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Coefficient rings for beta function classes of Markov chains

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Abstract. To each function we associate a period, a polytope, a group Γ, a subgroup $\Delta \subset \Gamma$ and a coset of Γ/Δ, and show these objects to be universal for the class of Markov chains with that beta function. One consequence is that polynomial matrices employed in coding problems of Markov chains can be chosen canonically.

1. *Introduction*

The association of matrices of exponential functions, and then matrices of polynomials, to Markov chains has been the key to a number of coding problems, most recently in [**MT4**]. While the passage to exponential functions is straightforward, the subsequent passage to polynomials depends on the choice of a finitely generated multiplicative group of exponentials or, equivalently, subgroup of the positive reals. For a nonnegative matrix A of exponential functions, the triple $\Gamma(A)$, $\Delta(A)$, $c_A^{d(A)}\Delta(A)$ studied in [**PS, S2, MT1**] and elsewhere is natural from various points of view. For example, it yields the coefficient ring of the dimension module associated to A in [**T2**]. It would be the natural choice in passing to polynomial rings, except that it is not invariant under a number of coding relations involving finite-to-one block codes. This drawback has forced the use of larger, noncanonical rings and meant that a polynomial matrix could not be associated with A until after we knew which other matrix B (representing another Markov chain) was being considered along with A.

The problem was resolved in [**MT1**] for beta function classes of Bernoulli shifts by showing that each Bernoulli shift yields universal objects for its class. (See Theorems 2.1 and 2.4 below.)

In this paper we associate a period $d(\beta)$, a polytope $W(\beta)$ and a triple $\Gamma(\beta)$, $\Delta(\beta)$, $c_\beta^{d(\beta)}\Delta(\beta)$ to an arbitrary beta function β, and prove these objects to be universal for the class of Markov chains having β as their beta function. Specifically, we prove that if the beta function β_A of A equals β then:

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(i) the weight-per-symbol polytope $W(A)$ of A equals $W(\beta)$;
(ii) $\Gamma(A)$ is a finite-index subgroup of $\Gamma(\beta)$;
(iii) $\Delta(A)$ is a finite-index subgroup of $\Delta(\beta)$; and
(iv) $d(\beta)$ divides the period $d(A)$ of A, and $c_A^{d(A)}\Delta(\beta) = c_\beta^{d(A)}\Delta(\beta)$.

Since the beta function is invariant under many codes and classifications [**T1, S2**], this has the consequence that we can canonically associate to any Markov chain with beta function β a matrix over the positive cone $\mathbb{Z}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$ of a Laurent polynomial ring determined by β, and put this matrix to use in coding problems involving the Markov chain. (See Remark 3.6 for a detailed discussion of this point.)

Let $\mathbb{Z}[\exp]$ denote the ring of integral combinations of exponential functions, and let $\mathbb{Z}^+[\exp]$ be its positive cone consisting of nonnegative integral combinations of exponential functions. In the Bernoulli case we have $\beta \in \mathbb{Z}^+[\exp]$, and the 1×1 matrix (β) corresponding to the Bernoulli shift presents itself as the minimal candidate for the class. The difficulty with the general case is that there is no obvious nonnegative minimal candidate. We will show that β has a minimal polynomial over $\mathbb{Z}[\exp]$ and derive the objects associated with β from the companion matrix of the minimal polynomial of $\beta^{d(\beta)}$.

The consideration of the period $d(\beta)$ of β appears to be new, even in the case of nonnegative integral matrices, where β reduces to the spectral radius μ of the matrix. We will observe that if an irreducible nonnegative matrix A has spectral radius μ then $d(\mu)$ divides the period $d(A)$ of A, and that there exists an irreducible nonnegative matrix of period $d(\mu)$ and spectral radius μ.

The basic definitions and facts needed in the paper will be given in §2. §3 contains the definitions of the objects we associate to β, the statements of the main results, and remarks related to these. The proofs of the main results are given in §§4 and 5 where the properties of $W(\beta)$, $\Gamma(\beta)$, $\Delta(\beta)$ and $c_\beta^{d(\beta)}\Delta(\beta)$ are developed.

2. *Markov chains and groups of weights*

A nonnegative integral matrix M determines a directed graph $G(M)$ and a shift of finite type (Σ_M, σ_m). For directed graphs, shifts of finite type and their block maps, we take [**LM**] as the standard reference and make free use of the definitions, notation and basic facts therein. We let $\mathbb{R}^{++} = \{a \in \mathbb{R} : a > 0\}$. We write exp for the set of exponential functions of a real variable t:

$$\exp = \{a^t : a \in \mathbb{R}^{++}\}.$$

Let $\mathbb{Z}[\exp]$ and $\mathbb{Z}^+[\exp]$ be the ring and semiring consisting, respectively, of integral and nonnegative integral combinations of elements of exp. Elements of $\mathbb{Z}[\exp]$ are functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(t) = \sum_{i=1}^k n_i a_i^t$, where $k \in \mathbb{N}$, $a_i \in \mathbb{R}^{++}$ and $n_i \in \mathbb{Z}$; we obtain elements of $\mathbb{Z}^+[\exp]$ by further insisting that $n_i \in \mathbb{Z}^+$. For $S \subset \mathbb{R}^{++}$, we put $S^t = \{a^t : a \in S\}$.

Let A be an irreducible matrix over $\mathbb{Z}^+[\exp]$. Putting $t = 0$, we obtain a nonnegative integral matrix $A(0)$. Let us write $G(A) = G(A(0))$ and $(\Sigma_A, \sigma_A) = (\Sigma_{A(0)}, \sigma_{A(0)})$. For states (vertices) I, J of $G(A)$, consider the entry $A(I, J)$ of A. If $A(I, J) \neq 0$, write

$$A(I, J) = \sum_{i=1}^k a_i^t,$$

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where $a_i \in \mathbb{R}^{++}$ need not be distinct. Then $G(A)$ contains exactly k edges starting at I and ending at J; assign $a_1, \ldots, a_k$ bijectively to these k edges. In this way each edge e of $G(A)$ is assigned a positive number called the *weight* of e and denoted $\mathrm{wt}(e) = \mathrm{wt}_A(e)$. The *weight* of a $G(A)$ word $\gamma = e_1 \ldots e_m$ is then defined to be

$$\mathrm{wt}(\gamma) = \mathrm{wt}_A(\gamma) = \prod_{i=1}^{m} \mathrm{wt}_A(e_i).$$

For $t \in \mathbb{R}$, let $\beta_A(t)$ be the spectral radius of the irreducible nonnegative matrix $A(t)$. This gives an analytic function $\beta_A : \mathbb{R} \to \mathbb{R}^{++}$ called the *beta function* of A. Taking $t = 1$, we apply the Perron–Frobenius theorem [**Se**] to see that $\beta_A(1)$ is a simple eigenvalue of $A(1)$, and find a strictly positive row vector l and a strictly positive column vector r such that $l\,A(1) = \beta_A(1)\,l$, $A(1)\,r = \beta_A(1)\,r$ and $lr = 1$. For a cylinder set

$$[e_0 e_1 \ldots e_m]_k = \{x = (x_n) \in \Sigma_A : x_k = e_0, x_{k+1} = e_1, \ldots, x_{k+m} = e_m\}$$

corresponding to a $G(A)$ path $e_0 e_1 \ldots e_m$ from state I to state J, we put

$$\mu_A([e_0 e_1 \ldots e_m]_k) = \frac{l(I)\,\mathrm{wt}_A(e_0 e_1 \ldots e_m) r(J)}{\beta_A(1)^{m+1}}$$

to obtain a σ_A-invariant Borel measure μ_A on (Σ_A, σ_A). We refer to $(\Sigma_A, \sigma_A, \mu_A)$ as the Markov chain and to μ_A as the Markov measure defined by A. This construction is easily related to the traditional definition of Markov chains: if M is an irreducible nonnegative matrix, in particular if M is stochastic, the Markov chain defined by M is the same as the one associated with the $\mathbb{Z}^+$[exp] matrix A with

$$A(I, J) = \begin{cases} 0, & \text{if } M(I, J) = 0, \\ M(I, J)^t, & \text{otherwise.} \end{cases}$$

The dynamical interpretation of the association of $\mathbb{Z}^+$[exp] matrices to Markov chains can be found in [**PT, T3**]; see [**B, BT, MT1, MT2, MT3, MT4, PT, T1, T2, T3**] for some fruits of this approach.

