and the estimates

$$|V_m| \asymp N^m = \rho^{-\alpha m}, \tag{4·18}$$

$$|K_w \cap V_m| \asymp N^{m-n} \text{ for every } w \in W_n. \tag{4·19}$$

For $x \in f(V_m), z \in f(V_n)$, one can naturally fix a decreasing sequence of cells K_{w_k} with $|w_k| = k$ for $k = n, \dots, m$ and vertices $x_k(x,z) \in V_{w_k}$ such that, $z = x_n(x,z)$, $x = x_m(x,z)$, so by Hölder's inequality we have

$$|u(z)-u(x)|^p \le \left(\sum_{k=n}^{m-1} N^{(n-k)q/p} \right)^{p/q} \left(\sum_{k=n}^{m-1} N^{k-n} |u(x_k(x,z)) - u(x_{k+1}(x,z))|^p \right)$$

$$\le C_3 \sum_{k=n}^{m-1} N^{k-n} |u(x_k(x,z)) - u(x_{k+1}(x,z))|^p, \tag{4·20}$$

where $q = p/(p-1)$. Note that for every pair $(a,b) \in f(K_{w'} \cap V_k) \times f(K_{w'} \cap V_{k+1})$, the cardinality of $(x,z) \in f(K_w \cap V_m) \times f(V_w)$ with $(x_k(x,z), x_{k+1}(x,z)) = (a,b)$ is no greater than $C' N^{m-k}$ for some $C' > 0$, due to the p.c.f. structure. Plugging (4·20) into (4·17), we obtain

$$I^F_{m,n}(u) \le C_4 N^{-(m+n)} \sum_{|w|=n} \sum_{f\in F} \sum_{(x,z)\in f(K_w\cap V_m)\times f(V_w)} \sum_{k=n}^{m-1} N^{k-n} |u(x_k(x,z)) - u(x_{k+1}(x,z))|^p$$

$$\le C_4 N^{-(m+n)} \sum_{|w|=n} \sum_{k=n}^{m-1} \sum_{f\in F} \sum_{\substack{|w'|=k \\ f(K_{w'})\subset f(K_w)}} \sum_{\substack{(x,z)\in f(K_w\cap V_m)\times f(V_w), \\ (a,b)\in f(K_{w'}\cap V_k)\times f(K_{w'}\cap V_{k+1})}} N^{k-n} |u(a)-u(b)|^p$$

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$$\le C_5 N^{-(m+n)} \sum_{|w|=n} \sum_{k=n}^{m-1} \sum_{f\in F} \sum_{\substack{|w'|=k\\ f(K_{w'})\subset f(K_w)}} \sum_{a,b\in f(K_{w'}\cap V_{k+1})} N^{k-n}\cdot N^{m-k}|u(a)-u(b)|^p$$

$$= C_5 \rho^{2n\alpha} \sum_{k=n}^{m-1} \sum_{|w|=n} \sum_{f\in F} \sum_{\substack{|w'|=k\\ f(K_{w'})\subset f(K_w)}} \sum_{a,b\in f(K_{w'}\cap V_{k+1})} |u(a)-u(b)|^p$$

$$\le C_6 \rho^{2n\alpha} \sum_{k=n}^{m-1} \sum_{f\in F} E_{k+1}^{(p)}(u\circ f) \le C_6 \rho^{2n\alpha} \sum_{k=n}^{m} E_k^{(p),F}(u), \tag{4·21}$$

where we use (1·18) in the last line. It follows that

$$I_{m,n}^F(u) \le C_6 \rho^{2n\alpha} \rho^{n(p\sigma-\alpha)} \sum_{k=n}^{m} \rho^{-n(p\sigma-\alpha)} E_k^{(p),F}(u) \le C_7 \rho^{n(p\sigma+\alpha)} \sup_{k\ge n} \mathcal{E}_k^{\sigma,F}(u),$$

which completes the proof by letting $m\to\infty$ and using Lemma 4·5.

The following simple proposition is required.

PROPOSITION 4·8. *For all $u\in B_{p,\infty}^{\sigma,F}$, we have*

$$\liminf_{n\to\infty} \rho^{-n(p\sigma+\alpha)} I_{\infty,n}^F(u) \ge \rho^{p\sigma+\alpha} C_H^{p\sigma+\alpha} \liminf_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n), \tag{4·22}$$

$$\limsup_{n\to\infty} \rho^{-n(p\sigma+\alpha)} I_{\infty,n}^F(u) \ge \rho^{p\sigma+\alpha} C_H^{p\sigma+\alpha} \limsup_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n). \tag{4·23}$$

Proof. Since $C_H\in(0,1)$, assume that $\rho^{s+1}\le C_H<\rho^s$ for some nonnegative integer s. Then, for any integer n,

$$\begin{aligned}
&\rho^{-n(p\sigma+\alpha)} I_{\infty,n}^F(u)\\
&\quad= \rho^{-n(p\sigma+\alpha)} \int_{K^F}\int_{B(x,C_H\rho^n)} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x)\\
&\quad\ge \rho^{-n(p\sigma+\alpha)} \int_{K^F}\int_{B(x,\rho^{n+s+1})} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x) \qquad (4·24)\\
&\quad= \rho^{(s+1)(p\sigma+\alpha)} \rho^{-(n+s+1)(p\sigma+\alpha)} \int_{K^F}\int_{B(x,\rho^{n+s+1})} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x)\\
&\quad> \rho^{p\sigma+\alpha} C_H^{p\sigma+\alpha} \rho^{-(n+s+1)(p\sigma+\alpha)} \int_{K^F}\int_{B(x,\rho^{n+s+1})} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x).
\end{aligned}$$

Taking $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ on both sides, we obtain (4·22) and (4·23).

Denote

$$I_n^F(u) := \int_{K^F}\int_{\{\rho^{n+1}\le d(x,y)<\rho^n\}} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x).$$

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LEMMA 4·9. *Let K^F be a fractal glue-up. If $\sigma > \alpha/p$, then for any $\delta \in (0, p\sigma - \alpha)$ and $u \in B^{\sigma,F}_{p,\infty}$,*

$$\mathcal{E}_n^{\sigma,F}(u) \le C \sum_{k=0}^{\infty} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I^F_{k+n} \le C' \sup_{k\ge 0} \Phi_u^{\sigma,F}(\rho^{n+k}). \tag{4·25}$$

Proof. For $a, b \in f(K_w)$, we have $|u(a) - u(b)|^p \le 2^{p-1}(|u(a) - u(x)|^p + |u(x) - u(b)|^p)$, where $x \in f(K_w)$ and $f \in F$. Approximate F with increasing finite sets F_j so that $F = \lim_{j\to\infty} F_j$. Integrating with respect to x and dividing by $\mu^F(f(K_w))$, we have

$$\begin{aligned} E_n^{(p),F_j}(u) &= \sum_{f\in F_j} \sum_{a,b\in f(V_w),|w|=n} |u(a) - u(b)|^p \\ &\le 2^{p-1} \sum_{f\in F_j} \sum_{a,b\in f(V_w),a\ne b,|w|=n} \\ &\quad \times \left(\frac{1}{\mu^F(f(K_w))} \int_{f(K_w)} |u(a) - u(x)|^p + |u(x) - u(b)|^p d\mu^F(x) \right) \\ &\le 2^{p-1}|V_0| \sum_{f\in F_j} \sum_{a\in f(V_w),|w|=n} \frac{1}{\mu^F(f(K_w))} \int_{f(K_w)} |u(a) - u(x)|^p d\mu^F(x). \end{aligned} \tag{4·26}$$

