Rectangles and spirals

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Introduction

Every reader knows about the Golden Rectangle (see [1, pp. 85, 119], [2, 3]), and that it can be subdivided into a square and a smaller copy of itself, and that this process can be continued indefinitely, converging towards the intersection point of diagonals of any two successive rectangles in the sequence. The circumscribed logarithmic spiral passing through the vertices and converging to the same point is also familiar (see [3, 4]), and is analogous to the circumcircle of a regular polygon or a triangle. The approximate logarithmic spiral obtained by drawing a quarter-circle inside each of the squares is equally well known [3, p. 64]. Perhaps slightly less familiar is the inscribed spiral, which is tangential to a side of every rectangle, like the incircle of a triangle or a regular polygon. It does not (quite) coincide with the spiral passing through the point of subdivision of each side, as discussed in [3, pp. 73-77]. The Golden Rectangle, its subdivisions, and the circumscribed and inscribed spirals are illustrated in Figure 1.

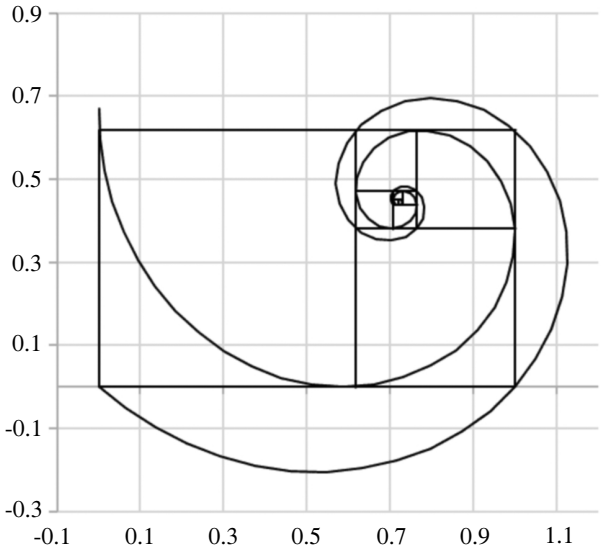


FIGURE 1

Popular accounts usually also mention the aesthetic qualities of the Golden Rectangle, and the importance of this focal point inside it. However, many canvases by the French artist Claude Lorrain [5], although not of golden proportions, show the sun rising or setting at a similarly placed focal

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point, which is therefore not exclusive to golden rectangles. Of course, many other artists do the same, and even amateur digital photographers are given the opportunity to improve the composition of their pictures with the aid of the three horizontal and vertical bands into which the viewfinder divides the field of vision. This article was inspired by consideration of what modifications can or must be made to the spiral process in order to adapt it to rectangles of other proportions, and what consequences they bring.

Notation and preliminaries

The rectangles considered will be in the proportion $1 : \alpha$, where $0 < \alpha \leqslant 1$, and will be located in the complex plane, with vertices at the points 0, 1, $1 + i\alpha$ and $i\alpha$. The golden rectangle occurs when $\alpha = \frac{1}{2}(-1 + \sqrt{5})$, the golden mean, approximately 0.618, which we shall denote by ψ. The golden ratio $\varphi = 1 + \psi = \psi^{-1}$.

The continued fraction expansion for a general α will later be necessary, where $\alpha_0 = \alpha$, $q_0 = 0$, and, for $n \geqslant 1$, if α_{n-1} is non-zero, then $\alpha_{n-1}^{-1} = q_n + \alpha_n$, where q_n is a positive integer and $0 \leqslant \alpha_n < 1$. If $\alpha_{n-1} = 0$ for any $n \geqslant 1$, then the process stops. The convergents to α are the rationals h_n/k_n, where $h_0 = 0$, $k_0 = 1$, $h_1 = 1$, $k_1 = q_1$, and $h_n = q_n h_{-1} + h_{n-2}$ and $k_n = q_n k_{n-1} + k_{n-2}$ for $n \geqslant 2$. If the process stops, then the last convergent is equal to α, which is therefore rational, and the procedure corresponds to the Euclidean algorithm. The numerators and denominators of the convergents to ψ are Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. See [6] for more details.

