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COMPUTABILITY IN UNCOUNTABLE BINARY TREES

REESE JOHNSTON

Abstract. Computability, while usually performed within the context of ω, may be extended to larger ordinals by means of α-recursion. In this article, we concentrate on the particular case of ω_1-recursion, and study the differences in the behavior of Π^0_1-classes between this case and the standard one.

Of particular interest are the Π^0_1-classes corresponding to computable trees of countable width. Classically, it is well-known that the analog to König's Lemma—"every tree of countable width and uncountable height has an uncountable branch"—fails; we demonstrate that not only does it fail effectively, but also that the failure is as drastic as possible. This is proven by showing that the ω_1-Turing degrees of even isolated paths in computable trees of countable width are cofinal in the Δ^1_1 ω_1-Turing degrees.

Finally, we consider questions of nonisolated paths, and demonstrate that the degrees realizable as isolated paths and the degrees realizable as nonisolated ones are very distinct; in particular, we show that there exists a computable tree of countable width so that every branch can only be ω_1-Turing equivalent to branches of trees with $\aleph_2$-many branches.

§1. Introduction. Computability theory is, in principle, the study of computation. For most purposes, it suffices to consider computations performed on natural numbers, within ω; this is a broad enough and flexible enough domain to capture most of what we mean when we say something may be "computed". At the same time, the notions that are essential to computability—the idea of a computation, of a bounded search, or of recursion—seem to make natural sense in a number of other contexts for which ω is not an appropriate universe. The most natural example is $\mathbb{R}$, the domain of all real numbers; many mathematicians consider addition and multiplication over the reals to be "simple" operations, despite being very difficult to describe and perform in the domain of traditional computability theory.

It is this and related interests which motivated the development of computability on domains other than ω. For the purposes of this article, the development of particular interest is that of α-recursion, in which the domain is replaced with a larger (usually admissible) ordinal α.

Arguably the field of α-recursion began with Kreisel and Sacks [13]; they studied the case in which α is taken to be the first admissible ordinal beyond ω. As was pointed out by Kreisel [12], an essential component of any such study was necessarily the determination of what might be called *finite* in the expanded setting—it could not be, as in the standard ω setting, that every bounded set qualified as "finite", because for countably infinite α this would result in more "finite" sets than Gödel numbers

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to code them. The natural domain, as it turned out, was Gödel's constructible universe—those sets which appear in the constructible hierarchy before the αth level behave with respect to α very similarly to the way finite sets behave with respect to ω.

The field was soon after expanded upon by Kripke [14], Platek [18], and Sacks [19], among many others, extending the study to larger admissible ordinals α. The fine structure theory of the constructible hierarchy developed by Jensen [9] provided the key techniques of the study. The interested reader may find an excellent treatment of α-recursion in general in [2].

In general, α-recursion is a complex subject; very little that holds at ω holds for all α. To take an extreme example, S.D. Friedman [4] showed that when $\alpha = \aleph_{\omega_1}$ the α-Turing degrees are well-ordered by Turing reducibility above $\mathbf{0}'$. This result and others suggest that if a close analogue of ω is desired, α should be taken to be a regular cardinal.

These significant changes to the structure of the α-Turing degrees depending on the choice of α led to a crucial question characterized by Greenberg [6] as "the role of true finiteness". Which results of traditional computability theory depended on genuine finiteness, as opposed to the general properties of finiteness that were captured in the notion of α-finiteness? Greenberg demonstrated a sweeping class of distinctions, showing that many key constructions in the r.e. degrees hinged on true finiteness in a nontrivial way and failed to hold in any admissible α that did not have cofinality ω in a certain effective fashion.

Similarly, rich ground for the subject of true finiteness comes with the study of computable binary (or finitely-branching) trees and the sets of their unbounded branches, called Π^0_1-classes. Central to the study of Π^0_1-classes in the setting of ω is *König's Lemma*, which states that any infinite finitely-branching tree must have an infinite branch; this has led to a wide range of important results, notably the Low Basis Theorem of Jockusch and Soare [10] which concludes that any computable, infinite, computably-bounded tree has a low infinite branch. A simplified form of König's Lemma which applies only to binary trees, known as WKL_0, is a central object of study in reverse mathematics, and is given a thorough treatment in [21].

Due to the richness of Π^0_1-classes, and the critical importance of the finite/infinite distinction in König's Lemma, a natural arena for the further investigation of true finiteness lies in the study of Π^0_1-classes in α-recursion. The most natural analogue of König's Lemma does not in general hold, simply because for an ordinal $\alpha > \omega$ it is often straightforward to construct an α-finitely-branching tree which is α-infinite or even uncountable at height ω. As a consequence, substantial effort has been put into modifying König's Lemma or WKL_0 so that they may apply to the generalized setting. Shore, in [20], proposes one appropriate analogue: any binary tree on a cardinal κ which is unbounded and continuous in a particular sense at limit levels has a branch of length κ.

Trees of ordinal height are, however, of interest to another field: set theory. In classical set theory, the existence or nonexistence of paths through trees subject to a "smallness condition" known as the *countable chain condition* (or, more generally, the κ-chain condition) are of critical importance to the technique of forcing. An overview of these techniques and results may be found in Kunen [15]. Importantly, however, the finite character condition proposed by Shore does not hold of most

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trees of set-theoretic interest. This article aims to study the computability-theoretic implications of this distinction.

For the purposes of this article, we make two restrictions to our study: we concentrate on the particular case of $\alpha = \omega_1$, and we operate entirely within the constructible universe L. Throughout the article, we take a particular weaker analogue to the finitely-branching condition required by König's Lemma. The trees of interest to us will be those trees such that each level is itself countable. Section 2 of this article lays out the background in α-recursion and other key definitions necessary for the remainder of the article. Section 3 presents definitions specific to this article, and a number of basic results that will form the foundation of the rest of the work.

For the bulk of the article, we will be concerned with the question of Π^0_1-classes generated by countable-width trees and consisting of just one element. In the setting of ω, this element must always be computable. In Section 4, we show that this does not hold in ω_1, but that the spectrum of possible singletons is impacted by restricting to trees of countable width. The major result of this article is, informally, that the ω_1-Turing degrees of these elements in fact cover the Δ^1_1 ω_1-Turing degrees, in the sense that any Δ^1_1 ω_1-Turing degree is bounded above by the unique member of a singleton Π^0_1-class generated by a tree of countable width.

In Section 5, we study the degree-theoretic structure of the class of degrees achievable as the sole member of a Π^0_1-class generated by a tree of countable width. We demonstrate that, under certain conditions, jump inversion holds within this class of degrees; we also establish that no interval $[\mathbf{a}, \mathbf{a}']$ is entirely contained in this class of degrees.

In Section 6, we extend our investigation to consider cases of Π^0_1-classes with more than one member, still generated by trees of countable width. In particular, we demonstrate that there exists a computable Kurepa tree (a tree of countable width with $\aleph_2$-many branches), a computable Kurepa tree K so that if a branch of K is Turing-equivalent to a branch of another computable tree T, then T is likewise Kurepa, and a computable Kurepa tree in which every branch has minimal ω_1-Turing degree.

Finally, in Section 7, we outline some key questions that this article leaves open, and some avenues of further research that may be of interest.

§2. Background and notation.

In this section, we present a minimal introduction to ω_1-recursion sufficient to understand the remainder of the article. Further details on ω_1-recursion than presented here may be found in [3] and [7].

The following definitions are slight rephrasings of those originally set out by Kripke [14]; definitions specific to this work will be presented in the next section.

DEFINITION 2.1. A set is *hereditarily countable* if it and all elements of its transitive closure are countable. Let HC be the collection of all hereditarily countable sets. Throughout this work, we will operate under the set-theoretic assumption that $L_{\omega_1} = HC$.

We consider some special classes of elementary first order formulas in the language of set theory.

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(a) A formula is Δ^0_0 if it has only bounded quantifiers $(\exists x \in y)$ or $(\forall x \in y)$.
(b) A formula is Σ^0_1 if it has the form $(\exists x)\varphi$, where φ is Δ^0_0, and it is Π^0_1 if it has the form $(\forall x)\varphi$, where φ is Δ^0_0.
(c) A formula is Σ^0_{n+1} if it has the form $(\exists x)\varphi$, where φ is Π^0_n, and it is Π^0_{n+1} if it has the form $(\forall x)\varphi$, where φ is Σ^0_n.

A subset $X \subseteq L_{\omega_1}$ is *computably enumerable* if there exists a Σ^0_1 formula φ and a parameter $c \in L_{\omega_1}$ so that for each $x \in L_{\omega_1}$, $x \in X$ iff $\varphi(x, c)$ is true.

A subset $X \subseteq L_{\omega_1}$ is *computable* if both X and $L_{\omega_1} \setminus X$ are computably enumerable. In general, a set is Σ^0_n (resp. Π^0_n) if it is definable by a Σ^0_n (Π^0_n) formula; it is Δ^0_n if it is both Σ^0_n and Π^0_n. Thus, X is computable iff X is Δ^0_1.

There are many other definitions which are equivalent under the assumption that $L_{\omega_1} = HC$; perhaps the most compelling one, laid out in [11], defines a set to be computable if it is computable by a Turing machine with a tape of length ω_1 and permitted to run for any countable number of stages. In light of this, a useful intuition is that a set is *computable* if it is "intuitively" computable by a computer that can hold any hereditarily countable set in memory and is permitted to run for any countable length of time before halting.

Note that we now have two definitions of "computable"; the standard definition used in the setting of ω, and the newly defined one in the context of ω_1. Henceforth, we will use *computable* to refer exclusively to the uncountable setting; on the occasions in which we need to refer to the more usual definition of computability, we will use phrases such as "ω-computable" or "the standard setting".

It is also useful to define a notion of *relative* computability: X is *computable from* Y if it is computable using information in Y. More precisely, if X and its complement can both be defined by Σ^0_1 formulas using Y as a parameter. We say such a formula is $\Sigma^0_1[Y]$. If X is computable from Y and Y is also computable from X, we say that they are *Turing-equivalent* and we denote this $X \equiv_T Y$; the *Turing degree* of X, $\deg X$, is the set of all sets Turing-equivalent to X.

DEFINITION 2.2. Given $X \subseteq \omega_1$, let X' denote the set of indices for $\Sigma^0_1[X]$ formulas φ with parameters $c \in L_{\omega_1}$ so that $L_{\omega_1} \models \varphi(c)$. Let $X^{(0)} = X$, $X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$, and for a countable limit ordinal α let $X^{(\alpha)} = \{(\beta, \gamma) \mid \gamma \in X^{(\beta)}\}$. The generalization of $X^{(\alpha)}$ to ordinals $\alpha \geq \omega_1$ can and will be defined in a later section.

It will shortly become useful to extend the hierarchy noted above to higher order levels.

DEFINITION 2.3. We say $X \subseteq \omega_1$ is Σ^1_1 if there exists a first-order formula φ with a parameter $c \in L_{\omega_1}$ so that $x \in X \iff (\exists Y \subseteq L_{\omega_1})\varphi(Y, x, c)$.

X is Π^1_1 iff $\omega_1 \setminus X$ is Σ^1_1, and X is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

When considering these higher order quantifiers, another set-theoretic assumption is relevant: $L_{\omega_2} = H(\aleph_1)$, where $H(\aleph_1)$ is the collection of sets hereditarily of size $\leq \aleph_1$. For many of the results of this work, this assumption is not necessary; where it is, it will be noted as such.

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One of the most important facts about ω_1-recursion for our purposes is that there is a computable bijection between ω_1 and L_{ω_1}; we will therefore interchange freely between the two.

Because we will be working within L, we will be able to make use of the canonical ordering on L, $<_L$. For a complete definition of this ordering, see [5]; for our purposes, the only important properties are the following.

- $<_L$ is a well-ordering.
- $<_L \restriction L_{\omega_1}$ is (ω_1)-computable.

Certain notation conventions we will preserve from the standard setting. For countable ordinals e, Φ_e or φ_e denotes the eth computable function; for $X \subseteq \omega_1$, Φ_e^X denotes the eth computable function with oracle X. A boldface lower-case letter $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$ will denote an ω_1-Turing degree unless otherwise noted.

By convention, we will use e, i, j for countable ordinals that represent indices of computable functions. The letters n and m will be finite ordinals, and lower-case Greek letters $\alpha, \beta, \gamma, \ldots$ will denote ordinals in general (usually countable). The lower-case Greek letters $\sigma, \tau, \rho, \theta$ will usually denote binary strings of countable length, and the symbol $\frown$ will be used to denote concatenation of these. Upper-case Greek letters $\Gamma, \Delta, \ldots$ will denote ω_1-Turing functionals; Λ will usually denote a set of symbols. Upper-case letters $(A, B, C, \ldots)$ will be used for subsets of ω_1 or for requirements (as in a priority argument). Almost exclusively, T will refer to a binary tree (defined below).

As usual, $\Phi_{e,s}(x)$ denotes the eth computable function evaluated at x within the first s stages; the precise interpretation of this is not relevant, but one possible definition would be that $\Phi_{e,s}(x)$ is $\Phi_e(x)$ as evaluated over L_s rather than L_{ω_1}, in which case the computation is said to diverge if no witness to the relevant Σ_1^0 formulas exists in L_s.

By convention, when σ is a countable string, Φ_e^σ is evaluated only up to stage $|\sigma|$ and considered to diverge if it requires information past $|\sigma|$ from the oracle.

Throughout, we will use the ω_1 analogue of Church's article freely and present constructions largely informally.

§3. Thin classes.

In this article, we will study a particular flavor of Π_1^0 class: those arising from trees that are *thin* in a particular sense. To begin, we must lay out a number of definitions that are the direct analogues of the corresponding notions in ω-computability.

DEFINITION 3.1. $2^{<\omega_1}$ is the collection of all countable binary strings. For $\sigma, \tau \in 2^{<\omega_1}$, $\sigma \prec \tau$ if σ is a proper initial segment of τ; likewise, $\sigma \prec X$ for $X \subseteq \omega_1$ if σ is a (proper) initial segment of the characteristic function of X. For $\sigma \in 2^{<\omega_1}$, $|\sigma|$ denotes its *length*: $\sup\{\alpha + 1 \mid \alpha \in \mathrm{dom}(\sigma)\}$.

When σ is a string, and α is an ordinal, σ^α is the string produced by concatenating σ with itself α times; for example, 0^α is the sequence of length α consisting entirely of 0.

A *binary tree* is a subset of $2^{<\omega_1}$ that is downward-closed under $\prec$. If T is a binary tree, then a *path* through T is a set $X \subseteq \omega_1$ so that $\{\sigma \in 2^{<\omega_1} \mid \sigma \prec X\} \subseteq T$. $[T]$ denotes the set of paths through T. A path X is *isolated* if there is an initial

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segment σ of X so that X is the only path through T passing through σ. The *unique path* of a tree T is a path X so that $[T] = \{X\}$.

A binary tree is *computable* if it is computable as a subset of L_{ω_1}. A Π^0_1-class is a subset of $\mathscr{P}(\omega_1)$ that can be described as $[T]$ for some computable binary tree T.

In the countable setting, two distinct classes of trees are studied: finitely branching trees, which are equivalent to binary trees; and infinitely branching trees, equivalent to downward-closed subsets of $\omega^{<\omega}$. We could likewise define the latter in this setting, considering trees that are subsets of $\omega_1^{<\omega_1}$; however, it is evident that there is no difference between these and binary trees. The principal obstacle is that even in a binary tree, the set of elements of a particular length (e.g., length ω) may have size ω_1, so an ω_1-branching tree may be effectively "modeled" in a binary one. For this reason, we henceforth will not distinguish between binary trees and ω_1-branching trees, and will use the word *tree* to refer to either or both of them.

However, most of the results in ω-computability regarding finitely-branching trees might be considered to depend less on the finiteness of the branching than on the finiteness of each level of the tree; while these are of course equivalent, their analogues in the setting of ω_1 are not. It seems natural to suppose, then, that the "correct" analogues of binary or finitely-branching trees in the countable setting might be trees of *countable width* in the uncountable setting.

DEFINITION 3.2. If T is a tree, T has *countable width* if $|T \cap 2^\alpha|$ is countable for every countable α.

It should be noted that this is an imperfect analogue at best—for example, a finitely-branching tree in the countable setting is permitted to have levels of cardinality unbounded below ω, while a tree of countable width in our present setting must have levels of cardinality bounded below $\aleph_1$.

Before moving on to the principal definitions of this section, we prove a simple result that will be useful throughout the section.

PROPOSITION 3.3. *There is a uniformly (partial) computable sequence $\langle T_i \rangle_{i<\omega_1}$ of subsets of $2^{<\omega_1}$ so that the following conditions hold*:

(i) *T_i is a tree for every i, and*
(ii) *for every computable tree T, there is an i (uniformly in an index for T) so that $T = T_i$.*

PROOF. For any e, let T_e be the subset of $2^{<\omega_1}$ enumerated by Φ_e, except for the following:

(i) If $\Phi_{e,s}(\sigma) \downarrow = 1$ and an initial segment of σ has been excluded from T_e, then exclude σ from T_e; and
(ii) If $\Phi_{e,s}(\sigma) \downarrow = 1$ and no initial segment of σ has been excluded from T_e, include all undecided initial segments of σ in T_e. ⊣

On the other hand, the natural extension of Proposition 3.3 to trees of countable width does not hold; that is, there is no uniformly computable sequence of trees of countable width which realizes every computable tree of countable width as above.

