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Additive arithmetic functions on intervals

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§1. A real-valued arithmetic function f is *additive* if it satisfies the relation $f(ab) = f(a)+f(b)$ for all mutually prime positive integers a, b. In the present paper I establish three theorems concerning the value distribution of such functions on intervals.

For real $x \geqslant y \geqslant 1$ and z the distribution function

$$\nu_{x,y}(n; f(n) \leqslant z)$$

is defined to be the frequency amongst the integers n, in the interval $x-y < n \leqslant x$, of those for which the inequality $f(n) \leqslant z$ is satisfied.

THEOREM 1. *Assume that* $1 \leqslant y \leqslant x, \log y/\log x \to 1$ *as* $x \to \infty$. *Then there are constants* $\gamma = \gamma(x, y)$ *so that the frequencies*

$$\nu_{x,y}(n; f_x(n)-\gamma \leqslant z) \tag{1}$$

approach the improper law at the origin, as $x \to \infty$, *if and only if there are further constants* λ *so that* $\lambda y/x \to 0$, *and*

$$\min_{\mu}\left(\mu + \sum_{q \leqslant x} \frac{1}{q} \min\left(1, |f(q)-(\lambda+\mu)\log q|^2\right)\right) \to 0 \tag{2}$$

as $x \to \infty$.

Remarks. Here $f_x(n)$ is an additive function, defined anew for each value of x, and q denotes a typical prime-power.

If $y = x$, then the constants λ can be omitted from the formulation, since λ^2 and $(\lambda+\mu)^2$ can only approach zero together. This particular case of the theorem was established by Ruzsa[**10**]. I shall apply Ruzsa's result towards the end of the proof of Theorem 1 itself. However, the reduction of the general case to the special case is implicitly quite as elaborate as Ruzsa's original proof.

Although γ might be viewed as a function of x, y, I have adopted the traditional nomenclature of the theory of probability, and regarded it as a translation constant for the distribution function $\nu_{x,y}(n; f(n) \leqslant z)$. In Theorems 2, 3 the variable x may be arbitrarily confined to a prescribed (unbounded) set of values. The terminology *constant*, or occasionally *parameter*, then appears appropriate.

THEOREM 2. *Assume that* $1 \leqslant y \leqslant x, \log y/\log x \to 1$, *as* $x \to \infty$ *in some manner. Then there are constants* $\gamma = \gamma(x, y)$ *so that for the same* x

$$\nu_{x,y}(n; f_x(n)-\gamma \leqslant z) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du \tag{3}$$

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if and only if there are further constants $\lambda = \lambda(x)$, *for the same* x*-values, so that* $\lambda y/x \to 0$ *and*

$$\nu_{x,x}(n; f_x(n) - \lambda \log n + \lambda \log x - \gamma \leqslant z) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du. \tag{4}$$

Remarks. The weak convergence indicated by $\Rightarrow$ denotes standard convergence at every point z which is a point of continuity of the limiting distribution.

When considering convergence to a general normal law, we may renormalize and reduce ourselves to the present situation. At the moment we do not have a simple criterion for the weak convergence of the frequencies $\nu_{x,x}$ unless something more is known about the additive function $f_x(n)$. See, for example, Elliott[2], [3].

Theorems 1 and 2 are deduced from the following more general result. For two distribution functions F, G, $F * G$ will denote the convolution

$$(F * G)(z) = \int_{-\infty}^{\infty} F(z-u)\, dG(u).$$

Let $F_x(z)$ denote the distribution function $y^{-1}\{\min(x, e^z) - \min(x-y, e^z)\}$. Let H be a distribution function whose characteristic function does not vanish on the real line.

THEOREM 3. *Assume that* $1 \leqslant y \leqslant x, \log y/\log x \to 1$, *as* $x \to \infty$ *in some manner. Then there are constants* $\gamma = \gamma(x, y)$ *so that for these* x*-values*

$$\nu_{x,y}(n; f_x(n) - \gamma \leqslant z) \Rightarrow H(z) \tag{5}$$

as $x \to \infty$, *if and only if there are further constants* $\lambda = \lambda(x), |\lambda| \leqslant x^{\frac{1}{3}}$, *so that*

$$F_x(\lambda^{-1}(z+\gamma)) * \nu_{x,x}(n; f_x(n) - \lambda \log n \leqslant z) \Rightarrow H. \tag{6}$$

Remark. The class of possible limit laws H, in this theorem, includes the infinitely divisible laws.

This paper is in eight sections. The next four contain background material and preparatory steps for the proofs of these theorems.

§2. During this section and what follows, g will denote a complex-valued multiplicative function which satisfies $|g(n)| \leqslant 1$ for all positive n. It will be convenient to use von Mangoldt's function $\Lambda(n)$, which is $\log p$ if n is a power of the prime p, and zero otherwise. For $n > 1$ define $\Lambda_1(n) = \Lambda(n)/\log n$, and set $\Lambda_1(1) = 0$. For $x \geqslant 1$,

$$M(g, x) = x^{-1} \sum_{n \leqslant x} g(n).$$

LEMMA 1. *There is an absolute constant* c_1 *so that*

$$|M(g, x)| \leqslant c_1 \left(\exp\left(-\tfrac{1}{5} w\left(1 + \frac{1}{\log x}\right)\right) + T^{-\frac{1}{5}} \right)$$

with

$$w(\sigma) = \min_{|\tau| \leqslant T} \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^{\sigma}} \left(1 - \operatorname{Re} \frac{g(n)}{n^{i\tau}}\right)$$

holds uniformly for all $x \geqslant 2$, $T \geqslant 2$, g. *The variable* τ *is confined to real values.*

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Proof. A form of this lemma is due to Halász [5]. It represents a typical result which may be obtained by applying his method of estimating the coefficient sums of Dirichlet series which have Euler products. Variants of it are possible. For a proof of the present inequality see Elliott ([3], lemma 6·10).

LEMMA 2. *Let* $|g(n)| = 1$ *for all* n, *let* b *be a real number, and*

$$L = \sum_{p \leqslant x} \frac{|g(p)-1|}{p}.$$

Then for $x \geqslant 2$

$$\sum_{n \leqslant x} g(n)\, n^{ib} = \frac{x^{ib}}{1+ib} \sum_{n \leqslant x} g(n) + O(xe^{L/2}(\log x)^{-\frac{1}{2}}(\log(|b|+\log x))^2),$$

with an absolute implied constant.

Proof. This result is due to Ruzsa ([11], lemma 8·6). Ultimately, it forms an essential ingredient of his proof of the case $y = x$ of Theorem 1. It is derived by the method of Halász.

Remark. In order that the error term not swamp the main term, it is required that L and b not be too large in comparison with x.