For every $a \in f(V_w)$ with $|w| = n$, we can fix a decreasing sequence of cells $\{f(K_{w_k})\}_{k=n}^m$ where $|w_k| = k$ for all large enough $m > n$, such that $a \in \cap_{k=n}^m f(K_{w_k})$ with $w_n = w$. We choose $x_k \in f(K_{w_k})$ for $k = n, n+1, \dots, m$. Let $\delta \in (0, p\sigma - \alpha)$. By Hölder's inequality,

$$\begin{aligned} &|u(a) - u(x_n)|^p \\ &\le 2^{p-1}|u(a) - u(x_m)|^p + 2^{p-1} \left(\sum_{k=n}^{m-1} \rho^{\delta(k-n)q/p} \right)^{p/q} \left(\sum_{k=n}^{m-1} \rho^{\delta(n-k)} |u(x_k) - u(x_{k+1})|^p \right), \end{aligned} \tag{4·27}$$

where $q = p/(p-1)$. Integrating (4·27) with respect to $x_k \in K_{w_k}$ and dividing by $\mu^F(f(K_{w_k}))$,

$$\begin{aligned} E_n^{(p),F_j}(u) &\le C_1 \sum_{f\in F_j} \sum_{a\in f(V_w),|w|=n} \frac{2^{p-1}}{\mu^F(f(K_{w_m}))} \int_{f(K_{w_m})} |u(a) - u(x_m)|^p d\mu^F(x_m) \\ &\quad + C_2 \sum_{f\in F_j} \sum_{k=n}^{m-1} \frac{\rho^{\delta(n-k)}}{\mu^F(f(K_{w_k}))\mu^F(f(K_{w_{k+1}}))} \\ &\quad \times \int_{f(K_{w_k})} \int_{f(K_{w_{k+1}})} |u(x_k) - u(x_{k+1})|^p d\mu^F(x_{k+1}) d\mu^F(x_k). \end{aligned}$$

Using Lemma 4·4, (4·7) and (4·18), the first term of the right-hand side above vanishes as $m \to \infty$ since

$$\begin{aligned} &\sum_{a\in f(V_w),|w|=n} \frac{2^{p-1}}{\mu^F(f(K_{w_m}))} \int_{f(K_{w_m})} |u(a) - u(x_m)|^p d\mu^F(x_m) \\ &\le C|f(V_n)|2^{p-1}\rho^{(p\sigma-\alpha)m}[u]^p_{B^\sigma_{p,\infty}(K^F)} \le C'|f(V_n)|2^{p-1}\rho^{(p\sigma-\alpha)m}[u]^p_{B^{\sigma,F}_{p,\infty}} \\ &\le C_3\rho^{(p\sigma-\alpha)m-\alpha n}[u]^p_{B^{\sigma,F}_{p,\infty}} \to 0. \end{aligned}$$

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Letting $m \to \infty$, we have

$$\begin{aligned} E_n^{(p),F_j}(u) &\le C_4 \sum_{k=n}^{\infty} \rho^{\delta(n-k)} \rho^{-2\alpha k} \int_{K^F} \int_{B(x,\rho^k)} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x) \\ &= C_4 \sum_{k=n}^{\infty} \rho^{\delta(n-k)} \rho^{-2\alpha k} \sum_{l=k}^{\infty} I_l^F(u) = C_4 \sum_{k=0}^{\infty} \rho^{-\delta k} \rho^{-2\alpha(n+k)} \sum_{l=k}^{\infty} I_{l+n}^F(u) \\ &= C_4 \sum_{l=0}^{\infty} \left(\sum_{k=0}^{l} \rho^{-\delta k} \rho^{-2\alpha(n+k)} \right) I_{l+n}^F(u) \\ &= C_4 \sum_{l=0}^{\infty} \left(\sum_{k=0}^{l} \rho^{\delta(l-k)} \rho^{2\alpha(l-k)} \right) \rho^{-\delta l} \rho^{-2\alpha(l+n)} I_{l+n}^F(u) \\ &\le C_5 \sum_{l=0}^{\infty} \rho^{-\delta l} \rho^{-2\alpha(l+n)} I_{l+n}^F(u) = C_5 \sum_{k=0}^{\infty} \rho^{-\delta k} \rho^{-2\alpha(k+n)} I_{k+n}^F(u). \end{aligned} \tag{4·28}$$

Letting $j \to \infty$, we have

$$\begin{aligned} \mathcal{E}_n^{\sigma,F}(u) &= \rho^{-(p\sigma-\alpha)n} E_n^{(p),F}(u) \le C_5 \sum_{k=0}^{\infty} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I_{k+n}^F(u) \\ &\le C_5 \sum_{k=0}^{\infty} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} \int_{K^F} \int_{B(x,\rho^{n+k})} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x) \\ &\le C_5 \sum_{k=0}^{\infty} \rho^{(p\sigma+\alpha-2\alpha-\delta)k} \sup_{k\ge 0} \Phi_u^{\sigma,F}(\rho^{n+k}) \\ &= C_5 \left(\sum_{k=0}^{\infty} \rho^{(p\sigma-\alpha-\delta)k} \right) \sup_{k\ge 0} \Phi_u^{\sigma,F}(\rho^{n+k}) = \frac{C_5}{1-\rho^{p\sigma-\alpha-\delta}} \sup_{k\ge 0} \Phi_u^{\sigma,F}(\rho^{n+k}), \end{aligned}$$

where the second inequality follows from enlarging the integral region.

We need the following proposition to manage the transitions between different levels of vertex energies.

PROPOSITION 4·10. *Let K be a connected homogeneous p.c.f. self-similar set, then there exists $C > 0$ such that for all $u \in L^p(K, \mu)$ and nonnegative integer n,*

$$E_n^{(p)}(u) \le C E_{n+1}^{(p)}(u).$$

Moreover, there exists $C > 0$ such that for all $u \in L^p(K^F, \mu^F)$,

$$E_n^{(p),F}(u) \le C E_{n+1}^{(p),F}(u). \tag{4·29}$$

Proof. We start with the basic case $n = 0$. For any $x, y \in V_0$, we can fix a vertex-disjoint path $x = z_1(x,y), z_2(x,y), \ldots, z_k(x,y) = y$ such that $z_j(x,y)$ and $z_{j+1}(x,y)$ $(1 \le j \le k)$ belong to the same V_w for some $w \in W_1$, and $2 \le k \le |V_1|$. Using Jensen's inequality,

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$$|u(x)-u(y)|^p \le (k-1)^{p-1}\sum_{j=1}^{k-1}|u(z_j(x,y))-u(z_{j+1}(x,y))|^p$$
$$\le |V_1|^{p-1}\sum_{j=1}^{k-1}|u(z_j(x,y))-u(z_{j+1}(x,y))|^p.$$

It follows that

$$E_0^{(p)}(u) \le \sum_{x,y\in V_0}|V_1|^{p-1}\sum_{j=1}^{k-1}|u(z_j(x,y))-u(z_{j+1}(x,y))|^p$$
$$\le |V_0|^2|V_1|^{p-1}\sum_{a,b\in V_w,|w|=1}|u(a)-u(b)|^p$$
$$= |V_0|^2|V_1|^{p-1}E_1^{(p)}(u),$$

since for each pair $(a,b)\in V_w$ with $|w|=1$, there exists at most one $j\in\{1,2,...k\}$ (depending on (x,y)) with $(z_j(x,y),z_{j+1}(x,y))=(a,b)$ for any $x,y\in V_0$. Thus, we show the desired for $n=0$ with $C:=|V_0|^2|V_1|^{p-1}$ for any u. The rest simply follows from that

$$E_n^{(p)}(u)=\sum_{x,y\in V_w,|w|=n}|u(x)-u(y)|^p=\sum_{|w|=n}E_0^{(p)}(u\circ\phi_w)$$
$$\le C\sum_{|w|=n}E_1^{(p)}(u\circ\phi_w)=C\sum_{a,b\in V_{wi},|w|=n,|i|=1}|u(a)-u(b)|^p=CE_{n+1}^{(p)}(u).$$

The second inclusion follows from the definition of K^F.

