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PERFECT PREDICTIONS IN ECONOMIC DYNAMICAL SYSTEMS WITH RANDOM PERTURBATIONS

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The paper studies multivariate non linear economic dynamical systems with an expectations feedback subjected to exogenous perturbations. In these systems, agents form expectations on future variables based on subjective transition probabilities given by a Markov kernel. The notion of a perfect Markov kernel that generates rational expectations along all orbits of the system is proposed. Conditions are provided under which perfect Markov kernels exist. Applications are given to models of the Cobweb type, to multivariate affine-linear systems, and to the stochastic OLG model of economic growth. For the latter two models, it is shown when a globally attracting random fixed point with rational expectations exists.

Keywords: Dynamics, Random Perturbations, Rational Expectations, Unbiased Forecasting Rules, Random Fixed Points

1. INTRODUCTION

A key feature of economic dynamical systems is the interplay between realizations of economic variables and agents' expectations about the future development of these variables. It has long been recognized that the actual evolution of an economy is generated by its basic underlying economic forces and by the forecasting rules that agents use to make predictions about the future. A theoretical description of such systems will therefore be complete only if the underlying economic law, including the nature of possible stochastic perturbations *and* the form of an expectations formation procedure of the participating agents, is specified.

One branch of macroeconomics employs the standard approach to close the model by assuming that expectations are rational and predictions are made accordingly; see, for example, Muth (1961) and Sargent (1993). This is done without defining an explicit expectations formation procedure that generates the desired predictions. For many multivariate and nonlinear models, it is not known whether

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this approach can be applied successfully and whether it leads to unique and/or stable solutions. Much effort has been spent to characterize those systems for which unique rational expectations equilibria exist. A second branch of the literature on adaptive expectations deals with the issue of how the formation of expectations can be made explicit and to what extent these forecasts are consistent with the actual development of the economy. A large body of the literature on learning explores updating rules and other econometric methods to predict rationally as the economy evolves and as more and more of its structure may be revealed by observations. Starting with the early work by Bray (1982) and Bray and Savin (1986), this theory is quite developed and to a large extent understood for linear economic models with additive perturbations (most of it i.i.d.); see, for example, the recent survey by Evans and Honkapohja (1999).

It is apparent that the techniques used for multivariate linear models are not easily transferable to nonlinear situations with random perturbations other than i.i.d. Moreover, important dynamical economic models with exogenous perturbations (productivity shocks in growth models, policy shocks, or innovative shocks in development) often are of a nonlinear nature. For this reason, a general nonlinearly structured theory should be designed to address the question whether rational expectations are possible in such environments. What causes the problem is that nonlinear economic forces, possibly nonlinear forecasting rules, and nonadditive random perturbations interact simultaneously. Hence, the resulting models may become very complex, and standard tools from the theory of stochastic processes do not seem to be capable of unlocking this complexity.

In an earlier paper [Böhm and Wenzelburger (1999)], we investigated the problem of the existence of perfect forecasting rules in a deterministic setting in which the point forecasts of agents interact with the underlying market forces described by a mapping called an economic law. It was shown that the existence and nature of perfect forecasting rules depend in a special way on the structure of a globally defined (and time-invariant) error function. This function is exclusively determined by the fundamentals of the economy and is independent of any expectations formation procedure. The insights obtained from the treatment of the deterministic situation seem to indicate that the structural features of economic laws that allow perfect forecasting rules can be generalized to a stochastic setting, if the sequential formulation of the model is chosen appropriately. The recent development of the mathematical theory of dynamical systems with random perturbations [cf. Arnold (1998)], which joins dynamical systems theory with the theory of stochastic processes, provides the natural formal framework for most economic systems with exogenous random perturbations. First applications in the theory of financial markets can be found in Böhm and Chiarella (2000), Böhm et al. (2000), and Wenzelburger (2001b), who study models with an endogenously generated random asset-pricing process.

This paper extends the basic geometric approach presented by Böhm and Wenzelburger (1999) for multivariate and nonlinear deterministic economic laws to the situation in which the law is subject to exogenous stochastic perturbations.

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The paper generalizes the conceptual framework of the deterministic setup by introducing forecasting rules as Markov kernels which, combined with a (random) economic law, define an economic *random dynamical system* with expectations feedback. This formulation encompasses a large class of recursive economic systems with random perturbations. We propose the notion of a perfect Markov kernel that generates rational expectations along all orbits of the system in the sense that true and subjective distributions of future variables coincide. This concept is specialized to unbiased Markov kernels that generate orbits along which the true and subjective means of future variables coincide. We show that the existence and uniqueness as well as all structural features of perfect and unbiased Markov kernels hinge essentially on global properties of an error function associated with the economic law and hence on the fundamental market forces of the economy under investigation.

Section 2 presents the basic mathematical model with a one-step forward-looking expectations feedback demonstrating the interaction between the exogenous perturbations, the economic law, and the forecasting rules. Section 3 introduces the concepts of perfect and unbiased Markov kernels, whereas Section 4 investigates the problem of existence of such Markov kernels. Section 5 specializes the concepts to the case of additive perturbations, Section 6 relates the proposed setup to two standard examples in the literature. Finally, in Section 7, we analyze the existence of unbiased Markov kernels in an OLG model of economic growth and show under which conditions a unique, globally attracting, random fixed point exists.

2. ECONOMIC RANDOM DYNAMICAL SYSTEMS

This section is devoted to the general setup of stochastic dynamic economic models with an expectations feedback. It generalizes the *nonlinear* deterministic case developed in Böhm and Wenzelburger (1999) to systems that are subjected to exogenous perturbations. In particular, many of the linear models treated, for example, in Evans and Honkapohja (1999) or Zenner (1996), are special cases of the structure proposed in this section. Let $X \subset \mathbf{R}^d$ denote the space of endogenous variables of an economy and $\mathrm{Prob}(X)$ be the set of all probability measures on X. An *economic law* describing all basic forces of the economy under investigation is assumed to be given by a continuous map,

$$S : \Sigma \times X \times \mathrm{Prob}(X) \to X, \qquad (\xi, x, \nu^e) \mapsto S(\xi, x, \nu^e), \tag{1}$$

where $\xi \in \Sigma \subset \mathbf{R}^m$ is a vector of parameters and/or exogenous variables, $x \in X$ is the current state of the economy, and $\nu^e \in \mathrm{Prob}(X)$ denotes the subjective probability distribution for the future state $S(\xi, x, \nu^e)$ of the economy.[1] The parameter ξ is subjected to exogenous stochastic perturbations that are driven by a stationary ergodic process $\{\xi_t\}_{t\in\mathbf{Z}}$ with values in Σ.

To fix the notation, let $\vartheta : \Omega \to \Omega$ be a measurable invertible map on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that is measure preserving with respect to $\mathbf{P}$ and whose

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inverse ϑ^{-1} is again measurable. Assume that $\mathbf{P}$ is ergodic with respect to ϑ and let ϑ^t denote the tth iterate of the map ϑ. The collection $(\Omega, \mathcal{F}, \mathbf{P}, \{\vartheta^t\}_{t\in\mathbf{Z}})$ is called an *ergodic metric dynamical system*; cf. Arnold (1998). It is well known that any stationary ergodic process $\{\xi_t\}_{t\in\mathbf{Z}}$ can be represented by an ergodic dynamical system. This implies that there exists a measurable map $\xi : \Omega \to \Sigma$ such that, for each fixed $\omega \in \Omega$, a sample path of the noise process is given by

$$\xi_t(\omega) = \xi(\vartheta^t\omega), \qquad t \in \mathbf{Z}. \tag{2}$$

The concept of modeling the noise in a system by means of a metric dynamical system amounts essentially to using the (equivalent) canonical realization process of a stochastic process. For example, if $\{\xi_t\}_{t\in\mathbf{Z}}$ is a sequence of Σ-valued i.i.d. random variables whose probability distribution is μ, then $(\Omega, \mathcal{F}, \mathbf{P}) = (\Sigma^{\mathbf{Z}}, \mathcal{B}(\Sigma)^{\mathbf{Z}}, \mu^{\mathbf{Z}})$, where Ω is the space of all sample paths of the process, $B(\Sigma)^{\mathbf{Z}}$ is the Borel σ-algebra of all cylinder sets, and $\mu^{\mathbf{Z}}$ is the product measure. Each element $\omega = \{\omega(s)\}_{s\in\mathbf{Z}} \in \Omega$ is a doubly infinite series describing a sample path of the process. The map $\vartheta : \Omega \to \Omega$, $\omega \mapsto \vartheta\omega$ is defined by $(\vartheta\omega)(s) := \omega(s+1)$, $s \in \mathbf{Z}$ and is called the left shift such that $(\vartheta^t\omega)(s) = \omega(s+t)$ for all $t, s \in \mathbf{Z}$. With the help of the evaluation map, $\xi : \Omega \to \Sigma$, $\xi(\omega) := \omega(0)$, a representation of the form (2) is obtained; see Arnold (1998, App. A) for more details.

For simplicity of exposition, we assume from now on that the noise is governed by a stationary ergodic Markov process. The case of an i.i.d. noise process is implicitly included. To complete the description of the economy, it is necessary to specify the way in which predictions are made, that is, how ν^e is determined at time t. This specification depends essentially on the type of information that is available for agents at time t. Consider the case in which agents at time t know the state of the economy x_t and have observed exogenous perturbations $\xi_\tau = \xi(\vartheta^\tau\omega)$ only up to time $t-1$, that is, for all $\tau \le t-1$. This means that, for these agents, the future realization $x_{t+1} = S(\xi_t, x_t, \nu^e)$ becomes a random variable whose distribution depends solely on x_t, ν^e, and—in the Markovian case—on ξ_{t-1}. Let agents be endowed with subjective transition probabilities given by a Markov kernel

$$\Psi : \Sigma \times X \times \mathcal{B}(X) \to [0, 1], \tag{3}$$

with $\mathcal{B}(X)$ denoting the σ-algebra of Borel sets; Ψ describes subjective transition probabilities for the two consecutive periods $(t, t+1)$ of the economy conditioned on information available at date t according to which agents form expectations. In the sequel, it is useful to view a Markov kernel (3) as a function $\Psi : \Sigma \times X \to \mathrm{Prob}(X)$, such that Ψ becomes a *predictor* or a *forecasting rule* for distributions. Then $\nu^e_{t+1} := \Psi(\xi_{t-1}, x_t) \in \mathrm{Prob}(X)$ is the subjective conditional probability distribution of the realization x_{t+1}. Associated with each Markov kernel Ψ is the (*conditional*) *mean-value predictor* ψ, defined by the function

$$\psi : \Sigma \times X \to X, \qquad (\xi, x) \mapsto \int_X y\, \Psi(\xi, x, dy), \tag{4}$$

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such that $x^e_{t+1} := \psi(\xi_{t-1}, x_t)$ becomes the predicted *mean value* of the realization x_{t+1} conditioned on information available at date t. In other words, based on the *subjective belief*, $\Psi(\xi_{t-1}, x_t), x^e_{t+1}$ is the best least-squares prediction for x_{t+1} given the history of realizations $\{\xi_{\tau-1}, x_\tau\}_{\tau \le t}$. Observe that the mean-value predictor ψ is a *point predictor* in the usual sense rather than a forecasting rule for distributions.