LEMMA 3. *Let* $|g(n)| \leqslant 1$ *for all* $n, x \geqslant y \geqslant 3$. *Then*

$$y^{-1} \sum_{x-y<n\leqslant x} g(n) = M(gn^{i\alpha}, x)\, y^{-1} \int_{x-y}^{x} t^{-i\alpha}\, dt + O(R(x,y)),$$

where

$$R(x,y) = \log\left(\frac{2\log x}{\log 2x/y}\right)^{-\frac{1}{4}}$$

and α *is any real number,* $|\alpha| \leqslant x$, *for which* $|M(gn^{i\alpha}, x)|$ *is maximal.*

Proof. This result is due to Hildebrand [6]. It does not apply the method of Halász. Instead, by employing an appropriate Turán–Kubilius dual, an approximate functional equation is introduced, a multiplicative analogue of that which appears in Elliott [1], chapter 9.

Remark. The parameter α depends upon g as well as x. Moreover, we cannot directly estimate $M(gn^{i\alpha}, x)$ by lemma 2, since L may be too big.

§3. In this section I prove that if

$$\operatorname{Re} \sum_{n \leqslant x} \frac{\Lambda_1(n)}{n}(1-n^{-i\tau})$$

is small, then so is τ. The result is slightly more ambitious than needed here, but is of independent interest.

LEMMA 4. *Let* $c > 0$. *Then*

$$\sum_{n \leqslant x} \Lambda(n)\, n^{-1-i\tau} = \frac{x^{i\tau}}{i\tau} - \frac{\zeta'}{\zeta}(1+i\tau) + O(\exp(-(\log x)^{\frac{1}{2}}))$$

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uniformly for real $\tau, 0 < |\tau| \leqslant x^c$. *If* $\tau = 0$, *the two leading terms in this estimate are to be replaced by* $\log x + c_2$, *for a constant* c_2.

Proof. The error-term in this estimate can be improved. The result asserted here may be obtained, with simple adjustments, from Titchmarsh [**12**], §3·14, p. 56, by employing a Vinogradov-type estimate for the zero-free region of the Riemann zeta function $\zeta(s)$, as is provided in §6·15, p. 114 of the same reference.

LEMMA 5. $\operatorname{Sin}\theta \leqslant \theta - \theta^2/24$, *uniformly for* $0 \leqslant \theta \leqslant \frac{5}{4}$.

Remark. The essential feature of this inequality is that the uniformity in θ allows values $\theta > 1$.

Proof. From its Taylor series expansion at the origin

$$\operatorname{Sin}\theta \leqslant \theta - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\left(1 + \frac{\theta^2}{2!} + \dots\right).$$

Apart from a factor of $\theta^2/2$, the third term in the upper bound is

$$\leqslant \frac{\theta^2}{12} e^{\theta} \leqslant \left(\frac{5}{4}\right)^2 \frac{e^{\frac{3}{2}}}{12} < \frac{5}{6},$$

since $e < 3$.

LEMMA 6. *Let* $c > 0, 0 < \epsilon \leqslant \frac{1}{8}$. *If*

$$\operatorname{Re} \sum_{n \leqslant x} \frac{\Lambda(n)}{n}\left(1 - \frac{1}{n^{i\tau}}\right) \leqslant \epsilon \log x \tag{7}$$

and $|\tau| \leqslant x^c$, *then either* x *does not exceed some function of* c *alone, or*

$$|\tau| \log x \leqslant 25\epsilon + \exp(-(\log x)^{\frac{1}{8}}).$$

Proof. Denote by $h(\tau)$ the *sum* which appears at (7). By Lemma 4 it has the estimate

$$\log x - \frac{x^{i\tau}}{i\tau} + \frac{\zeta'}{\zeta}(1 + i\tau) + c_2 + O(\exp(-(\log x)^{\frac{1}{8}})).$$

Suppose first that $|\tau| \geqslant \delta > 0$. Then by an estimate of Vinogradov (e.g. Titchmarsh [**12**], §6·15, p. 144), $\zeta'/\zeta \ll (\log x)^{\frac{2}{3}}$, so that for large x, depending only upon δ, c, a contradiction ensues.

If now $|\tau| < \delta$, then

$$\frac{\zeta'}{\zeta}(1 + i\tau) = \frac{1}{i\tau} + c_3 + O(|\tau|)$$

for some constant c_3. Thus

$$h(\tau) = \log x - \left(\frac{x^{i\tau} - 1}{i\tau}\right) + c_4 + O(|\tau| + \exp(-(\log x)^{\frac{1}{8}}))$$

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for some constant c_4. Letting $\tau \to 0$, and noting that $h(0) = 0$ for all $x \geqslant 2$, we see that $c_4 = 0$. Then

$$\operatorname{Re} h(\tau) = \log x - \frac{\operatorname{Sin} \tau \log x}{\tau} + O(|\tau| + \exp(-(\log x)^{\frac{1}{5}})).$$

Without loss of generality $\tau \geqslant 0$. If $|\tau| \log x > \frac{5}{4}$, then $\operatorname{Re} h(\tau) \geqslant \frac{1}{4} \log x + O(1)$, and a contradiction again ensues.

Otherwise $|\tau| \log x \leqslant \frac{5}{4}$, so that by lemma 5

$$\operatorname{Re} h(\tau) \geqslant \tfrac{1}{24}\tau(\log x)^2 + O(|\tau| + \exp(-(\log x)^{\frac{1}{5}})),$$

giving the asserted bound.

LEMMA 7. *Let $c > 0$. Assume that*

$$\operatorname{Re} \sum_{n \leqslant x} \frac{\Lambda_1(n)}{n}(1 - n^{-i\tau}) \leqslant \epsilon,$$

where $|\tau| \leqslant x^c$. Then for $x \geqslant x_0$,

$$|\tau| \log x \ll \epsilon + \sqrt{\epsilon}\,(\log x)^{-1} + \exp(-(\log x)^{\frac{1}{5}}),$$

the implied constant being absolute. The parameter x_0 can be given the form $\exp(\exp K(\epsilon+1))$ for a constant K depending only upon c.

Proof. From the hypothesis

$$\operatorname{Re} \sum_{p \leqslant x} \frac{\Lambda(p)}{p}(1 - p^{-i\tau}) \leqslant \epsilon \log x.$$

Moreover, if $'$ denotes summation over prime-powers p^k, with $k \geqslant 2$, then an application of the Cauchy–Schwarz inequality shows that

$$\left| \sum_{n \leqslant x}{}' \frac{\Lambda(n)}{n}(1 - n^{-i\tau}) \right| \leqslant \left(\sum_{n \leqslant x} \frac{\Lambda_1(n)}{n} |1 - n^{-i\tau}|^2 \sum_{n}{}' \frac{\Lambda(n)(\log n)^2}{n} \right)^{\frac{1}{2}}.$$

Since $|1 - n^{-i\tau}|^2 = 2 \operatorname{Re}(1 - n^{-i\tau})$, and the second majorizing sum is convergent, this upper bound is $\ll \epsilon^{\frac{1}{2}}$. We have

$$\operatorname{Re} \sum_{n \leqslant x} \frac{\Lambda(n)}{n}(1 - n^{-i\tau}) \leqslant K_1 \left(\epsilon + \frac{\sqrt{\epsilon}}{\log x} \right) \log x = K_2 \log x,$$

say. If $K_2 \leqslant \frac{1}{8}$, then the desired result follows from Lemma 6. Otherwise $\epsilon \geqslant 1$, and it will suffice to prove that $|\tau| \log x \ll 1$.