The primary topic of study in this document is the case of Π^0_1-classes comprised of a single element.

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DEFINITION 3.4. A Π^0_1 *singleton* is the unique path in a computable tree. A *thin* Π^0_1 *singleton* is the unique path in a computable tree of countable width.

For ease of notation, we introduce some symbols to represent important classes of degrees.

DEFINITION 3.5. Let $\mathcal{P} = \{\deg(D) : D \text{ is a } \Pi^0_1 \text{ singleton}\}$, and $\mathcal{P}_{\text{thin}} = \{\deg(D) : D \text{ is a thin } \Pi^0_1 \text{ singleton}\}$.

A degree in $\mathcal{P}_{\text{thin}}$ is called *thin*.

A degree $\boldsymbol{d}$ is *wide* if no member of $\boldsymbol{d}$ is contained in any thin Π^0_1-class (of any cardinality).

Observe that the notions of *thin* and *wide* are not complementary; a degree cannot be both thin and wide, but we will show in Section 2.6 that it can be neither. In particular, it might be that a degree appears as a path in a computable tree of countable width, but never as the *unique* path.

An essential tool in the study of Π^0_1-classes in the standard setting is König's Lemma, which for our purposes states the following:

PROPOSITION 3.6 (König's Lemma). *If T is an infinite, finitely-branching tree, then T has an infinite branch.*

Among its most notable consequences is the following result:

PROPOSITION 3.7 (Folklore). *If X is an isolated path of an ω-computable binary-branching tree T, then X is also ω-computable.*

The analogous statement to König's Lemma in the uncountable setting might be formulated as *"A binary-branching uncountable tree has an uncountable path"*. However, the following example demonstrates that this fails:

PROPOSITION 3.8. *There is a computable binary tree that is uncountable but has no uncountable path.*

PROOF. $2^{\leq\omega}$, the set of binary strings of length $\leq\omega$, suffices. ⊣

The triviality of this example suggests a re-formulation: *"A countably-branching tree of height ω_1 has an uncountable branch"*. This also has an immediate counterexample:

PROPOSITION 3.9. *There is a computable binary tree with elements of every countable length but having no uncountable path.*

PROOF. Note that there is a computable bijection $f : 2^\omega \to \omega_1$. Let $T = 2^{\leq\omega} \cup \{\sigma \frown 0^\alpha \mid \sigma \in 2^\omega \wedge \alpha \leq f(\sigma)\}$. Then T is the desired tree. ⊣

The final, and most plausible, attempt at a rephrasing of König's Lemma comes from reinterpreting the original statement as a claim about *width* instead of *branching*. That is to say, we interpret the original Lemma as stating that *"An infinite tree in which every level is finite has an infinite path"*. This is a particularly compelling interpretation because the proof of König's Lemma relies only on a pigeonhole principle argument. This suggests a new analogue to the Lemma in the uncountable case: *An uncountable tree of countable width has an uncountable path*. This notion was originally examined by Aronszajn as reported by Kurepa [16], leading to the following definition:

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DEFINITION 3.10. An ω_1-*Aronszajn tree* is a tree of countable width and height ω_1 with no uncountable path. Since κ-Aronszajn trees for $\kappa \neq \omega_1$ are not considered in this work, we will refer to an ω_1-Aronszajn tree as simply an *Aronszajn tree*.

That is to say, this latest analogue to König's Lemma states *There is no Aronszajn tree*. Before we demonstrate the falsity of this claim, we present another pair of definitions that will be useful:

DEFINITION 3.11 (Kunen [15]). An Aronszajn tree T is *well-pruned* if every element of T has uncountably many extensions in T.

DEFINITION 3.12. An *antichain* in a tree T is a set $C \subseteq T$ so that for all $\sigma, \tau \in C$, $\sigma \not\leq \tau$ and $\tau \not\leq \sigma$.

LEMMA 3.13. *There is a computable well-pruned Aronszajn tree with a computable uncountable antichain*.

PROOF. The following is an effectivized version of the construction presented in Kunen's *Set Theory* [15], augmented to produce the desired antichain.

We construct the tree T as a subtree of $\omega^{<\omega_1}$, ordered by extension. We define a sequence $\langle s_\alpha \rangle_{\alpha<\omega_1}$ of countable injections $s_\alpha : \alpha \to \omega$ so that for $\beta < \alpha$ the set $\{\gamma < \beta : s_\beta(\gamma) \neq s_\alpha(\gamma)\}$ is finite, and the range of s_α is coinfinite in ω. At the same time, we build a computable set of *constraints* $(\beta, n) \in \omega_1 \times \omega$ for β a limit ordinal; we say s_α *satisfies* the constraint if $\alpha \leq \beta$ or $s_\alpha(\beta) \neq n$, and otherwise it *violates* the constraint. We will ensure the following condition:

(*) If s_α violates the constraint (β, n), then α is at most $\beta + \omega$.

The construction proceeds by induction on α, in three cases: either α is a successor, α takes the form $\beta + \omega$ for β a limit ordinal, or α is a limit of limit ordinals.

The desired antichain will be $\{s_\alpha : (\exists \beta \text{ limit}) \alpha = \beta + \omega\}$. It is evident that this set is uncountable and computable, since the quantifier ranges only over ordinals less than α, so we need only ensure that it is an antichain.

Let s_0 be the empty function.

Let $\alpha = \beta + 1$. By construction, s_β violates at most one constraint (γ, n). If it does violate a constraint (γ, n), let $s_\alpha(\delta) = s_\beta(\delta)$ for all $\delta < \beta$ other than γ; otherwise, let $s_\alpha(\delta) = s_\beta(\delta)$ for all $\delta < \beta$. If s_α is not yet defined on γ, let $s_\alpha(\gamma)$ be the first m not already in the range of s_α other than n. Either way, let $s_\alpha(\beta)$ be the next m not already in the range of s_α.

Let $\alpha = \beta + \omega$ for β a limit ordinal. By construction, s_β violates at most one constraint (γ, n); if it does violate that constraint, we define $s^*_\alpha(\delta) = s_\beta(\delta)$ for all $\delta < \beta$ other than γ. Otherwise, let $s^*_\alpha(\delta) = s_\beta(\delta)$ for all $\delta < \beta$. Let $\langle m_i \rangle_{i<\omega}$ enumerate the complement of the range of s^*_α not including n; by construction, this is infinite. Let $\langle \delta_i \rangle_{i<\omega}$ enumerate the complement of the domain of s^*_α. Let $s_\alpha = s^*_\alpha$ on the domain of s^*_α, and let $s_\alpha(\delta_i) = m_{2i}$ for each i. Finally, place the constraint $(\beta, s_\alpha(\beta))$.

Finally, let α be a limit of limit ordinals. Let $\langle \alpha_i \rangle_{i<\omega}$ be the $<_L$-least ω-sequence of limit ordinals cofinal in α. By construction, each s_{α_i} violates at most one constraint (γ_i, n_i). Define a sequence of partial injections $s^*_{\alpha_i} : \alpha_i \to \omega$ inductively as follows:

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(i) $s^*_{\alpha_0}(\delta) = s_{\alpha_0}(\delta)$ if $\delta \neq \gamma_0$.
(ii) $s^*_{\alpha_{i+1}}(\delta) = s^*_{\alpha_i}(\delta)$ for $\delta < \alpha_i$, and $s^*_{\alpha_{i+1}}(\delta) = s_{\alpha_{i+1}}(\delta)$ for $\delta \geq \alpha_i$ and $\delta \neq \gamma_{i+1}$ such that $s_{\alpha_{i+1}}(\delta)$ is not in the range of $s^*_{\alpha_i}$.

Let $s^*_\alpha = \bigcup s^*_{\alpha_i}$. Let β_i be the first $\alpha_i < \beta < \alpha_{i+1}$ so that $s^*_\alpha(\beta)$ is defined. Let $\langle \delta_i \rangle_{i<\omega}$ enumerate the β_i and the complement of the domain of s^*_α in α, let $\langle m_i \rangle_{i<\omega}$ enumerate $s^*_\alpha(\beta_i)$ and the complement of the range of s_{α^*}, and define s_α as follows:

(a) For β not in the sequence $\langle \delta_i \rangle_{i<\omega}$, let $s_\alpha(\beta) = s^*_\alpha(\beta)$.
(b) Let $s_\alpha(\delta_i) = m_{2j}$ for j least such that $m_{2j} \neq s_\alpha(\delta_k)$ for any $k < i$ and no constraint (δ_i, m_{2j}) is in place.

This completes the construction of the sequence of s_α. Let T be the subtree of $\omega^{<\omega_1}$ consisting of countable injections σ so that $\{\alpha < |\sigma| : \sigma(\alpha) \neq s_{|\sigma|}(\alpha)\}$ is finite, and let $C = \{s_{\beta+\omega} : \beta \text{ limit}\}$.

Claim 3.14. *If $\beta < \alpha$, then $\{\gamma < \beta : s_\beta(\gamma) \neq s_\alpha(\gamma)\}$ is finite.*

Proof. By induction on α. Suppose the claim holds for all $\delta < \alpha$.

If $\alpha = \delta + 1$, then s_α differs from s_δ on at most one point; if the claim holds for δ, then it holds for α as well.

If δ is a limit and $\alpha = \delta + \omega$, then s_α differs from s_δ on at most one point, so if $\beta \leq \delta$ then s_α differs from s_β at most finitely often. If $\delta < \beta < \alpha$, then $\beta = \delta + n$ for some n and s_β differs from s_δ at most finitely often. s_α differs from s_β at most finitely often before δ, and possibly everywhere after δ; but this is still finite.

If α is a limit of limit ordinals, let $\langle \alpha_i \rangle_{i<\omega}$ be the cofinal sequence specified in the construction of s_α. Fix $\beta < \alpha$, and let i be least so that $\alpha_i > \beta$. By construction, $s_\alpha(\beta) = s_{\alpha_i}(\beta)$ except in one of three possible cases: (1) $(\beta, s_{\alpha_i}(\beta))$ is the one constraint violated by s_{α_i}; (2) $s_{\alpha_i}(\beta)$ is in the range of $s_{\alpha^*_{i-1}}$; or (3) $\beta = \beta_{i-1}$. Cases (1) and (3) apply in at most one case each. Only finitely many β fall under Case (2), because the range of $s^*_{\alpha_i}$ is a subset of the range of s_{α_i}, and s_{α_i+1} differs at most finitely from s_{α_i}. So s_α differs from each s_{α_i} at most finitely. Since each s_{α_i} differs only finitely from s_β for $\beta < \alpha_i$, s_α satisfies the claim as well. ⊣

T is therefore a tree of unbounded height; since it is a subtree of the tree of partial injections from ω_1 to ω, it cannot have an unbounded chain. It only remains to show that C is the desired antichain.

Claim 3.15. *C is an antichain.*

Proof. For each $s_\alpha \in C$, a constraint was placed during the construction of s_α of the form (β, n), where $s_\alpha(\beta) = n$. By condition (*), all future $s_{\alpha'}$ satisfy this constraint, and so in particular cannot extend s_α. ⊣

This completes the proof of Lemma 3.13. ⊣

As a consequence, the classical proof of Proposition 3.7 will not hold in the uncountable setting, even when we restrict to trees of countable width; we will need to employ other tools in the effort to characterize the possible degrees of isolated paths, regardless of the type of tree we consider.

In fact, the only immediate upper bound on the complexity of isolated paths is the following:

Proposition 3.16. *The unique path in a computable tree (countable width or otherwise) is Δ^1_1.*

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PROOF. Let T a computable tree, X the unique path through T. Then $\sigma \in 2^{<\omega_1}$ is an initial segment of X iff

$$(\exists Y)\left((\forall n) Y \upharpoonright n \in T \wedge \sigma \prec Y\right)$$

or equivalently iff

$$(\forall Y)\left((\forall n) Y \upharpoonright n \in T \rightarrow \sigma \prec Y\right).$$

These definitions are, respectively, Σ^1_1 and Π^1_1, so X is Δ^1_1. ⊣

The central result of this work is that, even with the additional assumption of countable width, this upper bound is optimal:

THEOREM 3.17. *For every Δ^1_1 degree $\boldsymbol{d}$, there is (uniformly in a Δ^1_1 index for a representative of $\boldsymbol{d}$) a degree $\boldsymbol{c} \geq \boldsymbol{d}$ so that $\boldsymbol{c} \in \mathcal{P}_{\text{thin}}$.*

Theorem 3.17, which will be proven in Section 4, guarantees only a member of $\mathcal{P}_{\text{thin}}$ above each Δ^1_1 degree; we also present several theorems aimed at characterizing the degrees that are actually members of $\mathcal{P}_{\text{thin}}$.

THEOREM 3.18. $\mathbf{0}^{(\alpha)} \in \mathcal{P}_{\text{thin}}$ *for every hyperarithmetic ordinal α.*

THEOREM 3.19. $\deg \mathcal{O} \in \mathcal{P}_{\text{thin}}$, *where $\mathcal{O}$ is an appropriate analogue of Kleene's $\mathcal{O}$.*

In fact, Theorems 3.18 and 3.19 can be relativized, subject to a technical condition that will be presented in Section 3.

All of these results present a sharp contrast from the case of ω-recursion, as evidenced by the Low Basis Theorem of Jockusch and Soare [10]:

THEOREM 3.20 (Low Basis Theorem). *In the standard setting, every Π^0_1-class has an element of low Turing degree.*

In Section 2, we demonstrate that $\mathcal{P}_{\text{thin}} \subsetneq \mathcal{P}$, even restricted to relatively small degrees. In Section 3, we prove relativized forms of Theorems 3.18 and 3.19, and present the definitions required to express the theorems in full generality. In Section 4, we show Theorem 3.17 by introducing a large class of members of $\mathcal{P}_{\text{thin}}$ characterized by a combinatorial property. In Section 5, we examine the structure of $\mathcal{P}_{\text{thin}}$ in more detail, concentrating on results relating to jump inversion and minimal degrees. In Section 6, we diverge from the theme of isolated paths to consider thin Π^0_1-classes with more than one member; of principal interest will be the Π^0_1-classes arising from computable Kurepa trees, which are thin but have cardinality $\aleph_2$. Finally, in Section 7, we discuss the notable questions left open by the results of this chapter.

§4. Singletons.

4.1. Separation. In light of the results of the previous section, it is reasonable to ask whether the property of countable width makes any difference at all in relation to isolated paths. The following result answers that question in the affirmative:

THEOREM 4.1. *There is a Δ^0_2 Π^0_1-singleton that is not the degree of any thin Π^0_1 singleton.*

To prove this result, we require the following lemma.

LEMMA 4.2. *Suppose T is a tree and $f : 2^{<\omega} \rightarrow T$ is an embedding such that for all $x \in 2^{\omega}$, $\lim_{n<\omega} f(x \upharpoonright n) \in T$. Then T does not have countable width.*

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PROOF. Suppose T and f are as given in the hypothesis of the lemma. Since the domain of f is countable, the range of f is likewise countable. Since ω_1 is regular, there exists a countable ordinal α so that $|f(\sigma)| < \alpha$ for all $\sigma \in 2^{<\omega}$. Then for each $x \in 2^\omega$, $|\lim_{n<\omega} f(x \upharpoonright n)| = \lim_{n<\omega} |f(x \upharpoonright n)| \leq \alpha$. There are uncountably many such x, and each one yields a different element of T: if x and y are distinct members of 2^ω, then for some n we have that $x \upharpoonright n \neq y \upharpoonright n$ and so $f(x \upharpoonright n) \neq f(y \upharpoonright n)$. So we have that $T \cap 2^{\leq\alpha}$ is uncountable. However, $T \cap 2^{\leq\alpha}$ has only countably many levels, so by the Pigeonhole Principle, there is some $\beta \leq \alpha$ so that $T \cap 2^\beta$ is uncountable. Therefore T does not have countable width. ⊣

PROOF OF THEOREM 4.1. Fix a recursive enumeration $\langle T_i \rangle_{i<\omega_1}$ of computable trees. Also fix a computable Aronszajn tree A.

We construct in stages a tree $T \subseteq 2^{<\omega_1}$ with a unique path P. The tree T will be $\bigcup_s T_s$, where each T_s is itself a tree, $T \cap 2^{<s} = T_s \cap 2^{<s}$, and $T_t \subseteq T_s$ whenever $t \leq s$. The path P will be $\lim_s P_s$, where each P_s is a branch of T_s.

At any stage s, some strings in T_s may be designated as *Aronszajn roots*. While σ is an Aronszajn root, strings will automatically be added to each successive T_t in order to extend an embedding from A into the subset of T_t extending σ. The effect of this is to ensure that while σ is designated an Aronszajn root, there will always be an extension of σ that can be added to T_t, but there will be no path extending σ.

A string σ is considered *available* at stage s if $|\sigma| \geq s$ and every initial segment of σ is in T_s; this means that σ is permitted to enter T_{s+1}.

The argument will be a variation on the infinite-injury priority argument frequently used in ω-computability. Let $\Lambda = \{0, 1\} \cup \{\langle 0, n\rangle : n < \omega\} \cup \{\langle 1, \alpha\rangle : \alpha < \omega_1\}$; this will be the set of *outcomes*. We will pursue the following requirements:

D_e : $P \neq \Phi_e$.
$R_{e,i}$: If T_i is a tree of countable width, then either Φ_e^P is not total, $\Pi_e^P \leq_T \emptyset$, or $\Phi_e^P \notin [T_i]$.