Clearly, with non-golden rectangles, it is not possible both to remove a square and to be left with a rectangle that is similar to the original one. We shall start with preserving similarity alone, and deal with removing squares later.

Preserving similarity — universal properties

If each rectangle has sides in the ratio $1 : \alpha$, then the dimensions of the rectangles will be $1 \times \alpha$, $\alpha \times \alpha^2$, … , $\alpha^{n-1} \times \alpha^n$, … . Since there is also an anticlockwise rotation at each stage, it follows that each new rectangle has a vertex at a point which is a new partial sum of the geometric series $1 + i\alpha + (i\alpha)^2 + \ldots$. The rectangles therefore converge to the sum to infinity of the geometric series, that is, to the pole or focal point

$$c = 1 + i\alpha + (i\alpha)^2 + \ldots = \frac{1}{1 - i\alpha} = \frac{1 + i\alpha}{1 + \alpha^2} = x_c + iy_c,$$

which lies on a diagonal of every rectangle.

If $v_0, v_1, v_2, \ldots$ denote the partial sums that form the vertices of the rectangular spiral above, then $c - v_0, c - v_1, c - v_2, \ldots$ are the tails of the series, from which it can be seen that $c - v_n = i\alpha(c - v_{n-1})$. Thus $c - v_n = (i\alpha)^n(c - v_0) = (i\alpha)^n c$, so $v_n = c(1 - (i\alpha)^n)$, which is the

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point where $t = n$ on the continuous logarithmic spiral

$$z = z(t) = c + re^{kt},$$

where $r = -c$ and $k = \ln(i\alpha) = \ln\alpha + i\frac{\pi}{2}$.

For any α between 0 and 1, therefore, the full spiral circumscribes the pattern of nested similar rectangles, as shown in Figure 1 for $\alpha = \psi$. When $\alpha = 1$, the spiral is an arc of the circle passing through the vertices of the unit square.

As α varies between 0 and 1, the focal point c moves continuously on a quarter-circle between 1 and $\frac{1}{2}(1 + i)$. This can be seen from the fact that the mapping $\alpha \mapsto (1 - i\alpha)^{-1}$ is a Möbius transformation, or, more simply, by observing that if $c = x_c + iy_c$, then $x_c^2 + y_c^2 = x_c$.

Figure 2 illustrates the case for $\alpha = \frac{1}{2}$, where the successive partial sums are joined to form a piecewise linear rectangular spiral. The fact that the limit lies on the intersection of any two successive diagonals of the rectangles, and also on the quarter-circle traced out by all such focal points, is shown.

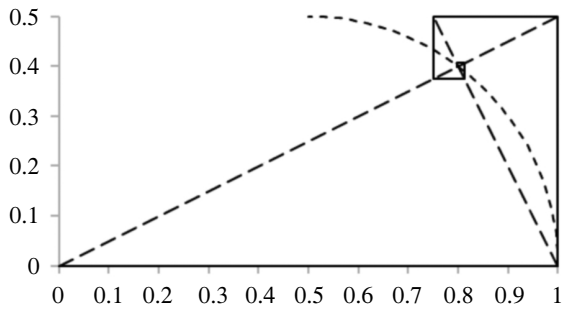


FIGURE 2

The inscribed spiral will also have an equation of the form $z = c + re^{kt}$, where c and k are as above, but r will be different. It is necessary for the tangent $\dot{z} = \frac{dz}{dt} = kre^{kt}$ to be parallel to each side at the point where they touch. Starting on the imaginary axis, say, with $t = 0$, we must find r such that both $z = c + r$ and $\dot{z} = kr$ have zero real part. If $r = x_r + iy_r$, this gives $x_r = -x_c = \dfrac{-1}{1+\alpha^2}$ and also $0 = \mathrm{Re}(kr) = x_r \ln\alpha - y_r\frac{\pi}{2}$, so $y_r = -\dfrac{2}{\pi}\dfrac{\ln\alpha}{1+\alpha^2}$. Thus the inscribed spiral also exists for any value of α strictly between 0 and 1. When $\alpha = 1$, the spiral is an arc of the circle inscribed in the unit square.