In [**23**, theorem 1·4], for a homogeneous p.c.f. self-similar set K, we established the equivalence

$$\sup_{n\ge0}\rho^{-n(p\sigma-\alpha)}E_n^{(p)}(u)\asymp[u]^p_{B^\sigma_{p,\infty}(K)}\quad(p\sigma>\alpha),\tag{4·30}$$

where $E_n^{(p)}(u)$ is from (1·17).

Using Lemma 4·7 and Lemma 4·9, we immediately derive such equivalence for K^F. Define the discrete local p-energy for K^F by

$$\mathcal{E}^{\sigma,F}_{p,\infty}(u):=\sup_{n\ge0}\rho^{-n(p\sigma-\alpha)}E_n^{(p),F}(u)=\sup_{n\ge0}\mathcal{E}_n^{\sigma,F}(u).$$

COROLLARY 4·11. *Let K^F be a fractal glue-up. If $\sigma>\alpha/p$, then for all $u\in B^{\sigma,F}_{p,\infty}$,*

$$\limsup_{n\to\infty}\mathcal{E}_n^{\sigma,F}(u)\asymp\limsup_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n),\ \mathcal{E}^{\sigma,F}_{p,\infty}(u)\asymp[u]^p_{B^{\sigma,F}_{p,\infty}}.\tag{4·31}$$

Proof. Taking limsup and sup (of n) on both sides of (4·16) and (4·25), we have

$$\limsup_{n\to\infty}\rho^{-n(p\sigma+\alpha)}I^F_{\infty,n}(u)\le C\limsup_{n\to\infty}\mathcal{E}_n^{\sigma,F}(u)\le C'\limsup_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n),\tag{4·32}$$

$$\sup_{n\ge0}\rho^{-n(p\sigma+\alpha)}I^F_{\infty,n}(u)\le C\sup_{n\ge0}\mathcal{E}_n^{\sigma,F}(u).\tag{4·33}$$

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By (4·23), we have

$$\limsup_{n\to\infty} \rho^{-n(p\sigma+\alpha)} I^F_{\infty,n}(u) \geq \rho^{p\sigma+\alpha} C_H^{p\sigma+\alpha} \limsup_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n),$$

thus showing

$$\limsup_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \asymp \limsup_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n)$$

by (4·32). On the other hand, we have from (4·10) and (4·5) that

$$\sup_{n\geq 0} \rho^{-n(p\sigma+\alpha)} I^F_{\infty,n}(u) = C_H^{p\sigma+\alpha} \sup_{n\geq 0} \Phi_u^{\sigma,F}(C_H\rho^n) = C_H^{p\sigma+\alpha} [u]^p_{B^{\sigma,F}_{p,\infty}}. \tag{4·34}$$

Since $C_H \in (0,1)$, assume that $\rho^{s+1} \leq C_H < \rho^s$ for some nonnegative integer s. Replacing n by $n+s+1$ in (4·25), we obtain

$$\mathcal{E}^{\sigma,F}_{n+s+1}(u) \leq C \sup_{k\geq 0} \Phi_u^{\sigma,F}(\rho^{n+s+1+k}) \leq C' \sup_{k\geq 0} \Phi_u^{\sigma,F}(C_H\rho^{n+k}). \tag{4·35}$$

By (4·29),

$$\begin{aligned}\mathcal{E}^{\sigma,F}_{n+s+1}(u) &= \rho^{-(n+s+1)(p\sigma-\alpha)} E^{(p),F}_{n+s+1}(u) \geq C^{-s-1}\rho^{-(n+s+1)(p\sigma-\alpha)} E^{(p),F}_n(u)\\ &= C^{-s-1}\rho^{-(s+1)(p\sigma-\alpha)} \mathcal{E}^{\sigma,F}_n(u),\end{aligned}$$

thus by (4·35),

$$\mathcal{E}_n^{\sigma,F}(u) \leq C'' C_H^{p\sigma-\alpha} \sup_{k\geq 0} \Phi_u^{\sigma,F}(C_H\rho^{n+k}). \tag{4·36}$$

We conclude from (4·33), (4·34) and (4·36) that

$$\mathcal{E}^{\sigma,F}_{p,\infty}(u) \asymp [u]^p_{B^{\sigma,F}_{p,\infty}}.$$

The proof is complete.

4·2. *Property (E) holds for nested fractals*

For a set V, let $l(V) = \{u : u \text{ maps V into } \mathbb{R}\}$.

Definition 4·12 ([**23**, definition 3·1]). We say that a connected homogeneous p.c.f. self-similar set K satisfies property (E), if there exist $\sigma > \alpha/p$ and a positive constant C such that:

(i) for any $u \in B^\sigma_{p,\infty}$ and for all $n \geq 0$, $\mathcal{E}_0^\sigma(u) \leq C\mathcal{E}_n^\sigma(u)$;
(ii) for any $u \in \ell(V_0)$, there exists an extension $\tilde{u} \in B^\sigma_{p,\infty}$.

We know from [**23**, remark 3·2] that, if property (E) holds with σ, then $\sigma = \sigma_p^*(K)$, and for all $n \geq m$, $\mathcal{E}_m^{\sigma_p^*}(u) \leq C\mathcal{E}_n^{\sigma_p^*}(u)$. Thus we omit the index σ. The following proposition shows the importance of property (E).

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PROPOSITION 4·13. *If a connected homogeneous p.c.f. self-similar set K satisfies property (E), then*

$$\sigma_p^*(K) = \sigma_p^\#(K) = \sigma_p^\#(K^F). \tag{4·37}$$

Moreover, K^F *satisfies property* $(VE)_{\sigma_p^\#(K)}$*, and* $B_{p,\infty}^{\sigma_p^\#(K),F}$ *contains non-constant functions.*

Proof. The fact $\sigma_p^*(K) = \sigma_p^\#(K)$ follows from [**23**, the proof of Proposition 3·4]. It is also clear that $\sigma_p^\#(K) \geq \sigma_p^\#(K^F)$, since any non-constant function $u \in l(K^F)$ in $B_{p,\infty}^{\sigma,F}$ naturally gives a non-constant function in $B_{p,\infty}^{\sigma}(K)$ for any $\sigma > 0$: just restrict u to a tile $f(K)$ where it is non-constant (for some $f \in F$).

For the reverse inclusion $\sigma_p^\#(K) \leq \sigma_p^\#(K^F)$ to hold, we use property (E)(ii) to find a non-constant function in $B_{p,\infty}^{\sigma_p^\#(K),F}$. To see this, we pick a contraction $f \in F$, and determine the values of u on $f(V_1)$ by $u(f(V_0)) = 0$ while $u(f(V_1 \setminus V_0)) = 1$. We can then extend u from $f(V_1)$ to $f(K)$ and make sure that $u \circ f \in B_{p,\infty}^{\sigma_p^\#(K)}$ by property (E)(ii), and simply define $u = 0$ outside of $f(K)$ on K^F. Then u is an extended non-trivial function in $B_{p,\infty}^{\sigma_p^\#(K),F}$ since by (4·31) and (4·30),

$$[u]^p_{B_{p,\infty}^{\sigma_p^\#(K),F}} \asymp \mathcal{E}_{p,\infty}^{\sigma_p^\#(K),F}(u) = \mathcal{E}_{p,\infty}^{\sigma_p^\#(K)}(u \circ f) \asymp [u \circ f]^p_{B_{p,\infty}^{\sigma_p^\#(K)}(K)} < \infty.$$

Thus we show (4·37) and that $B_{p,\infty}^{\sigma_p^\#(K),F}$ contains non-constant functions.