Effectively order the D_e and $R_{e,i}$ requirements as $\langle Q_j \rangle_{j<\omega_1}$ in some manner so that each D_e and each $R_{e,i}$ appears exactly once as some Q_j. For each $j < \omega_1$, assign to each string in Λ^j a strategy pursuing requirement Q_j; let $Q(\sigma)$ denote the strategy assigned to σ. At any stage, a given strategy may or may not have been *initialized*; once initialized, it has an *outcome* taken from Λ.

We say that $Q(\sigma)$ is *strictly higher-priority* than $Q(\tau)$ iff $\sigma \prec \tau$.

The *execution path at stage s* is the string $\eta(s) \in \Lambda^{\leq s}$ inductively defined as follows: for any $\alpha < \omega_1$, if $Q(\eta(s) \upharpoonright \alpha)$ has been initialized, then $\eta(s)(\alpha)$ is its *outcome* (to be defined later) at stage s. Otherwise, $\eta(s)(\alpha)$ is not defined. A strategy is said to be *along the execution path* at stage s if it is $Q(\eta(s) \upharpoonright \alpha)$ for some $\alpha < |\eta(s)|$.

At any stage s, a strategy that has been initialized may *require attention*. The first strategy (that is, the one with least height) along the execution path that requires attention is *permitted to act*. If no strategy along the execution path requires attention, then $Q(\eta(s))$ is initialized.

Every requirement, once initialized, has an *anchor* in T; it will be the case that whenever $\sigma \preccurlyeq \tau$, the anchor of $Q(\tau)$ extends that of $Q(\sigma)$.

Throughout the construction, we will maintain the following conditions:

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(i) If Q is a strategy along the execution path at stage s, then P_s extends the anchor of Q.
(ii) If a strategy Q acts to add elements to the tree, then those elements extend the anchor of Q.

4.1.1. The D requirements. When initialized at stage s, D_e sets its anchor to an available extension σ of P_s, puts $\sigma \frown 0$ and $\sigma \frown 1$ into T_{s+1}, designates $\sigma \frown 1$ as an Aronszajn root, and sets $P_{s+1} = \sigma \frown 0$. D_e begins with outcome 0.

Once initialized, D_e *requires attention* at stage s if its outcome at stage s is 0 and $\Phi_{e,s}(|\sigma|)\downarrow = 0$. When permitted to act, D_e selects an available extension τ of $\sigma \frown 1$, sets $P_{s+1} = \tau$, and switches to outcome 1.

Observe that any fixed D_e strategy will eventually cease to act, because it will act at most once.

CLAIM 4.3. *If Q is a D_e strategy and lies along the execution path at unboundedly many stages, and if all higher priority strategies eventually cease to act, then D_e is satisfied.*

PROOF. Suppose otherwise. Then $\Phi_e = P$. Let σ be the anchor of Q. Then $\Phi_e(|\sigma|) = P(|\sigma|)$. Let s_0 be least so that $\Phi_{e,s_0}(|\sigma|) = P(|\sigma|)$. Let $s_1 > s_0$ be least so that all strategies of strictly higher priority than Q have ceased to act. If $\Phi_e(|\sigma|)\downarrow = 1$, then Q will never act at all, and will remain in outcome 0; therefore $P(|\sigma|) = 0$, contradicting our supposition. Therefore, $\Phi_e(|\sigma|)\downarrow = 0$. Let $s_2 > s_1$ be the least stage at which Q lies along the execution path. At this stage, Q requires attention, and since no strictly higher priority strategy will act, Q must be the first strategy along the execution path which requires attention. The strategy Q is therefore permitted to act, and changes P so that $P(|\sigma|) = 1$ and hence $P \neq \Phi_e$. By construction, $P \upharpoonright (|\sigma| + 1)$ will not change again. ⊣

4.1.2. The R requirements.

DEFINITION 4.4. An *e-splitting tree* is a partial map $f : 2^{<\omega} \to T$ with the following properties:

(i) For each $\sigma, \tau \in \mathrm{dom}(f)$, if $\sigma \mid \tau$ then $\Phi_e^\sigma \mid \Phi_e^\tau$.
(ii) $f(\sigma) \prec f(\tau)$ iff $\sigma \prec \tau$.
(iii) If $\sigma \in \mathrm{dom}(f)$, then $\sigma \frown 0 \in \mathrm{dom}(f)$ iff $\sigma \frown 1 \in \mathrm{dom}(f)$.

An e-splitting tree f is *complete* if $\mathrm{dom}(f) = 2^{<\omega}$. A *leaf node* of f is $\sigma \in \mathrm{dom}(f)$ with no extensions in $\mathrm{dom}(f)$. Note that an e-splitting tree is complete iff it has no leaf nodes.

When initialized, an $R_{e,i}$ strategy sets its anchor to be an available extension σ of P_s and includes σ and $\sigma \frown 0$ in T. It takes outcome $\langle 0, 0\rangle$ and begins assembling an e-splitting tree $f_{e,i}$ by taking $f_{e,i}(\emptyset)$ to be σ.

At any stage after initialization, $R_{e,i}$ may be in either of two phases, *phase zero* and *phase one*.

While $f_{e,i}$ is not complete, $R_{e,i}$ remains in phase zero. During phase zero, $R_{e,i}$ is in some outcome of the form $\langle 0, n\rangle$ for $n \in \omega$. $R_{e,i}$ maintains a string τ_s so that τ_s is a leaf node of $f_{e,i}$. $R_{e,i}$ *requires attention* if there exists $t, u \leq s$ with $P_t, P_u \succcurlyeq f_{e,i}(\tau_s)$ and $\Phi_{e,s}^{P_t} \mid \Phi_{e,s}^{P_u}$. If $R_{e,i}$ is then permitted to act, it sets $f_{e,i}(\tau_s \frown 0) = P_t$ and $f_{e,i}(\tau \frown 1) = P_u$. It then takes τ_{s+1} to be the first string (in the standard ordering)

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that is still a leaf node of $f_{e,i}$, sets P_{s+1} to be some available extension of $f_{e,i}(\tau_{s+1})$, designates P_s an Aronszajn root, and takes on outcome $\langle 0, n+1\rangle$.

If at any stage there exists a $\tau \in 2^\omega$ so that $f_{e,i}(\tau \upharpoonright n)$ is defined for every $n \in \omega$, then include $\lim_{\rho \prec \tau} f_{e,i}(\rho)$ in T and designate it as an Aronszajn root. Note that this must occur regardless of whether the strategy is permitted to act or even along the execution path at all.

Once $f_{e,i}$ is complete, $R_{e,i}$ enters phase one, and takes outcome $\langle 1, 0\rangle$. $f_{e,i}$ induces a map $g_{e,i} : 2^\omega \to 2^{<\omega_1}$, given by $g_{e,i}(\sigma) = \lim_{n<\omega} f_{e,i}(\sigma \upharpoonright n)$; $R_{e,i}$ takes P_{s+1} to be the first available extension of $g_{e,i}(\sigma_0)$, where σ_0 is the $<_L$-least member of 2^ω, and designates P_s an Aronszajn root.

While in phase one, $R_{e,i}$ *requires attention* if it has not yet declared phase one *complete* and there is a $\tau \in 2^\omega$ such that $\Phi_{e,s}^{g_{e,i}(\tau)} \notin T_i$. If permitted to act, $R_{e,i}$ changes its outcome to the outcome $\langle 1, \alpha\rangle$ with α the index of τ in the standard enumeration of 2^ω, sets P_{s+1} to the first available extension of $g_{e,i}(\tau)$, and designates P_s an Aronszajn root. Once this has occurred, $R_{e,i}$ declares that phase one is complete.

Claim 4.5. *For a fixed $R_{e,i}$ strategy η, if all strictly higher priority requirements eventually cease to act, so does η.*

Proof. There are two cases: either η remains in phase zero indefinitely or it eventually enters phase one.

If η remains in phase zero indefinitely, then it may only act countably many times, because each action adds an element to the domain of the e-splitting tree maintained by η, and that domain must be a subset of $2^{<\omega}$. So at some countable stage, η will no longer act.

If η eventually enters phase one, then it performed a countably infinite number of actions in phase zero, and at most two actions in phase one (once to advance to phase one, and once when permitted to act). So η acts at most countably often and therefore eventually ceases to act. ⊣

Claim 4.6. *If η is an $R_{e,i}$ strategy and lies along the execution path at unboundedly many stages, and $P = \lim_s P_s$ exists, then η is satisfied.*

Proof. Note that, inductively by Claim 4.5, all strictly higher priority strategies eventually cease to act. Let s_0 be a stage large enough that all strictly higher priority strategies that will ever act have already finished doing so; note that since changing the outcome of a strategy requires an action, at this stage η will either always be along the execution path or will never be. Since η lies along the execution path unboundedly often, the former case must hold.

There are three possible cases.

(a) η never proceeds to phase one (so remains in phase zero indefinitely).
(b) η proceeds to phase one, but never declares it completed.
(c) η eventually declares phase one complete.

In Case (a), η does not completely fill out its e-splitting tree f, so there must be some string τ that eventually becomes a leaf node of f and remains so for the rest of the construction. Let $s_1 > s_0$ be a stage sufficiently large that τ is a leaf node of f. Since τ remains a leaf node thereafter, η cannot act after stage s_1. So η does not require attention after stage s_1; this means that for all $t, u > s_1$, it must be that $\Phi_e^{P_t}$ and $\Phi_e^{P_u}$ are always compatible. Then Φ_e^P is either computable or nontotal.

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In Case (b), since η never completes phase one, it must be that either T_i is nontotal or for every $\sigma \in 2^\omega$, $\Phi_e^{g_{e,i}(\sigma)}$ eventually enters T_i. In the latter case, $\Phi_e^{f_{e,i}(\cdot)} \cup \Phi_e^{g_{e,i}(\cdot)}$ is an embedding from $2^{\leq\omega}$ into T_i, so T_i is not a tree of countable width.

In Case (c), the action η took to complete phase one ensures that there is an initial segment σ of P so that $\Phi_e^\sigma \notin T_i$. So Φ_e^P is either nontotal or not a path through T_i; in either case, the requirement is satisfied. ⊣

4.1.3. Limit stages. At any limit stage s, take $P_s = \lim_{t<s} P_t$ if that limit exists. The outcome of any strategy η at stage s is taken to be the limit of its outcomes at previous stages, provided that value stabilizes below s. If not, then η is an $R_{e,i}$ strategy that acted infinitely often in phase zero before stage s (because the D_e strategies act at most once, and $R_{e,i}$ strategies in phase one act at most twice). Then, according to the $R_{e,i}$ strategy, η meets the conditions to transition into phase one at stage s; if η is the first such strategy along the execution path, permit it to do so.

If $\lim_{t<s} P_t$ does not exist, include all limit points of the $\{P_t\}$ sequence in T and designate them all as Aronszajn roots. By construction, some strategy along the execution path transitions to phase one at this stage and specifies a new value of P_s independent of the limit.

4.1.4. Verification.

Claim 4.7. *$P = \lim_s P_s$ exists and is a path through T.*

Proof. By construction, neither D strategies nor R strategies will revisit a previously abandoned outcome, so once a strategy leaves the execution path it will never be along the execution path again. It can only be that $P_t \not\succeq P_s$ for $t > s$ if some strategy acted between stages s and t; since every action precipitates a change of outcome, that means that the execution path at stage t is not an extension of the execution path at stage s. Furthermore, note that any strategy η only changes P at points above the length of its anchor, and that when a strategy is freshly initialized it takes the current value of P_s as its anchor. Hence, once η changes its outcome at stage s, any further change to $P_t \upharpoonright s$ must be caused by η or by a strictly higher priority strategy. Since the strategy order is well-founded, and since each strategy acts only boundedly often, this means that $P_t \upharpoonright s$ eventually stabilizes as t goes to ω_1. Because this holds for all s, $\lim_s P_s$ exists. Furthermore, since every P_s lies in T, P is a path through T. ⊣

Claim 4.8. *P is the only path through T.*

Proof. Let X be a path through T; we will show that $X = P$.

Since X is a path, its initial segments cannot have been added solely on the behalf of Aronszajn roots (that is, only through additions to the tree not made by strategies). So unboundedly often a strategy must add an initial segment of X to the tree. But each strategy adds only countably many elements (since each addition involves an action, and no strategy acts unboundedly often) and therefore unboundedly many distinct strategies must contribute to X. So unboundedly many strategies are anchored along X, because every strategy adds only extensions of its anchor. But since abandoned strategies cannot be revisited, this uncountable sequence of strategies must be the strategies along the true path (that is, strategies that are eventually always along the execution path). But these strategies are all anchored along P,

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so arbitrarily long initial segments of P are also initial segments of X. Therefore $X = P$. ⊣

Claim 4.9. *P is not computable.*

Proof. By Claim 4.3, every D_e requirement is eventually satisfied. Therefore, $\Phi_e \neq P$ for each e. ⊣

Claim 4.10. *P is not Turing-equivalent to the unique path through any computable tree of countable width.*

Proof. In fact, we prove something somewhat stronger; P does not compute any such path. By Claim 4.6, every $R_{e,i}$ requirement is eventually satisfied. So for every e, i, if T_i is a computable tree of countable width and Φ_e^P is total and noncomputable, then $\Phi_e^P \notin [T_i]$. So P does not compute any noncomputable path through any computable tree of countable width. Since P is itself not computable, P cannot be equivalent to any path through a computable tree of countable width, regardless of uniqueness. ⊣

This concludes the proof of Theorem 4.1. ⊣

4.2. State systems. In light of Theorem 4.1, constructing a computable tree of countable width in which the unique path is of a particular degree is nontrivial. In order to determine which degrees can be represented by these unique paths, it is useful to develop some general machinery for constructing trees with various properties.

The following definition is well-known but presented here for completeness:

Definition 4.11. In an ordered set A, a *club* is a set $C \subseteq A$ that is closed and unbounded, in that:

- If $a_0 < a_1 < a_2 < \cdots$ is an increasing sequence in C that has an upper bound in A, then the least upper bound is in C; and
- For all $a \in A$, there is a $c \in C$ with $c > a$.

For our purposes, all clubs will be clubs in ω_1 with the usual ordering.

Definition 4.12. A *system of states* is a partial order $(\mathbb{S}, \leq)$, with $\mathbb{S} \subseteq L_{\omega_1}$. The system is computable if both $\mathbb{S}$ and $\leq$ are computable.

A *filter* for $(\mathbb{S}, \leq)$ is a downward-closed set $X \subseteq \mathbb{S}$ so that the elements of X are pairwise comparable and there is no $x \in \mathbb{S}$ above every element of X.

A *state function* is an injective function $s : \omega_1 \to \mathbb{S}$ so that $s(t) \leq s(u)$ only if $t \leq u$. s is an *X-true state function* if $\{t < \omega_1 : s(t) \in X\}$ is a club in ω_1 and the range of s is unbounded in X.

Note that if s is a state function, then the range of s in X is well-ordered. Note also that s need not be completely order-preserving; in particular, $s(t)$ and $s(u)$ need not be comparable for $t \neq u$.

Definition 4.13. If $\mathbb{P}$ and $\mathbb{Q}$ are partially ordered sets, then $f : \omega_1 \to \mathbb{P}$ is said to be *smooth over* $g : \omega_1 \to \mathbb{Q}$ if the following conditions hold:

(i) If $f(s) \leq f(t)$, then $g(s) \leq g(t)$.
(ii) If $r < s < t$ is such that $f(r) \leq f(t)$ and $g(s) \leq g(t)$, then $f(r) \leq f(s)$.

The second requirement in Definition 4.13 is a technical condition only of use in the development of the analogue of α-true relations introduced by Montalbán

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[17] based on the priority framework of Ash [1]. Nevertheless, we include it here in anticipation of its future usefulness and because it does not significantly complicate the proof.

It is also convenient at this point to introduce a technique of tree construction that will be of general use, in this section and later.

DEFINITION 4.14. Let T be a computable tree. A *companion map* for T is a computable function $f : \omega_1 \to T$ so that the following conditions hold:

(i) For all $\alpha < \omega_1$, $|f(\alpha)| = \alpha$;
(ii) For any $\alpha, \beta < \omega_1$, if $f(\alpha) \preccurlyeq f(\beta)$ then $\alpha \leq \beta$; and
(iii) There is an $X \in 2^{\omega_1}$ so that $\{\alpha < \omega_1 : f(\alpha) \prec X\}$ is a club.

Observe that if f is a companion map for T and X is the witness to Condition (iii), then X is a path through T.

THEOREM 4.15 (State theorem). *Let $(\mathbb{S}, \leq)$ be a computable system of states, X a filter, and $\mathbf{s} : \omega_1 \to \mathbb{S}$ a computable X-true state function. Then there exists a computable tree T of countable width with a companion map f and unique path Y so that $Y \equiv_T X$ and f is smooth over $\mathbf{s}$. Furthermore, this holds with all possible uniformity.*

4.2.1. Proof of the Theorem. We construct T as $\bigcup_s T_s$, so that $T \cap 2^{<\alpha} = \bigcup_{\beta<\alpha} T_\beta \cap 2^{<\alpha}$ (so the first α levels are decided by stage α). We will suppress references to the particular levels of T, and instead refer simply to "adding σ to T" or "removing σ from T".

As we build T, we will build f simultaneously; the path Y will be defined as the unique path which f visits on a closed and unbounded set.

Throughout the construction, some nodes will be designated *Aronszajn roots*; at most countably many of these will appear below any fixed height α, but uncountably many such designations may be present at any stage. While σ is designated an Aronszajn root, at every stage $\alpha > |\sigma|$, extensions of σ will be added to T to copy a fixed computable Aronszajn tree rooted at σ up to height α.