We continue our discussion of the $\mathbb{Z}^+$[exp] matrix A and the Markov chain $(\Sigma_A, \sigma_A, \mu_A)$. We will denote the period of A by $d(A)$ and the length of a $G(A)$ word γ by $l(\gamma)$. Recall that periodic orbits of (Σ_A, σ_A) correspond to cycles of $G(A)$. The *weight-per-symbol* of a cycle γ is defined to be

$$\mathrm{wps}(\gamma) = \mathrm{wps}_A(\gamma) = \frac{1}{l(\gamma)} \log(\mathrm{wt}_A(\gamma)).$$

The rational convex hull of $\{\mathrm{wps}_A(\gamma) : \gamma \text{ is a cycle of } G(A)\}$ was shown in [**MT1**] to be a polytope. It is called the *weight-per-symbol polytope* of A, and will be denoted $W(A)$. Let $\Gamma(A)$ be the multiplicative subgroup of $\mathbb{R}^{++}$ generated by $\{\mathrm{wt}_A(\gamma) : \gamma \text{ is a cycle of } G(A)\}$, and let $\Delta(A) = \{\mathrm{wt}_A(\gamma)/\,\mathrm{wt}_A(\gamma') : \gamma, \gamma' \text{ are cycles of } G(A) \text{ with } l(\gamma) = l(\gamma')\}$. Picking cycles η, γ of $G(A)$ such that $l(\eta) = l(\gamma) + d(A)$, put

$$c_A = \left(\frac{\mathrm{wt}_A(\eta)}{\mathrm{wt}_A(\gamma)}\right)^{1/d(A)}.$$

Clearly, $\Delta(A)$ is a subgroup of $\Gamma(A)$. It is shown in [**PS**] that $\Gamma(A)/\Delta(A)$ is a cyclic group generated by $c_A^{d(A)}\Delta(A)$. The following description of the behaviour of these objects under finite-to-one block codes is culled from results of [**T1, JKKMS, PT, MT1, PS**].

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THEOREM 2.1. *Let A, B be irreducible $\mathbb{Z}^+[\exp]$ matrices. If there exists a finite-to-one measure-preserving block code of $(\Sigma_A, \sigma_A, \mu_A)$ onto $(\Sigma_B, \sigma_B, \mu_B)$, then*
(i) $\beta_A = \beta_B$;
(ii) $W(A) = W(B)$;
(iii) $\Gamma(A)$ *is a finite-index subgroup of* $\Gamma(B)$;
(iv) $\Delta(A)$ *is a finite-index subgroup of* $\Delta(B)$; *and*
(v) $d(B)|d(A)$ *and* $c_A^{d(A)}\Delta(B) = c_B^{d(A)}\Delta(B)$.

It was proved in [**MT1, T3**] that $\beta_A = \beta_B$ implies $W(A) = W(B)$. The following two propositions are contained in (5.2)–(5.5) of [**MT1**]: although the statements in [**MT1**] have the assumption $\beta_A = \beta_B$, their proofs only use the weaker hypothesis $W(A) = W(B)$.

PROPOSITION 2.2. *If A, B are irreducible $\mathbb{Z}^+[\exp]$ matrices with $W(A) = W(B)$, then there exists $k \in \mathbb{N}$ such that $d(A)|k$, $d(B)|k$ and $c_A^k\Delta(A) \cap c_B^k\Delta(B) \neq \emptyset$.*

PROPOSITION 2.3. *Suppose A, B are irreducible $\mathbb{Z}^+[\exp]$ matrices with $W(A) = W(B)$.*
(a) $\Gamma(A) \cap \Gamma(B)$ *has finite index in* $\Gamma(A)$ *and* $\Gamma(B)$.
(b) $\Delta(A) \cap \Delta(B)$ *has finite index in* $\Delta(A)$ *and* $\Delta(B)$.

An element $B \in \mathbb{Z}^+[\exp]$ may be regarded as a 1×1 matrix and it is easy to see that the Markov chain corresponding to $B \in \mathbb{Z}^+[\exp]$ is, in fact, a Bernoulli shift. In this case (5.5) and (6.1) of [**MT1**] provide:

THEOREM 2.4. *Let A be an irreducible $\mathbb{Z}^+[\exp]$ matrix with $\beta_A = B \in \mathbb{Z}^+[\exp]$. Then:*
(i) $\Gamma(A)$ *is a finite-index subgroup of* $\Gamma(B)$;
(ii) $\Delta(A)$ *is a finite-index subgroup of* $\Delta(B)$; *and*
(iii) $c_A^{d(A)}\Delta(B) = c_B^{d(A)}\Delta(B)$.

Consider a beta function $\beta = \beta_B$. If A is an irreducible $\mathbb{Z}^+[\exp]$ matrix such that $\beta_A = \beta = \beta_B$, then Proposition 2.3(a) implies that each element of $\Gamma(A)$ is a root of an element of $\Gamma(B)$. So, the group $\langle\Gamma(A), \Gamma(B)\rangle$ may be obtained from $\Gamma(B)$ by adjoining finitely many roots. What happens as we vary A? Is there a group $\Gamma(\beta) \subset \mathbb{R}^{++}$ such that $\Gamma(A)$ is a finite-index subgroup of $\Gamma(\beta)$ whenever A is an irreducible $\mathbb{Z}^+[\exp]$ matrix with $\beta_A = \beta$? Note that Theorem 2.4 says that when $\beta \in \mathbb{Z}^+[\exp]$ the objects associated with the corresponding Bernoulli shift have such universal properties. The unavailability of similar objects (even in the case $\beta \in \mathbb{Z}[\exp]\backslash\mathbb{Z}^+[\exp]$) has been a sticking point in the theory. In §3, we will define $d(\beta)$, $\Gamma(\beta)$, $\Delta(\beta)$ and $c_\beta^{d(\beta)}\Delta(\beta)$ for an arbitrary beta function β and, with these objects, generalize Theorem 2.4.

We end this section by relating the objects of A^n to those of A. For further properties of $\Gamma(A)$, $\Delta(A)$ and $c_A^{d(A)}\Delta(A)$, see [**PS, MT1**].