Employing the Euler product representation of the Riemann zeta function, with $\sigma = 1 + 1/\log x$, we have

$$\left| \frac{\zeta(\sigma)}{\zeta(\sigma + it)} \right| = \exp\left(\operatorname{Re} \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^{\sigma}}(1 - n^{-i\tau}) \right).$$

Estimates from elementary prime number theory show that in this exponential the

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terms with $n > x$ contribute an amount bounded uniformly in $x \geqslant 2$. The hypothesis shows that the terms $n \leqslant x$ contribute $O(\epsilon)$. Thus

$$e^{-K_3(\epsilon+1)} \leqslant \left|\frac{\zeta(\sigma+i\tau)}{\zeta(\sigma)}\right|.$$

If $|\tau| \geqslant \delta > 0$, then by a Vinogradov estimate the ratio of the zeta functions is $\ll (\sigma-1)(\log x)^{\frac{2}{3}}$, so that x is bounded in terms of ϵ, c, δ.

If $|\tau| < \delta$, and δ is fixed at a sufficiently small value, then

$$\frac{\zeta(\sigma)}{\zeta(\sigma+i\tau)} = \left(\frac{1}{\sigma-1}+O(1)\right)\left(\frac{1}{\sigma-1+i\tau}+O(1)\right)^{-1} \gg \frac{|\tau|}{\sigma-1},$$

which gives the desired bound for $|\tau| \log x$.

Remarks. This last argument would not give any improvement if ϵ were assumed much smaller than 1. To obtain such an improvement the more detailed argument of Lemma 6 was employed.

Consider now the function

$$\beta(\theta, \tau) = \operatorname{Re} \sum_{n \leqslant x} \frac{\Lambda_1(n)}{n^\sigma}(1-\theta(n)\, n^{-i\tau}),$$

where σ is real and the arithmetic function $\theta(n)$ satisfies $|\theta(n)| = 1$ for all prime powers n. As usual, $\theta_1\theta_2$ will denote the function defined by $(\theta_1\theta_2)(n) = \theta_1(n)\,\theta_2(n)$.

LEMMA 8. *For real τ_j, $j = 1, \ldots, k$,*

$$\beta(\theta_1 \ldots \theta_k, \tau_1 + \ldots + \tau_k) \leqslant k \sum_{j=1}^{k} \beta(\theta_j, \tau_j).$$

Proof. This inequality is obtained by appropriate application, for each integer $n \leqslant x$, of the trigonometric inequality

$$(\operatorname{Sin}(\phi_1 + \ldots + \phi_k))^2 \leqslant k \sum_{j=1}^{k} (\operatorname{Sin} \phi_j)^2,$$

valid for all real ϕ_j, and integers $k \geqslant 1$. Typically, $2\phi_j = \tau \log n + \arg \theta_j(n)$.

When $\theta(n) = 1$ identically, $\beta(\theta, \tau)$ differs by a bounded amount from the function $\operatorname{Re} h(\tau)$ appearing in Lemma 6. Clearly $\operatorname{Re} h(\tau)$ is an even function of τ. By Lemmas 7 and 8 it is almost a scalar multiplication on the reals. This property will be employed to obtain approximate functional equations.

LEMMA 9. *Let the complex-valued function $v(y)$ satisfy*

$$|v(y_1+y_2) - v(y_1) - v(y_2)| \leqslant M$$

whenever y_1, y_2, y_1+y_2 belong to $[-a, a]$, $a > 0$. Then

$$|v(y) - ya^{-1}v(a)| \leqslant 3M$$

for $-a \leqslant y \leqslant a$.

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Proof. By considering the function $v(az), |z| \leqslant 1$, we may reduce ourselves to the case $a = 1$. For this value the assertion becomes essentially lemma (5·11) of Ruzsa[11], who deduces it from a generalization of an earlier result of Hyers[7].

§4. This section contains results in the theory of probability.

A *median* of a distribution function F is a constant m so that $F(m) \geqslant \frac{1}{2}$, $1-F(m-) \geqslant \frac{1}{2}$. A median of a random variable is a median of its associated distribution function. Medians exist, but need not be uniquely defined.

LEMMA 10. *Let ξ, η be independent, identically distributed random variables, with a median m. Then for every $\alpha > 0$,*

$$P(|\xi-m| > \alpha) \leqslant 2P(|\xi-\eta| > \alpha).$$

Proof. See, for example, Loève[8], p. 245.

LEMMA 11. *Let $\phi(t)$ be the characteristic function of $F(z)$. Let τ, y be positive real numbers such that $\tau y > 1$, and $\pm y$ are continuity points of F. Then*

$$F(y)-F(-y) > \frac{\tau y\left|\frac{1}{2\tau}\int_{-\tau}^{\tau}\phi(t)\,dt\right|-1}{\tau y-1}.$$

Moreover, this inequality remains valid if ϕ is replaced by $\operatorname{Re}\phi$.

Proof. See Gnedenko and Kolmogorov[4], chapter 2, theorem 2 and following, pp. 53–54.

Let $\{F_x\}$ be a sequence of distribution functions, each with associated median m_x, and characteristic function ϕ_x. Let I be the improper law with jump at the origin: $I(z) = 1$ if $z \geqslant 0$, $= 0$ if $z < 0$.

LEMMA 12. *$F_x(z+m_x)$ converges weakly to I as $x \to \infty$, if and only if $|\phi_x(t)| \to 1$ uniformly on each bounded set of t-values.*

Remark. It is easy to see that these conditions are also equivalent to $F_x(z+\gamma_x)$ converging to an improper law for some constants γ_x, and to $\lim|\phi_x(t)| = 1$ almost surely in t.

Proof. If $F_x(z+m_x) \Rightarrow I$, as $x \to \infty$, then $\phi_x(t)\exp(-itm_x) \to 1$ uniformly on bounded t-sets. This gives half of the result.

Let ξ_x, η_x be independent random variables distributed according to F_x. Suppose that $|\phi_x(t)| \to 1$ as $x \to \infty$, uniformly on $|t| \leqslant \tau, \tau\epsilon > 1$. Then $|\phi_x(t)|^2$ is the characteristic function of $\xi_x-\eta_x$, so that by Lemma 11

$$P(|\xi_x-\eta_x| \leqslant \epsilon) \geqslant 1+o(1), \quad x \to \infty.$$

Since τ may be taken arbitrarily large, by Lemma 10

$$P(|\xi_x-m_x| > \epsilon) \to 0$$

for each fixed positive ϵ. This is equivalent to $F_x(z+m_x) \Rightarrow I, x \to \infty$.

