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109.38 A visual proof of convergence of a hyperpower sequence

In 2004, Azarpanah ([1]) wordlessly established that

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}} = 2.$$

Recently, Mukherjee ([2]) gave a wordless proof of convergence of a hyperpower sequence:

$$\left(n^{\frac{1}{n}}\right)^{\left(n^{\frac{1}{n}}\right)^{\left(n^{\frac{1}{n}}\right)^{\cdot^{\cdot}}}} = -\frac{nW_0\left(-\frac{\ln n}{n}\right)}{\ln n} \text{ for } n \in \mathbb{N}\setminus\{1\},$$

where W_0 denotes the principal branch of the Lambert W function ([3]). Note that for $n = 2$, $-\frac{nW_0\left(-\frac{\ln n}{n}\right)}{\ln n} = 2$. In this Note, we visually prove a surprising result:

$$\left(a^{\frac{1}{a}}\right)^{\left(a^{\frac{1}{a}}\right)^{\left(a^{\frac{1}{a}}\right)^{\cdot^{\cdot}}}} = a, \text{ where } 1 < a \leqslant e.$$

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Proof:

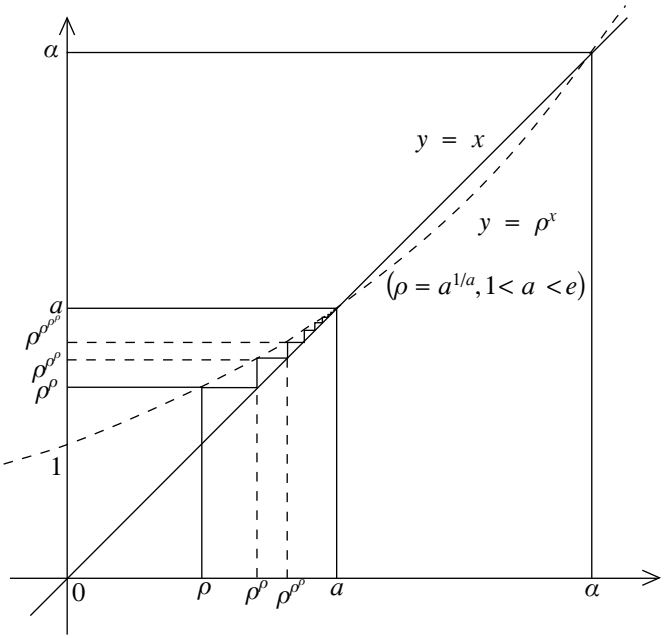


FIGURE 1: Visual proof for the case $1 < a < e$.

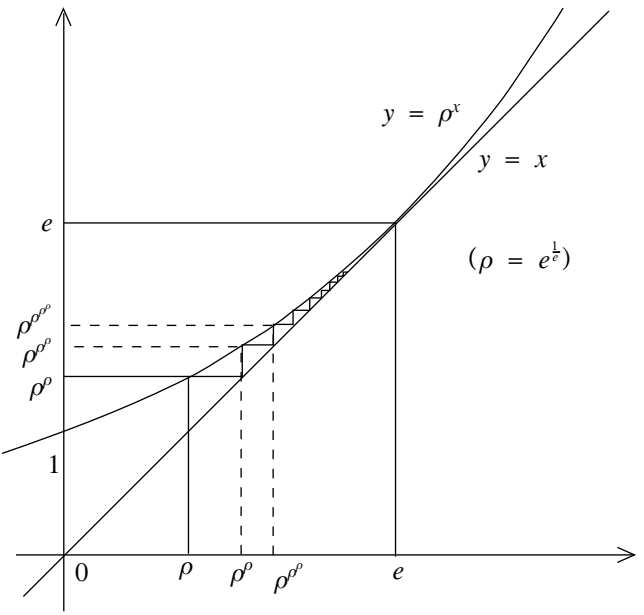
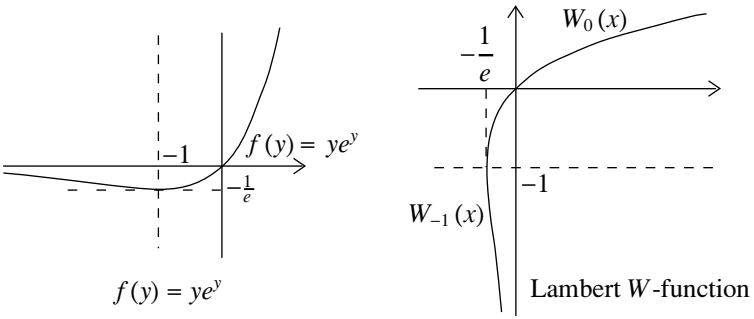


FIGURE 2: Visual proof for the case $a = e$.