PROPOSITION 2.5. *Let A be an irreducible $\mathbb{Z}^+[\exp]$ matrix, and $n \in \mathbb{N}$. Putting $g = \text{g.c.d.}\{n, d(A)\}$, we have $d(A^n) = d(A)/g$, $\Gamma(A^n) \subset \Gamma(A)$, $\Delta(A^n) = \Delta(A)$ and*

$$c_{A^n}^{d(A^n)}\Delta(A) = c_A^{d(A)n/g}\Delta(A).$$

Proof. The first two statements are easy to check. The third is proved in (1.6) of [**MT1**]. The final equality corrects (1.8)(5) of [**MT1**], which was stated without proof. To verify

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this equality, let η, γ be $G(A^n)$ cycles which pass through a common state and have lengths $l_{A^n}(\gamma) = kd(A^n)$ and $l_{A^n}(\eta) = (k+1)d(A^n)$ for some $k \in \mathbb{N}$. As $G(A)$ cycles their lengths are $knd(A^n)$ and $(k+1)nd(A^n)$, whose difference equals $nd(A^n) = d(A)n/g$. It follows that

$$c_{A^n}^{d(A^n)} = \frac{\mathrm{wt}_A(\eta)}{\mathrm{wt}_A(\gamma)} \pmod{\Delta(A)}$$

and

$$c_A^{d(A)\frac{n}{g}} = \frac{\mathrm{wt}_A(\eta)}{\mathrm{wt}_A(\gamma)} \pmod{\Delta(A)}.$$

Hence, $c_{A^n}^{d(A^n)} = c_A^{d(A)n/g} \pmod{\Delta(A)}$. □

3. *Defining the basic objects for a beta function*

We will define $d(\beta)$, $\Gamma(\beta)$, $\Delta(\beta)$ and $c_\beta^{d(\beta)}\Delta(\beta)$ for a beta function β. Let $\mathcal{R} = \mathbb{Z}[\exp]$ and $\mathcal{R}^+ = \mathbb{Z}^+[\exp]$. If $f \in \mathcal{R}$, write $f = \sum_{i=1}^k n_i a_i^t$ with $a_i \in \mathbb{R}^{++}$, $n_i \in \mathbb{Z}$ to define an element

$$|f| = \sum_{i=1}^k |n_i| a_i^t$$

of $\mathcal{R}^+$. More generally, for an $\mathcal{R}$ matrix B, we define an $\mathcal{R}^+$ matrix $|B|$ of the same size by letting

$$|B|(I, J) = |B(I, J)|.$$

LEMMA 3.1. *A nonzero element f of $\mathcal{R} = \mathbb{Z}[\exp]$ has finitely many roots.*

Proof. Express $f(t) = \sum_{i=1}^k n_i a_i^t$ with distinct $a_i \in \mathbb{R}^{++}$ and nonzero $n_i \in \mathbb{Z}$. Since the maximum and minimum elements of $\{a_i : 1 \le i \le k\}$ will dominate a_i^t as $t \to \infty$ and $t \to -\infty$, there exists $M > 0$ such that all roots of $f(t)$ lie in $[-M, M]$. Since $f(t)$ is analytic, its roots are isolated. The compactness of $[-M, M]$ does the rest. □

Consider a beta function β, and let A be an irreducible $\mathcal{R}^+$-matrix such that $\beta_A = \beta$. Let $\chi_A(\lambda) = \det(\lambda I - A) \in \mathcal{R}[\lambda]$ be the characteristic polynomial of A, and note that $\chi_A(\beta(t)) \equiv 0$. Thus, $\beta(t)$ is an integral function over $\mathcal{R}$. We show that a function which is integral over $\mathcal{R}$ has a minimal polynomial over $\mathcal{R}$.

Suppose U is an open interval, $\beta : U \to \mathbb{C}$ is analytic and there exists monic $\chi(\lambda) \in \mathcal{R}[\lambda]$ such that $\chi(\beta(t)) = 0$ for $t \in U$. Since $\mathcal{R}$ is far from being a unique factorization domain, we consider its field of fractions $F = \mathcal{R}/\mathcal{R}$. We define an ideal $J \subset F[\lambda]$ as follows. Let $f \in F[\lambda]$. Express $f = s/q$ with $s \in \mathcal{R}[\lambda]$ and nonzero $q \in \mathcal{R}$. Declare that $f \in J$ if and only if $s(\beta(t)) \equiv 0$ on U. By Lemma 3.1 and the analyticity of $s(\beta(t))$, this definition does not depend on the choice of $s \in \mathcal{R}[\lambda]$ and $q \in \mathcal{R}$ such that $s/q = f$. It is easy to check that J is an ideal of $F[\lambda]$. We summarize its definition by writing

$$J = \{f(\lambda) \in F[\lambda] : f(\beta(t)) \equiv 0\}.$$

As $F[\lambda]$ is a principal ideal domain, we can find a monic polynomial $p \in F[\lambda]$ such that $J = pF[\lambda]$. We show that, in fact, $p \in \mathcal{R}[\lambda]$: noting $\chi \in \mathcal{R}[\lambda] \cap J$, let $s \in F[\lambda]$ be such

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that $\chi = ps$. The polynomial s is monic, since χ and p are monic. Write

$$\chi(\lambda) = \lambda^l - q_1\lambda^{l-1} - \cdots - q_{l-1}\lambda - q_l,$$
$$p(\lambda) = \lambda^d - p_1\lambda^{d-1} - \cdots - p_{d-1}\lambda - p_d,$$
$$s(\lambda) = \lambda^m - s_1\lambda^{m-1} - \cdots - s_{m-1}\lambda - s_m,$$

with $q_i \in \mathcal{R}$ and $p_i, s_i \in F$. Expressing each p_i and s_i as a ratio of elements of $\mathcal{R} = \mathbb{Z}[\exp]$, let S be the subgroup of $\mathcal{R}^{++}$ generated by the bases a of the exponentials a^t appearing in the q_i and (the numerators and denominators of) the p_i, s_i. The group ring $\mathbb{Z}[S^t]$ is then a unique factorization domain. In fact, letting k be the rank of the finitely generated free Abelian group S, the ring $\mathbb{Z}[S^t]$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]$. Find $p_0, \widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_d \in \mathbb{Z}[S^t]$ such that $p_i = \widetilde{p}_i/p_0$ for $i = 1, \ldots, d$ and $(p_0, \widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_d) = 1$. Similarly, find coprime $s_0, \widetilde{s}_1, \ldots, \widetilde{s}_m \in \mathbb{Z}[S^t]$ such that $s_i = \widetilde{s}_i/s_0$ for $i = 1, \ldots, m$. From $\chi = ps$ we obtain

$$p_0 s_0(\lambda^l - q_1\lambda^{l-1} - \cdots - q_0) = (p_0\lambda^d - \widetilde{p}_1\lambda^{d-1} - \cdots - \widetilde{p}_d)(s_0\lambda^m - \widetilde{s}_1\lambda^{m-1} - \cdots - \widetilde{s}_m),$$

an equation over $\mathbb{Z}[S^t]$. Since $\mathbb{Z}[S^t]$ is a unique factorization domain, each irreducible factor of $p_0 s_0$ must divide one of the two terms on the right-hand side: if it divides the first term then it divides each of $p_0, \widetilde{p}_1, \ldots, \widetilde{p}_d$, and is invertible since $p_0, \widetilde{p}_1, \ldots, \widetilde{p}_d$ are coprime. If it divides the second term, it is invertible because $s_0, \widetilde{s}_1, \ldots, \widetilde{s}_m$ are coprime. So, p_0 is a monomial and $p \in \mathcal{R}[\lambda]$. In particular, this establishes p as the minimal polynomial of β over $\mathcal{R}[\lambda]$.