A distribution function F is a *factor* of another, H, if for some further distribution function $G, F*G = H$. Such factorizations are not unique.

A sequence F_j of distribution functions is *compact* if there is a subsequence of j-

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values on which the F_j converge weakly to a distribution. These limiting distributions will be called *limit laws* of the F_j.

LEMMA 13. *Suppose that $F_x * G_x$ converges weakly to H, as $x \to \infty$. Then for any unbounded sequence of x-values, the corresponding sequence of distribution functions $F_x(z+m_x)$ is compact. Furthermore, each of its limit laws is a factor of H.*

Proof. Let $h(t)$ be the characteristic function of H, and $\phi_x(t), \psi_x(t)$ those of F_x, G_x respectively. Then, uniformly on each bounded interval of t-values, $\phi_x \psi_x \to h, x \to \infty$.

Let $\epsilon > 0$. Since $h(t)$ is continuous at $t = 0$, and $h(0) = 1$, we can find a positive δ so that $|\phi_x \psi_x| > 1-\epsilon$ uniformly for $|t| \leqslant \delta$, and all sufficiently large x. With the same uniformity $|\phi_x| > 1-\epsilon$.

In the notation used in the proof of Lemma 12, we apply Lemma 11, this time with $\tau(=\delta)$ small, and y large. For large enough (but fixed) y

$$P(|\xi_x - \eta_x| \leqslant y) > 1-3\epsilon.$$

Another application of Lemma 10 gives

$$P(|\xi_x - m_x| \leqslant y) \ > 1-6\epsilon,$$

and the compactness of the corresponding $F_x(z+m_x)$ follows from a well-known result of Helly.

Suppose now that $\phi_x \exp(-itm_x) \to \phi$ as $x \to \infty$. By restricting to a further subsequence of x-values, we may also assert that $\psi_x \exp(-itn_x) \to \psi$, where n_x is a median of G_x. In view of our hypothesis,

$$\phi\psi \exp(it(m_x+n_x)) \to h, \quad x \to \infty,$$

uniformly on bounded t-intervals.

There is a proper interval $|t| \leqslant t_0$ on which ϕ, ψ do not vanish, and

$$\lim_{x\to\infty} \exp(it(m_x+n_x))$$

exists uniformly. By raising the exponential to integral powers we can extend this uniformity to any bounded set of t-values. Thus the distribution functions which are jumps at the points m_x+n_x converge weakly to a distribution function whose characteristic function w satisfies $|w(t)| = 1$ for all t. It is easy to see that w must have the form $\exp(it\alpha)$ for some α, and then that $m_x+n_x \to \alpha$ as $x \to \infty$.

We now have

$$\phi(t)\,\psi(t) \exp(it\alpha) = h(t),$$

and the limit law of the $F_x(z+m_x)$, which corresponds to ϕ, is a factor of H.

§5. I next study the function

$$J(iw) = y^{-1} \int_{x-y}^{x} u^{iw}\, du$$

which appears in the statement of Lemma 3.

LEMMA 14. *Let $P_j(x)$ be polynomials in x with complex coefficients, and d_j distinct real numbers, $j = 1, \ldots, k$. If*

$$\theta(t) = \sum_{j=1}^{k} P_j(x)\, e^{td_j t} = 0$$

on a proper interval of real t-values, then the polynomials are identically zero.

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Proof. Without loss of generality $0 = d_1 > d_2 > \ldots > d_k$. As a function of the *complex variable* t, $\theta(t)$ is everywhere analytic. After the hypothesis, analytic continuation shows that $\theta(t)$ is identically zero. If $P_1(x)$ has degree m, we consider $\lim y^{-m}\theta(-iy), y \to \infty$.

The terms $P_j(-iy)\exp(d_j y)$ with $j \geqslant 2$ converge exponentially to zero, whilst $(-iy)^{-m}P_1(-iy)$ approaches the coefficient of x^m in P_1. Since the value of this limit is zero, $P_1(x)$ is identically zero. An induction argument completes the proof of the lemma.

LEMMA 15. *Assume that for an unbounded increasing sequence of values $x = x_j$, there are numbers $y, 1 \leqslant y \leqslant x, \lambda$, possibly depending upon y, x, so that $\lambda y/x \to 0$ as $j \to \infty$. Then $x^{-i\lambda t}J(i\lambda t) \to 1$, uniformly on every bounded set of real t-values.*

If $\rho = \lim \lambda y/x, \psi = \lim y/x$ exist as $j \to \infty$, then with the same uniformity so does $\lim x^{-i\lambda t}J(i\lambda t)$. It has the value

$$\left.\begin{aligned} &1 \text{ if } \psi = 0 = \rho; \quad \frac{1-e^{-it\rho}}{it\rho} \text{ if } \psi = 0, \quad \rho \neq 0, \\ &\frac{1-(1-\psi)^{1+it\rho/\psi}}{\psi + it\rho} \text{ if } 0 < \psi < 1; \quad \frac{1}{1+i\rho t} \text{ if } \psi = 1. \end{aligned}\right\} \tag{8}$$

Proof. Direct integration shows that

$$x^{-tw}J(iw) = \frac{1-(1-y/x)^{1+tw}}{(1+iw)\,y/x}.$$

I begin with the second of the two assertions. Since the case $\psi = 1$ is clear, I shall assume that $\psi < 1$. In this case

$$\lim_{j\to\infty} \lambda \log\left(1-\frac{y}{x}\right) = \frac{\rho}{\psi}\log(1-\psi)$$

if $\psi > 0$, and otherwise has the value $-\rho$. Indeed, without loss of generality $y/x \leqslant 1-c < 1$ may be assumed for all (permitted) x. Employing the Taylor expansion of $-\log(1-z)$,

$$\left|\lambda \log\left(1-\frac{y}{x}\right) + \frac{\lambda y}{x}\sum_{m=1}^{k}\frac{1}{j}\left(\frac{y}{x}\right)^{m-1}\right| \leqslant \frac{|\lambda| y}{x}\sum_{m=k+1}^{\infty}\left(\frac{y}{x}\right)^{m-1} \ll (1-c)^k$$

holds for each positive integer k. If $\psi > 0$, the hypothesis guarantees that, for each fixed m, $\lambda(y/x)^m \to \rho\psi^{m-1}$. Thus

$$\limsup_{x\to\infty}\left|\lambda \log\left(1-\frac{y}{x}\right) - \frac{\rho}{\psi}\log(1-\psi)\right| \ll (1-c)^k,$$

and we may let $k \to \infty$. A similar argument holds when $\psi = 0, \rho \neq 0$. The case $\rho = 0 = \psi$ is included in the first assertion of Lemma 15, to be proved next.