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The Lambert W-function

The inverse of the real function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(y) = ye^y$, is called the Lambert W-function ([3, 4]).



Clearly f is not injective over all of $\mathbb{R}$. But, f is injective in each sub-interval, $(-\infty. -1]$ and $[-1, \infty)$. Again, the global minimum for f is at $y = -1$ and $f(-1) = -\frac{1}{e}$.

Thus the inverse of f on the interval $[-1, \infty)$ is a single-valued function defined on $[-\frac{1}{e}, \infty)$, denoted by W_0, and called the principal branch of the Lambert-W function, and the inverse of f on the interval $(-\infty, -1]$ is again a single-valued function defined on $[-\frac{1}{e}, 0)$, denoted by W_{-1}, and called the negative principal branch of the Lambert-W function.

At $x = -\frac{1}{e}$, the two branches of the function give the same value, i.e. $W_0(-\frac{1}{e}) = W_{-1}(-\frac{1}{e}) = -1$. Now, from the definition, it is clear that

$$W_0\left(xe^x\right) = x \text{ for } x \geqslant -1 \text{ and } W_{-1}\left(xe^x\right) = x \text{ for } x \leqslant -1.$$

Next, substituting $-\ln x$, we get

$$W_0\left(-\frac{\ln x}{x}\right) = -\ln x, \text{for} 0 < x \leqslant e \text{and} W_{-1}\left(-\frac{\ln x}{x}\right) = -\ln x, \text{for} x \geqslant e. \quad (1)$$

Let $1 < a \leqslant e$. Since $\left(a^{\frac{1}{a}}\right)^x = x$ can be written as

$$\left(-\frac{\ln a}{a}x\right)e^{-\frac{\ln a}{a}x} = \left(-\frac{\ln a}{a}\right).$$

Thus $\left(a^{\frac{1}{a}}\right)^x = x$ has the solutions $\alpha' = -\frac{a}{\ln a}W_0\left(-\frac{\ln a}{a}\right)$ and $\alpha = -\frac{a}{\ln a}W_{-1}\left(-\frac{\ln a}{a}\right)$. If $a = e$, then $\alpha' = e = \alpha$ and, if $1 < a < e$, then

$$\alpha' = -\frac{a}{\ln a}W_0\left(-\frac{\ln a}{a}\right) = -\frac{a}{\ln a}(-\ln a) = a.$$

For example, if we consider $a = 2$, then the solutions of the equation $x = \sqrt{2}^x$ are $-\frac{2}{\ln 2}W_0\left(-\frac{\ln 2}{2}\right)$ and $-\frac{2}{\ln 2}W_{-1}\left(-\frac{\ln 2}{2}\right)$. Now, from (1), we can

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calculate that $-\frac{2}{\ln 2}W_0\left(-\frac{\ln 2}{2}\right) = 2$ and

$$-\frac{2}{\ln 2}W_{-1}\left(-\frac{\ln 2}{2}\right) = -\frac{4}{\ln 4}W_{-1}\left(-\frac{\ln 4}{4}\right) = 4.$$

Thus, here $\alpha = 4$.

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109.39 Depressing cubics and barycentres

Introduction

The saga chronicling the resolution of cubic equations unravelled when Cardano observed that any cubic polynomial, $p(x) = ax^3 + bx^2 + cx + d$, can be depressed* (see [1] for a captivating summary) by performing the substitution

$$x = y - \frac{b}{3a}. \tag{1}$$

Equations involving depressed cubics, i.e. cubics whose quadratic term is null, are easier to solve both geometrically and algebraically and were already understood by Tartaglia and del Ferro.

When introducing cubic equations, transformation (1) is often presented as a *deus ex machina*. Given the importance of depression in solving equations of arbitrary degree (in the quadratic case, it is equivalent to completing the square) it is surprising that it is generally described as an *ad hoc* algebraic transformation without further ado. Let us explore the mechanism behind this transformation.

* Latin *deprimere*, to press down. The term was coined by Viète.

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