PROPOSITION 3.2. *The polynomial $p \in \mathcal{R}[\lambda]$ is such that if $\chi(\lambda) \in \mathcal{R}[\lambda]$ and $\chi(\beta(t)) \equiv 0$ then p divides χ in $\mathcal{R}[\lambda]$.*

For each $t \in \mathbb{R}$, we evaluate the coefficients of p at t to obtain a polynomial in $\mathbb{R}[\lambda]$ whose roots we denote by $\beta_1(t), \ldots, \beta_d(t)$. Note that the discriminant of p, that is

$$D(t) = \prod_{i<j} (\beta_i - \beta_j)^2,$$

is a symmetric polynomial in $\beta_1, \ldots, \beta_d$. Hence, D is a polynomial in the coefficients $p_1, \ldots, p_d$ of p. In particular, $D \in \mathcal{R}$ and, by Lemma 3.1, the roots of $D(t)$ form a finite set $V \subset \mathbb{R}$. It follows that, for $t \in \mathbb{R}\backslash V$, the roots $\beta_1(t), \ldots, \beta_d(t)$ are distinct. We order $\beta_1(t), \ldots, \beta_d(t)$ so that $\beta_1 = \beta$ on U and each $\beta_i(t)$ is analytic on $\mathbb{R}\backslash V$. In the case when β is a beta function, the application of the Perron–Frobenius theorem [**Se**] to an irreducible $\mathcal{R}^+$ matrix with $\beta_A = \beta$ yields $m \in \mathbb{N}$ and roots of unity $\theta_2, \ldots, \theta_m$, so that the ordering may be chosen to also have $\beta_i = \theta_i\beta(t)$ for $i = 2, \ldots, m$ and $|\beta_i(t)| < \beta(t)$ for $i = m+1, \ldots, d$. Letting $\theta_1 = 1$, we define the *period* $d(\beta)$ of β to be the smallest positive integer such that $\theta_i^{d(\beta)} = 1$ for $i = 1, \ldots, m$. It is easy to see that if $\delta \in \mathbb{N}$ is such that $\theta_i^{\delta} = 1$ for $i = 1, \ldots, m$ then $d(\beta)$ divides δ. In particular, $d(\beta)|d(A)$ for every irreducible $\mathcal{R}^+$ matrix A with $\beta_A = \beta$. The beta function is called *aperiodic* if $d(\beta) = 1$.

The notion of the period of β_A appears to be new even in the case of integral matrices: if A is over $\mathbb{Z}^+$ then $\beta_A(t)$ reduces to the spectral radius μ of A in that $\beta_A(t)$ is a constant function equal to μ. Following the above definition, the period $d(\mu)$ of μ is the smallest

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positive integer $d(\mu)$ such that $\mu^{d(\mu)}$ is a Perron number. By the main result of [**L**] (given in [**LM**] as Theorem 11.1.4), there exists an aperiodic $\mathbb{Z}^+$ matrix whose spectral radius equals $\mu^{d(\mu)}$, which easily leads to the following refinement of Theorem 11.1.5 of [**LM**].

THEOREM 3.3. *Let $\mu > 0$. The following are equivalent.*

(a) *μ is the spectral radius of a nonnegative integral matrix.*
(b) *μ is the spectral radius of an irreducible nonnegative integral matrix.*
(c) *μ^δ is a Perron number for some positive integer δ.*

If μ satisfies these equivalent conditions then:

(i) *$d(\beta)$ divides the period of every irreducible nonnegative integral matrix of spectral radius μ; and*
(ii) *there exists an irreducible nonnegative integral matrix of period $d(\beta)$ and spectral radius μ.*

We also remark that, by a result of Boyd [**Bd**], the conjugates of the spectral radius μ of a $\mathbb{Z}^+$-matrix are invariant under multiplication by $d(\mu)$th roots of unity. In fact, the set $\{\beta_1(t), \ldots, \beta_d(t)\}$ of conjugates of a beta function β is invariant under multiplication by $d(\beta)$th roots of unity as an immediate consequence of the following generalization of the result in [**Bd**]. I thank Boris Solomyak for bringing [**Bd**] to my attention.

THEOREM 3.4. *Let $\beta(t)$ be a positive algebraic function over $\mathcal{R}$ and let $\beta_1(t) = \beta(t)$, $\beta_2(t), \ldots, \beta_d(t)$ be its conjugates. Suppose there exist $m \in \mathbb{N}$, $1 \le m \le d$, and an open interval U such that the following hold on U. Each $\beta_i(t)$ is analytic, $|\beta_i(t)| = \beta(t)$ for $i = 1, \ldots, m$ and $|\beta_i(t)| < \beta(t)$ for $i = m+1, \ldots, d$. Then m divides d and there exists a polynomial $q \in \mathcal{R}[\lambda]$ of degree d/m such that the minimal polynomial $p(\lambda)$ of β is given by $p(\lambda) = q(\lambda^m)$. In the case when β is a beta function, m equals the period $d(\beta)$ of β.*

The proof of Theorem 3.4 is a straightforward adaptation of Boyd's proof [**Bd**]; the statement about beta functions follows from the definition of $d(\beta)$ and the other statements of Theorem 3.4.

Fix a beta function β. Let $C = C_{\beta^{d(\beta)}}$ be the companion matrix of the minimal polynomial of $\beta^{d(\beta)}$, and let $|C|$ be as defined at the beginning of this section. It is easy to see that, as a consequence of the definition of $d(\beta)$, the $\mathbb{Z}^+$[exp] matrix $|C|$ is aperiodic. We let

$$W(\beta) = \frac{1}{d(\beta)} W(|C|),$$
$$\Gamma(\beta) = \Gamma(|C|),$$
$$\Delta(\beta) = \Delta(|C|).$$

Considering $c_{|C|}\Delta(|C|)$, we let

$$c_\beta = (c_{|C|})^{1/d(\beta)}$$

to obtain a coset $c_\beta^{d(\beta)}\Delta(\beta)$ of $\Gamma(\beta)/\Delta(\beta)$ which generates the cyclic group $\Gamma(\beta)/\Delta(\beta)$. Our main result is as follows.

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THEOREM 3.5. *Let A be an irreducible $\mathbb{Z}^+$[exp] matrix such that $\beta_A = \beta$. Then:*

(i) $W(A) = W(\beta)$;
(ii) $\Gamma(A)$ *is a finite-index subgroup of* $\Gamma(\beta)$;
(iii) $\Delta(A)$ *is a finite-index subgroup of* $\Delta(\beta)$; *and*
(iv) $d(\beta)|d(A)$ *and* $c_A^{d(A)}\Delta(\beta) = c_\beta^{d(A)}\Delta(\beta)$.

Theorem 3.5 should be compared with with Theorems 2.1 and 2.4, which it generalizes. Properties of $\Gamma(\beta)$, $\Delta(\beta)$ and $c_\beta^{d(\beta)}\Delta(\beta)$ will be developed in §§4 and 5. The proofs of (i) and (ii) of Theorem 3.5 are contained in §4, those of (iii) and (iv) in §5. See the end of §5 for an example.