Suppose now that $\beta = |\lambda| y/x \to 0$ as $x \to \infty$. If $1 - y/x \geqslant \theta$ for some $\theta, 0 < \theta \leqslant \frac{1}{2}$, then the above argument yields

$$-\lambda \log\left(1-\frac{y}{x}\right) = \frac{\lambda y}{x} + O\left(\frac{\beta y}{\theta x}\right),$$

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from which
$$\left(1-\frac{y}{x}\right)^{1+t\lambda t} = \left(1-\frac{y}{x}\right)\exp\left(\frac{-it\lambda y}{x}+O\left(\frac{\beta y}{\theta x}\right)\right)$$
$$= 1-\frac{y}{x}-\frac{it\lambda y}{x}+O\left(\frac{\beta y}{\theta x}\right).$$

Employing this estimate in the representation for $J(iw)$ gives

$$x^{-t\lambda t}J(i\lambda t) = 1+O(\beta\theta^{-1}).$$

In the complementary situation $1-y/x < \theta$ we have almost at once that

$$x^{-t\lambda t}J(i\lambda t) = 1+O(\beta+\theta).$$

If β (and so λ) $= 0$, then this last relation holds with θ replaced by zero. Otherwise we set $\theta = \sqrt{\beta}$. This completes the proof of Lemma 15, the uniformity with respect to bounded t being clear.

Consider next the function $G(\rho,\psi)$ defined by the value of the limit (8) in Lemma 15, $0 \leqslant \psi \leqslant 1, \rho, t$ real. Suppose that $G(\rho,\psi) = A(t)$ on the interval $|t| \leqslant t_0, t_0 > 0$. Then ρ is determined. Moreover, if $\rho \neq 0$, then ψ is determined too. Suppose, for example, that $G(\rho_1,\psi_1) = G(\rho_2,\psi_2)$, with $0 < \psi_j < 1$; then

$$(\psi_2+i\rho_2 t)(1-(1-\psi_1)^{1+it\rho_1/\psi_1}) = (\psi_1+i\rho_1 t)(1-(1-\psi_2)^{1+it\rho_2/\psi_2}).$$

If $\rho_1 = 0 = \rho_2$, this identity reduces to $\psi_2\psi_1 = \psi_1\psi_2$, and nothing further may be concluded. Otherwise we may apply Lemma 14, first to show that if at least one of the ρ_j is non-zero, then both are, then that the exponents $\rho_j\psi_j^{-1}\log(1-\psi_j)$ are equal, finally that $\psi_1+i\rho_1 t = \psi_2+i\rho_2 t$. Thus $\psi_1 = \psi_2$ and $\rho_1 = \rho_2$.

A slightly more complicated version of this argument shows that $|G(\rho,\psi)|^2$, and so $|G(\rho,\psi)|$, as a function of t, determines $|\rho|$. Moreover, if $|\rho| \neq 0$, then once again ψ is determined.

I continue with the notation of Lemma 15.

LEMMA 16. *The following three conditions are equivalent:*

(i) $|J(i\lambda t)| \to 1$ *for* $|t| \leqslant t_0, t_0 > 0$,

(ii) $x^{-t\lambda t}J(i\lambda t) \to 1$ *for* $|t| \leqslant t_0, t_0 > 0$,

(iii) $\dfrac{\lambda y}{x} \to 0$ *as* $x_j \to \infty$.

Proof. I give a cyclic proof. If (i) is valid, then for $t = t_0/2$, and all sufficiently large values of x,

$$\tfrac{1}{2} \leqslant |J(i\lambda t)| \leqslant 2\left(|1+i\lambda t|\frac{y}{x}\right)^{-1},$$

so that the $|\lambda|\,y/x$ are uniformly bounded. The assertion (iii) now follows from the last of the above remarks, since all limit points of the sequence $|\lambda|\,y/x$ have the same value, which must be zero.

Assuming (iii), (ii) follows from Lemma 15; whilst (ii) clearly implies (i).

LEMMA 17. $\operatorname{Lim}|J(i\lambda t)|$ *exists for* $|t| \leqslant t_0, t_0 > 0$, *with a limit that is not identically one, and is non-zero for at least one non-zero value of* t, *if and only if* $\lim|\lambda|\,y/x$ *exists and is non-zero, and* $\lim y/x$ *exists.*

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Proof. Suppose that $\lim |J(i\lambda t)|$ exists. Since the value of the limit is non-zero for at least one t, we may follow the argument used in the step from (i) to (iii) in the proof of Lemma 16. This shows that $\delta = \lim |\lambda| y/x$ exists, and Lemma 16 itself shows that $\delta > 0$. Our earlier remarks then also imply the existence of $\lim y/x$.

The converse proposition follows readily from Lemma 15.

LEMMA 18. *Let parameters γ, possibly depending upon x, y, λ be given. Then the following assertions are equivalent as $x(=x_j) \to \infty$:*

(i) $\lim J(i\lambda t) \exp(-i\gamma t)$ *exists uniformly for an interval* $|t| \leqslant t_0$, $t_0 > 0$, *with a limiting function not identically one in absolute value,*

(ii) $F_x(\lambda^{-1}(z+\gamma))$ *converges weakly to a proper law,*

(iii) $\lim \lambda y/x = \rho \neq 0, \lim y/x = \psi, \mu = \lim(\gamma - \lambda \log x)$ *exist; or* $\lim |\lambda| y/x = \delta > 0$ *exists,* $\lim y/x = 0$, *and all subsequences of the* $\gamma - \lambda \log x$ *possess at most two limit points,* $\mu, \mu + \delta$, *as* $x \to \infty$.

The characteristic functions of the limit laws are $e^{-it\mu} G(\rho, \psi)$, *with the various possibilities for* $G, \rho \neq 0$, *listed at* (8).

Proof. I give a cyclic proof, assuming first that (i) is valid and deducing (iii). Let

$$J(i\lambda t) \exp(-i\gamma t) \to B(t), x \to \infty,$$

uniformly for $|t| \leqslant t_0, t_0 > 0$. Since $J(i\lambda t)$ is a characteristic function, so is $B(t)$. By continuity $B(t) \neq 0$ on some interval of t-values containing the origin. We can now apply Lemma 17 to gain the existence of $\lim |\lambda| y/x = \delta > 0, \psi = \lim y/x$.

Suppose that on a (sub)sequence of x-values, $\lambda y/x \to \delta$. Then on this same sequence

$$A = \lim J(i\lambda t) x^{-i\lambda t}$$

exists, with a value which may be found in the table (8). On this same sequence

$$A \exp(-it(\gamma - \lambda \log x)) \to B(t),$$

uniformly for $|t| \leqslant t_0$. Again A is a characteristic function, and $\lim(\gamma - \lambda \log x) = \mu$ exists. Thus $B(t)$ will have the form $\exp(-it\mu) G(\delta, \psi)$ in the notation of the remarks following Lemma 15.

Likewise, if on a suitable (sub)sequence of x-values, $\lambda y/x \to -\delta$, the only other possible limiting value, then $B(t)$ will have the form $\exp(-it\mu_1) G(-\delta, \psi)$. To be consistent

$$e^{-it\mu} G(\delta, \psi) = e^{-it\mu_1} G(-\delta, \psi)$$

must hold for all real t. In view of the table of forms (8), applications of Lemma 14 show that this will not allow $0 < \psi \leqslant 1$. For these values of ψ, $\lim \lambda y/x$ must exist, and so $\lim(\gamma - \lambda \log x)$.