Finally, to investigate the approximate spirals for a general α, we need to consider the portion cut off from each rectangle to create the next. From the first rectangle the portion removed has width $1 - \alpha^2$ and height α, and this ratio is preserved at later subdivisions. Thus, in order to construct an

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approximate spiral, it is sufficient to replace the quarter-circles by quarter-ellipses with semi-axes in the ratio $1 - \alpha^2 : \alpha$. These approximately logarithmic spirals are continuous and smooth, but generally have an abrupt change of curvature where two quarter-ellipses meet.

It is thus clear that most of the so-called aesthetic properties of golden rectangles are shared by all other rectangles, which is perhaps why I have found most attempted justifications rather laboured and unconvincing. The geometrical construction of the focal point on any canvas is also simple, since it is the intersection of one diagonal with the perpendicular from one of the other vertices. It would be interesting to find out if this technique was taught to apprentices in Renaissance or earlier artists' studios.

In passing, it is worth remarking that i could be replaced by some other root of unity, say ε. For example, with $\varepsilon = \exp\left(\frac{2}{3}i\pi\right)$, we obtain similar subdivisions of equilateral triangles, as shown for $\alpha = \psi$ in [1, Figure 41, p. 119] and [7, Figure 6, p. 69]. In general, the procedure leads to a nested sequence of polygons converging to $1/(1 - \varepsilon\alpha)$, and circumscribed and inscribed logarithmic spirals exist as before.

Preserving similarity — specific properties

Having discussed the properties shared by rectangles of all proportions, it is worth investigating properties that hold only for a particular value of α. Firstly, for which rectangles does the focal point c coincide with the amateur photographer's guideline, of splitting the screen into nine congruent blocks? From the expression $c = \dfrac{1 + i\alpha}{1 + \alpha^2}$, it is clear that this will occur when $\dfrac{1}{1 + \alpha^2} = \dfrac{2}{3}$, that is when $\alpha = \dfrac{1}{\sqrt{2}}$, so the ratio $1 : \alpha = \sqrt{2} : 1$. The ISO216 range of paper sizes, including the familiar A4, are, of course, of this shape, which was chosen because similarity is preserved when sheets are bisected.

As discussed at length in [3, pp 73–77], the points of tangency of the exact inscribed logarithmic spiral in the golden rectangle do not coincide with the points of subdivision of the sides. This is visible at the starting point of the spiral in Figure 1, which is at the point of tangency to the imaginary axis, and is clearly outside the largest rectangle. For $\alpha = \psi$, the calculated value of the starting point is

$$c + r = \frac{i\left(\psi - \dfrac{2}{\pi}\ln\psi\right)}{1 + \psi^2} \approx 0.6689i,$$

whereas $i\psi \approx 0.6180i$. For the point of tangency on the imaginary axis to coincide with the subdivision point, we therefore require $\dfrac{(\alpha - \frac{2}{\pi}\ln\alpha)}{1 + \alpha^2} = \alpha$. This simplifies to $\alpha^3 = -\frac{2}{\pi}\ln\alpha$, and the solution is $\alpha \approx 0.6498$.

Finally, with the approximately logarithmic inscribed spiral it is

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possible to choose α so that the curvature varies continuously, even at the junctions between quarter-ellipses. For an ellipse with semi-major axes a and b, the curvatures at the ends of the major and minor axes are respectively a/b^2 and b/a^2. The values of a and b are $1 - \alpha^2$ and α for the first quarter-ellipse in the spiral, and $\alpha(1 - \alpha^2)$ and α^2 for the second. For the curvature to be continuous we therefore require

$$\frac{\alpha}{(1 - \alpha^2)^2} = \frac{\alpha(1 - \alpha^2)}{\alpha^4},$$

which simplifies to $\alpha^2 + \alpha^{4/3} - 1 = 0$. The solution is $\alpha \approx 0.6559$.

Removing squares

We now drop the requirement that each new rectangle be similar to its predecessor, and require instead that the portions removed all be squares. The only difference from the golden rectangle is that if the ratio of the smaller to the larger side of any rectangle is less than 1 : 2, then two or more squares need to be removed in order to end up with a smaller rectangle (at right angles to the previous one) with the correct proportions to be able to continue the process.