Remark 3.6. We remark on the significance of Theorem 3.5 to the theory of coding for Markov chains. Note that if Γ is a finitely generated subgroup of $\mathbb{R}^{++}$, the ring $\mathbb{Z}[\Gamma^t]$ is isomorphic to a Laurent polynomial ring: picking a basis $a_1, \ldots, a_k$ for the free Abelian group Γ, an isomorphism of $\mathbb{Z}[\Gamma^t]$ with $\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]$ results from putting $x_1 = a_1^t, \ldots, x_k = a_k^t$. This enables us to replace the defining matrix of a Markov chain by a $\mathbb{Z}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$-matrix, which has the dynamical significance of making apparent a family of Markov measures indexed by $\mathbb{R}^k$ and invariantly associated to a single (generic) member of the family. (See §2 of [**T3**].) This approach has been useful in the study of codes and classifications of Markov chains. (See, for example [**B, BT, P, PT, T2, MT1, MT2, MT3, MT4**].) In making the transition to polynomial matrices it is important to pick a subgroup of $\mathbb{R}^{++}$ that is canonically associated to $(\Sigma_A, \sigma_A, \mu_A)$. For a variety of reasons [**T2, PS, S2**], the groups $\Gamma(A)$ and $\Delta(A)$ present themselves. Let us focus on $\Gamma(A)$; the situation with $\Delta(A)$ is similar. By (1.9) of [**MT1**], there is a diagonal matrix D over exp such that the entries of $D^{-1}AD$ belong to $\mathbb{Z}^+[\Gamma(A)^t]$. Since none of the objects under consideration is altered upon replacing A with $D^{-1}AD$, we assume A is over $\mathbb{Z}^+[\Gamma(A)^t]$. The problem is that $\Gamma(A)$ is not invariant, in general, under finite-to-one block codes. The beta function, on the other hand, is invariant. That is, for two Markov chains $(\Sigma_A, \sigma_A, \mu_A)$ and $(\Sigma_B, \sigma_B, \mu_B)$ to be related by a finite-to-one block code, or in a number of other ways, it is necessary [**T1, S1, S2**] that $\beta_A = \beta_B$. Prior to Theorem 3.5, the approach was to work over the group generated by $\Gamma(A) \cup \Gamma(B)$ which, by (7.4) of [**PS**], amounted to adjoining a finite number of roots to $\Gamma(A)$. In particular, we could not effectively associate a polynomial matrix with $(\Sigma_A, \sigma_A, \mu_A)$ until after we knew which Markov chains were being considered with it. It was not clear what would happen if we considered several Markov chains. For instance, would the number of roots adjoined to $\Gamma(A)$ keep growing? Such issues extend beyond aesthetics. In the case $\beta \in \mathbb{Z}^+$[exp], the answer provided by Theorem 2.4 was fundamental to the results of [**MT4**]. For the general case, Theorem 3.5(ii) shows that there is a single finite extension of $\Gamma(A)$, namely $\Gamma(\beta)$, which is universal for β. Picking a basis $a_1, \ldots, a_k$ for $\Gamma = \Gamma(\beta)$ and putting $x_1 = a_1^t, \ldots, x_k = a_k^t$, we can canonically associate to any Markov chain with beta function β a $\mathbb{Z}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$ matrix, which can then be put to use in coding problems involving the Markov chain. By (5.1) of [**PS**] and (iii) and (iv) of Theorem 3.5, similar remarks are valid for $\Delta(\beta)$.

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4. *The group* Γ

For the remainder of the paper, we let β be a beta function, $p(\lambda) = \lambda^d - p_1\lambda^{d-1} - \cdots - p_d \in \mathcal{R}[\lambda]$ the minimal polynomial of β given by Proposition 3.2, and C_β the companion matrix of p. We also let A be an irreducible matrix over $\mathcal{R}^+ = \mathbb{Z}^+[\exp]$ such that $\beta_A = \beta$. That $W(A) = W(\beta)$ is an easy consequence of the following formulation of (12) of [**T3**].

THEOREM 4.1. *If A is an irreducible matrix over $\mathbb{Z}^+[\exp]$ and $\beta_A = \beta$, then $W(A) = W(|C_\beta|)$.*

Proof. The corresponding result for matrices over $\mathbb{Z}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$ is proved in (12) of [**T3**]. Let Γ be a finitely generated subgroup of $\mathbb{R}^{++}$ such that A and $|C_\beta|$ are both over $\mathbb{Z}^+[\Gamma^t]$. Choose a basis for Γ to pass to matrices over $\mathbb{Z}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$, and use (12) of [**T3**]. □

COROLLARY 4.2. *If A is an irreducible matrix over $\mathbb{Z}^+[\exp]$ and $\beta_A = \beta$, then, for any $k \in \mathbb{N}$, we have*

$$\frac{1}{k}W(|C_{\beta^k}|) = \frac{1}{k}W(A^k) = W(A) = W(|C_\beta|) = W(\beta).$$

Proof. The fact that $(1/k)W(A^k) = W(A)$ is immediate from the definition of the weight-per-symbol polytope. The rest follows from 4.1 and the definition of $W(\beta)$. □

In particular, Corollary 4.2 proves Theorem 3.5(i). The rest of our proof of Theorem 3.5 will generalize that of (6.1) of [**MT1**]. Writing

$$C_\beta = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ p_d & p_{d-1} & \cdots & p_2 & p_1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ \beta \\ \vdots \\ \beta^{d-1} \end{bmatrix},$$

we have $C_\beta b = \beta b$. Note that $|C_\beta|$ is an irreducible matrix since $p_d \neq 0$. Recall that F denotes the field of fractions of $\mathcal{R} = \mathbb{Z}[\exp]$.

LEMMA 4.3. *Suppose A is $n \times n$. For column vectors $r_0, r_1, \ldots, r_{d-1} \in F^n$, let $r = r_0 + r_1\beta + \cdots + r_{d-1}\beta^{d-1}$ and let $R = [r_0\, r_1 \ldots r_{d-1}]$ be the $n \times d$ matrix with columns $r_0, r_1, \ldots, r_{d-1}$. We have $Ar = \beta r$ if and only if $AR = R\,C_\beta$.*

Lemma 4.3 is a straightforward consequence of the fact that p is the minimal polynomial of β over F.

PROPOSITION 4.4. *If A is an irreducible $\mathbb{Z}^+[\exp]$ matrix with $\beta_A = \beta$, then $\Gamma(A) \subset \Gamma(|C_\beta|)$.*

Proof. Write $\Gamma_0 = \Gamma(|C_\beta|)$, and let Γ be the subgroup of $\mathbb{R}^{++}$ generated by $\Gamma(A) \cup \Gamma_0$. By Theorem 4.1 and Proposition 2.3(a), Γ_0 has finite index in Γ. Conjugating A by a suitable diagonal matrix over exp, we assume that the entries of A belong to $\mathbb{Z}^+[\Gamma(A)^t] \subset \mathbb{Z}^+[\Gamma^t]$. Let r be a nontrivial vector over $\mathbb{Z}[\Gamma^t][\beta]$ such that $Ar = \beta r$; for example, take r to

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be a column of the adjoint $\mathrm{Adj}(\beta I - A)$. Since p is the minimal polynomial of β and its coefficients $p_1, p_2, \ldots, p_d$ belong to $\mathbb{Z}[\Gamma^t]$, we can find vectors $r_0, r_1, \ldots, r_{d-1}$ over $\mathbb{Z}[\Gamma^t]$ such that

$$r = r_0 + r_1\beta + \cdots + r_{d-1}\beta^{d-1}.$$

Letting $R = [r_0\, r_1 \ldots r_{d-1}]$, we see from Lemma 4.3 that $AR = R\,C_\beta$. Considering the finite group Γ/Γ_0, write

$$A = \sum_{g\in\Gamma/\Gamma_0} A_g, \quad R = \sum_{g\in\Gamma/\Gamma_0} R_g,$$

where the matrices A_g and R_g are over $\mathbb{Z}[g^t]$. Let $\widehat{A}$ be the $\Gamma/\Gamma_0 \times \Gamma/\Gamma_0$ block matrix whose (h,k) block equals

$$\widehat{A}_{(h,k)} = A_{h^{-1}k}.$$

For each $g \in \Gamma/\Gamma_0$, let R^g be the $\Gamma/\Gamma_0 \times 1$ block matrix whose $(k,1)$ block equals

$$(R^g)_k = (R^g)_{(k,1)} = R_{k^{-1}g}.$$

A simple calculation shows that

$$\widehat{A}R^g = R^g C_\beta.$$

Multiplying by b,

$$\widehat{A}(R^g b) = R^g C_\beta b = \beta(R^g b).$$

Now let $\widehat{I}$ be the $1 \times \Gamma/\Gamma_0$ block matrix consisting of $|\Gamma/\Gamma_0|$ copies of the $n \times n$ identity matrix I, and observe that