If $\psi = 0$, there remains the possibility $\mu = \mu_1 - \delta$. In this case

$$B(t) = e^{-it\mu} \left(\frac{1 - e^{-it\delta}}{it\delta} \right).$$

Altogether we have established (iii).

Assuming (iii), the validity of (ii) follows from Lemma 15 together with a standard theorem in the theory of probability. This same theorem shows that (ii) guarantees (i) for every t_0. (See Gnedenko and Kolmogorov [**4**], §13.)

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The following result sums up the properties of the function J that I presently need.

LEMMA 19. *There are parameters λ, γ and real numbers σ, $\beta \geqslant 0$ so that*

$$J(i\lambda t)\exp(-it\gamma) \to \exp(-\beta t^2 - i\sigma t), x \to \infty, \tag{9}$$

for all real t, if and only if $\lambda y/x \to 0$. In this case $\beta = 0$, and $\gamma - \lambda \log x \to \sigma$.

Proof. If (9) holds with $\beta > 0$, then so do (ii) and (iii) of Lemma 18. None of the characteristic functions listed at (8) is a proper normal law. Thus $\beta = 0$ in fact holds, and we may apply Lemma 16. Standard arguments from the theory of probability complete the proof.

§6. *Proof of Theorem* 3. Let $\phi_{x,y}(t)$ denote the characteristic function of the frequency $\nu_{x,y}(n; f_x(n) - \gamma \leqslant z)$, and $h(t)$ that of H. Assume that (5) holds. Then $\phi_{x,y} \to h$ uniformly on every bounded set of t-values.

Define the multiplicative function $g(n) = \exp(itf_x(n))$. A straightforward calculation shows that

$$\phi_{x,y}(t) = ([x]-[x-y])^{-1} \sum_{x-y<n\leqslant x} g(n)\exp(-it\gamma).$$

We apply Lemma 3 to obtain an estimate

$$\phi_{x,y} = \exp(-it\gamma)J(i\alpha)M(gn^{-i\alpha}, x) + R, \tag{10}$$

where $R \to 0$ and $\log y/\log x \to 1$, as $x \to \infty$, uniformly in all real t. Of course the parameter α, known to satisfy $|\alpha| \leqslant x$, depends upon both g and t.

For $w > 0$ set

$$\delta(w) = \min_{|t|\leqslant w} |h(t)|.$$

I assert that there are $t_x, t_x \to \infty$ as $x \to \infty$, so that

$$|R| + |\phi_{x,y} - h| \leqslant \tfrac{1}{4}\delta(t_x), \tag{11}$$

$$\delta(t_x) \geqslant \exp(-(\log\log x)^{\frac{1}{2}}), \tag{12}$$

uniformly for $|t| \leqslant t_x$, and all (allowable) x sufficiently large. To satisfy (11) we note that for each fixed k, $|R| + |\phi_{x,y} - h| \to 0$ uniformly for $|t| \leqslant 2^k$. Moreover, h is continuous and non-zero, so that $\delta(2^k) > 0$. Suppose therefore that

$$|R| + |\phi_{x,y} - h| \leqslant \tfrac{1}{4}\delta(t_0),$$

uniformly for $|t| \leqslant t_0$, where $t_0 \to \infty$, as $x \to \infty$. We take for t_x the greatest value, not exceeding t_0, so that (12) holds. This $t_x \to \infty$ as $x \to \infty$. Otherwise there would be an unbounded increasing sequence x_j, and a uniformly bounded sequence t_j, so that $|h(t_j)| \leqslant \exp(-(\log\log x_j)^{\frac{1}{2}})$. Since h is continuous, any limit point t_∞ of these t_j would satisfy

$$h(t_\infty) \leqslant \limsup_{j\to\infty} |h(t_j)| \leqslant \lim_{j\to\infty} \exp(-(\log\log x_j)^{\frac{1}{2}}) = 0,$$

contradicting an hypothesis.

A particular consequence of the estimates (10), (11) is that with $\delta = \delta(t_x)$

$$|J(i\alpha)M(gn^{-i\alpha})| \geqslant \delta/2 \tag{13}$$

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holds uniformly for $|t| \leqslant t_x$. Clearly $|J(i\alpha)| \leqslant 1$, so that $|M(gn^{-i\alpha}, x)| \geqslant \delta/2$ holds with the same uniformity. We next apply Lemma 1 to the multiplicative function $gn^{-i\alpha}$, setting $T = \log x$. In view of the lower bound (12), we obtain a real number τ, $|\tau| \leqslant \log x$, for which

$$\sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^{\sigma}} (1 - \operatorname{Re} g(n)\, n^{-i\alpha - i\tau}) \leqslant 5(\log\log x)^{\frac{1}{2}} + O(1), \tag{14}$$

where $\sigma = 1 + 1/\log x$. We regard α, τ as functions of t, and set $v(t) = \alpha + \tau$. We may avoid the use of the axiom of choice in the definition of τ by employing the continuity in τ of the function $w(\sigma)$ which appears in Lemma 1.

Consider values t_1, t_2 in the interval $|t| \leqslant t_x$ so that $t_1 + t_2$ lies in this same interval. According to Lemma 8, applied with $k = 3$, $\theta_j = \exp(it_j f_x(n))$, $j = 1, 2$, $\theta_3 = \exp(-i(t_1 + t_2) f_x(n))$,

$$\sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^{\sigma}} (1 - \operatorname{Re} n^{-i\beta}) \ll (\log\log x)^{\frac{1}{2}}$$

with

$$\beta = v(t_1) + v(t_2) - v(t_1 + t_2).$$

According to our upper bounds on α, τ, $|v(t)| \leqslant x + \log x$ for $|t| \leqslant t_x$, and since $n^{\sigma - 1} \leqslant e$ when $n \leqslant x$, we may apply Lemma 7, to obtain

$$v(t_1) + v(t_2) - v(t_1 + t_2) \ll \frac{(\log\log x)^{\frac{1}{2}}}{\log x}.$$

We can essentially solve this approximate functional equation, applying Lemma 9. Thus

$$v(t) - \lambda t \ll \frac{(\log\log x)^{\frac{1}{2}}}{\log x},$$

with $\lambda = v(t_x)/t_x$, holds uniformly on the interval $|t| \leqslant t_x$. Note that since $t_x \to \infty$, for all large x we may set $t = 1$, to show that $|\lambda| \leqslant x + 2\log x$.