Finally, since K satisfies property (E), that is for all n,

$$\mathcal{E}_n^{\sigma_p^\#}(u) \leq C \liminf_{k\to\infty} \mathcal{E}_k^{\sigma_p^\#}(u),$$

summing over F gives that for all n,

$$\mathcal{E}_n^{\sigma_p^\#,F}(u) \leq C \sum_{f\in F} \liminf_{k\to\infty} \mathcal{E}_k^{\sigma_p^\#}(u \circ f) \leq C \liminf_{k\to\infty} \sum_{f\in F} \mathcal{E}_k^{\sigma_p^\#}(u \circ f) = C \liminf_{k\to\infty} \mathcal{E}_k^{\sigma_p^\#,F}(u),$$

where we use Fatou's lemma to obtain the desired.

When property (E) fails, $\sigma_p^\# = \sigma_p^*$ does not always hold for connected homogeneous p.c.f. self-similar sets (see [**28**]). Moreover, we mention in passing that, the critical domain $B_{p,\infty}^{\sigma_p^*}$ can be trivial for some metric measure spaces (bounded or unbounded).

We present the following key lemma for *nested fractals*, a class of connected homogeneous p.c.f. self-similar sets defined in [**38**] satisfying conditions $(A\text{-}0) \sim (A\text{-}3)$ and $|V_0| \geq 2$ therein.

LEMMA 4·14. *For a nested fractal, property (E) holds and* $\sigma_p^\# > \alpha/p$ *for all* $1 < p < \infty$.

Proof. In this proof, we use the same terminology and notions as in [**16**]. Let K be a nested fractal. We fix the components r_i of $\mathbf{r}$ in [**16**, theorem 6·3] to be the same number r, so it is '$\mathcal{G}$-symmetric' (see [**16**, section 3] for definition). Then [**16**, theorem 6·3] states that, 'condition (**A**)' (see [**16**, the beginning of Section 4] for definition) holds for affine nested fractals, including K. By multiplying $\mathbf{r}$ with a constant (still denoted by $\mathbf{r}$), [**16**, theorem 4·2]

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guarantees that

$$\mathcal{T}E = E \tag{4·38}$$

(see [16, definition 2·8, definition 3·1] for related definitions), thus showing 'condition $(\mathbf{A}')$' (see [16, the beginning of Section 5]). So by [16, lemma 5·4], we can fix

$$r < 1 \tag{4·39}$$

to further satisfy condition $(\mathbf{A}')$ on K.

The equivalence of E and $E_0^{(p)}$ (for functions $u \in l(V_0)$, see the statement of [16, theorem 5·1]) is stated in the proof of [16, proposition 5·3 (b)], which implies that

$$\Lambda^n E(u) \asymp \Lambda^n E_0^{(p)}(u) = r^{-n} E_n^{(p)}(u) \tag{4·40}$$

for all $n \geq 0$ and all functions $u \in l(V_n)$ by the definition of Λ in [16, definition 3·1] and $E_n^{(p)}$ given in (1·17).

Now fix

$$\sigma = \frac{\log_\rho r + \alpha}{p}, \tag{4·41}$$

so that $r = \rho^{p\sigma-\alpha}$.

Property (E)(i) holds with this σ, by using the monotonicity property in [16, proposition 5·3 (a)] and (4·40) that

$$E_0^{(p)}(u) \asymp E(u) \leq \Lambda E(u) \leq \cdots \leq \Lambda^n E(u) \asymp r^{-n} E_n^{(p)}(u) = \rho^{-n(p\sigma-\alpha)} E_n^{(p)}(u) = \mathcal{E}_n^\sigma(u). \tag{4·42}$$

To see Property (E)(ii), we show the existence of piecewise harmonic functions (see definition in [16, section 5·1]) using a standard iterative construction as in [31]. The iteration part is that, for any boundary value of a cell $u_w \in l(f_w(V_0))$, by (4·38), we can take its next-level harmonic extension $\tilde{u}_w \in l(f_w(V_1))$ such that $\Lambda E(\tilde{u}_w) = E(u_w)$ and $\tilde{u}_w|_{f_w(V_0)} = u_w$. The iteration starts from V_0 to V_1, and from each level-1 cell to level-2 cells in the above way, then finally to V_*. Such an extension satisfies the definition of piecewise harmonic functions in [16, section 5·1], so we know by [16, section 5·2] that it embeds into $C(K)$ and this gives an extension to K instead of V_*.

Therefore, we conclude from above that K satisfies property (E) with σ defined as in (4·41). By (4·37), we have $\sigma = \sigma_p^\#$, thus

$$r = \rho^{p\sigma-\alpha} = \rho^{p\sigma_p^\#-\alpha}. \tag{4·43}$$

The inclusion $\sigma_p^\# > \alpha/p$ follows from (4·39).

Remark 4·15. This lemma also indicates that, $E_{p,\infty}^{\sigma_p^\#}(u)$, $[u]^p_{B_{p,\infty}^{\sigma_p^\#}}$ and the homogeneous discrete p-energy in [16, theorem 5·1] are equivalent for a nested fractal K. It is known from [4, theorem 8·18] that K admits a heat kernel satisfying (1·9), so (3·1) holds by Lemma 3·1. The critical exponents $\sigma_p^\#(K) = \sigma_p^*(K)$ due to property (E) by Proposition 4·13. Using $\Lambda^n E_0^{(p)}(u) \asymp \Lambda^n E(u)$ in (4·40), the p-energy in [16, theorem 5·1] with $r_i = r$ is equivalent to $\lim_{n\to\infty} \Lambda^n E(u)$ (which exists by [16, proposition 5·3]), where $r = \rho^{p\sigma_p^\#(K)-\alpha}$ is given by

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(4·43). By (4·30) and (4·42),

$$[u]^p_{B^{\sigma\#}_{p,\infty}} \asymp \lim_{n\to\infty} \Lambda^n E(u).$$

4·3. *Equivalence between* $(VE)_\sigma$ *and* $(NE)_\sigma$

Our goal is to show the equivalence between $(VE)_\sigma$ and $(NE)_\sigma$ for K^F. It is quite different and more delicate to obtain the lower limit equivalence, compared with the equivalence in Corollary 4·11 using the upper limit.