DEFINITION 4.16. *Anchoring the stage-s state at σ* is to perform the following sequence of steps.

(i) Include σ in T, if it isn't already present.
(ii) Add an Aronszajn tree A_σ rooted at σ.
(iii) Designate a computable uncountable antichain $Q_\sigma \subseteq A_\sigma$.
(iv) Designate each member of Q_σ as an Aronszajn root.
(v) Set $f(s) = \sigma$.

At the beginning of the construction, we anchor the stage-0 state at the root node λ. As a result, we will have defined an Aronszajn tree A_λ rooted at λ and a computable antichain $Q_\lambda \subseteq A_\lambda$, each member of which is designated as an Aronszajn root.

To *junk the stage-s state* is to perform the following sequence of steps.

(i) Let σ be the first member of Q_λ that is still designated as an Aronszajn root.
(ii) Remove the designation of σ as an Aronszajn root.
(iii) Select an extension τ of σ of maximal length up to s.
(iv) Anchor the stage-s state at τ.

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Stage $s > 0$: There are several cases. Let $A = \{t < s : \mathbf{s}(t) \leq \mathbf{s}(s)\}$, so that A is the set of previous stages at which the state was compatible with the current state.

CASE 1. Suppose that A is unbounded below s and $\lim_{t\in A} f(t)$ exists. Then anchor the stage-s state at $\lim_{t\in A} f(t)$.

CASE 2. Suppose that A is unbounded below s, but $\lim_{t\in A} f(t)$ does not exist. Then junk the stage-s state.

CASE 3. Suppose that A is bounded below s and has a maximal element t. Then let σ be the first member of $Q_{f(t)}$ that is still designated as an Aronszajn root. Remove that designation, and select $\tau \in T$ extending σ of length s. Anchor the stage-s state at τ.

CASE 4. Suppose that A is bounded below s but has no maximal element. Then junk the stage-s state.

Observe that if $f(s) \preccurlyeq f(t)$, then $\mathbf{s}(s) \leq \mathbf{s}(t)$.

4.2.2. Verification.

CLAIM 4.17. *T has at least one path which computes X.*

PROOF. Let $C = \{t : \mathbf{s}(t) \in X\}$. By hypothesis, this is a club. Let $\alpha : \omega_1 \to C$ be the function enumerating C in order. We show by induction that for every t and every $u < t$, $f(\alpha(u)) \prec f(\alpha(t))$. The claim is trivial for $t = 0$.

Suppose $t = x + 1$ and the claim holds for x. At stage $\alpha(t)$, the construction proceeds as follows: $A = \{y < \alpha(t) : \mathbf{s}(y) \leq \mathbf{s}(\alpha(t))\}$ consists entirely of stages with state below $\mathbf{s}(\alpha(t)) \in X$. Since X is a filter and therefore downward closed, $\mathbf{s}(y) \in X$ for every $y \in A$. So A is the initial segment of C below $\alpha(t)$, and so by our hypothesis on t has a maximal element, namely $\alpha(x)$. Then the stage-$\alpha(t)$ state is anchored above $\alpha(x)$, and by construction $f(\alpha(x)) \prec f(\alpha(t))$.

Suppose t is a limit ordinal and the claim holds for every $x < t$. Again, $A = \{x < \alpha(t) : \mathbf{s}(x) \leq \mathbf{s}(\alpha(t))\}$ is the initial segment of C below $\alpha(t)$. Since C is a club and t is a limit, A is unbounded below $\alpha(t)$. By the induction hypothesis, the $f(x)$ are comparable for every $x \in A$, so this stage falls into case 1. Then $f(\alpha(t)) = \lim_{x\in A} f(\alpha(x))$ is well-defined.

This completes the induction. Therefore $\bigcup_{t\in C} f(t)$ is a path Y through T. By the choice of C, $\boldsymbol{s}(t) \in X$ for each $t \in C$, and by construction each of these states is anchored along Y. Because $f(t)$ is increasing in length, these anchor points are unbounded along Y. Since X is downward-closed in $\mathbb{S}$, and since we have that $\boldsymbol{s}(s) \leq \boldsymbol{s}(t)$ whenever $f(s) \preccurlyeq f(t)$, it follows that the only states anchored along Y are members of X. These are unbounded in X; so, to compute whether a state x is in X, Y need only wait for a state to be anchored along it that is either above or incomparable with x. ⊣

CLAIM 4.18. *T has at most one path.*

PROOF. Let $Z \in [T]$. Let B be the set of stages at which the current state is anchored along Z. Observe that, by construction, B is unbounded and $\mathbf{s}(s) < \mathbf{s}(t)$ for every $s, t \in B$ with $s < t$.

Suppose that B is not closed. Let $t_0 < t_1 < \cdots < t_i < \cdots$ be an increasing sequence of members of B so that $t = \lim_i t_i \notin B$. At stage t, the state was not anchored at $\lim_t f(t_i)$; since $f(t_i) \prec Z$ for every i, this cannot be because we fell

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in Case 2. Instead we must fall in Case 3 or 4. In either case, $\lim_i f(t_i)$ is not added to T, so Z cannot be a path through T, contradicting our supposition.

Therefore B is a club in ω_1. Let C be as specified in the previous claim. Since B and C are both clubs, they intersect unboundedly often. But then Z and Y intersect unboundedly often, so $Z = Y$. ⊣

CLAIM 4.19. *f is smooth over* $\mathbf{s}$.

PROOF. Condition (i) in Definition 4.13 is clear from the construction. It remains to show Condition (ii).

Suppose that $r < s < t$ are such that $f(r) \leq f(t)$ and $\mathbf{s}(s) \leq \mathbf{s}(t)$. We aim to show that $f(r) \leq f(s)$.

Suppose that the claim fails; so, $f(r) \not\leq f(s)$. Further suppose, without loss of generality, that s is least so that this is the case.

Observe that since Condition (i) holds, we have that $\mathbf{s}(r) \leq \mathbf{s}(t)$. Since $\mathbb{S}$ is a tree, $\mathbf{s}(s)$ and $\mathbf{s}(r)$ are comparable; since $\mathbf{s}$ is order-respecting, $\mathbf{s}(r) \leq \mathbf{s}(s)$.

Consider the construction at stage s. Let $A = \{u < s : \mathbf{s}(u) \leq \mathbf{s}(s)\}$. By the minimality of s we supposed above, if A is unbounded below s then $\lim_{u \in A} f(u)$ exists and extends $f(r)$. Thus, the construction at this stage falls in either Case 1, Case 3, or Case 4.

CASE 1. The stage-s state was anchored at $\lim_{u \in A} f(u) \succcurlyeq f(r)$, so $f(s) \geq f(r)$, contradicting our assumption.

CASE 3. Say A has maximal element u. By the minimality of s, $f(u) \geq f(r)$; by construction, $f(s) \geq f(u)$. This contradicts our assumption.

CASE 4. The stage-s state is junked.

Therefore, the construction must fall into Case 4 at stage s. However, since $f(t) \geq f(r)$, there must be a later stage at which this happens; let t_0 be least $>s$ so that $f(t_0) \geq f(r)$. By the same argument as above, we must be in Case 4 again at stage t_0. But junking anchors a state at a position incompatible with all previous states except the stage-0 state. Since $f(t_0)$ was the result of junking and $f(t_0) \geq f(r)$, it must be that $r = 0$. But the stage-0 state was anchored at the root element; $f(0) \leq f(u)$ for every u. This completes the contradiction. ⊣

4.2.3. Applications. Theorem 4.15 gives as largely straightforward consequences a number of lemmas that are useful for characterizing the members of $\mathcal{P}_{\text{thin}}$. In particular, unique paths of computable trees of countable width with companion maps are closed under the Turing jump and under joins that are uniformly computable in such a path. At the end of this section, we will relate this to the analogue of the hyperarithmetic hierarchy.

LEMMA 4.20. *Let T be a computable tree of countable width with unique path X and companion map f. Then there is, uniformly in an index for T and f, a computable tree T' with a companion map g and unique path $Y \equiv_T X'$, with g smooth over f.*

PROOF. We prove the lemma by appealing to Theorem 4.15, so we will build a state system $\mathbb{S}$ and an appropriate filter F.

Let $\mathbb{S}$ be the subset of $2^{<\omega_1} \times 2^{<\omega_1}$ consisting of elements (σ, τ) with $|\sigma| = |\tau|$, where $(\sigma_1, \tau_1) \leq (\sigma_2, \tau_2)$ iff $\sigma_1 \preccurlyeq \sigma_2$ and $\tau_1 \preccurlyeq \tau_2$. Observe that $(\mathbb{S}, \leq)$ is a system of states.

Let $F = \{(\sigma, \tau) \in \mathbb{S} : \sigma \prec X, \tau \prec X'\}$. Then F is a filter of $\mathbb{S}$.

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DEFINITION 4.21. For $\sigma \in 2^{<\omega_1}$, σ' is the sequence τ of length $|\sigma|$ so that $\tau(e) = 1$ iff $\Phi^{\sigma}_{e,t}(e) \downarrow$ converges for some $t < |\sigma|$.

Let $\mathbf{s}(s) = (f(s), f(s)')$. By the properties of the companion map, $\mathbf{s}$ is immediately a state function.

CLAIM 4.22. $\mathbf{s}$ *is* F*-true.*

PROOF. Let $t_0 < t_1 < \cdots$ be an increasing sequence so that $\mathbf{s}(t_i) \in F$ for each i. Let $t = \lim_i t_i$.

For each i, $f(t_i) \prec X$. By the properties of the companion map, $f(t) = \lim_i f(t_i) \prec X$. Let $\tau = f(t)'$. $\tau(e) = 1$ iff $\Phi^{f(t)}_{e,t}(e) \downarrow$ for some $t < |f(t)| = \lim_i |f(t_i)|$. Equivalently, there is an i with $t < |f(t_i)|$, or equivalently there is an i so that $f(t_i)'(e) = 1$. So $f(t)' = \lim_i f(t_i)'$. Therefore $\mathbf{s}(t) \in F$.

Therefore $\{t : \mathbf{s}(t) \in F\}$ is closed. It remains to show that it is unbounded.

For any α, let $h(\alpha)$ be the least β so that for all $e < \alpha$, $\Phi^{X}_{e}(e)$ converges by stage β or not at all. Observe that h is continuous, so it has a club C of fixed points. By the conditions on the companion map, the set $\{s : f(s) \prec X\}$ is a club D. $C \cap D$ is then an unbounded subset of ω_1 so that for every $s \in C \cap D$, $f(s) \prec X$ and $f(s)' \prec X'$, so $\mathbf{s}(s) \in F$. ⊣

By Theorem 4.15, then, there is a computable tree of countable width T' with a companion map f' and unique path Y so that $Y \equiv_T F$. Clearly $F \equiv_T X'$, so $Y \equiv_T X'$. ⊣

LEMMA 4.23. *Let U be a computable tree of countable width with unique path X and companion map f. Let $F : U \to \omega_1$ be a computable function so that for every $\sigma \prec X$, $F(\sigma)$ is the index for a computable tree of countable width T_σ with unique path X_σ with companion map f_σ. Then there is a computable tree T with unique path Y and companion map g so that $Y \equiv_T X \oplus \bigoplus_{\sigma \prec X} X_\sigma$ and g is smooth over f and every f_σ.*

PROOF. We again show the lemma by constructing a state system $\mathbb{S}$ and an appropriate filter G, and appealing to Theorem 4.15.

Let $\mathbb{S}$ be the subset of $2^{<\omega_1} \times (2^{<\omega_1})^{<\omega_1}$ consisting of elements of the form $(\sigma, \langle \tau_i \rangle_{i<\alpha})$ so that $\alpha = |\sigma| = |\tau_i|$ for all $i < \alpha$. Then $\mathbb{S}$ is a system of states, ordered by componentwise extension. The requirement that the τ_i have length α is not necessary for the argument, but is included for clarity.

Let $G = \{(\sigma, \langle \tau_i \rangle_{i<|\sigma|}) \in \mathbb{S} : \sigma \prec X, (\forall i)\tau_i \prec X_{X\restriction i}\}$. Then G is a filter.

Let $\mathbf{s}(s) = (f(s), \langle f_{f(s)\restriction t} \rangle_{t<s})$, where $f_{f(s)\restriction t}$ is evaluated at stage s. Then $\mathbf{s}$ is a state function.

CLAIM 4.24. $\mathbf{s}$ *is* G*-true.*

PROOF. Let $C = \{s : \mathbf{s}(s) \in G\}$. We show that C is a club.

Let $t_0 < t_1 < \cdots$ be an ω-sequence of elements of C with limit t. Let $\mathbf{s}(t) = (\sigma_t, \langle \tau^t_i \rangle_{i<\alpha_t})$. $\sigma_t = f(t) = \lim_i f(t_i) \prec X$, by the properties of the companion map. Fix $i < \alpha_t$, and let $j < \omega$ be such that $t_j > i$. Then $\tau^t_i = f_{X\restriction i}(t) = \lim_k f_{X\restriction i}(t_k)$. For $k > j$, $f_{X\restriction i}(t_k) \prec X_{X\restriction i}$, so $\tau^t_i \prec X_{X\restriction i}$. So $\mathbf{s}(t) \in G$.

C is therefore closed; it remains to show that C is unbounded. $f(t) \prec X$ on a club D. For each α, note that $f_{X\restriction\alpha}(t) \prec X_{X\restriction\alpha}$ on a club C_α. Let $C_{<\alpha} = \bigcap_{\beta<\alpha} C_\beta$. Let $h(\alpha)$ be the least member of $D \cap C_{<\alpha}$. h is continuous, so has a club E of fixed points. $E \subseteq C$, so C is unbounded. ⊣

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The lemma follows by Theorem 4.15. ⊣

Developing the hyperarithmetic hierarchy in the context of ω_1-recursion is somewhat complicated by the fact that well-ordering is a Π^0_1 property; as a result, $\emptyset'$ computes more ordinals than $\emptyset$ does. The following definition is due to Greenberg and Turetsky in notes [8] that are to the author's knowledge presently unpublished.

DEFINITION 4.25. We define $\mathcal{O}$ as the smallest set satisfying the following conditions, while simultaneously defining sets H_a for $a \in \mathcal{O}$, a rank function $|\cdot| : \mathcal{O} \to ORD$, and an ordering $<_{\mathcal{O}}$ on $\mathcal{O}$.

(i) $0 \in \mathcal{O}$, $H_0 = \emptyset$, and $|0| = 0$.
(ii) If $a \in \mathcal{O}$, then $b = \langle \text{succ}, a\rangle \in \mathcal{O}$. $|b| = |a| + 1$, $H_b = H'_a$, and $c <_{\mathcal{O}} b$ iff $c <_{\mathcal{O}} a$ or $c = a$.
(iii) If $a_i \in \mathcal{O}$ and $a_i <_{\mathcal{O}} a_{i+1}$ for $i < \omega$, then $b = \langle \omega, \langle a_i\rangle_{i<\omega}\rangle \in \mathcal{O}$. $H_b = \bigoplus_{i<\omega} H_{a_i}$ and $|b| = \sup_{i<\omega} |a|$. $c <_{\mathcal{O}} b$ iff there exists an i so that $c <_{\mathcal{O}} a_i$.
(iv) Suppose $a \in \mathcal{O}$, $\{e\}^{H_a}$ is total, $\{e\}^{H_a}(i) \in \mathcal{O}$ and $\{e\}^{H_a}(i) <_{\mathcal{O}} \{e\}^{H_a}(j)$ for all $i < j < \omega_1$. Then $b = \langle \omega_1, a, e\rangle \in \mathcal{O}$. $|b| = \sup_{i<\omega_1} |\{e\}^{H_a}(i)|$, and $H_b = \bigoplus_{i<\omega_1} H_{\{e\}^{H_a}(i)}$. $c <_{\mathcal{O}} b$ iff there is an $i < \omega_1$ with $c <_{\mathcal{O}} \{e\}^{H_a}(i)$.

A *hyperarithmetic ordinal* is an ordinal α so that there is $a \in \mathcal{O}$ with $|a| = \alpha$.

PROPOSITION 4.26 (Greenberg, Turetsky). *If $a, b \in \mathcal{O}$ and $|a| = |b|$, then $H_a \equiv_T H_b$. Thus $\mathbf{0}^{(\alpha)}$ can be defined as the degree of H_a where $a \in \mathcal{O}$ and $|a| = \alpha$.*

The above can be relativized in a straightforward manner to any $X \subseteq \omega_1$, yielding a definition of $\mathcal{O}^X$, an *ordinal hyperarithmetic in X* and the degree $X^{(\alpha)}$ for any such ordinal α.

We can now prove Theorem 3.18:

THEOREM 3.18. $\mathbf{0}^{(\alpha)} \in \mathcal{P}_{\text{thin}}$ *for every hyperarithmetic ordinal α.*

In fact, the theorem can be stated more precisely as a corollary to the above lemmas:

COROLLARY 4.27. *Let X be the unique path in a computable tree of countable width T with a companion map f, and let α be an ordinal hyperarithmetic in X. Then, uniformly in indices for T and f and an X-notation for α, there is a computable tree $\tilde{T}$ of countable width with a companion map $\tilde{f}$ so that the unique path Y of $\tilde{T}$ is equivalent to $X^{(\alpha)}$.*

PROOF. By effective induction on α, using Lemma 4.20 for successor steps and Lemma 4.23 for limit steps. ⊣

We can also show Theorem 3.19:

THEOREM 3.19. $\deg \mathcal{O} \in \mathcal{P}_{\text{thin}}$, *where $\mathcal{O}$ is an appropriate analogue of Kleene's $\mathcal{O}$.*

Formulated more precisely, we in fact show the following.