The next step is to apply Lemma 2 with the function g there replaced by $gn^{-i\lambda t}$, and the constant b by $-(\alpha - \lambda t)$. In view of our estimate for τ, regarded as a function of t, $b \ll \log x$ uniformly for $|t| \leqslant t_x$. In the notation of Lemma 2,

$$L = \sum_{p \leqslant x} p^{-1} |1 - g(p)\, p^{-i\lambda t}|.$$

This we estimate by applying Lemma 8 with $k = 2$, $\theta_1(n) = g(n)\, n^{-iv(t)}$, $\theta_2(n) = n^{i(v(t) - \lambda t)}$. In the notation of that lemma, the bound (14) guarantees that $\beta(\theta_1, 0) \ll (\log\log x)^{\frac{1}{2}}$, using $\sigma = 1 + 1/\log x$. Moreover, from the simple inequality $|e^{i\psi} - 1| \leqslant |\psi|$, which is valid for all real ψ,

$$\beta(\theta_2, 0) \leqslant \sum_{n \leqslant x} \frac{\Lambda_1(n)}{n^{\sigma}} |v(t) - \lambda t| \log n \ll \left(\sum_{p \leqslant x} \frac{\log p}{p} + O(1) \right) \frac{(\log\log x)^{\frac{1}{2}}}{\log x} \ll (\log\log x)^{\frac{1}{2}},$$

a similar bound. Thus $L \ll (\log\log x)^{\frac{1}{2}}$, and by Lemma 2

$$M(gn^{-i\alpha}, x) = \frac{x^{-i(\alpha - \lambda t)}}{1 - i(\alpha - \lambda t)} M(gn^{-i\lambda t}, x) + O((\log x)^{-\frac{1}{3}}). \tag{15}$$

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Combining our earlier lower bound (13) with the elementary bound $|M(gn^{-it\alpha}, x)| \leqslant 1$ gives $|J(i\alpha)| \geqslant \delta/2$. Since $|J(i\alpha)| \leqslant (|1+i\alpha|\, y/x)^{-1}$, the restriction upon the growth of y ensures that $|\alpha| \leqslant 2x/(\delta y) < x^{\frac{1}{4}}$ for all sufficiently large values of x.

Let us temporarily denote the mean value of $gn^{-it\alpha}$ in (15) by S. Since α is chosen to maximize $|M(gn^{-it\alpha}, x)|$ subject to $|\alpha| \leqslant x$, and $|\lambda| \leqslant |\alpha| + |\tau| + O((\log\log x)^{\frac{1}{3}}/\log x) < x^{\frac{1}{2}}$ for all sufficiently large values of x, a particular consequence of the estimate (15) is that

$$|S| \leqslant |1 - i(\alpha - \lambda t)|^{-1}\, |S| + O((\log x)^{-\frac{1}{3}}).$$

Here $|S| \geqslant \frac{1}{2}\exp(-(\log\log x)^{\frac{1}{3}})$, so that

$$\alpha - \lambda t \ll (\log x)^{-\frac{1}{7}}. \tag{16}$$

I next show that as $x \to \infty$, $N = x^{-it\alpha}J(i\alpha) - x^{-i\lambda t}J(i\lambda t) \to 0$. Suppose first that $1 - y/x \leqslant \theta \leqslant \frac{1}{2}$. Then

$$N = \frac{x}{y}\frac{-i(\alpha-\lambda t)}{(1+i\alpha)(1+i\lambda t)} + O\left(\frac{\theta x}{y}\right) \ll (\log x)^{-\frac{1}{7}} + \theta.$$

However, if $1 - y/x > \theta$, then

$$\left|\left\{1 - \left(1 - \frac{y}{x}\right)^{1+t\alpha}\right\}\left(\frac{1}{1+i\alpha} - \frac{1}{1+i\lambda t}\right)\frac{x}{y}\right| = |J(i\alpha)|\left|1 - \frac{1+i\alpha}{1+i\lambda t}\right| \ll |\alpha - \lambda t| \ll (\log x)^{-\frac{1}{7}},$$

whilst

$$\gamma = \left|\exp\left(i(\lambda t - \alpha)\log\left(1 - \frac{y}{x}\right)\right) - 1\right| \leqslant |\lambda t - \alpha|\left|\log\left(1 - \frac{y}{x}\right)\right|$$

$$\ll (\log x)^{-\frac{1}{7}}\left(1 - \frac{y}{x}\right)^{-1} \ll (\log x)^{-\frac{1}{7}}\theta^{-1}.$$

In this case

$$N \ll \frac{\gamma}{|1+i\lambda t|\, y/x} + (\log x)^{-\frac{1}{7}}.$$

If $\psi = |t\lambda|\, y/x > (\log x)^{-\frac{1}{14}}$, we set $\theta = (\log x)^{-\frac{1}{14}}\psi^{-\frac{1}{2}}$ to obtain $N \ll (\log x)^{-\frac{1}{28}}$. Otherwise $\psi \leqslant (\log x)^{-\frac{1}{14}}$, and a similar upper bound for N may be obtained by applying with $\beta = |\alpha|\, y/x, |t\lambda|\, y/x$ in turn, the argument at the end of the proof of Lemma 15.

These results, together with the estimates (10), (15), show that, uniformly for $|t| \leqslant t_x$,

$$\phi_{x,y} = J(i\lambda t)\, M(gn^{-it\lambda t}, x)\exp(-it\gamma) + o(1) \tag{17}$$

as $x \to \infty$. To obtain the assertion (6) of Theorem 3, and so complete this half of its proof, we need only show that $J(it)$ is the characteristic function of the distribution function $F_x(z)$ defined earlier. The weak convergence (6) will then follow from a standard theorem in the theory of probability. This is readily done by noting that if I_β denotes the improper law with jump at β, then u^{it} is the characteristic function of $I_{\log u}$, so that $J(it)$ is the characteristic function of

$$\frac{1}{y}\int_{x-y}^{x} I_{\log u}\, du.$$

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Conversely, suppose that (6) holds as $x \to \infty$. This time

$$J(i\lambda t)\, M(gn^{-i\lambda t}, x) \exp(-it\gamma) \to h(t),$$

as $x \to \infty$, holds uniformly on some interval $|t| \leqslant t_x$, where $t_x \to \infty$ as $x \to \infty$. The first part of the argument used earlier in this proof can be followed, with the rôle of α now played by λt, and we have $|M(gn^{-i\lambda t}, x)| \geqslant \delta/2$. Without loss of generality $t_x \leqslant x^{\frac{1}{2}}$ may be assumed.

Moreover, the estimate (10) remains valid. Since $\alpha, |\alpha| \leqslant x$, maximizes $|M(gn^{-i\alpha}, x)|$, we obtain $|M(gn^{-i\alpha}, x)| \geqslant \delta/2$ also. We can now apply the previous arguments completely, until we reach (17), which gives the desired weak convergence (5).

This completes the proof of Theorem 3.