LEMMA 4·16. *If a fractal glue-up* K^F *satisfies* $(\widetilde{VE})_\sigma$ *with* $\sigma > \alpha/p$, *then there exists* $C > 0$ *such that for all* $u \in B^{\sigma,F}_{p,\infty}$,

$$\liminf_{n\to\infty} \mathcal{E}^{\sigma,F}_n(u) \le C \liminf_{n\to\infty} \Phi^{\sigma,F}_u(\rho^n). \tag{4·44}$$

Proof. Since $C_H \in (0,1)$, we can assume that $\rho^{s+1} \le C_H < \rho^s$ for some $s \in \mathbb{N}$. For all $n > 0$, we have by (4·24) that

$$I^F_{\infty,n}(u) \ge \int_{K^F} \int_{B(x,\rho^{n+s+1})} |u(x)-u(y)|^p d\mu^F(y) d\mu^F(x) \ge I^F_{n+s+1}(u). \tag{4·45}$$

Note that there exists a positive integer n_0 such that

$$\sup_{k\ge n_0} \mathcal{E}^{\sigma,F}_k(u) \le 2 \limsup_{n\to\infty} \mathcal{E}^{\sigma,F}_n(u).$$

By (4·45), we have for all $n \ge n_0 + s + 1$ that

$$\begin{aligned} I^F_n(u) &\le I^F_{\infty,n-s-1}(u) \\ &\le C\rho^{(n-s-1)(p\sigma+\alpha)} \sup_{k\ge n-s-1} \mathcal{E}^{\sigma,F}_k(u) \quad \text{(using (4·46))} \\ &\le CC_H^{-(p\sigma+\alpha)} \rho^{n(p\sigma+\alpha)} \sup_{k\ge n_0} \mathcal{E}^{\sigma,F}_k(u) \le 2CC_H^{-(p\sigma+\alpha)} \rho^{n(p\sigma+\alpha)} \limsup_{n\to\infty} \mathcal{E}^{\sigma,F}_n(u) \\ &\le C_1 \rho^{n(p\sigma+\alpha)} \liminf_{n\to\infty} \mathcal{E}^{\sigma,F}_n(u) \quad \text{(by property } (\widetilde{VE})_\sigma). \end{aligned} \tag{4·46}$$

By Lemma 4·9, for any positive integer $L \ge n_0 + s + 1$ and $0 < \delta < p\sigma - \alpha$,

$$\begin{aligned} \mathcal{E}^{\sigma,F}_n(u) &\le C_2 \sum_{k=0}^{\infty} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I^F_{k+n}(u) \\ &= C_2 \sum_{k=0}^{L} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I^F_{k+n}(u) + C_2 \sum_{k=L}^{\infty} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I^F_{k+n}(u) \\ &\le C_2 \sum_{k=0}^{L} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I^F_{k+n}(u) + C_1 C_2 \sum_{k=L}^{\infty} \rho^{(p\sigma-\alpha-\delta)k} \liminf_{n\to\infty} \mathcal{E}^{\sigma,F}_n(u); \\ &= C_2 \left(\sum_{k=0}^{L} \rho^{(-2\alpha-\delta)k} \rho^{-(p\sigma+\alpha)n} I^F_{k+n}(u) + \frac{C_1 \rho^{(p\sigma-\alpha-\delta)L}}{1-\rho^{(p\sigma-\alpha-\delta)}} \liminf_{n\to\infty} \mathcal{E}^{\sigma,F}_n(u) \right), \end{aligned}$$

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where we use (4·46) in the third line. Taking $\liminf_{n\to\infty}$ in the right-hand side above, we have

$$C_3 \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \le \liminf_{n\to\infty} \sum_{k=0}^{L} \rho^{-(2\alpha+\delta)k}\rho^{-(p\sigma+\alpha)n} I_{k+n}^F(u),$$

where $C_3 := 1/C_2 - C_1\rho^{(p\sigma-\alpha-\delta)L}/(1-\rho^{(p\sigma-\alpha-\delta)})$.

Fix a large integer L such that $C_3 > 0$, then

$$\begin{aligned} C_3\rho^{(2\alpha+\delta)L} \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) &\le \rho^{(2\alpha+\delta)L} \liminf_{n\to\infty} \left(\sum_{k=0}^{L} \rho^{-n(p\sigma+\alpha)}\rho^{-(2\alpha+\delta)k} I_{n+k}^F(u) \right) \\ &\le \liminf_{n\to\infty} \rho^{-n(p\sigma+\alpha)} \sum_{k=0}^{L} I_{n+k}^F(u) \le \liminf_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n), \end{aligned}$$

thus showing (4·44).

Next, we show the reverse inequality of (4·44) by using (4·29) under property $(\widetilde{NE})_\sigma$.

LEMMA 4·17. *If property $(\widetilde{NE})_\sigma$ holds for a fractal glue-up K^F with $\sigma > \alpha/p$, then there exists $C > 0$ such that for all $u \in B_{p,\infty}^{\sigma,F}$,*

$$\liminf_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n) \le C \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u). \tag{4·47}$$

Proof. By (4·21), we have

$$\begin{aligned} \rho^{-n(p\sigma+\alpha)} I_{m,n}^F(u) &\le C\rho^{-n(p\sigma-\alpha)} \sum_{k=n}^{m} \rho^{k(p\sigma-\alpha)}\rho^{-k(p\sigma-\alpha)} E_k^{(p),F}(u) \\ &\le C \sum_{k=n}^{m} \rho^{(k-n)(p\sigma-\alpha)} \mathcal{E}_k^{\sigma,F}(u) \le C \sum_{k=0}^{\infty} \rho^{k(p\sigma-\alpha)} \mathcal{E}_{k+n}^{\sigma,F}(u). \end{aligned} \tag{4·48}$$

Note that there exists $n_0 > 0$ such that for all positive integer $L > n_0$,

$$\begin{aligned} \sup_{n\ge L} \mathcal{E}_n^{\sigma,F}(u) &\le 2 \limsup_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \\ &\asymp \limsup_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n) \text{ (by Corollary 4.11)} \\ &\le C \liminf_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n) \text{ (by property } (\widetilde{NE})_\sigma). \end{aligned}$$

It follows from (4·48) that

$$\begin{aligned} \rho^{-n(p\sigma+\alpha)} I_{m,n}^F(u) &\le C \sum_{k=0}^{\infty} \rho^{k(p\sigma-\alpha)} \mathcal{E}_{k+n}^{\sigma,F}(u) \\ &= C \sum_{k=0}^{L} \rho^{k(p\sigma-\alpha)} \mathcal{E}_{k+n}^{\sigma,F}(u) + C \sum_{k=L+1}^{\infty} \rho^{k(p\sigma-\alpha)} \mathcal{E}_{k+n}^{\sigma,F}(u) \\ &\le C \sum_{k=0}^{L} \rho^{k(p\sigma-\alpha)} \mathcal{E}_{k+n}^{\sigma,F}(u) + C \sum_{k=L+1}^{\infty} \rho^{k(p\sigma-\alpha)} \sup_{n\ge L} \mathcal{E}_n^{\sigma,F}(u) \end{aligned}$$

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$$\leq C\sum_{k=0}^{L}\rho^{k(p\sigma-\alpha)}\mathcal{E}_{k+n}^{\sigma,F}(u)+C'\sum_{k=L+1}^{\infty}\rho^{k(p\sigma-\alpha)}\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n)$$
$$=C\sum_{k=0}^{L}\rho^{k(p\sigma-\alpha)}\mathcal{E}_{k+n}^{\sigma,F}(u)+C'\frac{\rho^{(p\sigma-\alpha)(L+1)}}{1-\rho^{p\sigma-\alpha}}\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n).$$

Using Lemma 4·5, we have

$$\rho^{-n(p\sigma+\alpha)}I_{\infty,n}^{F}(u)\leq\rho^{-n(p\sigma+\alpha)}\liminf_{m\to\infty}I_{m,n}^{F}(u)$$
$$\leq C\sum_{k=0}^{L}\rho^{k(p\sigma-\alpha)}\mathcal{E}_{k+n}^{\sigma,F}(u)+\frac{C'\rho^{(p\sigma-\alpha)(L+1)}}{1-\rho^{p\sigma-\alpha}}\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n).$$