Inserting the Markov kernel (3) into the economic law (1) yields a time-one map $S_\Psi : \Omega \times X \to X$ of a discrete-time random dynamical system on X, defined by

$$x_{t+1} = S_\Psi(\vartheta^t\omega)x_t := S\big(\xi(\vartheta^t\omega), x_t, \Psi\big(\xi(\vartheta^{t-1}\omega), x_t\big)\big), \tag{5}$$

which governs the evolution of the economy. If the system started at x_0, then the iteration of the map S_Ψ under the perturbation ω induces a measurable map $\varphi : \mathbf{N} \times \Omega \times X \to X$, defined by

$$\varphi(t, \omega, x_0) = \begin{cases} \big(S_\Psi(\vartheta^{t-1}\omega) \circ \cdots \circ S_\Psi(\omega)\big)x_0 & \text{if } t > 0 \\ x_0 & \text{if } t = 0 \end{cases} \tag{6}$$

such that $x_t = \varphi(t, \omega, x_0)$ is the state of the system at time t. The map φ generated by S_Ψ describes a *random dynamical system* in the sense of Arnold (1998) (in forward time). For an arbitrary initial condition $x_0 \in X$ and any perturbation $\omega \in \Omega$, the sequence of points $\gamma(x_0) := \{x_t\}_{t\in\mathbf{N}}$ with $x_t = \varphi(t, \omega, x_0)$, $t \in \mathbf{N}$ is called an orbit of the random dynamical system S_Ψ. In the sequel, (5) or, equivalently, (6) is referred to as an *economic random dynamical system* (*ERDS*).

The primary aim of this paper is to characterize Markov kernels that are correct in a well-specified sense with a particular emphasis on those that generate rational expectations. To focus on the nonlinear effects of the expectations feedback, we restrict most of our analysis to either Markovian or i.i.d. noise with an emphasis on the former. The case of general stationary and ergodic perturbations is more technical but can be treated in a similar fashion. In fact, it is already implicit that some of our results hold for this class of more general noise processes as well. We analyze time-invariant forecasting rules Ψ thereby disregarding learning schemes and updating. Many adaptive learning schemes such as OLS or stochastic gradient methods may be formulated as time-invariant predictors in our sense when defined on an appropriate state space [see, e.g., Wenzelburger (2001a, App. B)]. For this reason, our approach includes updating schemes such as OLS. The situation in which agents at time t can observe all exogenous perturbations *including* ξ_t can be treated like the deterministic case; see Böhm and Wenzelburger (1999).

3. PERFECT PREDICTIONS IN RANDOM ECONOMIC SYSTEMS

3.1. Perfect Markov Kernels

A systematic analysis of the performance of a forecasting rule requires the comparison of the subjective probability distribution ν^e with the true distribution of the future state. For each fixed state $x \in X$ and any distribution $\nu^e \in \mathrm{Prob}(X)$ at an

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arbitrary date t, the economic law (1) combined with the noise process (2) defines a random variable

$$S\big(\xi_t(\cdot), x, \nu^e\big) : \Omega \to X, \qquad \omega \mapsto S\big(\xi_t(\omega), x, \nu^e\big) \tag{7}$$

on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in the state space X. The distribution of (7), denoted by $\mu(\xi_{t-1}, x, \nu^e)$, depends on the current state x, on ν^e, and on ξ_{t-1} assuming the noise to be Markovian. The relationship between true and subjective distributions is described by introducing the concept of an error function. A *conditional error function for distributions* associated with the economic law S is defined by

$$E_S : \Sigma \times X \times \mathrm{Prob}(X) \to \mathcal{M}(X), \qquad (\xi, x, \nu^e) \mapsto \mu(\xi, x, \nu^e) - \nu^e, \tag{8}$$

where $\mathcal{M}(X)$ denotes the space of all signed measures. Given $x_t \in X$ and $\nu^e_{t+1} \in \mathrm{Prob}(X)$, the signed measure $E_S(\xi_{t-1}, x_t, \nu^e_{t+1})$ is the difference between the true and the subjective distribution of the random variable $\omega \mapsto x_{t+1}(\omega) = S(\xi_t(\omega), x_t, \nu^e_{t+1})$ conditioned on the realization ξ_{t-1} of the exogenous noise.

In terms of deviations from the true probability, the map E_S encodes all possible forecast error distributions independently of any forecasting rule. The idea now is to compare the true and the subjective distribution of future outcomes of the economy by means of a *(pseudo)metric* ρ on $\mathrm{Prob}(X) \subset \mathcal{M}(X)$.[2] Given a state $(\xi, x) \in \Sigma \times X$ and any $\epsilon \geq 0$, the measure ν^e provides an ϵ-perfect prediction for $\mu(\xi, x, \nu^e)$ with respect to the pseudometric ρ, if $\rho(\mu(\xi, x, \nu^e), \nu^e) \leq \epsilon$. Note, in particular, that $E_S(\xi, x, \nu^e) = 0$ for $\epsilon = 0$, because ρ is a pseudometric. The set of all states $(\xi, x) \in \Sigma \times X$ and $\nu^e \in \mathrm{Prob}(X)$ that satisfy this criterion is given by

$$\mathcal{W}^{\rho,\epsilon} := \{(\xi, x, \nu^e) \in \Sigma \times X \times \mathrm{Prob}(X) | \rho(\mu(\xi, x, \nu^e), \nu^e) \leq \epsilon\}.$$

Hence, the following definition is natural.

DEFINITION 1. *Given a law S and a pseudometric ρ on* $\mathrm{Prob}(X)$, *a Markov kernel Ψ is called locally ϵ-perfect for S with respect to ρ if there exists a family of open subsets $\{U(\xi)\}_{\xi \in \Sigma}$ of X such that*

$$(\xi, x, \Psi(\xi, x)) \in \mathcal{W}^{\rho,\epsilon}, \qquad x \in U(\xi), \qquad \xi \in \Sigma.$$

If $U(\xi) = X$ for all $\xi \in \Sigma$, then Ψ is called a (globally) ϵ-perfect Markov kernel. For $\epsilon = 0$, Ψ is called (locally) perfect.

The set $\mathcal{W}^{\rho} \equiv \mathcal{W}^{\rho,0}$ is called the *constraint variety* of the economic law S. $\mathcal{W}^{\rho,\epsilon}$ defines a closed ϵ-neighborhood of the zero-level set of the error function E_S. From the theoretical point of view, the pseudometric ρ could, in particular, be any metric on $\mathrm{Prob}(X)$. In view of econometric applications, however, it will often suffice to compare the corresponding moments of true and subjective distributions by means of an appropriate pseudometric. Given an economic law S, an ϵ-perfect Markov kernel, if it exists, will not be uniquely determined, if either $\epsilon > 0$ or ρ is not a metric.

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A special case is the pseudometric $\rho = \rho_{\mathbf{E}}$, which measures the absolute difference between the first moments of two measures. In this case, a (locally) ϵ-perfect Markov kernel Ψ is called a (locally) ϵ-*unbiased* Markov kernel. An ϵ-unbiased Markov kernel is such that, for all possible states $x_t \in X$ and all times t, the predicted conditional mean value $\psi(\xi_{t-1}, x_t) = \int_X y\ \Psi(\xi_{t-1}, x_t, dy)$ coincides with the true conditional mean value $\mathbf{E}[S_\Psi(\xi_t(\cdot), x_t)|\xi_{t-1}]$ of the random variable $\omega \mapsto x_{t+1}(\omega) = S_\Psi(\xi_t(\omega), x_t)$ up to a deviation of size ϵ. (Recall that ξ_t is not observable at date t, whereas x_t is.)

For $\epsilon = 0$, $x^e_{t+1} = \psi(\xi_{t-1}, x_t)$ is the best least-squares prediction for x_{t+1}, given the history of realizations $\{\xi_{\tau-1}, x_\tau\}_{\tau \le t}$. This justifies the terminology ϵ-unbiased Markov kernel for a kernel Ψ associated with such a point predictor ψ.

The actual verification of the performance criterion of a forecasting rule given in Definition 1 will always be carried out along a particular orbit of the ERDS. Hence the following notion of perfect predictions and unbiased predictions along orbits of an ERDS is natural.

DEFINITION 2. *An orbit $\gamma(x_0) = \{x_t\}_{t\in\mathbf{N}}$ of the ERDS S_Ψ is an ϵ-perfect orbit with respect to the pseudometric ρ on* Prob(X), *if $(\xi_{t-1}, x_t, \Psi(\xi_{t-1}, x_t)) \in \mathcal{W}^{\rho,\epsilon}$ for all $t \in \mathbf{N}$. For $\rho = \rho_{\mathbf{E}}$, $\gamma(x_0)$ is called an ϵ-unbiased orbit of S_Ψ.*

For $\epsilon = 0$, an ϵ-perfect orbit is called a perfect orbit for short. Definition 2 characterizes orbits in the extended state space $X \times$ Prob(X). In particular, an unbiased orbit is, by definition, such that, given x_t, the subjective conditional and true conditional mean for x_{t+1} coincide. In this sense an unbiased orbit is nothing more but a rational expectations equilibrium in the classical sense. Comparing Definition 1 with Definition 2 reveals that perfect (unbiased) Markov kernels are precisely those forecasting rules that, when applied, generate rational expectations in the strong (weak) sense along all orbits of the system. Summarizing, one obtains the following Lemma.