$$\widehat{I}R^g b = Rb = r. \qquad (*)$$

In particular, each $R^g b$ is nontrivial. But, as was shown in (6.3) of [**MT1**], the matrix $\widehat{A}$ is irreducible. It follows from the Perron–Frobenius theorem that $\beta_{\widehat{A}} = \beta$ and the vectors $R^g b$ are multiples of one another. In fact, considering $(*)$, we must have $R^g b = R^h b$ for all $g, h \in \Gamma/\Gamma_0$. In order to draw our final conclusions, we let $(r_i)_g$ be vectors over $\mathbb{Z}[g^t]$ such that

$$r_i = \sum_{g\in\Gamma/\Gamma_0} (r_i)_g$$

for $i = 0, 1, \ldots, d-1$. Observe that each $R^g b$ is a block vector whose blocks are indexed by Γ/Γ_0 and that the block of $R^g b$ corresponding to $k \in \Gamma/\Gamma_0$ is given by

$$(R^g)_k\, b = R_{k^{-1}g} b = (r_0)_{k^{-1}g}\,\beta + \cdots + (r_{d-1})_{k^{-1}g}\beta^{d-1}.$$

So, the equation $R^g b = R^h b$ means that $(r_i)_g = (r_i)_h$ for all $g, h \in \Gamma/\Gamma_0$ and $i = 0, 1, \ldots, d-1$. This can only happen if Γ/Γ_0 is the trivial group. □

Proof of Theorem 3.5(ii). By Propositions 2.5 and 4.4 and the fact that $d(\beta)|d(A)$,

$$\Gamma(A) = \Gamma(A^{d(\beta)}) \subset \Gamma(|C_{\beta^{d(\beta)}}|) = \Gamma(\beta). \qquad \square$$

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5. *The group Δ and the coset $c^d\Delta$*

The following simple lemma is easily verified.

LEMMA 5.1. *Consider the minimal polynomial*

$$p(\lambda) = \lambda^d - p_1\lambda^{d-1} - \cdots - p_{d-1}\lambda - p_d$$

of β and its companion matrix C_β. For $a^t \in \exp$, the minimal polynomial of $a^t\beta$ is

$$\lambda^d - a^t p_1\lambda^{d-1} - a^{2t} p_2\lambda^{d-2} - \cdots - a^{(d-1)t} p_{d-1}\lambda - a^{dt} p_d$$

and it has companion matrix

$$C_{a^t\beta} = a^t U^{-1} C_\beta U,$$

where U is the diagonal matrix

$$U = \begin{bmatrix} a^{-t} & & & \\ & a^{-2t} & & \\ & & \ddots & \\ & & & a^{-dt} \end{bmatrix}.$$

Before directing our attention to the proofs of Theorem 3.5(iii) and (iv), we relate the objects associated with β^n to those of β.

PROPOSITION 5.2. *If β is aperiodic and $n \in \mathbb{N}$, then $\Gamma(|C_{\beta^n}|) \subset \Gamma(|C_\beta|^n)$, $\Delta(|C_{\beta^n}|) \subset \Delta(|C_\beta|)$ and $c_{|C_{\beta^n}|}\Delta(|C_\beta|) = (c_{|C_\beta|})^n\Delta(|C_\beta|)$.*

COROLLARY 5.3. *Let $n \in \mathbb{N}$ and $g = \text{g.c.d.}\{n, d(\beta)\}$. Then $d(\beta^n) = d(\beta)/g$,*

$$\Gamma(\beta^n) \subset \Gamma(|C_{\beta^{d(\beta)}}|^{n/g}) \subset \Gamma(\beta),$$

$\Delta(\beta^n) \subset \Delta(\beta)$ and $c_{\beta^n}^{d(\beta^n)}\Delta(\beta) = c_\beta^{d(\beta)n/g}\Delta(\beta)$.

Proof of Proposition 5.2. Let $\beta_1 = \beta, \beta_2, \ldots, \beta_d$ be the conjugates of β. By aperiodicity, we have $\beta_i^n \neq \beta^n$ for $i \neq 1$, and the field polynomial

$$q(\lambda) = \prod_{i=1}^{d}(\lambda - \beta_i^n)$$

is irreducible. So, q is the minimal polynomial of β^n over F. Express

$$q(\lambda) = \lambda^d - q_1\lambda^{d-1} - \cdots - q_{d-1}\lambda - q_d,$$

with $q_i \in \mathbb{Z}[\exp]$. Note that the aperiodicity of β implies that $|C_\beta|$ and $|C_{\beta^n}|$ are aperiodic matrices. The cyclic structure of $G(|C_\beta|)$ is easily described in terms of the coefficients p_i of the minimal polynomial p of β: the graph $G(|C_\beta|)$ contains a simple cycle of length i and weight a for each exponential term a^t of $|p_i|$, these simple cycles may be viewed as having a common base state and their free concatenations yield all cycles of $G(|C_\beta|)$. A similar statement is valid for $G(|C_{\beta^n}|)$ and $|q_i|$.

Clearly $\pm q_i$ equals the ith elementary symmetric polynomial in $\beta_1^n, \ldots, \beta_d^n$. So, q_i is a symmetric homogeneous polynomial of degree ni in $\beta_1, \ldots, \beta_d$. Symmetry implies

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that each q_i is a polynomial in the coefficients of p, and homogeneity shows that it is an integral combination of products $p_1^{k_1} p_2^{k_2} \dots p_d^{k_d}$ with $k_1 + 2k_2 + \cdots + dk_d = ni$. Note that $\Gamma(|p_1^{k_1} p_2^{k_2} \dots p_d^{k_d}|) \subset \Gamma(|p_1|^{k_1}|p_2|^{k_2} \dots |p_d|^{k_d})$ and that the condition $k_1 + 2k_2 + \cdots + dk_d = ni$ ensures that the terms of the product $|p_1|^{k_1}|p_2|^{k_2} \dots |p_d|^{k_d}$ correspond to $|C_\beta|$-cycles of length ni. Thus, $\Gamma(|q_i|) \subset \Gamma(|C_\beta|^n)$. Since $\Gamma(|C_{\beta^n}|)$ is generated by

$$\bigcup_{i=1}^{d} \Gamma(|q_i|),$$

it follows that $\Gamma(|C_{\beta^n}|) \subset \Gamma(|C_\beta|^n)$. Writing $c = c_{|C_\beta|}$, find a diagonal matrix D over exp such that

$$\bar{C} = c^{-t} D^{-1} |C_\beta| D$$

is over $\mathbb{Z}^+[\Delta(|C_\beta|)^t]$. Using Proposition 2.5, Lemma 5.1 and the inclusion just proved,

$$\Delta(|C_{\beta^n}|) = \Delta(|C_{c^{-nt}\beta^n}|) \subset \Gamma(|C_{c^{-nt}\beta^n}|) \subset \Gamma(|C_{c^{-t}\beta}|).$$

Using Lemma 5.1 again,

$$\Gamma(|C_{c^{-t}\beta}|) = \Gamma(c^{-t}|C_\beta|) = \Gamma(\bar{C}) = \Delta(\bar{C}) = \Delta(|C_\beta|).$$

This shows $\Delta(\beta^n) \subset \Delta(\beta)$. Using our description of the cyclic structures of $G(|C_{\beta^n}|)$ and $G(|C_\beta|)$ in terms of $|q_i|$ and $|p_i|$, and the above homogeneity argument, we have: for each $G(|C_{\beta^n}|)$-cycle of length l, there exists a cycle of the same weight and length nl in $G(|C_\beta|)$. Considering the ratio of the weights of two $G(|C_{\beta^n}|)$-cycles of lengths l and $l+1$, we see that

$$c_{|C_{\beta^n}|} = (c_{|C_\beta|})^n \pmod{\Delta(|C_\beta|)}. \qquad \square$$