We define $q_n = 0$ and $\alpha_n = \alpha$ for $n = 0$, and then, for $n \geqslant 1$, define q_n to be the number of squares removed at the nth subdivision and α_n to be the ratio of the smaller to larger side of the rectangle remaining. It is then clear that $\dfrac{1}{\alpha_{n-1}} = q_n + \alpha_n$, separated into integer and fractional parts. This coincides with the notation of continued fractions, given above, so the number of squares removed at the nth subdivision is precisely the nth partial quotient q_n.

Even though the ratio α_n at each subdivision is no longer necessarily constant, vertices of successive rectangles still occur on a piecewise linear spiral path, as partial sums of the power series

$$P(\alpha, i) = 1 + i\alpha_0 + i^2(\alpha_0\alpha_1) + \ldots + i^n(\alpha_0 \ldots \alpha_{n-1}) + \ldots, \quad (1)$$

which is then the limit or focal point. This can be compared with the geometric series obtained when preserving similarity.

If α is rational, say $\alpha = h/k$, then the procedure is finite, and is a geometrical version of the Euclidean algorithm for finding gcd (k, h). This is very energetically illustrated by Marcus du Sautoy in the delightful programme [8], and more conventionally in [9, pp. 44-46]. In this case, $P(\alpha, i)$ is a polynomial in i. More precisely, if there are n steps in the Euclidean algorithm, then $\alpha = h_n/k_n$, the nth convergent, and it is easy to show that

$$P\left(\frac{h_n}{k_n}, i\right) = \frac{k_n + h_n i + r_1 i^2 + r_2 i^3 + \ldots + r_{n-1} i^n}{k_n}, \quad (2)$$

where $r_1, r_2, \ldots, r_{n-1} (= 1)$ are the successive remainders in the Euclidean algorithm (and $r_n = 0$).

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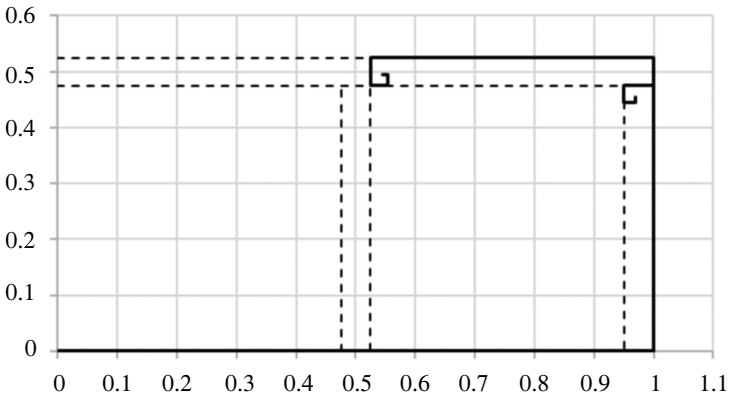


FIGURE 3

Figure 3 shows the finite rectangular spirals for two numbers near $\frac{1}{2}$. With $\alpha = \frac{48}{101}$, two squares (dashed) can be removed at the first subdivision, leaving a narrow strip with proportions 5 : 48 on the right-hand side, from which nine squares (not shown) can be removed. The process continues, and ends at

$$P\left(\frac{48}{101}, i\right) = \frac{101 + 48i - 5 - 3i + 2 + i}{101} = \frac{98 + 46i}{101} \approx 1 + \tfrac{1}{2}i = P\left(\tfrac{1}{2}, i\right).$$

However, with $\alpha = \dfrac{53}{101}$, only one square (dashed) can be removed at the first stage, and again at the second, leaving a narrow horizontal strip near the top right of the figure, and finally

$$P\left(\frac{53}{101}, i\right) = \frac{101 + 53i - 48 - 5i + 3 + 2i - 1}{101} = \frac{55 + 50i}{101} \approx \tfrac{1}{2} + \tfrac{1}{2}i,$$

which is not near $P\left(\frac{1}{2}, i\right)$.

If α is equal to any Fibonacci ratio F_n / F_{n+1}, then only one square is removed at each step (except the last), and the resulting subdivision is called a Fibonacci spiral (see [4]). A finite approximation to a golden spiral can be drawn by inscribing quarter-circles inside each square of the subdivision.