LEMMA 1. *Ψ is an ϵ-perfect Markov kernel for S iff all orbits of S_Ψ are ϵ-perfect orbits.*

3.2. Random Fixed Points

For many economic applications, it suffices to have good forecasts in the long run, that is, ϵ-perfect predictions on special (random) sets or along special orbits of the system. This is the case when the set $\{U(\xi)\}_{\xi\in\Sigma}$ appearing in Definition 1 contains a random attractor that is the random analogue of an attractor of a deterministic system [see Arnold (1998, p. 483)]. Each orbit starting from the corresponding domain of attraction will then eventually become an ϵ-perfect orbit. Otherwise, an orbit may loose this property. Typical candidates for these special orbits are generated by asymptotically stable *random fixed points* of an ERDS. The following definition of a random fixed point [Schmalfuss (1996, 1998)],[3] includes a stability notion, given as Definition 7.4.6 in Arnold (1998).

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DEFINITION 3. *A random fixed point of the ERDS S_Ψ is a random variable $x^* : \Omega \to X$ on $(\Omega, \mathcal{F}, \mathbf{P})$ such that*

$$x^*(\vartheta\omega) = S_\Psi(\omega)x^*(\omega) \quad \textit{for all} \quad \omega \in \Omega',$$

where $\Omega' \subset \Omega$ is a ϑ-invariant set $\Omega' \subset \Omega$ of full measure $\mathbf{P}(\Omega') = 1$.[4]

A random fixed point x^* is called asymptotically stable with respect to a norm $\|\cdot\|$ on X if there exists a random neighborhood $U(\omega) \subset X$, $\omega \in \Omega$ such that $\mathbf{P}$*–a.s.*

$$\lim_{t\to\infty} \left\|\varphi(t, \omega, x_0(\omega)) - x^*(\vartheta^t\omega)\right\| = 0 \quad \textit{for all} \quad x_0(\omega) \in U(\omega),$$

where φ is given in (6).

The first part in Definition 3 implies that $x^*(\vartheta^{t+1}\omega) = S_\Psi(\vartheta^t\omega)x^*(\vartheta^t\omega)$ for all times t. Hence, a random fixed point generates orbits $\{x^*(\vartheta^t\omega)\}_{t\in\mathbf{N}}$, $\omega \in \Omega$, that solve the random difference equation (5) induced by S_Ψ. If S_Ψ is independent of the perturbation ω, then Definition 3 coincides with the notion of a deterministic fixed point. A random fixed point x^* is asymptotically stable if, for almost all perturbations $\omega \in \Omega$, all orbits starting at sufficiently close points $x_0(\omega) \in U(\omega)$ eventually converge to orbits of the random fixed point. By stationarity and ergodicity of ϑ, the process $\{x^* \circ \vartheta^t\}_{t\in\mathbf{N}}$ is stationary and ergodic. Let $x^*\mathbf{P}$ denote the probability distribution of x^*. If, in addition, $\mathbf{E}\,\|x^*\| < \infty$, then the ergodicity and the stability property of x^* imply that

$$\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} 1_B(\varphi(t, \omega, x_0(\omega)) = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} 1_B(x^*(\vartheta^t\omega)) = x^*\mathbf{P}(B)$$

for all $x_0(\omega) \in U(\omega)$ $\mathbf{P}$–a.s. In other words, the empirical law of any "nearby" orbit $\{\varphi(t, \omega, x_0(\omega))\}_{t\in\mathbf{N}}$ is for $\mathbf{P}$-almost all $\omega \in \Omega$ well defined and converges to the true probability distribution of x^*. We are now ready to establish the following notion.

DEFINITION 4. *Given a law S, a Markov kernel Ψ is called asymptotically ϵ-perfect with respect to ρ if there exists an asymptotically stable random fixed point x^* of S_Ψ such that Ψ is locally ϵ-perfect along x^*; that is,*

$$(\xi(\vartheta^{-1}\omega), x^*(\omega), \Psi(\xi(\vartheta^{-1}\omega), x^*(\omega))) \in \mathcal{W}^{\rho,\epsilon} \quad \mathbf{P}\textit{–a.s.}$$

For $\rho = \rho_{\mathbf{E}}$, Ψ and the mean-value predictor ψ associated with Ψ are called asymptotically ϵ-unbiased.

Consider a Markov kernel Ψ that is asymptotically unbiased and let ψ denote the mean-value predictor associated with Ψ. Then there exists a random fixed point x^* such that the forecast errors $\zeta_{t+1} = x^*(\vartheta^{t+1}\omega) - \psi(\xi(\vartheta^{t-1}\omega), x^*(\vartheta^t\omega))$ satisfy $\mathbf{E}[\zeta_{t+1}|\xi_{t-1}] = 0$ for all times $t \in \mathbf{N}$. That is, all mean forecast errors conditioned on the maximum amount of information available prior to actual

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realizations vanish. This implies in particular that all forecast errors are zero in the mean, $\mathbf{E}\,\zeta_t = 0$ for all $t \in \mathbf{N}$. It follows from the stability property and the ergodicity of x^* that the sample mean errors and all sample autocorrelation coefficients of all sufficiently close orbits $\{x_t\}_{t\in\mathbf{N}}$ as well as those of the corresponding forecast errors $\{\zeta_t\}_{t\in\mathbf{N}}$ converge almost surely to the respective true means and the true autocorrelation coefficients associated with the random fixed point x^*.

This observation illustrates that an asymptotically unbiased Markov kernel induces a (stochastic) *consistent expectations equilibrium* (CEE) in the sense of Hommes and Sorger (1998, 2001), provided that agents in the model have a *perceived law of motion* whose hypothesized autocorrelation structure is confirmed by the autocorrelation structure of the realizations. Notice, however, that an unbiased Markov kernel is more precise than a forecasting rule that induces a CEE because the former provides the best least-squares prediction, given the history of the process. Moreover, the autocorrelation structure of the forecast errors associated with unbiased Markov kernels may well be nontrivial; see Section 5 for a more concrete example. Hence, learning schemes designed to find a CEE such as *consistent adaptive learning schemes* in the sense of Schönhofer (1999, 2001) might only be meaningful in a situation with additive i.i.d. noise.

The concept of an asymptotically stable random fixed point in Definition 3 is a natural extension of the notion of an asymptotically stable fixed point in the deterministic case. Combined with the property of ϵ-perfection, these criteria seem to be natural stability requirements for an ERDS. In fact, these properties can be established for a number of important economic models (see Sections 6 and 7), yielding a characterization of the long-run behavior of orbits in models with noise. This improves the purely statistical viewpoint of considering invariant distributions and Markov equilibria, as done in most economic applications; see, for example, Spear and Srivastava (1986), Spear (1998), Duffie et al. (1994), Bhattacharya and Majumdar (2001). For a discussion of the relationship between random fixed points and Markov equilibria, see Schenk-Hoppé and Schmalfuss (1998).

4. EXISTENCE OF PERFECT MARKOV KERNELS

The problem of existence and uniqueness of perfect Markov kernels is structurally very similar to that of perfect forecasting rules in the deterministic case. As was seen in the preceding section, perfect Markov kernels must satisfy properties embodied in the corresponding constraint varieties. A sufficient criterion for the existence of perfect Markov kernels is provided by the next theorem.

THEOREM 1. *Let S be an economic law and ρ be a pseudometric on* $\mathrm{Prob}(X)$. *Consider the function*

$$\mathcal{E}_S : \Sigma \times X \times \mathrm{Prob}(X) \to \Sigma \times X \times \mathcal{M}(X),\ (\xi, x, \nu^e) \mapsto \big(\xi, x, E_S(\xi, x, \nu^e)\big)$$

induced by (8). *Let $\mathcal{E}_S$ be invertible locally on $\mathcal{U} \subset \mathcal{W}^\rho$ with $\mathcal{U} = \{\mathcal{U}(\xi)\}_{\xi\in\Sigma}$. Then there exists a Markov kernel that is locally perfect with respect to ρ. If $\mathcal{E}_S$*

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is globally invertible, then there exists a Markov kernel Ψ^* that is perfect with respect to ρ. If, in addition, ρ is a metric, then Ψ^* is uniquely determined.

Proof. If $\mathcal{E}_S$ is globally invertible, then its inverse is a function of the form

$$\mathcal{G} : \Sigma \times X \times \mathcal{M}(X) \to \Sigma \times X \times \mathrm{Prob}(X), \qquad (\xi, x, \varrho) \mapsto (\xi, x, G(\xi, x, \varrho)),$$

with $E_S(\xi, x, G(\xi, x, \varrho)) = \varrho$ for all $(\xi, x) \in \Sigma \times X, \varrho \in \mathcal{M}(X)$. Then $\Psi^*(\xi, x) := G(\xi, x, 0)$ with $\xi \in \Sigma, x \in X$ is a globally perfect Markov kernel with respect to ρ.

If $\mathcal{E}_S$ is only locally invertible, then the inverse $\mathcal{G}$ is defined only locally around the image set $\mathcal{E}_S(\mathcal{U})$ in $\Sigma \times X \times \mathcal{M}(X)$. Consequently, Ψ^* is defined only locally and hence is locally perfect with respect to ρ.

The uniqueness of Ψ^* in case ρ is a metric follows from the definition of $\mathcal{W}^\rho$. ∎

Theorem 1 transforms the problem of existence of a perfect Markov kernel into an invertibility condition of the induced error function $\mathcal{E}_S$ which depends exclusively on the fundamentals of the economy. It thus determines the functional form of perfect Markov kernels. Notice that, for the existence of a perfect Markov kernel, it suffices to have invertibility of $\mathcal{E}_S$ on an open neighborhood of $\mathcal{W}^\rho$. It is straightforward to generalize Theorem 1 to the case of ϵ-perfect Markov kernels.

For a first application of Theorem 1, consider models of the Cobweb type, which are characterized by the fact that no state variable of the system enters as an argument into the economic law. Economic laws of the Cobweb type are formally defined by maps

$$S : \Sigma \times \mathrm{Prob}(X) \to X, \qquad (\xi, \nu^e) \mapsto S(\xi, \nu^e). \tag{9}$$

Observe that the transition probabilities associated with (9) and hence the corresponding error function do not depend on actual states $x \in X$. It is therefore immediate from Theorem 1 that perfect Markov kernels for models of the Cobweb type should not depend on past or current states at all. This is the essence of the following corollary, which extends and confirms the results known for their deterministic analogues.