Proof of Corollary 5.3. Clearly, $d(\beta^n) = d(\beta)/g$ and $nd(\beta^n) = d(\beta)n/g$. Since $n/g \in \mathbb{N}$, we can apply Proposition 5.2 to $\beta^{d(\beta)}$ to see that

$$\Gamma(\beta^n) = \Gamma(|C_{\beta^{nd(\beta^n)}}|) \subset \Gamma(|C_{\beta^{d(\beta)}}|^{n/g}) \subset \Gamma(|C_{\beta^{d(\beta)}}|) = \Gamma(\beta).$$

Similarly, $\Delta(\beta^n) \subset \Delta(\beta)$. Using the definition of the period and Proposition 5.2 again, modulo $\Delta(\beta)$ we have

$$c_{\beta^n}^{d(\beta^n)} = c_{|C_{\beta^{nd(\beta^n)}}|} = (c_{|C_{\beta^{d(\beta)}}|})^{n/g} = c_\beta^{d(\beta)n/g} = c_\beta^{nd(\beta^n)}. \qquad \square$$

Proof of Theorem 3.5(iii). Let $C = C_{\beta^{d(\beta)}}$. Apply Proposition 2.5, Theorem 4.1 and Proposition 2.2 to $A^{d(\beta)}$ and $|C|$ to find $k \in \mathbb{N}$ and $c > 0$ such that $d(A^{d(\beta)})|k$ and

$$c^k \in (c_{A^{d(\beta)}})^k \Delta(A) \cap (c_{|C|})^k \Delta(|C|).$$

Since we have already observed in §3 that $d(\beta)|d(A)$, we see from Proposition 2.5 that

$$(c_{A^{d(\beta)}})^{d(A)/d(\beta)} \Delta(A) = c_A^{d(A)} \Delta(A).$$

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So, $c^k\Delta(A) = c_A^{kd(\beta)}\Delta(A)$, and by Lemma 5.1 of [**PS**] (or (1.9) of [**MT1**]), we can find a diagonal matrix D over exp such that the matrix

$$\widetilde{A} = c^{-kt}D^{-1}A^{kd(\beta)}D$$

is over $\mathbb{Z}^+[\Delta(A)^t]$. In particular, $\Delta(\widetilde{A}) = \Gamma(\widetilde{A})$, so that

$$\Delta(A) = \Delta(c^{-kt}A^{kd(\beta)}) = \Delta(\widetilde{A}) = \Gamma(\widetilde{A}) = \Gamma(c^{-kt}A^{kd(\beta)}).$$

Using Theorem 3.5(ii) and Lemma 5.1, and applying Proposition 5.2 to the aperiodic beta function $c^{-t}\beta^{d(\beta)}$,

$$\Gamma(c^{-kt}A^{kd(\beta)}) \subset \Gamma((c^{-t}\beta^{d(\beta)})^k) \subset \Gamma(|C_{c^{-t}\beta^{d(\beta)}}|^k) = \Gamma(c^{-kt}|C|^k).$$

Since $\beta^{d(\beta)}$ is aperiodic, $|C|$ is an aperiodic matrix and

$$c^k\Delta(|C|) = (c_{|C|})^k\Delta(|C|) = c_{|C|^k}\Delta(|C|).$$

Thus the matrix $c^{-kt}|C|^k$ is conjugate, by a diagonal matrix over exp, to a matrix over $\mathbb{Z}^+[\Delta(|C|)^t]$. Combining these observations, we see that

$$\begin{aligned}\Delta(A) \subset \Gamma(c^{-kt}|C|^k) &= \Delta(c^{-kt}|C|^k)\\ &= \Delta(|C|^k) = \Delta(|C|) = \Delta(\beta).\end{aligned}$$ □

Proof of Theorem 3.5(iv). The proof is divided into three cases.

Case 1: $|\Gamma(\beta)/\Delta(\beta)| = \infty$. Noting that $W(|C_{\beta^{d(\beta)}}|) = W(A^{d(\beta)})$ by Corollary 4.2, we obtain from Proposition 2.2 a positive integer k such that $d(A)|k$ and $c_A^k\Delta(A) \cap c_\beta^k\Delta(\beta) \neq \emptyset$. By Theorem 3.5(iii), this implies

$$\left(\frac{c_A}{c_\beta}\right)^k \in \Delta(\beta). \qquad (*)$$

Using Theorem 3.5(ii) and the fact that $d(\beta)|d(A)$, we find that $c_A^{d(A)} \in \Gamma(A) \subset \Gamma(\beta)$ and $c_\beta^{d(A)} \in \Gamma(\beta)$. Moreover, by $(*)$, a power of $(c_A/c_\beta)^{d(A)} \in \Gamma(\beta)$ lies in $\Delta(\beta)$. Since the infinite cyclic group $\Gamma(\beta)/\Delta(\beta)$ is torsion free, it follows that $c_A^{d(A)}\Delta(\beta) = c_\beta^{d(A)}\Delta(\beta)$.

Case 2: $|\Gamma(\beta)/\Delta(\beta)| < \infty$ and β is aperiodic. Let $\delta = d(A)$ be the period of A, and $C = C_{\beta^\delta}$. By Proposition 5.2, $\Delta(|C|) \subset \Delta(\beta)$ and there exists $w \in \Delta(\beta)$ such that $c_{|C|} = c_\beta^\delta w$. Hence, we can find a diagonal matrix D over exp such that

$$c_\beta^{-\delta t}D|C|D^{-1}$$

is over $\mathbb{Z}^+[\Delta(\beta)^t]$. Then

$$\bar{C} = c_\beta^{-\delta t}DCD^{-1}$$

is over $\mathbb{Z}[\Delta(\beta)^t]$. Let $\bar{A}$ be a matrix over $\mathbb{Z}^+[\Delta(A)^t]$ such that $\bar{A}$ is conjugate by a diagonal matrix over exp to $c_A^{-\delta t}A^\delta$. Note that

$$\beta_{\bar{A}} = (c_A^{-t}\beta)^\delta,$$

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and that Lemma 5.1 provides us with a diagonal matrix U over exp such that

$$C_{\beta_{\bar{A}}} = c_A^{-\delta t} U^{-1} C U.$$

Let $r_0, r_1, \ldots, r_{d-1}$ be vectors over $\mathbb{Z}[\exp]$ such that

$$r = r_0 + r_1 \beta_{\bar{A}} + \cdots + r_{d-1} \beta_{\bar{A}}^{d-1}$$

satisfies $\bar{A} r = \beta_{\bar{A}} r$. Letting $R = [r_0 r_1 \ldots r_{d-1}]$ as in Lemma 4.3, we have

$$\bar{A} R = R C_{\beta_{\bar{A}}}.$$

Writing $\mu = (c_\beta / c_A)^\delta$,

$$\bar{A} R = c_A^{-\delta t} R U^{-1} C U = \mu^t R U^{-1} D^{-1} \bar{C} D U.$$