Combining this with (4·22), we have

$$\rho^{p\sigma+\alpha}C_H^{p\sigma+\alpha}\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n)\leq C\liminf_{n\to\infty}\sum_{k=0}^{L}\rho^{k(p\sigma-\alpha)}\mathcal{E}_{k+n}^{\sigma,F}(u)$$
$$+\frac{C'\rho^{(p\sigma-\alpha)(L+1)}}{1-\rho^{p\sigma-\alpha}}\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n).$$

For sufficiently large L ($>n_0$), we have

$$C'':=\frac{1}{C}\left(\rho^{p\sigma+\alpha}C_H^{p\sigma+\alpha}-\frac{C'\rho^{(p\sigma-\alpha)(L+1)}}{1-\rho^{p\sigma-\alpha}}\right)>0.$$

Then for this L, we obtain

$$C''\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n)\leq\liminf_{n\to\infty}\sum_{k=0}^{L}\rho^{k(p\sigma-\alpha)}\mathcal{E}_{k+n}^{\sigma,F}(u)$$
$$=\liminf_{n\to\infty}\sum_{k=0}^{L}\rho^{k(p\sigma-\alpha)}\rho^{-(k+n)(p\sigma-\alpha)}E_{k+n}^{(p),F}(u)$$
$$=\liminf_{n\to\infty}\sum_{k=0}^{L}\rho^{-n(p\sigma-\alpha)}E_{k+n}^{(p),F}(u).\tag{4·49}$$

Using (4·29), there exists $c>0$ such that for all positive integer n and $u\in B_{p,\infty}^{\sigma,F}$,

$$E_n^{(p),F}(u)\leq cE_{n+1}^{(p),F}(u).$$

Applying the above inequality to (4·49), we have

$$C''\liminf_{n\to\infty}\Phi_u^{\sigma,F}(\rho^n)\leq\liminf_{n\to\infty}C_L\,\rho^{-(n+L)(p\sigma-\alpha)}E_{n+L}^{(p),F}(u)$$
$$=C_L\liminf_{n\to\infty}\mathcal{E}_{n+L}^{(p),F}(u)=C_L\liminf_{n\to\infty}\mathcal{E}_n^{(p),F}(u),$$

where $C_L=((1-c^L)/(1-c))\rho^{L(p\sigma-\alpha)}$ does not depend on n, which gives (4·47).

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Proof of Theorem 1·7. Let $\sigma > \alpha/p$. By (4·31), (4·44) and (4·8), $(\widetilde{VE})_\sigma$ implies $(\widetilde{NE})_\sigma$ since

$$\limsup_{r\to 0} \Phi_u^{\sigma,F}(r) \le C \limsup_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \le C' \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \le C'' \liminf_{r\to 0} \Phi_u^{\sigma,F}(r). \tag{4·50}$$

By (4·31), (4·8) and (4·47), $(\widetilde{NE})_\sigma$ implies $(\widetilde{VE})_\sigma$ since

$$\limsup_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \le C_1 \limsup_{r\to 0} \Phi_u^{\sigma,F}(r) \le C_1' \liminf_{r\to 0} \Phi_u^{\sigma,F}(r) \le C_1'' \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u). \tag{4·51}$$

Thus, by (4·50) and (4·51), we show Theorem 1·7 (i).

By (4·31) and (4·44) adjoint with $(VE)_\sigma \Rightarrow (\widetilde{VE})_\sigma$, we immediately have from (4·6), (4·8) that

$$\sup_{r\in(0,C_H)} \Phi_u^{\sigma,F}(r) \le C_2 \sup_{n\ge 0} \mathcal{E}_n^{\sigma,F}(u) \le C_2' \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \le C_2'' \liminf_{r\to 0} \Phi_u^{\sigma,F}(r). \tag{4·52}$$

Similarly, by (4·31) and (4·47) adjoint with the fact $(NE)_\sigma \Rightarrow (\widetilde{NE})_\sigma$, we immediately have

$$\sup_{n\ge 0} \mathcal{E}_n^{\sigma,F}(u) \le C_3 \sup_{n\ge 0} \Phi_u^{\sigma,F}(C_H\rho^n) \le C_3' \liminf_{n\to\infty} \Phi_u^{\sigma,F}(\rho^n) \le C_3'' \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u). \tag{4·53}$$

Thus, by (4·52) and (4·53), we show Theorem 1·7 (ii).

Remark 4·18. The definition of discrete energies in (1·18) uses point-wise values of functions as opposed to cell-averaged values in [**37**, **39**]. When $p\sigma > \alpha$, the domain of p-energy $\mathcal{E}_{p,\infty}^{\sigma,F}$ is a subset of $C(K^F)$, and both definitions are equivalent for a connected homogeneous p.c.f. self-similar set by modifying the arguments in [**32**, lemma 3·1] and [**46**, section 5].

4·4. *Consequences of $(VE)_\sigma$: BBM type characterisation and Gagliardo-Nirenberg inequality*

Let us state the following embedding inequality given by Baudoin, using our notions.

LEMMA 4·19 ([**11**, theorem 4·3]). *Suppose that property $(NE)_{\sigma_p^\#}$ holds with $R_0 = \infty$ for the metric measure space (M, d, μ), and there exists $R > 0$ such that $\inf_{x\in M} \mu(B(x, R)) > 0$. When $p\sigma_p^\# \ne \alpha_1$, where α_1 is from (1·3), let $q = p\alpha_1/(\alpha_1 - p\sigma_p^\#)$. For $r, s \in (0, \infty]$, $\theta \in (0, 1]$ satisfying*:

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s},$$

there exists a constant $C > 0$ such that for any $f \in B_{p,\infty}^{\sigma_p^\#}$,

$$\|f\|_r \le C(\|f\|_p + [f]_{B_{p,\infty}^{\sigma_p^\#}})^\theta \|f\|_s^{1-\theta}. \tag{4·54}$$

By Theorem 1·5 and Theorem 1·7, we have the following corollary.

COROLLARY 4·20. *Suppose that K^F is a fractal glue-up that admits a heat kernel satisfying (1·9), where K is a connected homogeneous p.c.f. self-similar set. Let $R_0 = C_H$. Then properties $(\widetilde{KE})_\sigma$, $(\widetilde{NE})_\sigma$, $(\widetilde{VE})_\sigma$ are equivalent with the same $\sigma > \alpha/p$, and properties $(KE)_{\sigma_p^\#}$,$(NE)_{\sigma_p^\#}$,$(VE)_{\sigma_p^\#}$ are equivalent when $\sigma_p^\# > \alpha/p$ on K^F.*

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The following theorem can be regarded as a summary of our paper when the underlying space is a fractal.

THEOREM 4·21. *Let K be a connected homogeneous p.c.f. self-similar set with property (E), and let K^F be a fractal glue-up. Then K^F satisfies $(NE)_{\sigma_p^\#(K)}$ with $R_0 = C_H$ and $B^{\sigma_p^\#(K^F)}_{p,\infty}(K^F) := B^{\sigma_p^\#}_{p,\infty}(K^F)$ contains non-constant functions. Also, BBM type characterisation (2·7) holds for K^F with $R_0 = C_H$, that is, for $R_0 = C_H$, there exists a positive constant C such that for all $u \in B^{\sigma_p^\#}_{p,\infty}(K^F)$,*

$$C^{-1}[u]^p_{B^{\sigma_p^\#}_{p,\infty}(K^F)} \le \liminf_{\sigma\uparrow\sigma_p^\#(K^F)} \left(\sigma_p^\#(K^F) - \sigma\right)[u]^p_{B^\sigma_{p,p}(K^F)} \le \limsup_{\sigma\uparrow\sigma_p^\#(K^F)} \left(\sigma_p^\#(K^F) - \sigma\right)[u]^p_{B^\sigma_{p,p}(K^F)} \le C[u]^p_{B^{\sigma_p^\#}_{p,\infty}(K^F)}. \tag{4·55}$$

Furthermore, if K is a nested fractal and $K^F = K_\infty$, where K_∞ is defined in Example 4·3, then $(NE)_{\sigma_p^\#(K)}$ and $(KE)_{\sigma_p^\#(K)}$ hold for K_∞ with $R_0 = \infty$, and the Gagliardo-Nirenberg inequality (4·54) holds.