COROLLARY 1. *Let S be an economic law of the Cobweb type and ρ be a metric. Then, perfect Markov kernels for S, if they exist, are of the form*

$$\Psi^* : \Sigma \times \mathcal{B}(X) \to [0, 1].$$

The notion of a perfect Markov kernel is illustrated next for the case of the pseudometric $\rho_{\mathbf{E}}$, that is, for unbiased Markov kernels. Given $x \in X$, $\nu^e \in \mathrm{Prob}(X)$, the conditional expected value of the random variable $\omega \mapsto S[\xi_t(\omega), x, \nu^e]$ is

$$\mathbf{E}\big[S\big(\xi_t(\cdot), x, \nu^e\big) \,\big|\, \xi_{t-1} = \xi_{t-1}(\omega)\big] = \int_X y \, \mu\big(\xi_{t-1}(\omega), x, \nu^e, dy\big). \tag{10}$$

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Thus, all possible forecast errors in terms of deviations from the true expected value (10) are given by the *mean error function*

$$e_S^{\mathbf{E}} : \Sigma \times X \times \mathrm{Prob}(X) \to \mathbf{R}^d,\ (\xi, x, \nu^e) \mapsto \int_X y\mu(\xi, x, \nu^e, dy) - \int_X y\nu^e(dy). \tag{11}$$

The relationship between the error functions E_S and $e_S^{\mathbf{E}}$ then is

$$\rho_{\mathbf{E}}\big(E_S(\xi, x, \nu^e), 0\big) = \big\|e_S^{\mathbf{E}}(\xi, x, \nu^e)\big\|, \quad \xi \in \Sigma,\ x \in X,\ \nu^e \in \mathrm{Prob}(X).$$

In view of numerical or empirical applications, it is useful to follow standard practice and restrict the set of all possible subjective distributions $\mathrm{Prob}(X)$ to an appropriate subset $\{\nu_a\}_{a\in A}$ with a finite-dimensional parameter set $A \subset \mathbf{R}^l$. For example, $\{\nu_a\}_{a\in A}$ may be a family of distributions parameterized in mean values and variances. The elements of the constraint varieties can naturally be identified with the corresponding parameters. Clearly, any Markov kernel Ψ with distributions in $\{\nu_a\}_{a\in A}$ induces a map $\psi : \Sigma \times X \to A$ such that $\Psi(\xi, x) = \nu_{\psi(\xi,x)}$, $\xi \in \Sigma$, $x \in X$. A special case of Theorem 1 may now be reformulated as follows.

PROPOSITION 1. *Let S be an economic law. Assume that subjective probability distributions $\{\nu_a\}_{a\in A}$, are parameterized by subjective mean values $a \in A \subset \mathbf{R}^d$. Consider the function*

$$\tilde{\mathcal{E}}_S : \Sigma \times X \times A \to \Sigma \times X \times \mathbf{R}^d, \qquad (\xi, x, a) \mapsto \big(\xi, x, e_S^{\mathbf{E}}(\xi, x, \nu_a)\big)$$

induced by the mean error function (11). *Let $\tilde{\mathcal{E}}_S$ be locally invertible on some subset $V = \{V(\xi)\}_{\xi\in\Sigma}$, of $\Sigma \times X \times A$ on which $e_S^{\mathbf{E}}(\xi, x, \nu_a) = 0$. Then there exists a locally unbiased Markov kernel. If $\tilde{\mathcal{E}}_S$ is globally invertible, then there exists an unbiased Markov kernel Ψ^* with values in $\{\nu_a\}_{a\in A}$.*

The proof of Proposition 1 is analogous to the proof of Theorem 1. As in Böhm and Wenzelburger (1999), global inverse function theorems of the type found in Rheinboldt (1969) may be employed. The construction of perfect Markov kernels will, in general, be difficult if not impossible, since it amounts to computing an explicit inverse function. It requires full knowledge of the underlying economic law, including the stochastic nature of the perturbations.

5. ADDITIVE PERTURBATIONS

A frequently encountered situation is the case in which the endogenous variables are perturbed by additive noise. It is shown in this section that additive noise entails a particular functional form for perfect Markov kernels. Let $\Sigma, X \subset \mathbf{R}^d$, where X denotes the space of endogenous variables of an economy, as before. An economic law with additive perturbations $S : \Sigma \times X \times \mathrm{Prob}(X) \to X$ is of the form

$$x_{t+1} = S\big(\xi_t, x_t, \nu_{t+1}^e\big) := F\big(x_t, \nu_{t+1}^e\big) + \xi_t, \tag{12}$$

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where $F : X \times \mathrm{Prob}(X) \to X$ is an arbitrary map. Here, the deterministic part F of the law (12) may be thought of as defining a purely deterministic economic law. For this case, the notion of a perfect predictor was defined in Böhm and Wenzelburger (1999). Let $\mathbf{P}(\cdot)$ denote the stationary transition probabilities of the Markovian noise process and let $\tau_y(\xi) := y + \xi$ be a translation by $y \in X$. Then, for fixed $x \in X$, $\nu^e \in \mathrm{Prob}(X)$, and fixed time t, the conditional distribution of the economic law (12) taken as a random variable $\omega \mapsto S(\xi_t(\omega), x, \nu^e)$ is given by

$$\mu\big(\xi_{t-1}, x, \nu^e\big)(B) = \tau_{F(x,\nu^e)}\mathbf{P}(\xi_{t-1})(B) := \mathbf{P}(\xi_{t-1})\big(\tau^{-1}_{F(x,\nu^e)}(B)\big), \qquad B \in \mathcal{B}(X).$$

Thus, the true distribution of (12) is an additive shift of $\mathbf{P}(\xi_{t-1})$.

PROPOSITION 2. *Let S be an economic law with additive noise and ρ be a metric. Then, the constraint variety $\mathcal{W}^\rho$ is isomorphic to*

$$\mathcal{V} := \{(\xi, x, y) \in \Sigma \times X \times X \mid y = F(x, \tau_y\mathbf{P}(\xi))\}.$$

Moreover, $\Psi^* : \Sigma \times X \times \mathcal{B}(X) \to [0, 1]$ is a perfect Markov kernel for S iff there exists a function $g : \Sigma \times X \to X$ with $(\xi, x, g(\xi, x)) \in \mathcal{V}$ for all $\xi \in \Sigma$, $x \in X$, such that

$$\Psi^*(\xi, x, B) = \tau_{g(\xi,x)}\mathbf{P}(\xi)(B). \tag{13}$$

Proof. We show that the map

$$\hat{\tau} : \mathcal{V} \to X \times \mathrm{Prob}(X), \qquad (\xi, x, y) \mapsto (\xi, x, \tau_y\mathbf{P}(\xi))$$

is an isomorphism onto $\mathcal{W}^\rho$. Since $\mathbf{P}(\xi)$ is a probability distribution on Σ, $\hat{\tau}$ is injective. Let $(\xi, x, y) \in \mathcal{V}$ be arbitrary and set $\nu^e := \tau_y\mathbf{P}(\xi)$. Since $y = F(x, \tau_y\mathbf{P}(\xi))$, we have $\nu^e = \tau_y\mathbf{P}(\xi) = \tau_{F(x,\nu^e)}\mathbf{P}(\xi)$ and therefore $(\xi, x, \nu^e) \in \mathcal{W}^\rho$. Now, let $(\xi, x, \nu^e) \in \mathcal{W}^\rho$ be arbitrary. Since ρ is a metric, $\nu^e = \tau_{F(x,\nu^e)}\mathbf{P}(\xi)$. Setting $y := F(x, \nu^e)$ yields $\tau_y\mathbf{P}(\xi) = \nu^e$. This in turn implies that $y = F(x, \tau_y\mathbf{P}(\xi))$ such that $(\xi, x, y) \in \mathcal{V}$.

The second statement follows from the isomorphisms of the constraint varieties. ∎

Proposition 2 shows that, in the case of additive noise, perfect Markov kernels are essentially an additive shift of the conditional probability distribution of the noise process, where the shift depends in a possibly nonlinear and nonstationary manner on the state $x \in X$ described by the function g. For economic applications, this implies that agents believing in additive noise should use subjective probability distributions of the form $\{\tau_x\, \mathbf{P}(\xi)\}$ and therefore Markov kernels of the form (13), where a priori g is unknown.

Let Ψ^* be a perfect (or unbiased) Markov kernel and ψ^* denote the corresponding mean-value predictor. Then the dynamics under rational expectations are given by

$$x_{t+1} = S_{\Psi^*}(\xi_{t-1}, x_t) = \psi^*(\xi_{t-1}, x_t) - \mathbf{E}[\xi_t \mid \xi_{t-1}] + \xi_t,$$

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and the resulting forecast errors $\{\zeta_{t+1}\}_{t\in\mathbf{N}}$ by

$$\zeta_{t+1} := x_{t+1} - \psi^*(\xi_{t-1}, x_t) = \xi_t - \mathbb{E}[\xi_t \mid \xi_{t-1}]. \tag{14}$$

It follows from (14) that all conditional mean errors vanish: $\mathbf{E}[\zeta_{t+1} \mid \xi_{t-1}] = 0$. Moreover, under rational expectations, the autocorrelation structure of the forecast errors coincides with the autocorrelation structure of the exogenous noise process or, equivalently,

$$\mathbb{C}\mathrm{ov}(\zeta_{t+1}, \zeta_{t+\tau}) = \mathbb{C}\mathrm{ov}(\xi_t, \xi_{t+\tau-1}) \quad \text{for all } \tau \in \mathbf{N}. \tag{15}$$

This observation still holds true for exogenous noise other than the Markovian type and illustrates that an unbiased Markov kernel induces a (stochastic) *consistent expectations equilibrium* (CEE) in the sense of Hommes and Sorger (1998), provided that the dynamics under rational expectations are stable in the sense of Section 3 and that agents in the model have a *perceived law of motion*, the hypothesized autocorrelation structure of which is confirmed by the autocorrelation structure of the forecast errors. Notice again that an unbiased Markov kernel is more precise than a forecasting rule that induces a CEE.

The following lemma shows that point predictors, which are perfect for the unperturbed economic law in the deterministic sense, induce unbiased Markov kernels. To do so, we use the map $x^e \mapsto \delta_{x^e}$, which embeds X into $\mathrm{Prob}(X)$ with δ_{x^e} denoting the Dirac measure supported at $x^e \in X$. A perfect predictor $\psi : X \to X$ for the unperturbed economic law F is, by definition, such that $\psi(x) = F(x, \delta_{\psi(x)})$ for all $x \in X$; cf. Böhm and Wenzelburger (1999).