On the other hand, if α is irrational, then (1) is an infinite power series. However, convergence to the limit $P(\alpha, i)$ is assured, because the real and imaginary parts are both alternating series, with terms strictly decreasing in magnitude. An interesting special case occurs when $\alpha_n = \alpha$ for all n, because then the two methods of subdivision coincide. Each partial quotient (or number of squares removed) must be equal to the integer part of α^{-1}, say q. Thus $\alpha^{-1} = q + \alpha$, giving $\alpha = \frac{1}{2}\left(\sqrt{q^2 + 4} - q\right)$, for $q = 1, 2, 3, \ldots$, with $\alpha = \psi$ when $q = 1$. Numbers with periodic continued fractions, like these, are all quadratic irrationals (see [6, §13.1]).

The locus or trajectory of $P(\alpha, i)$ for $0 < \alpha < 1$ is, not surprisingly, very different from the previous smooth quarter-circle, because it involves

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the continued fraction expansion of α. As α increases from 0 to 1, $P(\alpha, i)$ moves irregularly within the triangle with vertices 1, $1 + i$ and $\frac{1}{2}(1 + i)$, being continuous at every irrational value of α, but discontinuous at every rational value. An outline proof follows.

Firstly, to determine the boundaries of the region: if $P(\alpha, i) = x_\alpha + iy_\alpha$, then clearly $x_\alpha \leqslant 1$, so it need only be shown that $x_\alpha \geqslant y_\alpha$ and that $x_\alpha + y_\alpha \geqslant 1$. Moreover, as can be seen from the way the rectangular spiral formed by joining successive partial sums of (1) coils inside itself towards the limit, it is actually sufficient to show that the top-left point on the spiral lies on or below the line $y = x$ and the bottom-left point (after $1 + i\alpha$) lies on or above the line $x + y = 1$. (The sufficiency also follows from the result that the limit of an alternating series, with terms strictly decreasing in magnitude, lies between any two successive partial sums.) Respectively, the two salient points are

$$1 + i\alpha_0 + i^2\alpha_0\alpha_1 = (1 - \alpha_0\alpha_1) + i\alpha_0$$

and

$$1 + i\alpha_0 + i^2\alpha_0\alpha_1 + i^3\alpha_0\alpha_1\alpha_2 = (1 - \alpha_0\alpha_1) + i\alpha_0(1 - \alpha_1\alpha_2).$$

(We may assume that $\alpha_1 \neq 0$, since in that case the result is trivial.) The required inequalities are therefore

$$1 - \alpha_0\alpha_1 - \alpha_0 \geqslant 0 \text{ and } 1 - \alpha_0\alpha_1 + \alpha_0(1 - \alpha_1\alpha_2) \geqslant 1.$$

From the fact that $\alpha_0^{-1} = q_1 + \alpha_1$ it follows that $1 - \alpha_0\alpha_1 = q_1\alpha_0$, so

$$1 - \alpha_0\alpha_1 - \alpha_0 = q_1\alpha_0 - \alpha_0 = (q_1 - 1)\alpha_0 \geqslant 0,$$

since $q_1 \geqslant 1$. This proves the first inequality. For the second inequality, we also need the analogous result $1 - \alpha_1\alpha_2 = q_2\alpha_1$. Then

$$\begin{aligned} 1 - \alpha_0\alpha_1 + \alpha_0(1 - \alpha_1\alpha_2) &= q_1\alpha_0 + \alpha_0q_2\alpha_1 \\ &= \alpha_0(q_1 + q_2\alpha_1) \geqslant \alpha_0(q_1 + \alpha_1) = 1, \end{aligned}$$

using the fact that $q_2 \geqslant 1$.

Secondly, we sketch the proof that the mapping $\alpha \mapsto P(\alpha,1)$ has a jump discontinuity at every rational value of α, though the limit from one side is equal to $P(\alpha, i)$. For example, the jump at $\alpha = \frac{1}{2}$ is suggested by the two spirals in Figure 3. The essential point is that if $\beta \leqslant \frac{1}{2}$, then at the first stage two squares can be removed, and $P(\beta, i) \approx 1 + \frac{1}{2}i$, whereas if $\beta > \frac{1}{2}$, then at the first stage only one square can be removed, and $P(\beta, i) \approx \frac{1}{2} + \frac{1}{2}i$.