That is,

$$\bar{A}(R U^{-1} D^{-1}) = \mu^t (R U^{-1} D^{-1}) \bar{C}.$$

This means that there is a nontrivial solution of $\bar{A}\bar{R} = \mu^t \bar{R} \bar{C}$ over F. Since $\mu \in \Gamma(\beta)$ and $\Delta(\bar{A}) = \Delta(A) \subset \Delta(\beta) \subset \Gamma(\beta)$, a linear algebra argument tells us that the above equation in fact has a nontrivial solution over the field of fractions of $\mathbb{Z}[\Gamma(\beta)^t]$. Clearing denominators, we find a nonzero matrix $\bar{R}$ over $\mathbb{Z}[\Gamma(\beta)^t]$ such that

$$A \bar{R} = \mu^t \bar{R} \bar{C}.$$

Writing $k = |\Gamma(\beta)/\Delta(\beta)|$ and identifying the cyclic group $\Gamma(\beta)/\Delta(\beta)$ with $\mathbb{Z}/k\mathbb{Z}$, we express $\bar{R} = \sum_{i=0}^{k-1} R_i$ with R_i over the ith coset of $\Gamma(\beta)/\Delta(\beta)$. Letting $0 \le j \le k-1$ be such that μ belongs to the jth coset of $\Gamma(\beta)/\Delta(\beta)$, we have

$$\bar{A} R_i = \mu^t R_{i-j} \bar{C} \qquad (**)$$

and, cycling through,

$$\bar{A}^k R_i = \mu^{kt} R_i \bar{C}.$$

Running backwards through the above calculation,

$$\begin{aligned} \bar{A}^k (R_i U D) &= \mu^{kt} (R_i U D)(U^{-1} D^{-1} \bar{C}^k U D) \\ &= \mu^{kt} (R_i U D) c_\beta^{-\delta k t} (U^{-1} C^k U) \\ &= (R_i U D) C_{\beta_{\bar{A}}}^k. \end{aligned}$$

Making sure that i is such that $R_i \neq 0$ and multiplying the last equation with the column vector $\bar{b} = [1, \beta_{\bar{A}}, \ldots, \beta_{\bar{A}}^{d-1}]^{tr}$, we see that

$$\bar{A}^k (R_i U D \bar{b}) = \beta_{\bar{A}}^k (R_i U D \bar{b}).$$

By the uniqueness of the Perron–Frobenius eigenvector we must have

$$\bar{A}(R_i U D \bar{b}) = \beta_{\bar{A}} (R_i U D \bar{b})$$

and, by Lemma 4.3,

$$\bar{A}(R_i U D) = (R_i U D) C_{\beta_{\bar{A}}}.$$

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This means that

$$\bar{A}R_i = \mu^t R_i \bar{C}.$$

Combining this with (∗∗), we find

$$R_i \bar{C} = R_{i-j}\bar{C}.$$

Since one side of this equality is over the ith coset and the other side over the $(i-j)$th coset (mod k), we must have $j = 0$ and $\mu = (c_\beta/c_A)^{d(A)} \in \Delta(\beta)$. That is,

$$c_A^{d(A)}\Delta(A) \subset c_A^{d(A)}\Delta(\beta) = c_\beta^{d(A)}\Delta(\beta).$$

Case 3: $|\Gamma(\beta)/\Delta(\beta)| < \infty$. Let $k \in \mathbb{N}$ be such that $d(A) = d(\beta)k$. Since $\beta^{d(\beta)}$ is aperiodic, $\Gamma(\beta^{d(\beta)}) = \Gamma(\beta)$ and $\Delta(\beta^{d(\beta)}) = \Delta(\beta)$, we can use Proposition 2.5 and the inclusion proved in case 2 to see that

$$c_A^{d(A)}\Delta(A) = (c_{A^{d(\beta)}})^k\Delta(A) \subset (c_{\beta^{d(\beta)}})^k\Delta(\beta).$$

But, by Theorem 3.5(iii) and Corollary 5.3, we have $\Delta(A) \subset \Delta(\beta)$ and

$$(c_{\beta^{d(\beta)}})^k\Delta(\beta) = c_\beta^{d(\beta)k}\Delta(\beta) = c_\beta^{d(A)}\Delta(\beta). \qquad \square$$

We end the paper with an example.

Example 5.4. Let a, b be multiplicatively independent elements of $\mathbb{R}^{++}$. Put

$$A = \begin{bmatrix} a^t & 1 \\ b^t & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ b^t - a^t & 1 + a^t \end{bmatrix},$$

and $\beta = \beta_A$. Let

$$p(\lambda) = \chi_A(\lambda) = \lambda^2 - (1 + a^t)\lambda + a^t - b^t$$

be the characteristic polynomial of A. Then C is the companion matrix of p. For $t \in \mathbb{R}$, the roots of p are given by

$$\beta(t) = \frac{1 + a^t + \sqrt{(1-a^t)^2 + 4b^t}}{2}, \quad \beta_2(t) = \frac{1 + a^t - \sqrt{(1-a^t)^2 + 4b^t}}{2}.$$

We claim p is the minimal polynomial of β over $\mathcal{R} = \mathbb{Z}[\exp]$. Suppose not. By Proposition 3.2 and the fact that p has degree two, we then have $\beta \in \mathbb{Z}[\exp]$, and therefore $\sqrt{(1-a^t)^2 + 4b^t} \in \mathbb{Z}[\exp]$. So, there exists $f(t) \in \mathbb{Z}[\exp]$ such that

$$(1 - a^t)^2 + 4b^t = f(t)^2.$$

Let Γ be a finitely generated subgroup of $\mathbb{R}^{++}$ with $a^t,\ b^t,\ f(t) \in \mathbb{Z}[\Gamma^t]$. Pick a basis $a_1, a_2, \ldots, a_k$ for Γ so that, for suitable $m, n \in \mathbb{N}$, we have $\langle a_1^m, a_2^n\rangle = \langle a, b\rangle$. Letting $u_1, u_2, v_1, v_2 \in \mathbb{Z}$ be such that

$$a = a_1^{mu_1}a_2^{nu_2}, \quad b = a_1^{mv_1}a_2^{nv_2}$$

and writing $x_1 = a_1^t, x_2 = a_2^t, \ldots, x_k = a_k^t$, we obtain an equation

$$4x_1^{mv_1}x_2^{nv_2} = (f - 1 + x_1^{mu_1}x_2^{nu_2})(f + 1 - x_1^{mu_1}x_2^{nu_2})$$

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over the unique factorization domain $\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]$. Hence, there exist $k_1, k_2 \in \mathbb{Z}$ and monomials $w_1, w_2 \in \mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]$ such that $k_1 k_2 = 4$, $w_1 w_2 = x_1^{mv_1} x_2^{nv_2}$ and

$$\begin{cases} f - 1 + x_1^{mu_1} x_2^{nu_2} = k_1 w_1 \\ f + 1 - x_1^{mu_1} x_2^{nu_2} = k_2 w_2. \end{cases} \tag{$*$}$$

Subtracting,

$$2 - 2x_1^{mu_1} x_2^{nu_2} = k_2 w_2 - k_1 w_1.$$

Hence

$$x_1^{mu_1} x_2^{nu_2} = w_1 w_2 = x_1^{mv_1} x_2^{nv_2},$$

which implies $a = b$. Since a, b are independent, this shows that p is the minimal polynomial of β. Since $|\beta_2| < \beta$, we have $d(\beta) = 1$. From

$$|C| = \begin{bmatrix} 0 & 1 \\ b^t + a^t & 1 + a^t \end{bmatrix}$$

we easily see that $\Gamma(\beta) = \Delta(\beta) = \langle a, b \rangle$, the subgroup of $\mathbb{R}^{++}$ generated by a, b. It follows from Theorem 3.5 that every $\mathbb{Z}^+[\exp]$ matrix B with $\beta_B = \beta$ will have $\Gamma(B) \subset \Gamma(\beta) = \langle a, b \rangle$. For example, $\sqrt{a}$ cannot occur as the weight of a cycle of such a matrix.

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