LEMMA 2. *Let S be an economic law with additive perturbations* (12) *and assume, in addition, that $\{\xi_t\}_{t\in\mathbf{Z}}$ is a martingale difference sequence. If $\psi : X \to X$ is a perfect predictor for the unperturbed law F, then the Markov kernel $\Psi(x) := \delta_{\psi(x)}$, $x \in X$ is unbiased for S.*

Proof. Since $\{\xi_t\}_{t\in\mathbf{N}}$ is a martingale difference sequence, the mean error function associated with (12) has the form

$$e_S^{\mathbf{E}}(\xi, x, \delta_{x^e}) = F(x, \delta_{x^e}) - x^e, \qquad \xi \in \Sigma, \qquad x, x^e \in X.$$

This gives the assertion. ∎

Proposition 2 and Lemma 2 show that the structural features in the case of additive perturbations are closely related to the deterministic case treated in Böhm and Wenzelburger (1999). This implies in particular that existence and uniqueness for unbiased Markov kernels follow from the corresponding statements there.

6. TWO APPLICATIONS

To relate the concepts introduced in the preceding sections to some of the more established results in the literature, two standard economic models are treated in

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this section: the affine (linear) case and models of the Cobweb type. A nonlinear example is discussed in Section 7.

6.1. Unbiased Predictions in Linear Models

Let $X = \Sigma = \mathbf{R}^d$ and consider the simplest version of a linear (affine) economic law $S : \mathbf{R}^{3d} \to \mathbf{R}^d$, given by

$$x_{t+1} = S\big(\xi_t, x_t, x^e_{t+1}\big) := Ax_t + Bx^e_{t+1} + \xi_t, \tag{16}$$

where A and B are $d \times d$ matrices, $x^e_{t+1} \in X$ is a point prediction for x_{t+1}, and $\{\xi_t\}_{t\in\mathbf{Z}}$ is an i.i.d. process with zero mean. The law (16) fits into the general setup (1) introduced in Section 2 by using the natural embedding $x^e \mapsto \delta_{x^e}$ of X into $\mathrm{Prob}(X)$ with δ_{x^e} denoting the Dirac measure supported at $x^e \in X$. Many linear economic models with rational expectations [see, e.g., Marcet and Sergent (1989), Zenner (1996), or Evans and Honkapohja (1999)] are of the form (16). The mean error function associated with (16) is simply

$$e^{\mathbf{E}}_S(x, x^e) = Ax + [B - I]x^e.$$

Hence, $e^{\mathbf{E}}_S$ is globally invertible iff $I - B$ is invertible. If this is the case, then

$$\psi^*(x) = ([I - B]^{-1}A)x \tag{17}$$

is the unique predictor that is globally unbiased on $\mathbf{R}^d$. Inserting (17) into the economic law (16) yields the time-one map

$$x_{t+1} = S_{\Psi^*}(\xi_t, x_t) := ([I - B]^{-1}A)x_t + \xi_t, \tag{18}$$

which defines the dynamics under rational expectations. Observe that these dynamics are defined globally on $\mathbf{R}^d$ and that all conditional mean forecast errors vanish; that is, $\mathbf{E}[(x_{t+1} - \psi^*(x_t))|\xi_{t-1}] = 0$ for all times t and all initial conditions $x_0 \in \mathbf{R}^d$. In other words, $\psi^*(x_t)$ is the best *linear* least-squares prediction for the future state x_{t+1}, given the history of the process $\{x_\tau\}_{\tau\le t}$.

It is well known that the map (18) possesses a unique random fixed point x^* that is globally asymptotically stable, if all eigenvalues of $[I - B]^{-1}A$ lie inside the unit circle; cf. Arnold (1998). In this stable case, Arnold (1998, Corollary 5.6.6) tells us that the random fixed point is given by the random variable

$$x^*(\omega) = \sum_{i=0}^{\infty} ([I - B]^{-1}A)^i \, \xi\big(\vartheta^{-(i+1)}\omega\big), \qquad \omega \in \Omega. \tag{19}$$

Here we made use of the fact that any i.i.d. process $\{\xi_t\}_{t\in\mathbf{Z}}$ can be represented by an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, \{\vartheta^t\}_{t\in\mathbf{Z}})$ such that $\xi_t(\omega) = \xi(\vartheta^t\omega)$; cf. Section 2. The global asymptotic stability property of the random fixed point (19) implies that all orbits $\gamma(x_0) = \{x_t\}_{t\in\mathbf{N}}$, $x_0 \in \mathbf{R}^d$ of (18) converge to orbits of

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the stationary process $\{x^*(\vartheta^t\omega)\}_{t\in\mathbf{N}}$ for $\mathbf{P}$-almost all $\omega\in\Omega$. In other words, (19) describes the long-run behavior of the system under rational expectations.

This shows that unbiased predictors for linear laws must themselves be linear maps and, as a consequence, the evolution of a linear system under rational expectations itself is driven by an affine time-one map. In classical terminology, (18) can be viewed as a state-space representation of an ARMA process, whereas (19) corresponds to the so-called *steady state solution* of (18); see Hannan and Deistler (1988, Ch. 1). The affine case is easily generalized to the situation in which the perturbations (not necessarily i.i.d.) act on all coefficients of the map (16); see Appendix B.

We close the section with a remark indicating that an error function may be estimated from times-series data. If $\zeta_{t+1} := x_{t+1} - x^e_{t+1}$ denotes the forecast error induced by some point predictor ψ, where $x^e_{t+1} = \psi(x_t)$, the corresponding error process induced by the economic law (16) and the predictor is

$$\zeta_{t+1} = Ax_t + [B - I]\psi(x_t) + \xi_t, \qquad t \in \mathbf{N}. \tag{20}$$

Hence, agents believing in a linear economic law (16) should try to estimate the coefficients A and B from the time series $\{x_t, x^e_{t+1}, \zeta_{t+1}\}_{t=0}^T$ of the stochastic process (20).

6.2. Models of the Cobweb Type

Models of the Cobweb type, as already defined in (9), are the most popular economic dynamical systems for demonstrating the role of expectations in market systems. In their simplest version, they describe a one-dimensional price process. The essential feature, however, that characterizes them independently of the dimension of the state space is the fact that no state of the system enters as an argument into the economic law. Hence, apart from exogenous perturbations, the formation of expectations is the only source for the dynamic behavior in these systems. Using the natural identification of points in a state space with Dirac measures in the associated space of probability measures, we specialize (9) to a one-dimensional map $S : \mathbf{R}_+^2 \to \mathbf{R}_+$,

$$x_{t+1} = S\left(\xi_t, x^e_{t+1}\right) = F\left(x^e_{t+1}\right) + \xi_t, \tag{21}$$

where $F : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous map, $x^e_{t+1} \in \mathbf{R}_+$ is a point forecast for x_{t+1}, and $\{\xi_t\}_{t\in\mathbf{Z}}$ is a stationary ergodic process with values in $\mathbf{R}_+$. The mean forecast errors are given by

$$e_S^{\mathbf{E}}\left(\xi_{t-1}, x^e_{t+1}\right) = F\left(x^e_{t+1}\right) + \mathbf{E}[\xi_t \mid \xi_{t-1}] - x^e_{t+1}.$$

If the map $F - I$ is invertible, then there exists an unbiased predictor $\psi^* = (F - I)^{-1}$ with

$$e_S^{\mathbf{E}}\left(\xi_{t-1}, \psi^*(\mathbf{E}[\xi_t \mid \xi_{t-1}])\right) = 0.$$

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Inserting ψ^* into (21) yields a time-one map given by

$$x_{t+1} = \psi^*(\mathbf{E}[\xi_t \mid \xi_{t-1}]) - \mathbf{E}[\xi_t \mid \xi_{t-1}] + \xi_t,$$

which governs the dynamics under rational expectations. Clearly, the dynamics of the Cobweb model with an unbiased predictor are those of a pure stochastic process. For an i.i.d. noise process with zero mean, the dynamics reduce to $x_{t+1} = \psi^*(0) + \xi_t$, where $\psi^*(0)$ is the unique fixed point of F.

Notice again that both the conditional and the unconditional mean errors are zero and that the autocorrelation structure of the forecast errors coincides with the autocorrelation structure of the exogenous noise; that is, equation (15) holds. Since the latter is assumed to be ergodic, all sample means and sample autocorrelation coefficients converge to their true values with probability 1.

7. STOCHASTIC GROWTH WITH RATIONAL EXPECTATIONS

Consider a stochastic version of an OLG model of economic growth [see, e.g., Spear and Srivastava (1986), Spear (1998), Wang (1994) and references therein] in which the stochastic perturbations affect the technology. Output per capita y_t in period t is produced from per-capita capital stock k_t via a production function subject to an exogenously perturbed parameter ξ_{t-1}, such that $y_t = \xi_{t-1} \cdot f(k_t)$. The shock process $\{\xi_t\}_{t\in\mathbf{Z}}$ is driven by a strictly positive stationary ergodic process $\xi(\vartheta^t\cdot) : \Omega \to \Sigma := [\underline{\xi}, \infty) \subset \mathbf{R}_+, \underline{\xi} > 0$ on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\xi_t = \xi(\vartheta^t\omega)$ for all $\omega \in \Omega$. The production function f is assumed to be twice differentiable, strictly monotonically increasing, strictly concave, and to satisfy the Inada conditions. The generations of workers grow at a constant rate $n > -1$ and capital depreciates at a rate $d \in [0, 1]$.

Consumers consist of two-period-lived overlapping generations with preferences over consumption in both periods. Each member of a generation receives wage income only in the first period and saves to consume in the second period of his life. His intertemporal consumption/savings decision at date t is based on maximizing expected utility with respect to subjective probability distributions ν^e_{t+1} for the actual rate of return R_{t+1}. The real wage is determined by the marginal product of labor

$$w_t = w(\xi_{t-1}, k_t) := \xi_{t-1}[f(k_t) - k_t f'(k_t)]$$

at full employment in each period. Old consumers receive all profit income from production, which determines the actual rate of return on their savings.