In general, suppose α is rational and is equal to its nth convergent h_n/k_n. Its last partial quotient is q_n, so the last line of the continued fraction algorithm is $\alpha_{n-1}^{-1} = q_n + 0$, and $\alpha_{n-1} = q_n^{-1}$. Suppose a neighbouring point β has the same first $n - 1$ partial quotients as α, and that β_{n-1} is very close to $\alpha_{n-1} = q_n^{-1}$. If $\beta_{n-1} < q_n^{-1}$, then the next line of the algorithm is $\beta_{n-1}^{-1} = q_n + \beta_n$, where β_n is small. If $\beta \to \alpha$ from this side, then $\beta_n \to 0^+$ and $\beta_m \to \alpha_m$ for all $m < n$, so $P(\beta, i) \to P(\alpha, i)$.

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On the other hand, if $\beta_{n-1} > q_n^{-1}$, then the next line of the algorithm is $\beta_{n-1}^{-1} = (q_n - 1) + \beta_n$, where β_n is close to 1. If $\beta \to \alpha$ from this side, then $\beta_n \to 1^-$ and $\beta_m \to \alpha_m$ for all $m < n$, so

$$P(\beta, i) \to P(\alpha, i) + i^{n+1}\alpha_0 \ldots \alpha_{n-1} \times 1 = P(\alpha, i) + \frac{i^{n+1}}{k_n}.$$

Thirdly, continuity at irrational values of α can be proved by considering the limit of $P(h_n/k_n, i)$ over the infinite sequence of convergents to α. The discontinuity at h_n/k_n is immaterial, because the jump of magnitude $1/k_n$ tends to zero as $n \to \infty$.

By pure serendipity, the analysis in general is very similar to that in [10], which, from a totally different direction, dealt with the real-valued function $P(\alpha, -1)$. The full proof of continuity and discontinuity is technical and notationally complicated, but can be constructed by adapting [10, Theorem 2.3].

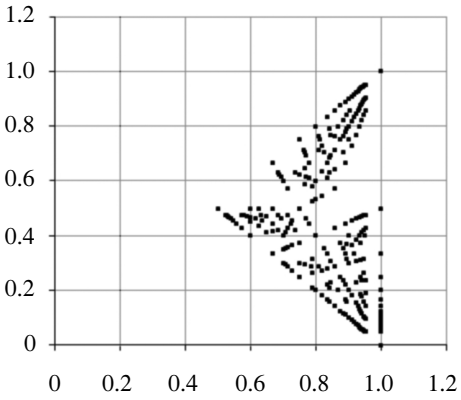


FIGURE 4

Figure 4 gives an impression of the trajectory of the focal points, by plotting the points at both ends of the jump of $P(\alpha, i)$ for each rational number α with denominator at most 21. These points are all cluster points of the trajectory, though only one end of each jump is actually in the set.

Future research

From Figure 4 it is apparent that the set of values of $P(\alpha, i)$ for $0 < \alpha \leqslant 1$ warrants further study. It is obvious from (1) that $P(\alpha, i) = 1 + i\alpha P(\alpha_1, i)$ if $\alpha_1 \neq 0$, so the transformation $P(\alpha, i) \mapsto P(\alpha_1, i)$, induced by the Gauss map $\alpha_n \mapsto \alpha_{n+1}$, has a simple geometrical interpretation. See [6, Dynamical systems] for more details. In [10, §3] this was used to demonstrate measure-theoretic and self-similarity properties of the graph of $P(\alpha, -1)$. It would be interesting to undertake a similar investigation for $P(\alpha, i)$.

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Another possible avenue of research would be to treat α as a parameter and study the function $P(\alpha, z)$ for a general complex variable z. In general, it is a power series, as in (1) with i replaced by z, and it converges at least for $|z| < 1 + \psi$, irrespective of α. This follows from a well-known result, also given in [10, p. 274, equation 3]. When α is rational, $P(\alpha, z)$ is a polynomial; when α is a quadratic irrational, $P(\alpha, z)$ is a rational function, because the continued fraction eventually repeats, as described in [6]; for other values, $P(\alpha, x)$ can even be an entire function. Further properties remain to be discovered.

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