Proof. By Proposition 4·13, K^F satisfies $(VE)_{\sigma_p^\#(K)}$ and the critical domain $B^{\sigma_p^\#}_{p,\infty}(K^F)$ contains non-constant functions, where $\sigma_p^\# = \sigma_p^\#(K) = \sigma_p^\#(K^F)$. Thus $(NE)_{\sigma_p^\#}$ holds true for K^F by Theorem 1·7(ii) with $R_0 = C_H$, and (4·55) holds by Theorem 2·4.

Since a nested fractal K satisfies property (E) by Lemma 4·14, these inclusions hold for nested fractals. Next, we state how to upgrade $R_0 = C_H$ to $R_0 = \infty$ for $(NE)_{\sigma_p^\#}$ on K_∞.

For $n \in \mathbb{N}$, define the scaling function $g_n : K^F \to \rho^{-n}K^F$ by $g_n(x) = \rho^{-n}x$. Due to the global self-similarity of K_∞, we have

$$g_n(K_\infty) = K_\infty. \tag{4·56}$$

For any $u \in B^{\sigma_p^\#,F}_{p,\infty}$, clearly $u \circ g_n \in B^{\sigma_p^\#,F}_{p,\infty}$. By definition, for all $r > 0$,

$$\begin{aligned}
\Phi^{\sigma_p^\#,F}_{u\circ g_n}(\rho^n r) &= (\rho^n r)^{-p\sigma_p^\# - \alpha} \int_{K_\infty}\int_{B(x,\rho^n r)} |u(g_n(x)) - u(g_n(y))| d\mu^F(y) d\mu^F(x) \\
&= (\rho^n r)^{-p\sigma_p^\# - \alpha} \int_{K_\infty}\int_{B(g_n(x),r)} |u(g_n(x)) - u(y)| d(\mu^F \circ g_n^{-1})(y) d\mu^F(x) \\
&= (\rho^n r)^{-p\sigma_p^\# - \alpha} \int_{g_n(K_\infty)}\int_{B(x,r)} |u(x) - u(y)| d\mu^F(\rho^n y) d\mu^F(\rho^n x) \\
&\asymp (\rho^n r)^{-p\sigma_p^\# - \alpha} \rho^{2n\alpha} \int_{K_\infty}\int_{B(x,r)} |u(x) - u(y)| d\mu^F(y) d\mu^F(x) \\
&= \rho^{-n(p\sigma_p^\# - \alpha)} \Phi^{\sigma_p^\#,F}_u(r),
\end{aligned} \tag{4·57}$$

where we use (4·56) and that μ^F is α-regular in the fourth line. For any $u \in B^{\sigma_p^\#,F}_{p,\infty}$, there exists $n < \infty$ such that

$$\sup_{r\in(0,\infty)} \Phi^{\sigma_p^\#,F}_u(r) \le 2 \sup_{r\in(0,C_H\rho^{-n})} \Phi^{\sigma_p^\#,F}_u(r).$$

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Combined with (4·57),

$$\sup_{r\in(0,\infty)} \Phi_u^{\sigma_p^\#,F}(r) \le C \sup_{r'\in(0,C_H)} \Phi_{u\circ g_n}^{\sigma_p^\#,F}(r')\rho^{n(p\sigma_p^\#-\alpha)}. \tag{4·58}$$

As $(NE)_{\sigma_p^\#}$ holds for $R_0 = C_H$,

$$\begin{aligned}\sup_{r'\in(0,C_H)} \Phi_{u\circ g_n}^{\sigma_p^\#,F}(r')\rho^{n(p\sigma_p^\#-\alpha)} &\le C \liminf_{r'\to 0} \Phi_{u\circ g_n}^{\sigma_p^\#,F}(r')\rho^{n(p\sigma_p^\#-\alpha)}\\ &= C \liminf_{r'\to 0} \Phi_u^{\sigma_p^\#,F}(\rho^{-n}r') \text{ (by (4·57))}\\ &= C \liminf_{r\to 0} \Phi_u^{\sigma_p^\#,F}(r),\end{aligned}$$

where the constant C does not depend on u and n. Thus $(NE)_{\sigma_p^\#}$ also holds with $R_0 = \infty$ by (4·58). Since μ^F is α-regular, we know by Lemmas 4·14 and 4·19 that (4·54) holds true. Finally, for a nested fractal K, it is known by [**38**] (see also [**21**, theorem 1·1]) that K_∞ admits a heat kernel satisfying (1·9), thus $(KE)_{\sigma_p^\#}$ is verified by Theorem 1·5. The proof is complete.

When $p = 2$, we use heat kernel estimates to verify weak-monotonicity properties in the following corollary, which is analytical (compared with the algebraic resistance-estimate arguments for $(VE)_\sigma$).

COROLLARY 4·22. *When $p = 2$, if a fractal glue-up K^F admits a heat kernel with sub-Gaussian estimates* (1·9), *then $(VE)_{\beta^*/2}$ and $(NE)_{\beta^*/2}$ automatically hold for K^F.*

Proof. By Remark 1·2, property $(KE)_{\beta^*/2}$ automatically holds for K^F when $p = 2$. The conclusion then follows from Corollary 4·20.

We use an example based on [**30**, section 4] to show that Corollaries 4·20 and 4·22 are also meaningful for some non-nested fractal blow-ups. That is, there exists a connected homogeneous p.c.f. self-similar set which is not a nested fractal, such that, its fractal blow-up admits a heat kernel satisfying (1·9).

EXAMPLE 4·23. *Let S be a unit equilateral triangle in $\mathbb{R}^2$. Its vertices are located at $q_0 = (0,0)$, $q_1 = (1,0)$, and $q_2 = (1/2, \sqrt{3}/2)$. Divide S into a mesh of sub-triangles of side length $1/12$, and pick 34 sub-triangles as shown in Figure 1. Let $\{\phi_i\}_{i=1}^{34}$ be the IFS on $\mathbb{R}^2$ that maps S to these 34 sub-triangles with attractor K. Then $K = \bigcup_{i=1}^{34} \phi_i(K)$ is a connected homogeneous p.c.f. self-similar set, which is also a uniformly finitely ramified fractal (u.f.r. fractal for short, see definition in [**30**, section 2]). However, K is not a nested fractal. Its Hausdorff dimension is $\alpha = \dim_H(K) = \log 34/\log 12$, and the α-dimensional Hausdorff measure exists since the open set condition is satisfied. Define $V_0 = \{q_k\}_{k=0}^2$ and $V_{n+1} = \bigcup_{i=1}^{34} \phi_i(V_n)$ as in (4·2). Let K_∞ be the fractal blow-up of K (defined in Example 4·3 as $\bigcup_{l=1}^{\infty} K_l$, where $K_l = 12^l K$). Let d_∞ be the shortest path length between any two points in K_∞. By the global self-similarity of blow-ups, d_∞ is bi-Lipschitz equivalent to the Euclidean metric $d_{\mathbb{R}^2}|_{K_\infty}$.*

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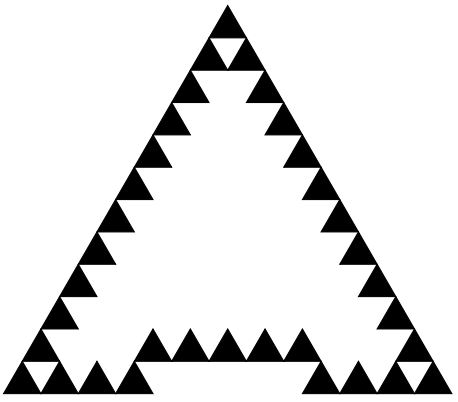


Fig. 1. A uniformly finitely ramified fractal.