Denoting by $s(w_t, \nu^e_{t+1})$ the savings function of a young consumer, one thus obtains two equations that describe an economic law in the sense of Section 2, namely,

$$\begin{cases} k_{t+1} &= S\big(\xi_{t-1}, k_t, \nu^e_{t+1}\big) := \frac{1}{1+n} s\big(w(\xi_{t-1}, k_t), \nu^e_{t+1}\big), \\ R_{t+1} &= 1 - d + \xi_t f'\big(S\big(\xi_{t-1}, k_t, \nu^e_{t+1}\big)\big). \end{cases} \tag{22}$$

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Two features are worth noting. Observe, first, that expectations affect the evolution of the capital intensities k_t only if the savings behavior depends on subjective beliefs about the expected rate of return. Second, both functions in (22) are independent of the actual rate of return R_t, implying that the two equations in (22) are only coupled via a forecasting rule.

Consider the case in which consumers use a once and for all fixed family of subjective probability distributions $\{\nu^e \equiv \nu_{R^e}\}_{R^e \in \mathbf{R}_+}$ parameterized by subjective means R^e. The savings function derived from maximizing expected utility may then be viewed as a function depending solely on the real wage w and R^e, such that $s(w, \nu_{R^e}) \equiv s(w, R^e)$. Observe that, from the consumer's perspective, the real wage w is a known quantity when deciding on the amount to save. By a slight abuse of notation, the mean error function $e_S^{\mathbf{E}}$ associated with the economic law (22) takes the form

$$e_S^{\mathbf{E}}\big(\mathbf{E}[\xi_t \mid \xi_{t-1}], w, R^e\big) = 1 - d + \mathbf{E}[\xi_t \mid \xi_{t-1}]\, f'\left(\frac{1}{1+n} s(w, R^e)\right) - R^e. \quad (23)$$

Since the actual rate of return does not enter $e_S^{\mathbf{E}}$, it is immediate from (23) that predictors that depend on R in a nontrivial manner cannot be unbiased. As in the deterministic case, it can be shown that uniquely determined unbiased predictors exist independently of the chosen technology, as long as the savings functions are nondecreasing in R^e. This result can be strengthened as follows.

THEOREM 2. *Consider the growth model as introduced above and assume that the following hypotheses hold*:

(i) *Consumers form expectations based on a fixed family of distributions* $\{\nu_{R^e}\}_{R^e \in \mathbf{R}_+}$.
(ii) *Consumption in both periods is of normal goods*.
(iii) *The production function* f *is twice differentiable, strictly monotonically increasing, strictly concave, satisfies the Inada conditions, and* $f(0) = 0$. *Moreover,* $E_{f'}(k) := kf''(k)/f'(k) > -1$ *for all* $k \in \mathbf{R}_+$; *that is, capital income* $kf'(k)$ *is increasing in* k.

Then there exists a unique unbiased Markov kernel Ψ^* *with values in* $\{\nu_{R^e}\}_{R^e \in \mathbf{R}}$.

Proof. Let $\xi, w \in \mathbf{R}_+, \xi \geq \underline{\xi}$ be arbitrary but fixed and for simplicity of notation let $n = 0$. Then $\lim_{R^e \to 0} e_S^{\mathbf{E}}(\xi, w, R^e) \geq 0$ and the concavity of f implies that

$$e_S^{\mathbf{E}}(\xi, w, R^e)\, s(w, R^e) \leq \xi f(s(w, R^e)) - [R^e - (1-d)] s(w, R^e).$$

Since both goods are normal, $R^e s(w, R^e)$ is monotonically increasing. Hence,

$$\lim_{R^e \to \infty} \left[e_S^{\mathbf{E}}(\xi, w, R^e) s(w, R^e)\right] \leq \lim_{R^e \to \infty} [\xi f(s(w, R^e)) - [R^e - (1-d)] s(w, R^e)] < 0,$$

where the negativity of the limit on the right-hand side follows from the boundedness of s and from $f(0) = 0$. Since $s \geq 0$, we have $\lim_{R^e \to \infty} e_S^{\mathbf{E}}(\xi, w, R^e) < 0$.

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This implies that, for each $\xi, w \in \mathbf{R}_+$, $\xi \geq \underline{\xi}$, there exists at least one solution $\bar{R}^e$ with $e_S^{\mathbf{E}}(\xi, w, \bar{R}^e) = 0$.

Suppose now that there exist two distinct solutions, say $\bar{R}_1^e > \bar{R}_2^e$ with $0 = e_S^{\mathbf{E}}(\xi, w, \bar{R}_1^e) = e_S^{\mathbf{E}}(\xi, w, \bar{R}_2^e)$. Then, $\xi[f'(s(w, \bar{R}_1^e)) - f'(s(w, \bar{R}_2^e))] = \bar{R}_1^e - \bar{R}_2^e > 0$ and the strict concavity of f implies that $s(w, \bar{R}_1^e) < s(w, \bar{R}_2^e)$. Since capital income is increasing according to Assumption (iii), one obtains

$$\begin{aligned} 0 &\geq \xi\left[f'\big(s\big(w, \bar{R}_1^e\big)\big)s\big(w, \bar{R}_1^e\big) - f'\big(s\big(w, \bar{R}_2^e\big)\big)s\big(w, \bar{R}_2^e\big)\right] \\ &> \bar{R}_1^e s\big(w, \bar{R}_1^e\big) - \bar{R}_2^e s\big(w, \bar{R}_2^e\big). \end{aligned}$$

This implies that $R^e s(w, R^e)$ cannot be monotonically increasing in R^e, which is a contradiction to Assumption (ii).

Since ξ and w were arbitrary, this implies that, for each $\xi_{t-1} \geq \underline{\xi}$ and each $w_t \in \mathbf{R}_+$, there exists uniquely an R_{t+1}^e such that $e_S^{\mathbf{E}}(\mathbf{E}[\xi_t \mid \xi_{t-1}], w_t, R_{t+1}^e) = 0$. This implies the existence of a unique unbiased mean-value predictor ψ^* with $R_{t+1}^e = \psi^*(\xi_{t-1}, w_t)$. Therefore, a unique unbiased Markov kernel Ψ^* with values in $\{\nu_e\}_{R^e \in \mathbf{R}_+}$ exists. ∎

COROLLARY 2. *Under the hypotheses of Theorem* 2, *if the savings function* s *is differentiable with respect to both arguments, then for fixed* $(\xi_{t-1}, k_t) \in \mathbf{R}_+^2$ *and* $w_t = w(\xi_{t-1}, k_t)$,

$$\begin{aligned} &\partial_w \psi^*(\xi_{t-1}, w_t) \\ &\quad = \frac{\mathbf{E}[\xi_t \mid \xi_{t-1}]\ f''\left(\dfrac{1}{1+n} s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)\right) \partial_w s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)}{1 + n - \xi_{t-1}\ f''\left(\dfrac{1}{1+n} s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)\right) \partial_R s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)} \end{aligned}$$

and

$$\begin{aligned} &\partial_k S_{\Psi^*}(\xi_{t-1}, k_t) \\ &\quad = \frac{\partial_w s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)\ \partial_k w(\xi_{t-1}, k_t)}{1 + n - \xi_{t-1}\ f''\left(\dfrac{1}{1+n} s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)\right)\ \partial_R s\big(w_t, \psi^*(\xi_{t-1}, w_t)\big)}, \end{aligned}$$

where S_{Ψ^*} *denotes the time-one map associated with* (22). *If, in addition* s *is nondecreasing in* R^e, *then* S_{Ψ^*} *is strictly increasing in* k.

Proof. The derivatives are obtained from (22) and implicit differentiation of (23). Observe that f is strictly concave and that s is strictly increasing in w, because both goods are normal. Therefore, S_{Ψ^*} is strictly increasing in k if s is nondecreasing in R^e. ∎

Theorem 2 shows that unbiased predictors exist under standard assumption on the preferences, the characteristics of goods, and the technology. The geometric

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intuition for the existence of unbiased predictors is analogous to the deterministic case given in Böhm and Wenzelburger (1999). To illustrate Theorem 2, consider consumers with a von Neumann–Morgenstern utility function of the form $u(c_1, c_2) = h(c_1) + c_2$, where h is a strictly monotonically increasing and strictly concave function. The savings function derived from maximizing expected utility with respect to a given subjective distribution ν_{R^e} is then given by

$$s(w(\xi, k), \nu_{R^e}) = \max\{0, \xi[f(k) - f'(k)k] - h'^{-1}(R^e)\}, \tag{24}$$

where R^e is the subjective mean of ν_{R^e}. Since h is strictly concave, (2) is nondecreasing in R^e. As a consequence of Theorem 2, consumers with additive separable utility functions of the form (24) may have unbiased predictions independently of the chosen technology *and* independently of the chosen family of probability distributions. This includes all distributions with mean preserving spread.

The next theorem shows that, under certain conditions, the stochastic growth model (22) under unbiased predictions admits a unique random fixed point that is globally attracting. Thus, the long-run behavior is described by a stable stationary stochastic process of the capital intensity. As a consequence, rates of return will eventually follow a stationary stochastic process.

THEOREM 3. *Let the hypotheses of Theorem* 2 *be satisfied and the savings function* s *be differentiable with respect to both arguments. Let* $\mathbf{E}\,\xi < \infty$ *and assume, in addition, that the following conditions hold*:

(i) *There exists a continuous function* $\underline{S} : \mathbf{R}_+ \to \mathbf{R}_+$ *with a unique positive fixed point* $\bar{k} = \underline{S}(\bar{k}) > 0$ *such that* $S_{\Psi^*}(\xi, k) \geq \underline{S}(k)$ *for all* $\xi \in \Sigma$, $k \in \mathbf{R}_+$. *The set* $\Sigma' := \{\xi \in \Sigma \mid S_{\Psi^*}(\xi, \cdot) > \underline{S}\}$ *has positive measure*, $\mathbf{P}(\Sigma') > 0$.

(ii) *There exists a random variable* c *with* $\mathbf{E}\,c < 0$, *such that*

$$\sup\{\log|\partial_k S_{\Psi^*}(\xi(\omega), k)| : \bar{k} \leq k < \infty\} \leq c(\omega), \qquad \omega \in \Omega.$$

Then there exists a unique nontrivial random fixed point $k^* : \Omega \to \mathbf{R}_+$ *which is asymptotically stable in the sense of Definition* 3 *globally on* $\mathbf{R}_{++}$. *Moreover,* k^* *is measurable with respect to the sub-*σ*-algebra of the past* $\mathcal{F}^- := \sigma\{\varphi(t, \vartheta^{-T}\cdot, k) \mid 0 \leq t \leq T,\ k \in \mathbf{R}_+\} \subset \mathcal{F}$, *where* φ *denotes the random dynamical system generated by* S_{Ψ^*}.