THEOREM 4.28. *Let T be a computable tree of countable width with unique path X and companion map f. Then there exists, uniformly in an index for T and f, a computable tree $T_{\mathcal{O}}$ with unique path $X_{\mathcal{O}}$ and companion map $f_{\mathcal{O}}$ so that $X_{\mathcal{O}} \equiv_T \mathcal{O}^X$.*

PROOF. We mimic the proofs of Lemmas 4.20 and 4.23 by constructing a state system $\mathbb{S}$ and a filter G, and appealing to Theorem 4.15.

Let $\mathbb{S}$ be the collection of tuples $\langle \sigma, \langle H_i\rangle_{\sigma(i)=1}, \langle D_i\rangle_{i<|\sigma|}\rangle$, where each H_i is a countable binary string and each D_i a function with domain a countable tree.

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Order $\mathbb{S}$ in the natural way. Note that we put no constraints on the size of the H_i and D_i (beyond requiring countability); in particular, we do not require that the sizes correlate in any way. The state function that we construct will handle that synchronization.

Note that, by Lemmas 4.20 and 4.23 we have a uniform map $a \to (T_a, f_a)$ so that if $a \in \mathcal{O}$, T_a is a tree of countable width with companion map f_a and unique path equivalent to H_a.

Define a map $\mathbf{s} : \omega_1 \to \mathbb{S}$ so that $\mathbf{s}(s) = \langle \sigma, \langle H_i \rangle_{\sigma(i)=1}, \langle D_i \rangle_{i<|\sigma|} \rangle$ as follows.

Let H_i (for each $i < |\sigma|$) be $f_i(s)$, evaluated at stage s. Define D_i by induction as follows:

(i) $D_i(\lambda) = i$.
(ii) If $D_i(\tau) = \langle \text{succ}, a \rangle$, then $D_i(\tau \frown 0) = a$.
(iii) If $D_i(\tau) = \langle \omega, \langle a_j \rangle_{j<\omega} \rangle$, then $D_i(\tau \frown n) = a_n$ for each $n < \omega$.
(iv) If $D_i(\tau) = \langle \omega_1, a, e \rangle$, and $a < s$, then $D_i(\tau \frown 0) = a$, and for each $\beta < s$, if $\Phi_{e,s}^{H_a}(\beta) \downarrow = c$ then $D_i(\tau \frown (1 + \beta)) = c$.

Then σ is the string of length s so that $\sigma(i) = 1$ iff D_i is well-founded.

Let G be the set of elements $\langle \sigma, \langle G_i \rangle_{\sigma(i)=1}, \langle D_i \rangle_{i<|\sigma|} \rangle$ so that $\sigma \prec \mathcal{O}$, $G_i \prec H_i$ for each i, and D_i is defined as above for each i. Note that G is clearly downward-closed, and its elements are pairwise compatible, so G is a filter in $\mathbb{S}$.

Claim 4.29. $\mathbf{s}$ *is a* G*-true state function*.

Proof. It is evident from the definition that $\mathbf{s}$ is a state function. It remains only to show that $\mathbf{s}$ is G-true.

We show first that the set $S = \{t : \mathbf{s}(t) \in G\}$ contains a club, then that it is closed.

For $a \in \mathcal{O}$, $f_a(t) \prec X_a$ on a club C_a, by the definition of the companion map. Let $C_{<a} = \bigcap_{b<a, b \in \mathcal{O}} C_b$, and let $C^+ = \{s : s \in C_{<s}\}$. Note that C^+ is closed and unbounded, and that at stages in C^+ all of the approximations to H-sets used by $\mathbf{s}$ are correct.

For $a \notin \mathcal{O}$, there is a first stage in C^+ at which this is recognized, either because D_a includes an element of nonnotation form or because an infinite branch has appeared in D_a. Observe that if this behaviour is observed at a stage in C^+, then a is not in $\mathcal{O}$, because all of the H-sets corresponding to true notations that were used in computing D_a at this stage were correct. Let $w(a)$ be this first stage. Let $C = \{s : s \in C^+ \wedge (\forall t < s) w(t) < s\}$. This is also a closed and unbounded subset of ω_1, and at these stages t, $\mathbf{s}(t) \in G$.

To show that S is closed, let $t_0 < t_1 < \cdots \in S$, with $t = \lim_i t_i$. At all of these stages, all of the approximations to H-sets are correct; in other words, $f_a(t_i) \prec X_a$ for $a \in \mathcal{O}$, $a < t_i$. By the conditions on the companion map, $f_a(t) = \lim_i f_a(t_i) \prec X_a$. Furthermore, at each of these stages, the first component is correct; so for each $a < t$ with $a \notin \mathcal{O}$, there is a $t_i > w(a)$. So $t \in C$ as defined above, and therefore $C \in S$. ⊣

We then obtain the desired tree by Theorem 4.15. ⊣

It is evident that similar approaches can push this hierarchy of thin degrees much further; combining the results of this section, for example, we can define a new $\mathcal{O}$-like notation system $\mathcal{N}$ by adding to the closure conditions of $\mathcal{O}$ the additional condition

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(v) Suppose $a \in \mathcal{N}$. Then $b = \langle \mathcal{O}, a \rangle \in \mathcal{N}$. $H_b = \mathcal{O}^{H_a}$, and $|b|$ is the least ordinal not hyperarithmetic in H_a. $c <_{\mathcal{N}} b$ iff $c <_{\mathcal{N}} d$ for some $d \in \mathcal{N}$ constructed from H_a using only Conditions (i)–(iv).

By a proof similar to Theorem 4.28, we can show that the degrees of paths of computable trees of countable width with companion maps are (uniformly) closed under the map $X \to \mathcal{N}^X$; and we can repeat this process further. But since this process can evidently be continued indefinitely, we must look elsewhere for an upper bound on the complexity of the thin degrees. In the next chapter, we will take a less constructive approach to building these degrees, culminating in the main theorem of this document.

4.3. Club approximation. In the previous chapter, the presence of a companion map is critical to the construction of paths of higher and higher complexity; however, it is straightforward to construct an example in which a companion map is not possible. In a certain sense the path constructed in this fashion still has a "good" approximating function; it simply doesn't obey the precise constraints placed on a companion map. In order to understand these approximating functions and their role in determining whether or not a path of a given complexity exists, we present in this chapter several extensions of the notion of a companion map, and use them to construct unique-path representations of a much larger class of degrees.

DEFINITION 4.30. Recall the definition of *club* given in Definition 4.11.

A function $f : A \to B$, with A and B ordered sets, is *order-respecting* if there exists no $x, y \in A$ with $x < y$ and $f(x) > f(y)$.

A set X is *weakly club-approximable* if there exists an order-respecting computable function $f : \omega_1 \to 2^{<\omega_1}$ so that $\{s : f(s) \preccurlyeq X\}$ contains a club.

X is *strongly club-approximable* if there exists an order-respecting computable function $f : \omega_1 \to 2^{<\omega_1}$ so that $\{s : f(s) \preccurlyeq X\}$ is a club.

X is *continuously club-approximable* if there exists a continuous order-respecting computable function $f : \omega_1 \to 2^{<\omega_1}$ so that $\{s : f(s) \preccurlyeq X\}$ is a club.

If X is weakly (strongly, continuously) club-approximable, then the computable function f witnessing that classification is a *weak (resp. strong, continuous) club-approximation* to X.

Observe that, in principle, these three notions differ widely in complexity. The most striking example is that the property "A is a club" is a Π^0_2 property (A *is unbounded and every countable increasing sequence in A has a limit in A*) while the property "A contains a club" is a strictly Σ^1_1 property [3]. It is reasonable to suppose, then, that the weakly club-approximable sets would comprise a much wider class than the strongly club-approximable sets, which in turn one would expect to be much more extensive than the continuously club-approximable sets.

The following is a straightforward consequence of the definitions:

PROPOSITION 4.31. *The unique path of any tree with a companion map is strongly club-approximable, and the unique path of any tree with a continuous companion map is continuously club-approximable.*

THEOREM 4.32. *Let X be a strongly club-approximable set. Then* $\deg X \in \mathcal{P}_{\text{thin}}$.

PROOF. Let $\mathbb{S} = 2^{<\omega_1}$ with the usual ordering; then $\mathbb{S}$ is a system of states, and X is a filter for $\mathbb{S}$.

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Fix a strong club approximation f to X. Let $\mathbf{s}(t) = f(t)$. Then $\mathbf{s}$ is X-true, by the definition of the companion map. The theorem follows from Theorem 4.15. ⊣

However, the same cannot be said of weakly club-approximable sets:

PROPOSITION 4.33. *The weakly club-approximable sets are closed downwards under Turing reduction.*

PROOF. Let f be a weak club approximation for X, and let $Y = \Phi_e^X$. $C = \{s : f(s) \prec X\}$ contains a club, by hypothesis. For each α, let $t(\alpha)$ be the least stage t so that $\Phi_{e,t}^X \upharpoonright \alpha \downarrow$; $t(\alpha)$ is then a continuous function of α, and hence has a club D of fixed points.

Let $g(s) = \Phi_{e,s}^{f(s)}$. For $s \in C \cap D$, $g(s) \prec Y$, so g is a weak club approximation to Y. ⊣

A somewhat weaker result holds of strongly club approximable sets, though as will be demonstrated later, the strongly club approximable sets cannot be downward-closed under Turing reduction.

THEOREM 4.34. *Let X be a strongly club-approximable set, and let $Y \equiv_T X$. Then Y is strongly club approximable, uniformly in an index for a strong club approximation to X and the reductions between Y and X.*

PROOF. Fix f a strong club-approximation to X, and let Γ and Δ be Turing functionals so that $\Gamma^X = Y$ and $\Delta^Y = X$.

Let C be the set of stages s so that the following conditions hold:

(i) $\Gamma_s^{f(s)} \upharpoonright s \downarrow$,
(ii) $\Delta_s^{\Gamma^{f(s)}} \upharpoonright s \downarrow$, and
(iii) $\Delta_s^{\Gamma^{f(s)}} \upharpoonright s = f(s)$.

Let $\langle s_\alpha \rangle_{\alpha<\omega_1}$ enumerate the members of C in increasing order, and let $g(t) = \Gamma^{f(s_t)} \upharpoonright s_t$.

CLAIM 4.35. *C is unbounded.*

PROOF. Let C_0 be the set of stages s with $f(s) \prec X$; by hypothesis, C_0 is a club. For each s, let $h(s)$ be the least stage t at which $\Gamma_t^X \upharpoonright s \downarrow$. h is continuous, so has a club C_1 of fixed points. The elements of $C_0 \cap C_1$ satisfy Conditions (i)–(iii), and therefore form a club inside C. ⊣

CLAIM 4.36. *g is a strong club approximation to Y.*

PROOF. $g(t) \prec Y$ unboundedly often; for t with $s_t \in C_0 \cap C_1$, $g(t) \prec Y$.

Let $t_0 < t_1 < \cdots$ be such that $g(t_i) \prec Y$, and let $t = \lim_i t_i$. For each i, $\Delta^{g(t_i)} = f(s_{t_i})$ by construction of C. Since $g(t_i) \prec Y$ for each i, $f(s_{t_i}) \prec X$. Since f is a strong club approximation to f, $u = \lim_i s_{t_i}$ is such that $f(u) \prec X$. $u \in C$, so $u = \lim_i s_{t_i}$, and $g(u) \prec Y$. ⊣

This concludes the proof of Theorem 4.34. ⊣

As a result, we can make the following definition:

DEFINITION 4.37. A Turing degree $\boldsymbol{a}$ is *strongly club-approximable* if either of the following equivalent conditions hold:

(i) $\boldsymbol{a}$ contains a strongly club-approximable set, or
(ii) every member of $\boldsymbol{a}$ is strongly club-approximable.

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Likewise, a Turing degree is *weakly club-approximable* if either it contains a weakly club-approximable set or, equivalently, it consists entirely of weakly club-approximable sets.

Based on these results and the main theorem of Chapter 2, we can conclude that the weakly club-approximable and strongly club-approximable degrees are different classes of degrees:

PROPOSITION 4.38. *There is a* Δ^0_2 Π^0_1*-singleton that is weakly club-approximable but not strongly club-approximable.*

PROOF. By Theorem 4.1, there is a Δ^0_2 set X that is a Π^0_1 singleton but not a thin Π^0_1 singleton. By Theorem 4.32, this set cannot be strongly club-approximable. By Proposition 4.31 and Lemma 4.20, $\emptyset'$ is weakly club-approximable. By Proposition 4.33, so is X. ⊣

Given that every example presented thus far of a degree that cannot be represented as the unique path in a computable tree of countable width has also failed to be strongly club-approximable, it is reasonable to ask whether the strongly club-approximable degrees are exactly the degrees of such paths. The following result answers that question in the negative, even for relatively low levels of complexity.

PROPOSITION 4.39. *There is a* Δ^0_2 *degree* $\mathbf{d} \in \mathcal{P}_{\text{thin}}$ *that is not strongly club approximable.*

PROOF. By Theorem 4.34, it suffices to construct a computable tree of countable width T with unique path X so that X is not a strongly club-approximable set.

We construct T in stages as $\bigcup T_s$, so that level s of the tree will be decided by stage $s+1$. X will be the limit of stage-s guesses X_s. At any stage, there will be at most countably many elements of T_s designated as Aronszajn roots; extensions will be added to the tree above these nodes as necessary so that, in the limit, an Aronszajn tree will be built above each one. This ensures that while σ is designated an Aronszajn root, there will always be an extension of σ sufficiently long that none of its extensions have yet been decided.

We prove the claim by bounded injury, satisfying the following requirements, with the natural priority order:

R_e: Φ_e is not a strong club approximation to X.

At any stage s, the highest-priority requirement that has been initialized and requires attention is permitted to act. If no initialized requirement requires attention, then the first uninitialized requirement is initialized.

When initialized at stage s, R_e fixes $\sigma_e = X_s$, adds $\sigma_e \frown 0$ and $\sigma_e \frown 1$ to T_{s+1}, designates $\sigma_e \frown 1$ as an Aronszajn root, and sets $X_{s+1} = \sigma_e \frown 0$. R_e declares itself as *in phase zero*, and sets $\tau_e = \sigma_e \frown 1$.

There are a number of conditions under which R_e requires attention. While R_e is in phase zero, we say R_e *requires attention* if $\Phi_{e,s}(t) \downarrow = \tau$ for some new $t \leq s$ with $|\tau| > |\sigma_e|$ and $\tau \preccurlyeq X_s$. When permitted to act, R_e sets X_{s+1} to a free extension of τ_e, sets τ_e to X_s, declares X_s an Aronszajn root, and deinitializes all lower priority requirements.

At a limit stage, if R_e was permitted to act unboundedly often, then include both limits of the X_s assignments made by R_e in T, declare both as Aronszajn roots. We then again say R_e *requires attention*, and when permitted to act it *enters phase one*

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and sets X_s to some free extension of the limit point above $\sigma_e \frown 0$ and deinitializes all lower priority requirements. Let u_e be the current stage.

While R_e is in phase one, R_e *requires attention* if $\Phi_{e,s}(u_e) \downarrow = \tau$ with $|\tau| > |\sigma_e|$ and $\tau \preccurlyeq X_s$. When permitted to act, R_e sets X_{s+1} to some available extension of $\sigma_e \frown 1$, deinitializes all lower priority requirements, and declares itself satisfied.

This completes the construction. Observe that, unless reinitialized by a higher priority requirement, every strategy acts at most $\omega + 2$ times; ω-many times in phase zero, and at most twice in phase one.

CLAIM 4.40. *For each* α, $\lim_s X_s \upharpoonright \alpha$ *exists and is in* T.

PROOF. Suppose otherwise. Then there is some $\beta < \alpha$ so that $X_s(\beta)$ changes uncountably often. Each change to X_s must be the result of an action on the part of some requirement. Since each requirement acts at most $\omega + 2$ times once its predecessors are done acting, it must be that an uncountable sequence of requirements change $X_s(\beta)$. But each requirement R_e causes changes only above $|\sigma_e|$; since X_s is increasing in length and newly initialized requirements take $\sigma_e = X_s$, only countably many requirements can cause a change at position β. ⊣

In fact, it is clear from the above argument that $X_s(\beta)$ changes at most $(\omega + 2)^\beta$-many times (where the exponentiation here is ordinal exponentiation, not cardinal), so the limit set X is in fact ω_1-c.e.; this is not, however, important for the result we intend to prove.