Define $V = \bigcup_{i=0}^{\infty} 12^n V_n$ and $E := \{(x, y) \in V \times V : d(x, y) = 1\}$. Consider the graph (V, E) with weights μ satisfying the p_0-condition in [**8**, formula (1·5)]. By substituting each edge in E with an isometric copy of the line segment [0, 1] (referred to as a *cable*), and connecting them in a natural way at the vertices, we obtain an unbounded connected closed set $X_0 \subset \mathbb{R}^2$, known as the *cable system* associated with (V, E). The metric d_0 is defined on $X_0 \times X_0$ by employing the Euclidean distance along each cable, and this is extended to a metric over $X_0 \times X_0$ that is consistent with the graph distance on V. We define $\Phi_0(r) := r \vee r^{\alpha}$ and $\Psi_0(r) := r^2 \vee r^{d_w}$. Denote by m_0 the Hausdorff measure on (X_0, d_0), then we have $m_0(B(x, r)) \asymp \Phi_0(r)$ by [**8**, lemma 2·1], where $B(x, r) := \{z \in X_0 : d_0(x, z) < r\}$. Define a regular Dirichlet form $(\mathcal{E}_0, \mathcal{F}_0)$ corresponding to the cable process on (X_0, d_0, m_0) as in [**8**, section 2], which is conservative and strongly local. By [**30**, corollary 4·14], there exist constants c_1, c_2, c_3 and $c_4 > 0$ such that for all $x, y \in V$, $k \geq d_0(x, y)$

$$\begin{aligned} p_k(x, y) &\leq c_1 k^{-\alpha/d_w} \exp\left(-c_2 \left(\frac{d_0(x, y)^{d_w}}{k}\right)^{1/(d_w-1)}\right), \\ p_k(x, y) + p_{k+1}(x, y) &\geq c_3 k^{-\alpha/d_w} \exp\left(-c_4 \left(\frac{d_0(x, y)^{d_w}}{k}\right)^{1/(d_w-1)}\right). \end{aligned} \tag{4·59}$$

Therefore, (4·59) implies [**8**, formulae (1·12) and (1·13)]. By [**8**, theorem 1·2, Corollary 2·5 and Lemma 2·6], the elliptic Harnack inequality (EHI) and the mean exit time estimate (E_{Ψ_0}) (see [**26**, definition 3·10]) hold for the cable process. Thus the metric measure Dirichlet space $(X_0, d_0, m_0, \mathcal{E}_0, \mathcal{F}_0)$ admits a heat kernel $\{p_t\}_{t>0}$ satisfying $\mathrm{HKE}(\Psi_0)$ by [**26**, theorem 5·15], where the condition $HKE(\Psi)$ (for the metric measure space (X,d,m)) is defined in [**18**, definition 1·1] as follows: there exist $c_1, c_2, c_3, \delta > 0$ and a heat kernel $\{p_t\}_{t>0}$ such that for any $t > 0$,

$$p_t(x, y) \leq \frac{c_1}{V(x, \Psi^{-1}(t))} \exp\left(-c_2 t \Phi\left(c_3 \frac{d(x, y)}{t}\right)\right) \qquad \text{for } m\text{-a.e. } x, y \in X \tag{4·60}$$

$$\text{and } \; p_t(x, y) \geq \frac{c_1^{-1}}{V(x, \Psi^{-1}(t))} \qquad \text{for } m\text{-a.e. } x, y \in X \text{ with } d(x, y) \leq \delta \Psi^{-1}(t), \tag{4·61}$$

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where $V(x,r) := m(\{z \in X : d(z,x) < r\})$ and

$$\Phi(s) := \Phi_\Psi(s) := \sup_{r>0}\left(\frac{s}{r} - \frac{1}{\Psi(r)}\right).$$

Next, we construct a sequence of metric measure spaces $(X_n, d_n, m_n, \mathcal{E}_n, \mathcal{F}_n)$ $(n \in \mathbb{N})$, by

$$X_n = X_0,\ d_n = 12^{-n} d_0 \text{ and } m_n := 12^{-\alpha n} m_0,$$
$$\mathcal{E}_n = 12^{(d_w-\alpha)n}\mathcal{E}_0,\ u, v \in \mathcal{F}_n := \mathcal{F}_0.$$

To apply the pointed Gromov-Hausdorff convergence results in [18], we need to verify that the spaces (X_n, d_n, m_n, q_0) satisfy Conditions (A1)-(A6) in [18, theorem 1·2]. For the scaled sequence (X_n, d_n, m_n), the heat kernel $p_t^{(n)}(x,y)$ is related to the original heat kernel $\{p_t\}_{t>0}$ by

$$p_t^{(n)}(x,y) = 12^{\alpha n} \cdot p_{12^{d_w n} t}(x,y).$$

This scaling preserves the sub-Gaussian form, so $(X_n, d_n, m_n, \mathcal{E}_n, \mathcal{F}_n)$ satisfies $HKE(\Psi_n)$ with constants independent of n and $\Psi_n(r) := 12^{-d_w n}\Psi_0(12^n r)$. Therefore, condition (A1) in [18, theorem 1·2] holds. It is easy to verify that conditions (A2-A5) in [18, theorem 1·2] also hold.

It remains to verify condition (A6) in [18, theorem 1·2]. Define

$$\widetilde{X}_n := 12^{-n} X_0 = \{12^{-n} x : x \in X_0\},\ n \geq 1$$

and

$$\widetilde{d}_n(x,y) := 12^{-n} d_0(12^n x, 12^n y),\ \forall x, y \in \widetilde{X}_n.$$

This scaling satisfies the consistency relation with d_∞:

$$d_\infty(x,y) = \widetilde{d}_n(x,y) \quad \forall x, y \in \widetilde{X}_n.$$

Note that $\bigcup_{n=1}^\infty X_n = K_\infty$ and the map $x \mapsto 12^{-n}x$ is a bijective isometry from (X_n, d_n) to $(\widetilde{X}_n, \widetilde{d}_n)$. Clearly, $(\widetilde{X}_n, \widetilde{d}_n)$ is a proper length space, since the intrinsic metric $\widetilde{d}_n$ coincides with the shortest path length between points, so $(\widetilde{X}_n, \widetilde{d}_n, q_0)$ is an *eventually proper sequence of pointed length spaces*. Similar to [18, example 8·1(6) on Page 41], we know that $(\widetilde{X}_n, \widetilde{d}_n, q_0)$ pointed Gromov-Hausdorff converge to $(K_\infty, d_\infty, q_0)$.

Finally, applying [18, theorem 1·2 and remark 1·3 (i)], we conclude that there is a measure m_∞ on (K_∞, d_∞) satisfies $m_\infty(B_\infty(x,r)) \asymp r^\alpha$, and there is a regular strongly local Dirichlet form on $L^2(K_\infty, d_\infty, m_\infty)$ that satisfies (1·9).

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