Remarks. The argument in this proof gives something more. For example, suppose that

$$y^{-1} \Big| \sum_{x-y<n\leqslant x} g(n) \Big| \geqslant \delta \geqslant \max(2|R|, \exp(-(\log\log x)^{\frac{1}{2}})),$$

with $g(n) = \exp(itf_x(n))$, holds uniformly on the interval $|t| \leqslant t_x$. Then on this same interval there is a representation

$$y^{-1} \sum_{x-y<n\leqslant x} g(n) = J(i\lambda t)\, M(gn^{-i\lambda t}, x) + O(|R| + (\log x)^{-\frac{1}{28}})$$

with a real $\lambda, \lambda \ll x/(y\delta)$, assuming that $\exp(2(\log\log x)^{\frac{1}{2}}) < y \leqslant x$.

A similar result can be obtained with the variable t confined to integer values. As a consequence it is possible to consider the value distribution, over intervals $x-y < n \leqslant x$, of additive functions (mod 1). The details of such an application I leave to another occasion, but the following result is typical. Assume that $1 \leqslant y \leqslant x$, and as $x \to \infty$, $\log y/\log x \to 1$. Then there is always a translation parameter $\gamma = \gamma(x)$ so that the frequencies

$$\nu_{x,y}(n; f(n) - \gamma \leqslant z (\mathrm{mod}\ 1))$$

converge weakly to a distribution (mod 1).

§7. *Proof of Theorem* 2. Suppose first that the frequencies (3) converge weakly to the normal law with mean zero, variance 1. Then according to Theorem 3, so do $F_x(\lambda^{-1}(z+\gamma)) * \nu_{x,x}(n; f_x(n) - \lambda \log n \leqslant z)$. I next prove that $\lambda y/x \to 0$.

Suppose, if possible, that x_j is an unbounded increasing sequence of x-values on which the ratio $|\lambda|\, y/x \to \infty$. According to Lemma 13, for suitable parameters m_x, the distribution functions $F_x(\lambda^{-1}(z+m_x+\gamma))$ are compact. On a suitable subsequence of the x_j-values, these same distributions converge. By Lemmas 16 and 18, $\lim |\lambda|\, y/x$ exists (finitely), a contradiction. Thus the ratios $\lambda y/x$ are in fact uniformly bounded.

Suppose next that for a suitable unbounded sequence x_j, $\lambda y_j/x_j \to \rho \neq 0$. We can argue as in the preceding paragraph, and obtain the convergence of the frequencies $F_x(\lambda^{-1}(z+m_x+\gamma))$ to a law $K(z)$, say, once again possibly on a subsequence of x-values. According to Lemma 13, $K(z)$ will be a factor of the initial normal law. A well known result, conjectured by Lévy and proved by Cramer (e.g. Lukacs [9], theorem 8·2·1), is that all factors of normal laws are again normal laws. Thus $K(z)$ will be a normal law, and by Lemma 19, $\lambda y/x \to 0$. This contradiction shows that ρ cannot be non-zero, and that indeed $\lambda y/x \to 0$ as $x \to \infty$ on our original sequence.

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We can now appeal to Lemma 16, to obtain $x^{-it\lambda}J(i\lambda t) \to 1$ as $x \to \infty$. Rewriting (6) in the equivalent form

$$J(i\lambda t)\exp(-it\gamma)M(gn^{-it\lambda}, x) \to e^{-t^2/2}, \quad x \to \infty, \tag{18}$$

we see that

$$\exp(-it(\gamma - \lambda\log x))M(gn^{-it\lambda}, x) \to e^{-t^2/2}, \quad x \to \infty. \tag{19}$$

This asymptotic relation is, in turn, equivalent to the weak convergence (4).

Conversely, given (4) and so (19), noting that by Lemma 16 the condition $\lambda y/x \to 0$ ensures that $x^{-it\lambda}J(i\lambda t) \to 1$, we see that (18) holds. According to Theorem 3 this is equivalent to the weak convergence (3).

This completes the proof of Theorem 2.

§8. *Proof of Theorem* 1. Assume that the frequencies (1) approach the improper law with jump at the origin. Then, by Theorem 3,

$$J(i\lambda t)\exp(-it\gamma)M(gn^{-it\lambda}, x) \to 1, \quad x \to \infty, \tag{20}$$

uniformly on each bounded interval of t-values. Here the mean value has absolute value at most 1, and $|J(i\lambda t)|$ likewise, so that $|J(i\lambda t)| \to 1$ as $x \to \infty$. From Lemma 16, $\lambda y/x \to 0$, and $x^{-it\lambda}J(i\lambda t) \to 1$. Together with (20) this gives

$$\exp(-it(\gamma - \lambda\log x))M(gn^{-it\lambda}, x) \to 1,$$

or equivalently

$$\nu_{x,x}(n; g(n) - \lambda\log n - (\gamma - \lambda\log x) \leqslant z) \tag{21}$$

converges weakly to the improper law with jump at the origin. A direct application of Ruzsa's theorem [**10**] shows that this is possible only if condition (2) of Theorem 1 is valid.

Conversely, if condition (2) is satisfied, then with an appropriate choice of $\gamma = \gamma(x)$, so is (21). Since $\lambda y/x \to 0$, once again Lemma 16 guarantees that $x^{-it\lambda}J(i\lambda t) \to 1$, so that (20) holds. This is equivalent to the weak convergence of the frequencies (1) to the improper law with jump at the origin, and the proof of Theorem 1 is complete.

Concluding Remarks. It is clear that the present approach to the study of the frequencies $\nu_{x,y}(n; f(n) \leqslant z)$ leads quite naturally to the consideration of the factorization of probability distribution functions.

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REFERENCES

[1] P. D. T. A. ELLIOTT. *Arithmetic Functions and Integer Products.* Grundlehren der math. Wiss. 272 (Springer-Verlag, 1984).

[2] P. D. T. A. ELLIOTT. General asymptotic distributions for additive arithmetic functions. *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 43–54.

[3] P. D. T. A. ELLIOTT. *Probabilistic Number Theory.* (Two volumes.) Grundlehren der math. Wiss. 239, 240 (Springer-Verlag, 1979/80).

[4] B. V. GNEDENKO and A. N. KOLMOGOROV. *Limit Distributions for Sums of Independent Random Variables.* Transl. from the Russian and annotated by K. L. Chung (Addison-Wesley, 1968).

[5] G. HALÁSZ. On the distribution of additive arithmetic functions. *Acta Arith.* **27** (1975), 143–152.

https://doi.org/10.1017/S0305004100064719 Published online by Cambridge University Press

[6] A. Hildebrand. Multiplicative functions in short intervals. *Canadian J. Math.* to appear.
[7] D. H. Hyers. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA*, **27** (1941), 222–224.
[8] M. Loève. *Probability Theory* (D. Van Nostrand, 1963).
[9] E. Lukacs. *Characteristic Functions* (second edition) (Griffin, 1970).
[10] I. Z. Ruzsa. The law of large numbers for additive functions. *Studia Sci. Math. Hungar.* **14** (1979), 247–253 (1982).
[11] I. Z. Ruzsa. On the concentration of additive functions. *Acta Math. Acad. Sci. Hungar.* **36** (1980), 215–232.
[12] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function* (Clarendon Press, Oxford, 1951).

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