CLAIM 4.41. *X is not strongly club-approximable.*

PROOF. Suppose that Φ_e is a strong club-approximation to X. Let s_0 be a stage large enough that R_i has finished acting by stage s_0 for all $i < e$, and suppose that R_e is not already satisfied. Note that it will be the case that $\Phi_e(s) \preccurlyeq X_s$ for unboundedly many s, because the initial segments of the X_s must stabilize; R_e will therefore get uncountably many opportunities to act. So R_e acts infinitely many times in phase zero, and proceeds to phase one at stage s_1. $\Phi_e(s) \prec X$ unboundedly often for $s < s_1$, and Φ_e is a strong club approximation to X; therefore $\Phi_e(s_1) \prec X$. But when this occurs, R_e changes X_s to a value incompatible with $\Phi_e(s_1)$, and no strategy is able to change it back. This is a contradiction. ⊣

CLAIM 4.42. *X is the only path through T.*

PROOF. Let $Y \in [T]$. Y is a path, so Y is not entirely contained in an Aronszajn tree. But the only strings that are not contained in Aronszajn trees are those that are initial segments of X_s uncountably often. Since the initial segments of the X_s stabilize, this can only happen for strings that are initial segments of X. Then every initial segment of Y is an initial segment of X, so $Y = X$. ⊣

⊣

On the other hand, the weakly club-approximable degrees are very extensive:

THEOREM 4.43. *A set X is Δ^1_1 if and only if it is weakly club-approximable. Furthermore, this holds with all possible uniformity.*

PROOF. Suppose X is weakly club-approximable, with weak club approximation f. Then $\alpha \in X$ if and only if

$$(\exists C)[C \text{ is a club} \wedge (\forall \alpha, \beta \in C)(\alpha < \beta \implies \alpha \in f(\beta))]$$

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or, equivalently,

$$(\forall C)[C \text{ is a club } \wedge (\forall \beta, \gamma \in C)(\beta < \gamma \implies f(\beta) \prec f(\gamma))] \implies (\forall \alpha, \beta \in C)(\alpha < \beta \implies \alpha \in f(\beta)).$$

These are, respectively, a Σ^1_1 and a Π^1_1 definition of X, so X is Δ^1_1.

Suppose instead that X is Δ^1_1, and φ and ψ are first-order formulas of $\mathcal{L}_{\omega_1\omega}$ so that $\alpha \in X$ iff $\exists Y \varphi(\alpha, Y)$ iff $\forall Z \psi(\alpha, Z)$. Define a function $f : \omega_1 \to 2^{<\omega_1}$ as follows:

(1) Suppose that there exists $\beta > \alpha$ such that the following hold:
   (i) $L_\beta \models V = L_{\omega_2}$,
   (ii) $\alpha = (\omega_1)^{L_\beta}$, and
   (iii) $L_\beta \models \forall \gamma \exists Y \varphi(\gamma, Y) \iff \forall Z \psi(\gamma, Z)$.
   Then $f(\alpha) = \sigma \in 2^{<\alpha}$, where for $\gamma < \alpha$, $\sigma(\gamma) = 1$ iff $L_\beta \models \exists Y \varphi(\gamma, Y)$. (Note that this is independent of the choice of the witnessing β.)
(2) Otherwise, $f(\alpha) = 0^\alpha$.

Let $M_0 \subset M_1 \subset \cdots$ be an increasing continuous ω_1-sequence of countable elementary substructures of L_{ω_2}, and let $\overline{M_i}$ be the transitive collapse of M_i. $\overline{M_i} \models ZF^- + V = L$, so $\overline{M_i} = L_{\gamma_i}$ for some countable ordinal γ_i. Let $\alpha_i = (\omega_1)^{L_{\gamma_i}}$; by the continuity of the original sequence, these α_i form a club. And $f(\alpha_i)$ is always a correct initial segment of X. ⊣

We now recall the main theorem, Theorem 3.17:

THEOREM 3.17. *For every Δ^1_1 degree $\boldsymbol{d}$, there is (uniformly in a Δ^1_1 index for a representative of $\boldsymbol{d}$) a degree $\boldsymbol{c} \geq \boldsymbol{d}$ so that $\boldsymbol{c} \in \mathcal{P}_{\text{thin}}$.*

As a result of Theorem 4.43, this theorem can be proven by concentrating on weak club-approximations rather than on Δ^1_1 definitions of sets.

We now proceed to prove Theorem 3.17, for which we will need the following lemma:

LEMMA 4.44 (Folklore). *Fix $\langle C_\alpha \rangle_{\alpha < \omega_1}$ a sequence of clubs. Let D be the* diagonal intersection *of the C_α; that is, $\alpha \in D$ iff $\alpha \in \bigcap_{\beta < \alpha} C_\beta$. Then D is a club.*

Though the lemma is well-known, we include a proof of it here for completeness.

PROOF. D is unbounded: Fix a countable ordinal δ. Let $\alpha_0 = \delta$. For each β, let α_β be the first member of $\bigcap_{\gamma < \beta} C_\gamma$ greater than α_γ for every $\gamma < \beta$. Observe that this sequence is continuous: for a limit ordinal β, every club C_γ for $\gamma < \beta$ contains a tail segment of the sequence of α_γ for $\gamma < \beta$, so every C_γ includes the supremum of that sequence, which will be α_β. The sequence therefore has a fixed point $\alpha = \alpha_\alpha$. This α is a member of D greater than δ.

D is closed: Let $\alpha_0 < \alpha_1 < \cdots$ an ω-sequence in D with limit α. Each C_β for $\beta < \alpha$ contains a tail segment of this sequence, so all of the C_β include α. Therefore $\alpha \in D$. ⊣

PROOF OF THEOREM 3.17. We will use Theorems 4.43 and 4.32, and show instead that every weakly club-approximable set is computable from a strongly club-approximable set.

Let Y be Δ^1_1. Then by Theorem 4.43, X is weakly club-approximable. Let f be a weak club approximation to X. The property "C is a club so that for all $\alpha \in C$, $f(\alpha) \prec X$" is an arithmetic one; the $<_L$-least witness is therefore Δ^1_1 (definable by

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statements of the form "there exists a well-founded model of $V = L_{\omega_2}\ldots$" and "for every well-founded model of $V = L_{\omega_2}\ldots$"). Let $C(X)$ be that $<_L$-least club. Note that $C(X) \geq_T X$, because $C(X)$ filters out all of the incorrect "guesses" from f.

Define $C^\alpha(X)$ inductively as follows for countable ordinals α:

(i) $C^0(X) = C(X)$,
(ii) $C^{\alpha+1}(X) = C^\alpha(X) \cap C(C^\alpha(X))$, and
(iii) For limit α, $C^\alpha(X) = \bigcap_{\beta<\alpha} C^\beta(X)$.

Note that for each α, $C^\alpha(X)$ is a set of stages on which the approximations to X and every $C^\beta(X)$ for $\beta < \alpha$ are simultaneously correct.

Without loss of generality, suppose $0 \in C(X)$.

Let f_β be the weak club approximation to $C^\beta(X)$. Note that by the uniformity of Theorem 4.43, and the uniformity in the selection of $C(\cdot)$, the sequence of indices for the f_β is computable.

Let $X^* = \bigoplus_{\alpha<\omega_1} C^\alpha(X)$; for the time being, consider X^* as a member of $2^{\omega_1\times\omega_1}$. We construct a computable function g from ω_1 to countably-supported members of $2^{\omega_1\times\omega_1}$ as follows:

(1) Suppose that the $f_\beta(\alpha)$ is closed and unbounded below α for every $\beta \leq \alpha$. Then let $g(\alpha)$ be the function $h : \alpha \times \alpha \to 2$ given by $h(\beta, \gamma) = f_\beta(\alpha)(\gamma)$.
(2) Otherwise, let $g(\alpha)$ be the function $h : \alpha \times \alpha \to 2$ given by $h(\beta, \gamma) = 0$.

Fix a bijective pairing function $\langle\cdot,\cdot\rangle : \omega_1 \times \omega_1 \to \omega_1$ so that $\langle 0, 0\rangle = 0$. Let $F : \omega_1 \to 2^{<\omega_1}$ be defined as follows:

(1) If $\langle\beta, \gamma\rangle < \alpha$ for all $\beta, \gamma < \alpha$, then $F(\alpha) = \sigma \in 2^{<\alpha}$, where $\sigma(\langle\beta, \gamma\rangle) = g(\alpha)(\beta, \gamma)$.
(2) Otherwise, $F(\alpha) = 0^\alpha$.

Identify X^* with its image under $\langle\cdot,\cdot\rangle$.

CLAIM 4.45. *F is a strong club approximation to X^*.*

PROOF. Let $A = \{\alpha | F(\alpha) \prec X^*\}$. We aim to show that A is closed and unbounded.

Let E be the set of α so that $\langle\beta, \gamma\rangle < \alpha$ for all $\beta, \gamma < \alpha$; note that E is a club.

For each α, let $C^\alpha(X)'$ be the set of limit points of $C^\alpha(X)$; note that $C^\alpha(X)'$ is also a club. Let D be the diagonal intersection of $C^\alpha(X)'$; that is, $\beta \in D$ iff $\beta \in C^\alpha(X)'$ for all $\alpha < \beta$. Then D is a club. Note that if $\beta \in D$ then $\beta \in C^\beta(X)'$, because the $C^\alpha(X)'$ are nested.

Let $\alpha \in D \cap E$. Then α falls under Case (1) in the definition of F, so $F(\alpha)$ is the image of $g(\alpha)$. Since $\alpha \in D$, α also falls under Case (1) in the definition of g. So $g(\alpha)$ is given by $h(\beta, \gamma) = f_\beta(\alpha)(\gamma)$. But $\alpha \in C^{\beta+1}(X)'$ for each $\beta < \alpha$; so $\alpha \in C^{\beta+1}(X)$; so $f_\beta(\alpha) \prec C^\beta(X)$. Therefore $g(\alpha)$ is a "correct square" of X^*, and hence $F(\alpha)$ is a correct initial segment of X^*.

Therefore $D \cap E \subseteq A$, so A contains a club; in particular, A is unbounded. It remains to show that A is closed.

Suppose $\alpha_0 < \alpha_1 < \cdots$ is an increasing ω-sequence in A with limit α. $F(\alpha_i) \prec X^*$ for each i. Note that by the simplifying assumptions $0 \in C(X)$ and $\langle 0, 0\rangle = 0$, the first bit of X^* is 1; so every α_i must fall under Case (1) in the construction of F. It's then clear that α likewise falls under Case (1).

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Likewise, every α_i must fall under Case (1) in the definition of g. Fix $\beta < \alpha$. By hypothesis, α_i is a limit point of $C^\beta(X)$ for all sufficiently large i; so $\alpha_i \in C^\beta(X)$ for all sufficiently large i. Since the $C^\beta(X)$ are nested, $\alpha_i \in C^\beta(X)$ for all i. $C^\alpha(X) = \bigcap_{\beta<\alpha} C^\beta(X)$, so α is a limit point of $C^\alpha(X)$. So α also falls under Case (1).

Since $g(\alpha_i)$ is correct for each i, $f_\beta(\alpha_i) \prec C^\beta(X)$ for each $\beta < \alpha$ and each sufficiently large i. But $\alpha \in C^{\beta+1}(X)$ for each $\beta < \alpha$, so $f_\beta(\alpha) \prec C^\beta$. Thus $g(\alpha)$ is also correct, so $F(\alpha) \prec X^*$ and $\alpha \in A$. ⊣

Therefore X^* is a strongly club-approximable set computing X. By Theorem 4.32, X^* is Turing-equivalent to a thin Π^0_1 singleton; so X is computable from a thin Π^0_1 singleton. ⊣

§5. Structure of $\mathcal{P}_{\text{thin}}$.

5.1. Jump inversion.

THEOREM 5.1. *Let $A \subseteq \omega_1$ be the unique path of a computable tree of countable width T_A, with a companion map f_A. Let T be a computable tree of countable width with unique path $X \geq_T A'$ and companion map f. Then, uniformly in an index for T, f, and the reduction from X to A', there is a computable tree of countable width S with unique path $Y \geq_T A$ and companion map g so that $Y' \equiv_T Y \oplus A' \equiv_T X$.*

PROOF. For clarity, we prove the claim when $A = \emptyset$, and discuss relativization afterward.

We build S in stages as $\bigcup_s S_s$, so that by the end of stage s, we will have decided level s in the tree S and $g(s)$. We also build a Δ^0_2 partial map $h : S \to T$ so that $\bigcup_\alpha h(Y \upharpoonright \alpha) = X$.

We perform the construction on a tree of strategies. Let $\Lambda = \omega_1 + 1$ with the natural ordering; this is the collection of outcomes for any strategy. The outcomes corresponding to countable ordinals are designated *coding outcomes*; the outcome corresponding to ω_1 is designated the *convergence outcome*. To each node of the tree $\Lambda^{<\omega_1}$ we assign a strategy η corresponding to the requirement $R_{|\eta|}$, defined below:

R_e: Either $\Phi^Y_e(e)\downarrow$, or there is an initial segment σ of Y such that $h(\sigma) = X \upharpoonright e$.

Strategies at nodes higher in the tree have lower priority.

At any stage s, the *execution path* $\delta(s)$ is the string in $\Lambda^{<\omega_1}$ of length s so that $\delta(s)(t)$ is the outcome at stage s of the strategy assigned to $\delta(s) \upharpoonright t$.

During stage s, if any strategy along the execution path requires attention, then the highest-priority strategy along the execution path that requires attention is permitted to act. If no strategy along the execution path requires attention, then the first uninitialized strategy along the execution path is initialized.

At any stage, some strategies may be *cancelled*. A cancelled strategy does not act or require attention, and no strategy of strictly lower priority than a cancelled strategy may be initialized. If the execution path at stage s passes through a cancelled strategy, and no higher priority requirement requires attention or is uninitialized, then set $g(s+1) = g(s) \frown 0$.

At every stage, if $g(s+1)$ is not compatible with $g(s)$, then $g(s)$ is designated an Aronszajn root.

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5.1.1. Strategy for R_e. Let η be a strategy pursuing requirement R_e. When η is initialized at stage s, set $\sigma_\eta = g(s)$. Include an Aronszajn tree A_η rooted at $g(s)$, with a computable uncountable antichain $\langle q_i^\eta \rangle_{i<\omega_1}$. Designate every q_i^η as an Aronszajn root, set $g(s+1) = q_0^\eta$, set $h(q_0^\eta) = f(s) \upharpoonright e$, and set η to outcome 0.

If η ever enters its convergence outcome, then it never requires attention from that point forward.

If η is in a coding outcome, and there is a $\tau \in S_s$, $\tau \succ \sigma_\eta$, so that $\Phi_{e,s}^\tau(e)\downarrow$, then η requires attention. If permitted to act, η switches to its convergence outcome and sets $g(s+1)$ to some extension of τ with length at least s.

If η is in coding outcome α and $f(s) \upharpoonright e \neq h(q_\alpha^\eta)$, then η requires attention. If there is an outcome β which η has previously visited so that $h(q_\beta^\eta) = f(s) \upharpoonright e$, then η returns to outcome β and sets $g(s+1)$ to a string consistent with the outcomes of all strategies along the new execution path. If no such β exists, then let β be the first coding outcome that η has not yet visited. Set $h(q_\beta^\eta) = f(s) \upharpoonright e$ and $g(s+1)$ to some extension of q_β^η of sufficient length, and switch η to outcome β.

5.1.2. Limit stages. Let s be a limit stage. Let U be the set of limit points of $g(t)$ for $t < s$. Observe that, since each corresponds to a particular configuration of outcomes, each corresponds to a particular limit point of $f(t)$ for $t < s$. Only countably many of these are in T; include only the corresponding members of U in S. Cancel all limit strategies corresponding to elements of U not included this way.

5.1.3. Verification.

CLAIM 5.2. *There exists $P \in \Lambda^{\omega_1}$ so that $\{s : \delta(s) \prec P\}$ is a club.*

PROOF. Let C be the set of stages at which $f(s) \prec X$; by hypothesis, this is a club. We inductively define P and a sequence $\langle s_\alpha \rangle_\alpha$ as follows: Let η be the strategy assigned to the node $P \upharpoonright \alpha$. If η ever enters its convergence outcome, then $P \upharpoonright (\alpha+1) = P \upharpoonright \alpha \frown \omega_1$ and s_α is the first stage after $\sup_{\beta<\alpha} s_\beta$ at which η is in its convergence outcome. Otherwise, η eventually assigns a correct initial segment of X to some outcome γ. Then $P \upharpoonright (\alpha+1) = P \upharpoonright \alpha \frown \gamma$, and s_α is the first stage after $\sup_{\beta<\alpha} s_\beta$ at which η enters outcome γ.

For limit α, $P \upharpoonright \alpha = \bigcup_{\beta<\alpha} P \upharpoonright \beta$ and $s_\alpha = \sup_{\beta<\alpha} s_\beta$. The map $\alpha \to s_\alpha$ is then continuous, so its set of fixed points forms a club.

Let $C' = \{\alpha \in C : \alpha = s_\alpha\}$. Then C' is a club, and for $t \in C'$ $\delta(t) \prec P$. It is evident by construction that whenever $\delta(t) \prec P$, $t \in C'$. ⊣

CLAIM 5.3. *There is a club C of stages so that for $s, t \in C$ with $s < t$, $g(s) \prec g(t)$. Then $Y = \bigcup_{s \in C} g(s)$ is a path through S.*

PROOF. The club C is precisely the club of stages s so that $\delta(s) \prec P$. ⊣

CLAIM 5.4. *Y is the only path through S.*

PROOF. Let Z be a path through S. This Z does not fall permanently inside an Aronszajn tree, so unboundedly often $g(s)$ is an initial segment of Z. Therefore, unboundedly many strategies η have σ_η along Z. But it is clear by construction that this can only happen if these strategies are assigned to compatible strings; in other words, Z corresponds to a path through the tree of strategies that is visited unboundedly often. Call this path P'. But then there are unboundedly many coding outcomes along P', which together must code a path through T. The only such

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path is X, so the coding outcomes of P' agree with the coding outcomes of P. Then by induction, the convergence outcomes must also match; so $P' = P$, and hence $Z = Y$. ⊣

CLAIM 5.5. $Y' \geq_T X$.