The proof is an application of Theorem A.1 due to Schmalfuss (1998, 1996) and a modification of the proof of Theorem 4.2 in Schenk-Hoppé and Schmalfuss (1998)[5] and given in Appendix A. Assumption (i) is satisfied if S_{Ψ^*} is increasing in $\xi \in \Sigma$ and $\underline{S} = S_{\Psi^*}(\underline{\xi}, \cdot)$. Assumption (ii) is called an average contractivity condition for S_{Ψ^*}. A sufficient condition for Assumption (ii) is $|\partial_k S_{\Psi^*}(\xi(\omega), k)| < 1 - \delta$, $\delta > 0$ for all $k \in [\bar{k}, \infty)$ **P**–a.e. According to Corollary 2, this is the case when the income effect is less than the interest-factor effect.

Figures 1A and B illustrate two possible situations, in which, first, a unique random fixed point and, second, two stable random fixed points may occur. Each diagram shows the graph of the time-one map for an arbitrary realization of the

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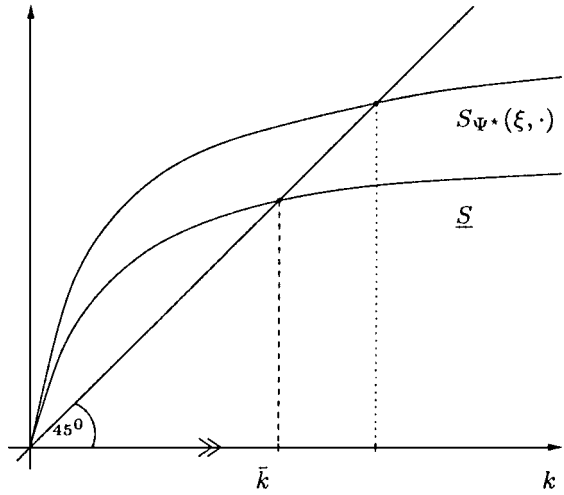


FIGURE 1A. Unique random fixed point.

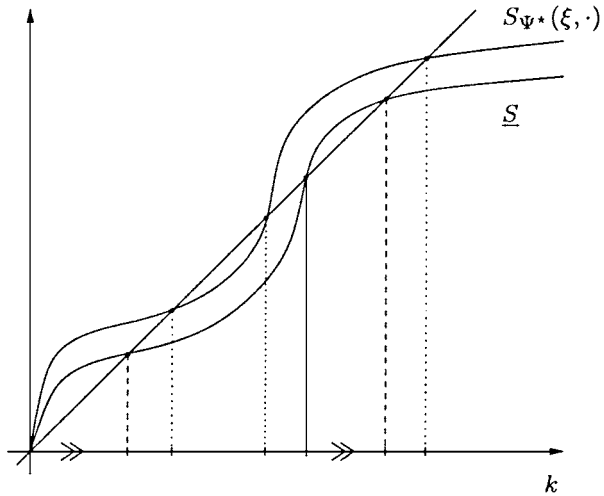


FIGURE 1B. Multiple random fixed points.

productivity shock and the graph of the lower bound $\underline{S}$, respectively. Figure 1A corresponds to the situation in which Assumptions (i) and (ii) are fulfilled, implying in particular that unique deterministic steady states exist for each realization of the shock. A violation of either condition may lead to the situation in Figure 1B. The corresponding domains of attraction are indicated by double arrows.

8. CONCLUSIONS

The theory of random dynamical systems is the appropriate framework for studying dynamic economic models with an expectations feedback which, in addition,

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are subject to exogenous noise. The careful distinction between an economic law and a forecasting rule was the necessary step to obtain a proper description of an economic random dynamical system. Modeling a forecasting rule as a Markov kernel is one key ingredient for obtaining a forward recursive structure. The interaction of such a forecasting rule with the economic law constitutes the fundamental conceptual difference between the global approach presented in this paper and the classical equilibrium analysis. Instead of being restricted to special solutions on a purely distributional level, it allows for an analysis of all possible sample paths of an economic model with exogenous noise. As in the deterministic case, the concept of an error function turned out to be the appropriate tool to evaluate the performance of forecasting rules along orbits of the system.

The results of this paper confirm that the property of rational expectations along orbits of an economic random dynamical system imposes structural restrictions on the economic law. The class of perfect predictors that generate rational expectations are themselves time-independent Markov kernels, that is, time-invariant updating procedures. The existence of perfect Markov kernels is equivalent to the existence of a global inverse of the error function associated with the system. This error function depends exclusively on the fundamentals of the economy and is independent of any expectations formation procedure. The amount of information necessary to construct a perfect Markov kernel requires detailed knowledge of the whole economic system. This in turn is tantamount to the ability to compute the global inverse of the corresponding error function. The discussion of the economic examples indicates that these requirements are not a universal feature of economic systems, such that the property of perfection may well be the exception rather than the rule. However, since the geometric features of the economic law are of a static nature, this opens up the possibility to estimate unbiased forecasting rules from time-series data. Hence, in principle, they can be learned from historical data, whenever they exist.

NOTES

1. The extension of this formulation to heterogeneous expectations is straightforward. To keep the presentation within a reasonable length, we restrict ourselves to systems with a single agent who may be thought of as a forecasting agency. As in the deterministic case [Böhm and Wenzelburger (2000)], it should be possible to deal with expectational leads in the same fashion.

2. A pseudometric on $\mathcal{M}(X)$ is, by definition, a real-valued function ρ defined on $\mathcal{M}(X) \times \mathcal{M}(X)$, which satisfies (i) $\rho(\mu, \nu) \geq 0$ and $\rho(\mu, \mu) = 0$, (ii) $\rho(\mu, \nu) = \rho(\nu, \mu)$, and (iii) $\rho(\mu, \delta) \leq \rho(\mu, \nu) + \rho(\nu, \delta)$ for all $\mu, \nu, \delta \in \mathcal{M}(X)$. If, in addition, (iv) $\rho(\mu, \nu) = 0$ iff $\mu = \nu$, then ρ is called a metric on $\mathcal{M}(X)$.

3. Random fixed points can also be defined via a special class of invariant measures for the random dynamical system φ, called the *random Dirac measures*; see Arnold (1998, p. 25).

4. In the context of random dynamical systems, the term almost surely (a.s.) is used in a nonstandard sense: A property holds a.s. if there exists a ϑ-invariant set $\Omega' \subset \Omega$ ($\vartheta \Omega' = \Omega'$) such that the property holds for all $\omega \in \Omega'$.

5. We would like to thank an anonymous referee for pointing out to us that Theorem 4.2 implies directly that there exists a unique globally attracting random fixed point in the optimal growth model

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of Brock and Mirman (1972). This illustrates that random fixed points will most likely exist in these types of growth models also.

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APPENDIX A: RANDOM FIXED-POINT THEOREM

This section briefly reviews a simplified version of an existence theorem for random fixed points due to Schmalfuss (1996, 1998). See Schenk-Hoppé and Schmalfuss (1998) for additional information and a first economic application. Let $G(\omega) \subset \mathbf{R}^d, \omega \in \Omega$ be a random set; that is, each $G(\omega)$ is closed a.s. and $\{\omega \in \Omega \mid G(\omega) \cap U \neq \emptyset\}$ is measurable for all open sets U [Arnold (1998, Prop. 1.6.2)]. Consider random variables $g : \Omega \to \mathbf{R}^d$ which are tempered, meaning that

$$\lim_{t\to\infty} e^{-\delta t}\|g(\vartheta^t\omega)\| = 0 \quad \text{for all } \delta > 0.$$

Let

$$\mathcal{G} := \{g : \Omega \to \mathbf{R}^d \mid g \text{ is tempered and } g(\omega) \in G(\omega),\ \omega \in \Omega\}$$

denote the set of all tempered random variables with values $g(\omega)$ in $G(\omega)$.

THEOREM A.1. *Let $\varphi : \mathbf{N} \times \Omega \times X \to X, X \subset \mathbf{R}^d$ be a random dynamical system over an ergodic dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, \{\vartheta^t\}_{t\in\mathbf{Z}})$ and assume, in addition, that the time-one map $x \mapsto \varphi(1, \omega, x)$ is continuously differentiable a.s. Suppose there exists a random set $G(\omega) \subset X, \omega \in \Omega$, such that $\mathcal{G}$ is nonempty and the following holds*:

(i) *The map $\omega \mapsto \varphi(1, \vartheta^{-1}\omega, g(\vartheta^{-1}\omega))$ is contained in $\mathcal{G}$ for all $g \in \mathcal{G}$;*
(ii) $\sup_{x\in G(\omega)} \log\|\partial_x\varphi(1, \omega, x)\| \leq c(\omega), \omega \in \Omega$ *for some random variable c with $\mathbf{E}c < 0$;*
(iii) *if, for some $g \in \mathcal{G}$, $\{\varphi[t, \vartheta^{-t}\omega, g(\vartheta^{-t}\omega)]\}_{t\in\mathbf{N}}$ is a Cauchy sequence for each $\omega \in \Omega$, then its limit is contained in $\mathcal{G}$.*

Then there exists a unique random fixed point $g^* \in \mathcal{G}$ of φ with

$$\lim_{t\to\infty} \|\varphi(t, \omega, g(\omega)) - g^*(\vartheta^t\omega)\| = 0 \qquad \text{for all } g \in \mathcal{G},\ \text{a.s.}$$

If the noise process is trivial, i.e., if φ is independent of ω and hence of ϑ, then Theorem A.1 becomes a Banach fixed-point theorem. Conditions (i)–(iii) are an invariance, an average contractivity, and a completeness assumption, respectively; $\mathcal{G}$ is a subset of the domain of attraction of g^*. The long-run behavior of all sample paths with initial values $g(\omega) \in G(\omega)$ is completely determined by the orbit $\{g^*(\vartheta^t\omega)\}_{t\in\mathbf{N}}$. In particular, the ergodicity of the noise process implies that the empirical means satisfy

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$$\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}\varphi[t,\omega,g(\omega)]=\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}g^*(\vartheta^t\omega)=\mathbf{E}\,g^*,\quad \mathbf{P}\text{–a.s.}$$

and thus are constant a.s., provided $\mathbf{E}\,g^*$ is finite.

Proof of Theorem 3. The proof is an application of Theorem A.1 and a modification of the proof of Theorem 4.2 in Schenk-Hoppé and Schmalfuss (1998). We check the assumptions of Theorem A.1.

1. Observe that each $S_{\Psi^*}(\xi,k),\xi\in\Sigma$ is bounded by $1/(1+n)\xi f(k)$. The concavity of f and Condition (i) then implies that $\underline{S}(k)>k$ for all $k\in(0,\bar{k})$. Condition (i) implies in particular that $\bar{k}$ is a lower bound for the deterministic fixed points of all the maps $S_{\Psi^*}(\xi,\cdot),\xi\in\Sigma$. Set $G:=[\bar{k},\infty)$. Then $G(\omega)\equiv G,\omega\in\Omega$ is a random set and the set of all tempered random variables $\mathcal{G}$ associated with G is nonempty since it contains constant functions with values above $\bar{k}$.
2. Check of Assumption (i). Clearly, G is forward invariant under S_{Ψ^*} meaning that $k\in G$ implies $S_{\Psi^*}(\xi(\omega),k)\in G$ for all $\omega\in\Omega$. Since f is concave,

$$S_{\Psi^*}(\xi(\omega),k)\le\frac{1}{1+n}\xi(\omega)f(k)\le\frac{1}{1+n}\xi(\omega)[f'(y)k+f(y)]\quad\text{for any } y>0, \tag{A.1}$$

   and $\xi(\cdot)$ is tempered [this follows from $\mathbf{E}\,\xi<\infty$; see Arnold (1998, Ch. 4)], the map $\omega\mapsto S_{\Psi^*}(\xi(\vartheta^{-1}\omega),g(\vartheta^{-1}\omega))$ is tempered because it is bounded by products and sums of tempered maps. Hence, Assumption (i) is satisfied.
3. Condition (ii) is a mere restatement of Assumption (ii) in Theorem A.1.
4. Check of Assumption (iii). Let $\{\varphi(t,\vartheta^{-t}\omega,g(\vartheta^{-1}\omega))\}_{t\in\mathbf{N}}$ be a Cauchy sequence for some $g\in\mathcal{G}$ and all ω, where φ denotes the random dynamical system (6) generated by S_{Ψ^*}. Then its limit $\lim_{t\to\infty}\varphi(t,\vartheta^{-t}\omega,g(\vartheta^{-1}\omega))$ is in G for all ω because G is forward invariant and complete. It remains for us to show that this limit is contained in $\mathcal{G}$. The right-hand side of (A.1) defines an affine random difference equation

$$k_{t+1}=\frac{\xi(\vartheta^t\omega)f'(y)}{1+n}k_t+\frac{\xi(\vartheta^t\omega)f(y)}{1+n}$$

   which has a globally attracting random fixed point for any $y>0$ with $\mathbf{E}\log(\frac{\xi f'(y)}{1+n})<0$; cf. Arnold (1998, Theorem 5.6.5). Since $\lim_{y\to\infty}f'(y)=0$, this condition can always be satisfied, and it follows from Theorem 5.6.5 of Arnold (1998) that this random fixed point is tempered and hence attracts all tempered random variables. Now, (A.1) implies that $\lim_{t\to\infty}\varphi(t,\vartheta^{-t}\omega,g(\vartheta^{-1}\omega))$ is bounded by a tempered random variable and hence must itself be tempered.
5. Global attraction. It follows from (1)–(4) and Theorem A.1 that there exists a unique random fixed point $k^*\in\mathcal{G}$ whose domain of attraction contains the set $G=[\bar{k},\infty)$. Since $\mathbf{P}(\Sigma')>0$ by assumption, the ergodicity of ϑ implies $\lim_{T\to\infty}1/T\sum_{t=0}^{T}1_{\Sigma'}(\xi(\vartheta^t\omega))=\mathbf{P}(\Sigma')>0$. Hence, $\xi_t=\xi(\vartheta^t\omega)\in\Sigma'$ for infinitely many t, $\mathbf{P}$–almost everywhere on Ω. This implies that, for each initial condition $k_0<\bar{k}$, there exists some finite time $T_0=T_0(\omega,k_0)<\infty$ such that $\varphi(t,\omega,k_0)>\bar{k}$ for all $t\ge T_0$. Therefore, k^* is a unique nontrivial random fixed point that is asymptotically stable globally on $\mathbf{R}_{++}$.

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6. Measurability with respect to the past. Since $\mathcal{G}$ contains all constant functions $g(\omega) = k$, $k \geq \bar{k}$ and k^* is globally attracting on $\mathbf{R}_+$ by (5), the stationarity of ϑ implies that $\varphi(T, \vartheta^{-T}\omega, k)$ converges a.s. to $k^*(\omega)$ as $T \to \infty$. Hence, k^* is measurable with respect to the sub-σ-algebra $\mathcal{F}^-$. ■

APPENDIX B: THE GENERAL AFFINE CASE

Let $X = \mathbf{R}^d$ (or $\mathbf{R}^d_+$) and consider an affine economic law

$$S : \Sigma \times X \times X \to X, \qquad ((A, B, b), x, x^e) \mapsto Ax + Bx^e + b, \tag{B.1}$$

where A and B are coefficient matrices, $b \in X$, and $x^e \in X$ is a point prediction for $Ax + Bx^e + b$. The law (B.1) fits into the general setup (1) introduced in Section 2 by identifying points x^e in X with Dirac measures δ_{x^e} in $\mathrm{Prob}(X)$. Notice that for fixed A, B, b, the map S is affine. Many linear economic models with rational expectations [see, e.g., Evans and Honkapohja (1999)] are special cases of (B.1). However, the perturbation here may act on A, B, and b simultaneously, including so-called Markov switching models.

Let $\{(A_t, B_t, b_t)\}_{t\in\mathbf{Z}}$ be a stationary and ergodic stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in an appropriately defined space Σ. Then, for any $x, x^e \in X$ and any arbitrary date t, the mean error function of S is given by

$$e_S^{\mathbf{E}}\left(A_{t-1}, B_{t-1}, \boldsymbol{b}_{t-1}, x, \boldsymbol{x}^e\right) = \mathbf{E}[A_t \mid A_{t-1}]\, x + [\mathbf{E}[B_t \mid B_{t-1}] - I]x^e + \mathbf{E}[b_t \mid b_{t-1}].$$

Clearly, $e_S^{\mathbf{E}}$ is $\mathbf{P}$–a.e. globally invertible iff $I - \mathbf{E}[B_t \mid B_{t-1}]$ is invertible $\mathbf{P}$–a.e. for all t. By invoking Proposition 1, this implies that

$$\psi^*(A_{t-1}, B_{t-1}, b_{t-1}, x) = [I - \mathbf{E}[B_t \mid B_{t-1}]]^{-1}(\mathbf{E}[A_t \mid A_{t-1}]x + \mathbf{E}[b_t \mid b_{t-1}]), \qquad x \in X$$

is a globally unbiased predictor for S. Note that $x^e_{t+1} = \psi^*(A_{t-1}, B_{t-1}, b_{t-1}, x_t)$ is an unbiased forecast for x_{t+1} such that the conditional forecast error vanishes, $\mathbf{E}[(x_{t+1} - x^e_{t+1})|A_{t-1}, B_{t-1}, b_{t-1}] = 0$. Inserting ψ^* into the economic law (B.1) yields the time-one map S_{ψ^*} in the sense of Section 2, given by

$$\begin{aligned} x_{t+1} = {} & \left(A_t + B_t[I - \mathbf{E}[B_t \mid B_{t-1}]]^{-1}\mathbf{E}[A_t \mid A_{t-1}]\right)x_t \\ & + \left(B_t[I - \mathbf{E}[B_t \mid B_{t-1}]]^{-1}\right)\mathbf{E}[b_t \mid b_{t-1}]\big) + b_t. \end{aligned} \tag{B.2}$$

The map (B.2) defines an affine random dynamical system in the sense of Arnold (1998, Ch. 5) describing the dynamics under rational expectations. Observe again that these dynamics are defined globally on $\mathbf{R}^d$ and that the conditional mean forecast errors vanish along all orbits of the system. It is well known [Arnold (1998, Ch. 5)] that the map (B.2) admits a unique random fixed point x^* which, in the stable case [see Arnold (1998, Corollary 5.6.6)], is given by the random variable

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$$x^*(\omega) = \sum_{n=0}^{\infty} \Phi(n, \vartheta^{-n}\omega)\, \eta\left(\vartheta^{-(n+1)}\omega\right), \qquad \omega \in \Omega, \tag{B.3}$$

where

$$\Phi(n, \omega) := \prod_{i=0}^{n} (A(\vartheta^i\omega) + B(\vartheta^i\omega)[I - \mathbf{E}[B(\vartheta^i\cdot)|B(\vartheta^{i-1}\omega)]]^{-1}\mathbf{E}[A(\vartheta^i\cdot)|A(\vartheta^{i-1}\omega)])$$

and

$$\eta(\omega) := (B(\omega)[I - \mathbf{E}[B(\cdot)|B(\vartheta^{-1}\omega)]]^{-1})\mathbf{E}[b(\cdot)|b(\vartheta^{-1}\omega)] + b(\omega).$$

Here again, we make use of the fact that a stationary ergodic process has a representation as an ergodic dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, \{\vartheta^t\}_{t\in\mathbf{Z}})$ with measurable maps $A, B, b : \Omega \to \Sigma$ such that $A_t = A(\vartheta^t\omega)$, $B_t = B(\vartheta^t\omega)$, and $b_t = b(\vartheta^t\omega)$. The global asymptotic stability property of the random fixed point (B.3) implies that all orbits $\gamma(x_0) = \{x_t\}_{t\in\mathbf{N}}$, $x_0 \in \mathbf{R}^d$ of (B.2) converge to orbits of the stationary process $\{x^*(\vartheta^t\omega)\}_{t\in\mathbf{N}}$ for $\mathbf{P}$-almost all $\omega \in \Omega$. In other words, (B.3) describes the long-run behavior of the system under rational expectations. This shows that unbiased predictors for affine laws must themselves be affine maps and, as a consequence, the evolution of a linear system under rational expectations itself is driven by an affine time-one map.

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