PROOF. Because h is Δ^0_2, Y' clearly computes h. The domain of h is unbounded along Y. Therefore Y' computes $h[Y] = X$. ⊣

CLAIM 5.6. $Y \oplus \emptyset' \geq_T Y'$.

PROOF. Once a strategy η is initialized, $\emptyset'$ is sufficient to determine whether η will enter its convergence outcome, which occurs if and only if $\Phi_e^Y(e) \downarrow$. Y is sufficient to determine where σ_η falls. ⊣

As a consequence, $X \equiv_T Y' \equiv_T Y \oplus \emptyset'$. ⊣

To relativize, we may *interleave* S with T_A, taking a new tree S_A to be $\{\sigma \oplus \tau \mid \sigma \in S \wedge \tau \in T_A \wedge |\sigma| = |\tau|\}$, and similarly interleave g and f_A to form a corresponding companion map.

5.2. Gaps.

PROPOSITION 5.7. *If $X \not\leq_T \emptyset$ is hyperimmune-free, then X is not strongly club-approximable, and is therefore not equivalent to the unique path in any computable tree with a companion map.*

PROOF. Suppose that X is both hyperimmune-free and strongly club-approximable. Let f be a computable function witnessing that X is strongly club-approximable; that is, $C = \{s : f(s) \prec X\}$ is a club. Clearly, $C \leq_T X$. Let $g \leq_T X$ be the function enumerating the elements of C in order. Because C is a club, g is continuous. Since X is hyperimmune-free, there is a computable function $h \geq g$. Without loss of generality, h is also continuous.

Let x be a fixed point of h with $x > \omega$. $x = h(x) \geq g(x) \geq x$, so this is also a fixed point of x. Then $x \cap C$ is a club in x of order type x.

CLAIM 5.8. *Let $\alpha < \omega_1$, $\alpha > \omega$, and suppose that A_1 and A_2 are closed and unbounded subsets of α with order type α. Then $A_1 \cap A_2$ is an unbounded subset of α.*

PROOF. Let g_1 and g_2 enumerate the members of A_1 and A_2 respectively.

$g_1 \circ g_2$ is a continuous function. Its fixed points are in both A_1 and A_2, and there are clearly unboundedly many below α. ⊣

As a result of the claim, there is only one string σ of length x so that the set $\{s < x : f(s) \prec \sigma\}$ is a club of order type x, and that σ is $X \upharpoonright x$. So X is computable. ⊣

PROPOSITION 5.9. *For any degree $\mathbf{a}$, there exists a degree $\mathbf{d}$ so that $\mathbf{a} < \mathbf{d} \leq \mathbf{a}'$ and the thin degrees computable from $\mathbf{d}$ are exactly the thin degrees computable from $\mathbf{a}$.*

PROOF. Fix a representative $A \in \mathbf{a}$; for example, the $<_L$-least, if canonicity is desired. We will construct by forcing with oracle A' a set G so that $D = A \oplus G$ is of the desired degree.

Fix also a standard effective bijection taking ω_1^3 to ω_1, denoted $\langle \cdot, \cdot, \cdot \rangle$.

Finally, fix a uniformly A-computable enumeration $\langle T_i \rangle$ of A-computable trees (countable width or otherwise).

Let $\sigma_0 = \langle\rangle$.

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Stage s: If s is a limit, let $\sigma_s = \lim_{t<s} \sigma_t$.

Stage $s+1 = \langle e+1, a, b\rangle + 1$*:* Let $t = \langle e, a, b\rangle$. Take $\sigma_s = \sigma_t \frown i$, where $i \neq \{e\}^A(|\sigma_t|)$.

Stage $s+1 = \langle 0, e, i\rangle + 1$*:* Let $t = \langle 0, e, i\rangle$. Search for one of the following:

(i) $\tau \succ \sigma_t$ and $x \in \omega_1$ such that $(\forall \rho \succ \tau)(\{e\}^{\rho \oplus A}(x)\uparrow$;

(ii) $\tau \succ \sigma_t$ such that

$$(\forall x \in \omega_1)(\forall \rho_0, \rho_1 \succ \tau)\left[\left(\{e\}^{\rho_0 \oplus A}(x)\downarrow \wedge \{e\}^{\rho_1 \oplus A}(x)\downarrow\right) \rightarrow \{e\}^{\rho_0 \oplus A}(x) = \{e\}^{\rho_1 \oplus A}(x)\right]$$

(iii) $\tau \succ \sigma_t$ such that $\{e\}^{\tau \oplus A} \notin T_i$; or

(iv) an order-preserving map $f : 2^{<\omega} \rightarrow 2^{<\omega_1}$ so that $\{e\}^{f(\cdot) \oplus A}$ is an order-preserving map $2^{<\omega} \rightarrow T_i$.

In Cases (i)–(iii), take $\sigma_{s+1} = \tau$. In Case (iv), note that $\{e\}^{f(\cdot) \oplus A}$ induces an injection $F : 2^\omega \rightarrow 2^{<\omega_1}$ with bounded range. If the range of F is entirely contained in T_i, then T_i does not have countable width, and we take $\sigma_{s+1} = \sigma_s \frown 0$. Otherwise, let $\rho \in 2^\omega$ be such that $F(\rho) \notin T_i$ and take $\sigma_{s+1} = \lim f(\rho)$.

5.2.1. Verification. It is clear that $A' \geq_T G$, so $A' \geq_T A \oplus G \geq_T A$.

Suppose that $\{e\}^{G \oplus A}$ is a path through T_i. This pair was considered at stage $s+1 = \langle 0, e, i\rangle + 1$. It is easy to see that one of the four cases was encountered. If the construction fell into Case (i), then $\{e\}^{G \oplus A}$ is not total. In Case (ii), $\{e\}^{G \oplus A}$ is A-computable. In Case (iii), $\{e\}^{G \oplus A}$ does not remain in T_i; and in Case (iv), either $\{e\}^{G \oplus A}$ does not remain in T_i or T_i does not have countable width.

Finally, by the other class of stages, G is not computable from A, so $G \oplus A >_T A$. ⊣

As a consequence of Proposition 5.9, we can obtain thin-free intervals corresponding to a wide class of jump-like operators. In particular, for any operator J with "enough" flexibility in inversion, there exists a degree $\mathbf{d}$ so that the interval $[\mathbf{d}, J(\mathbf{d})]$ includes no thin degrees. More formally, we have the following:

PROPOSITION 5.10. *Let J be a one-place operation on Turing degrees so that for every $\mathbf{a} > J(\mathbf{0})$, there exists $\mathbf{b} \not\leq J(\mathbf{0})$ such that $J(\mathbf{b}) = \mathbf{a}$. Then there exists a degree $\mathbf{d}$ so that the thin degrees computable from $J(\mathbf{d})$ are exactly the thin degrees computable from $\mathbf{d}$.*

PROOF. By Proposition 5.9, there is a degree $\mathbf{a}$ with $J(\mathbf{0}) < \mathbf{a} \leq J(\mathbf{0})'$ so that the thin degrees computable from $\mathbf{a}$ are exactly the thin degrees computable from $J(\mathbf{0})$. By the conditions on J, there exists a degree $\mathbf{d} \not\leq J(\mathbf{0})$ so that $J(\mathbf{d}) = \mathbf{a}$.

Suppose that there were a thin degree $\mathbf{c}$ with $\mathbf{d} \leq \mathbf{c} \leq \mathbf{a}$. By the choice of $\mathbf{a}$, $\mathbf{c} \leq J(\mathbf{0})$. But then $\mathbf{d} \leq J(\mathbf{0})$, contradicting the choice of $\mathbf{d}$.

Thus $\mathbf{d}$ is the desired degree. ⊣

5.3. Types of degrees. The results of the previous section were in a certain sense purely internal to $\mathcal{P}_{\text{thin}}$: the structure of $\mathcal{P}_{\text{thin}}$ itself was considered without reference to the underlying structure of the Turing degrees themselves. We now turn our attention to a more external question, the question of which sorts of Turing degrees appear in $\mathcal{P}_{\text{thin}}$. One particular example, that of minimal degrees, will be considered in detail.

THEOREM 5.11. *There is a computable tree T of countable width with exactly one path X, so that X has minimal Turing degree.*

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PROOF. Let A be a fixed computable Aronszajn tree.

Let $\Lambda = \{S\} \cup \omega_1$. Let $\mathscr{T} = \Lambda^{<\omega_1}$. To each string $\sigma \in \mathscr{T}$ we attach a *worker* η_σ. At any stage s, each worker which has been initialized has an outcome in Λ; let $\delta(s)$ be the maximal string so that $\delta(s)(i+1)$ is the stage-s outcome of $\eta_{\delta(s)\restriction i}$.

Each worker maintains one or more *scaffolds*, each one corresponding to an outcome of that worker. The *active* scaffold at stage s is the one corresponding to the worker's outcome at stage s.

DEFINITION 5.12. A *scaffold* is a partial function $f : 2^{<\omega_1} \to 2^{<\omega_1}$ together with a set $S \subseteq 2^{<\omega_1}$ such that the following conditions hold.

(i) $\mathrm{dom}(f) \subseteq S$;
(ii) For $\sigma, \tau \in \mathrm{dom}(f)$, $f(\sigma) \preccurlyeq f(\tau)$ iff $\sigma \preccurlyeq \tau$;
(iii) If $\sigma \frown i \in \mathrm{dom}(f)$, then $f(\sigma \frown i)(|f(\sigma)|) = i$; and
(iv) If $\sigma_0 \prec \sigma_1 \prec \cdots$ is a sequence in $\mathrm{dom}(f)$ and $\sigma = \lim_i \sigma_i \in \mathrm{dom}(f)$, then $f(\sigma) = \lim_i f(\sigma_i)$.

We denote the scaffold (S, f).

For scaffolds (S_0, f_0) and (S_1, f_1), say (S_0, f_0) is a *subscaffold* of (S_1, f_1) if $\mathrm{ran}(f_0) \subseteq \mathrm{ran}(f_1)$. Observe that there is no restriction on the relationship between S_0 and S_1.

It will be the case that every scaffold maintained by η_σ is a subscaffold of η_τ for each $\tau \prec \sigma$. The root worker $\eta_{\langle\rangle}$ maintains only one scaffold, and the corresponding function is the identity; this scaffold will form the final tree T.

At any stage s, a worker η_σ may issue a *request* for any $\theta \in 2^{<\omega_1}$ every initial segment of which is currently in the range of the active scaffold of every η_τ for $\tau \prec \sigma$. Provided that $|\theta| \geq s$, this request will be *honored* by every such η_τ: η_τ will add an assignment to its active scaffold (possibly adding to the domain of the scaffold) so that θ will be in the range, and then add to the domain of the scaffold so that the preimage of θ has a copy of A above it.

Successor workers pursue two sorts of requirements:

D_e: $\Phi_e \neq X$.
M_e: One of the following holds:
(i) $(\exists x)\Phi_e^X(x)\uparrow$,
(ii) $\Phi_e^X \leq_T \emptyset$, or
(iii) $\Phi_e^X \geq_T X$.

Divide the countable successor ordinals into two computable unbounded sets $\mathscr{D}$ and $\mathscr{M}$. Workers η_σ for $|\sigma|$ the eth member of $\mathscr{D}$ pursue the diagonalization requirement D_e; workers η_σ for $|\sigma|$ the eth member of $\mathscr{M}$ pursue the minimality requirement M_e. Workers $\eta_{\langle\rangle}$ and η_σ for σ of limit length behave differently and pursue no requirement.

For ease of notation, when σ is a string of successor length, denote by σ^- its immediate predecessor.

For any worker η_σ, let (S^σ, f^σ) be the currently active scaffold.

5.3.1. Strategy for limit workers. $\eta_{\langle\rangle}$ does nothing except respond to requests; we therefore concentrate on η_σ for $|\sigma|$ of limit length.

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When initialized at stage s, η_σ takes $S^\sigma = A$ and sets $f^\sigma(\langle\rangle)$ to be the limit of $f^\tau(\langle\rangle)$ for $\tau \prec \sigma$. It enters outcome 0, and remains there permanently.

At later stages, η_σ works to define f^σ on progressively higher and higher levels of S^σ. Limit levels are straightforward: η_σ simply defines the function as necessary to preserve continuity, requesting any strings not already present in the ranges of the previous scaffolds. At successor levels, suppose η_σ has defined $f^\sigma(\tau)$ and needs to define $f^\sigma(\tau \frown i)$. For each $\rho \prec \sigma$, let $\theta(\rho)$ be the element in the domain of f^ρ so that $f^\rho(\theta(\rho)) = f^\sigma(\tau)$. Let $f^\sigma(\tau \frown i) = \lim f^\rho(\theta(\rho) \frown i)$, requesting the string if necessary. This limit exists and has the property that every f^ρ has a sequence cofinal along it; so the request is a valid one.

5.3.2. Strategy for diagonalization requirements. Let η_σ be a successor worker pursuing requirement D_e. When initialized at stage s, η_σ looks for θ_0 and θ_1 in the range of f^{σ^-} with length at least s differing at some position $\alpha < s$. η_σ sets $S^\sigma = A$, puts $f^\sigma(\langle\rangle) = \theta_0$, and enters outcome 0.

If at some stage $t > s$, $\Phi_{e,t}(\alpha) \downarrow = \theta_0(\alpha)$, then η_σ requires attention. If it receives attention, it deletes its existing scaffold, sets $S^\sigma = A$ and $f^\sigma(\langle\rangle) = \theta_1$, and enters outcome 1.

At any successor stage, η_σ considers each leaf node τ of $\mathrm{dom}(f^\sigma)$. For each such τ, by induction there exists $\theta \in \mathrm{dom}(f^{\sigma^-})$ with $f^{\sigma^-}(\theta) = f^\sigma(\tau)$; if $\theta \frown 0, \theta \frown 1 \in \mathrm{dom}(f^{\sigma^-})$, then η_σ takes $f^\sigma(\tau \frown i) = f^{\sigma^-}(\theta \frown i)$ for each i. Otherwise, it simply waits until the necessary assignments have been made.

At a limit stage, η_σ considers each limit node $\tau \in S^\sigma$ so that f^σ has been defined on every initial segment of τ. f^σ induces an obvious choice of $f^\sigma(\tau)$; if this string is not already in the range of f^{σ^-}, then η_σ requests it. Regardless, it takes this as the assignment of $f^\sigma(\tau)$.

5.3.3. Strategy for minimality requirements. Let η_σ be a successor worker pursuing requirement M_e. When initialized at stage s, η_σ sets $S^\sigma = A$ and puts $f^\sigma(\langle\rangle)$ some element of $\mathrm{ran}(f^{\sigma^-})$ with length at least s (if none exists, η_σ waits until it does). This scaffold is designated the *splitting scaffold*; it will be the active scaffold whenever η_σ is in outcome S. The worker η_σ begins in outcome S.

The worker η_σ will maintain the condition that if (S, f) is its splitting scaffold and $\tau_0, \tau_1 \in S$ are incomparable, then $\Phi_e^{f(\tau_0)}$ and $\Phi_e^{f(\tau_1)}$ are incomparable. Call this condition (*).

While still in outcome S, η_σ attempts to extend f^σ while preserving (*). At a limit stage, it simply fills in any limit nodes in S^σ, requesting whatever is necessary. At a successor stage, η_σ will attempt to extend each leaf node of $\mathrm{dom}(f^\sigma)$ in both directions. For each leaf $\tau \in \mathrm{dom}(f^\sigma)$, if there exist $\rho_0, \rho_1 \in \mathrm{ran}(f^{\sigma^-})$ with length at least s and $x < s$ so that $\rho_i \succcurlyeq f^\sigma(\tau) \frown i$ and $\Phi_e^{\rho_0}(x) \downarrow \neq \Phi_e^{\rho_1}(x) \downarrow$, then η_σ takes $f^\sigma(\tau \frown i) = \rho_i$. If none exists, then let $\epsilon = f^\sigma(\tau)$. η_σ switches to a new ordinal outcome and associates with it a new scaffold as follows: $f^\sigma(\langle\rangle) = \theta$, where θ is the first extension of $\epsilon \frown 0$ in the range of f^{σ^-} (as usual, if none exists, wait until it does); S^σ is the translation of the part of S^{σ^-} extending that point.

While in an ordinal outcome, η_σ copies the scaffold of η_{σ^-}, just as the diagonalization workers do.

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If a splitting pair is found within this new scaffold (that is, $\rho_0, \rho_1 \in S^\sigma$ with $\Phi_e^{\rho_0}$ and $\Phi_e^{\rho_1}$ incomparable) then η_σ again switches to a new ordinal outcome and replaces the current scaffold with a new scaffold, defined the same way as before but rooting at an extension of $\epsilon \frown 1$.

If a spitting pair is found within this scaffold as well, then between the two sets of splitting pairs is a pair appropriate for extending the splitting scaffold at τ; η_σ returns to the S outcome, re-activating the splitting scaffold, and adds those elements.

To ensure that the search for strings of sufficient length is successful (that is, that higher priority scaffolds are not unexpectedly short) at every stage we allow the active workers to act *in order*, beginning from the root. Responses to requests are immediate, but propagate in the same manner.

This completes the construction.

Claim 5.13. *For any α, only countably many requests are made for strings of length $<\alpha$, and none are requested past stage α.*

Proof. Any worker initialized past stage α will, by construction, place its root at a string of length at least α. Again by construction, no worker ever makes requests that do not extend its root; therefore, only the workers initialized before stage α will ever request strings of length less than α.

Let η be such a worker, and suppose that it requests a limit string θ at stage s. In every case, this request is made as soon as the initial segments are available; that is, it must be that there was a sequence $s_0 < s_1 < \cdots$ cofinal in s at which η assigned strings in the domain of its scaffold to initial segments of θ. But by construction a worker can only make stage-t assignments of length at least t; so these initial segments of θ have lengths at least $s_0, s_1, \ldots$. Thus θ has length at least s.

Clearly no worker requests uncountably many strings in a single stage; therefore no individual worker requests uncountably many strings of length $<\alpha$. Since only countably many workers can request these strings at all, only countably many such requests are made. ⊣

Claim 5.14. *The tree T is computable and of countable width.*

Proof. This is an immediate consequence of Claim 5.13. ⊣

Claim 5.15. *All requests are honored.*

Proof. Again, this is an immediate consequence of Claim 5.13; any request made at stage s for a string with length at least s is honored, and by the claim, no requests not satisfying this condition are made. ⊣

Claim 5.16. *The set of strings in $\mathscr{T}$ that lie along the execution path unboundedly often forms a path.*

Proof. Observe that limit workers (including the root worker) never change outcomes; diagonalization workers change outcome at most once; and while minimalization workers may change outcome unboundedly often, the only outcome they may visit unboundedly often is S. So every worker visits exactly one outcome unboundedly often; the inductive argument is straightforward. ⊣

We call this the *true path*, and the workers along this path *true workers*; their outcomes that lie along the true path are *true outcomes*.

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Note that the set of strings $f^\sigma(\langle\rangle)$ evaluated during stages at which η_σ has its true outcome exactly defines a path through the tree T; we call this path X.

CLAIM 5.17. *X is the only path through T.*

PROOF. Suppose for contradiction that $Y \neq X$ is a path through T. Y cannot be "native"—that is, it must be obtained through uncountably many requests, because otherwise it would be part of an Aronszajn tree.

No worker makes requests incompatible with its root; it therefore cannot be that the requests for initial segments of Y are unbounded along the true path. As a consequence, there is a minimal worker η so that only countably many requests for initial segments of Y are made above the true outcome of η. Note that η is not a limit worker or the root worker, because neither class of worker changes outcomes. Note also that η is not a diagonalization worker; at some stage, a worker pursuing a diagonalization requirement will enter its true outcome and never leave, so only countably many requests for anything can be made above the other outcome. So η is a minimalization worker. Furthermore, η's true outcome must be S—otherwise, the same argument holds as for the diagonalization case.

Since η's true outcome is not one of the ordinal outcomes, it cannot be that uncountably many requests for initial segments of Y are made above only one of them; so it must be that an uncountable sequence of ordinal outcomes have at least one request each. The scaffolds for these outcomes must be rooted at points along Y; but all of them come from points at which the attempt to split in the splitting outcome temporarily failed. So all of these initial segments of Y are in the splitting scaffold—which means uncountably many requests for initial segments of Y must have been made above the S outcome, contradicting our assumption on η. ⊣

CLAIM 5.18. *X is not computable.*

PROOF. Suppose $\Phi_e = X$. Let η be the (unique) worker along the true path pursuing requirement D_e. When initialized, η selected some pair of strings and an α at which they differed. At some stage t, $\Phi_{e,t}(\alpha)\downarrow$; at that stage, η switched outcomes if necessary and directed construction along the string of the pair that did not agree with Φ_e at α. This is a contradiction. ⊣

CLAIM 5.19. *X has minimal Turing degree.*

PROOF. Suppose that Φ_e^X is total but not computable. Let η be the (unique) worker along the true path pursuing requirement M_e. Observe that if the true outcome of η were an ordinal, Φ_e^X would be either partial or computable, because a split would never be found; $\Phi_e^X(x)$ could be computed by taking the first computation $\Phi_e^\tau(x)$ to converge for τ in the range of the scaffold of η. So the true outcome of η is S, and therefore in the limit η constructs a scaffold (U, f) so that if σ and τ are incomparable members of U then $\Phi_e^{f(\sigma)}$ and $\Phi_e^{f(\tau)}$ disagree somewhere. But then given Φ_e^X we can determine a unique branch Y of U so that $\Phi_e^{f(Y)} = \Phi_e^X$; this branch must be the preimage of Y. Thus $\Phi_e^X \geq_T X$.

So for every e we have that Φ_e^X is either partial, computable, or above X; since we already have that $X >_T \emptyset$, X is of minimal Turing degree. ⊣

This completes the proof of Theorem 5.11. ⊣

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§6. Nonisolated paths.

6.1. General results. Once one has examined the behavior of thin Π^0_1-classes containing exactly one element, a natural follow-up question is what happens to thin Π^0_1 classes more generally.

The following basis theorem appeared in a more specialized form in the proof of Theorem 3.17, but is now more directly applicable.

THEOREM 6.1. *Let P be an arithmetic property of sets so that $(\exists X)P(X)$. Then in fact $(\exists X \in \Delta^1_1)P(X)$.*

PROOF. Let $\varphi(Y)$ be the formula expressing that Y is $<_L$-least such that $P(Y)$ holds, and let X be the unique set so that $\varphi(X)$ holds. Then

$$\sigma \prec X \iff (\exists Z)([Z \models ZF^- + V = L + ((\exists Y)\varphi(Y) \wedge \sigma \prec Y)] \wedge Z \text{ is well-founded}).$$

In other words, σ is an initial segment of X if and only if there is an L_α so that L_α models that σ is an initial segment of the $<_L$-least X such that $P(X)$ holds. Note that any uncountable well-founded model of $ZF^- + V = L$ must be isomorphic to some L_α for α uncountable, and that the $<_L$ least witness to $(\exists X)P(X)$ must appear in the first L_α that has any witness at all (and be correctly identified as such in this L_α).

Equivalently,

$$\begin{aligned}\sigma \prec X \iff (\forall Z)(([Z \models ZF^- + V = L + (\exists Y)P(Y)] \wedge Z \text{ is well-founded}) \\ \to Z \models (\forall Y)(\varphi(Y) \to \sigma \prec X)).\end{aligned}$$

In other words, σ is an initial segment of X if and only if every L_α in which there is a witness to $(\exists X)P(X)$ models that σ is an initial segment of the $<_L$-least witness.

We think of the Z in these definitions as codings of structures as subsets of ω_1, so that the quantifiers $(\exists Z)$ and $(\forall Z)$ are second-order quantifiers. Observe that the relation $Z \models T$ for Z a structure of cardinality $\aleph_1$ and T a countable first-order theory is arithmetic, because quantifiers are over members of Z, which have been coded as countable ordinals.

These are, respectively, Σ^1_1 and Π^1_1 definitions of X; X is hence Δ^1_1. ⊣

6.2. Kurepa trees. Throughout this section, we will operate under the additional assumption $L_{\omega_2} = H(\aleph_1)$.

DEFINITION 6.2 (Kurepa [16]). A *Kurepa tree* is a tree T with countable width and $|[T]| = \aleph_2$.

A *perfect Kurepa tree* is a Kurepa tree K so that $K(\sigma) = \{\tau \in K : \tau \succcurlyeq \sigma\}$ is also Kurepa for every $\sigma \in K$.

Related is the following definition, due to Jensen [9].

DEFINITION 6.3. A $\diamondsuit^+$-*sequence* (in ω_1) is a sequence $\{\mathcal{A}_\alpha\}_{\alpha<\omega_1}$ of sets so that $\mathcal{A}_\alpha$ is a countable set of subsets of α and, for any $X \subseteq \omega_1$, there is a club $C \subseteq \omega_1$ so that for all $\alpha \in C$, $X \restriction \alpha$ and $C \restriction \alpha$ are in $\mathcal{A}_\alpha$.

The following result was essentially shown by Jensen [9]; we include it here for completeness, with a proof presented in terms more closely matching the rest of this

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article. This proof is an effectivization of the analogous one given by Kunen [15]. We use the same notation, insofar as possible.

LEMMA 6.4. *There is a computable $\diamondsuit^+$-sequence.*

PROOF. Let $ZF - P$ be the collection of axioms of Zermelo-Frankel set theory without Choice and Power Set. For each $\alpha < \omega_1$, let α^* be least $>\alpha$ so that $L_{\alpha^*} \models ZF - P + |\alpha| = \aleph_0$. Let $\mathcal{A}_\alpha = L(\alpha^*) \cap \mathcal{P}(\alpha)$. This is certainly a computable sequence; we show that it is also a $\diamondsuit^+$ sequence.

Let $A \subseteq \omega_1$. By the supposition $L_{\omega_2} = H(\aleph_1)$, $A \in L_{\omega_2}$. For $\beta < \omega_1$, let $M_{A,\beta} = \mathfrak{H}(L_{\omega_2}, \{A\} \cup \beta)$ (which we define to be the set of elements of L_{ω_2} definable using parameters from $\{A\} \cup \beta$). Then $M_{A,\beta}$ is a countable elementary substructure of L_{ω_2} including A and the ordinals $<\beta$. Let $C = \{\beta < \omega_1 : M_{A,\beta} \cap \omega_1 = \beta\}$. C is clearly a club in ω_1.

Let $\alpha \in C$. The transitive collapse of $M_{A,\alpha}$ is some L_γ; since $\alpha = (\omega_1)^{L_\gamma}$, we have that $\alpha < \gamma < \alpha^*$. Then $A \cap \alpha$, which is preserved under the transitive collapse, is a member of L_{α^*} and hence of $\mathcal{A}_\alpha$.

Since $L_{\alpha^*} \models ZF - P$, we can define $\hat{C} = \{\beta < \alpha : \mathfrak{H}(L_\gamma, \{A \cap \alpha\} \cup \beta) \cap \alpha = \beta\}$ in L_{α^*}. Since $\hat{C} \in L_{\alpha^*}$, it remains to show that $\hat{C} = C \cap \alpha$. To show that $C \cap \alpha \in \mathcal{A}_\alpha$, it suffices to show that for $\beta < \alpha$, $\mathfrak{H}(L_\gamma, \{A \cap \alpha\} \cup \beta) \cap \alpha = \beta$ iff $\mathfrak{H}(L_{\omega_2}, \{A\} \cup \beta) \cap \omega_1 = \beta$.

Let $\zeta < \omega_1$, $\zeta \in \mathfrak{H}(L_{\omega_2}, \{A\} \cup \beta)$, and $\zeta \geq \beta$. Then ζ is definable in L_{ω_2} from $\{A\} \cup \beta$. Because $\alpha \in C$, the ordinals definable from ordinals less than α are exactly the ordinals less than α, so $\zeta < \alpha$. Since $M_{A,\alpha}$ is an elementary submodel of L_{ω_2}, ζ is likewise definable in $M_{A,\alpha}$. Since the transitive collapse fixes ordinals $<\alpha$ and takes A to $A \cap \alpha$, ζ is also definable in L_γ using parameters in $\{A \cap \alpha\} \cup \beta$, so $\mathfrak{H}(L_\gamma, \{A \cap \alpha\} \cup \beta) \cap \alpha \neq \beta$. The converse is similar. ⊣

LEMMA 6.5. *There is a computable perfect Kurepa tree.*

PROOF. Fix a computable $\diamondsuit^+$-sequence $\langle A_\alpha \rangle_{\alpha \in \omega_1}$ so that each A_α is closed under finite difference, includes the empty set, and satisfies

$$(\forall \beta < \alpha, A \in A_\alpha, B \in A_\beta)(\exists C \in A_\alpha)(C \upharpoonright (\beta + 1) = B \frown 1 \wedge C \upharpoonright (\alpha \setminus (\beta + 1)) = A \upharpoonright (\alpha \setminus (\beta + 1))).$$

In other words, each A_α is closed under replacing initial segments with elements of A_β for $\beta < \alpha$.

Let K be the set consisting of the *null elements* $\mathbf{0}^\alpha$ for each α and all pairs $(A, C) \in 2^\alpha \times 2^\alpha$ such that the following holds:

(i) $A, C \in A_\alpha$;
(ii) For all $\beta \in C$, $A \upharpoonright \beta, C \upharpoonright \beta \in A_\beta$; and
(iii) If $\langle \alpha_i \rangle_{i<\omega}$ is an increasing sequence in C that is bounded below α, then $\sup \alpha_i \in C$.

In the above conditions, we use the convention that for $C \in 2^\alpha$, $\beta \in C$ iff $C(\beta) = 1$.

Impose an ordering $<$ on K as follows:

(a) $\mathbf{0}^\alpha < \mathbf{0}^\beta$ iff $\alpha < \beta$.
(b) $\mathbf{0}^\alpha < (A, C)$ iff $\mathrm{dom} A > \alpha$.
(c) $(A_0, C_0) < (A_1, C_1)$ iff $A_0 \prec A_1$ and $C_0 \prec C_1$.

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$(K, <)$ is then obviously a computable tree of countable width, and is clearly Kurepa; it remains to show that it is perfect.

Fix $(A, C) \in K$, and let $\mathcal{A} \in 2^{\omega_1}$ so that $\mathcal{A} \succ A$. Because $\langle A_\alpha \rangle$ is a $\diamondsuit^+$-sequence, there exists a club $\mathcal{C}$ so that for every $\alpha \in \mathcal{C}$, $\mathcal{A} \upharpoonright \alpha, \mathcal{C} \upharpoonright \alpha \in A_\alpha$. Take $\mathcal{C}^*$ such that $\mathcal{C}^* \upharpoonright (\alpha + 1) = C \frown 1$ and $\mathcal{C}^*(\beta) = \mathcal{C}(\beta)$ for $\beta > \alpha + 1$. Then, by the closure property imposed on the $\diamondsuit^+$-sequence, $\mathcal{C}^*$ has the same property as $\mathcal{C}$; thus the initial segments of $(\mathcal{A}, \mathcal{C}^*)$ form a path in K extending (A, C). Since $\mathcal{A}$ was chosen to be an arbitrary extension of A, it is clear that there are ω_2-many such paths. ⊣

PROPOSITION 6.6. *There are $\aleph_2$-many non-thin, non-wide degrees.*

PROOF. There is a computable Kurepa tree K. This K has countable width, so every path in K is non-wide. K is Kurepa, so K has $\aleph_2$-many paths; but there are obviously only $\aleph_1$-many thin degrees, so $\aleph_2$-many of these paths are non-thin. ⊣

DEFINITION 6.7. A *Kurepa degree* is a degree $\mathbf{d}$ so that

(i) $\mathbf{d}$ is thin, but
(ii) If T is a computable tree of countable width and $\mathbf{d} \cap [T] \neq \emptyset$, then $|[T]| = \aleph_2$.

PROPOSITION 6.8. *There exists a thin Π^0_1-class consisting entirely of sets of Kurepa degree.*

The proposition is in fact a corollary of the following more powerful theorem:

THEOREM 6.9. *Let K be a computable perfect Kurepa tree. Then there exists, uniformly in an index for K, a Δ^0_3 functional Γ and a computable Kurepa tree T such that the following holds*:

(i) *For each $X \in [K]$, $\Gamma^X \in [T]$;*
(ii) *For each $Y \in [T]$, there is exactly one $X \in [K]$ so that $Y = \Gamma^X$; and*
(iii) *Every $Y \in [T]$ has Kurepa degree.*

PROOF. We will construct T in stages, so that $T = \bigcup_s T_s$. At the same time, we will define at each stage a partial map $F_s : K \to T_s$, so that $F = \liminf_s F_s$ will be the desired Γ.

6.2.1. The tree of strategies. This will be an argument by a modified form of unbounded injury. Let

$$\Lambda = \{\text{Kurepa}, \text{Halt}\} \cup \{\langle \text{Div}, \alpha \rangle | \alpha \in \omega_1\} \cup \{0, 1\}$$

with the attached ordering $\text{Halt} < \text{Kurepa} < \langle \text{Div}, \alpha \rangle < \langle \text{Div}, \beta \rangle$ for every $\alpha < \beta$; 0 and 1 are not included in the ordering.

Let $\mathfrak{S}$ be the subtree of $\Lambda^{<\omega_1}$ consisting of those σ satisfying properties (i) and (ii) below. For ease of notation, say an ordinal is *even* if it is of the form $\omega\beta + n$ for n even. Let $\text{even}(\sigma)$ denote the string τ so that $\tau(\omega\beta + n) = \sigma(\omega\beta + 2n)$ for all $\beta < \omega_1$ and $n < \omega$; in other words, $\text{even}(\sigma)$ is the string consisting of the even-indexed entries of σ.

(i) $\sigma(\alpha) \in \{0, 1\}$ iff n is even, and
(ii) $\text{even}(\sigma) \in K$.

In other words, the members of $\mathfrak{S}$ are those σ which take values 0 and 1 exactly on even indices, and so that the resulting embedded binary string is a member of K.

To each node σ of $\mathfrak{S}$ we attach a strategy η_σ. The strategy η_σ will attempt to satisfy the requirement at index $|\text{even}(\sigma)|$ in the priority order outlined below.

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6.2.2. The requirements. Let $\langle U_i \rangle_{i<\omega_1}$ be the computable enumeration of computable trees found in Proposition 3.3. We work to satisfy requirements of the following form:

$R_{e,i,j}$: For some string $\sigma \in T$, one of the following holds:
- (a) For every $X \in [T]$ with $\sigma \prec X$, $\Phi_e^X = Y \in [U_j]$ and $\Phi_i^Y = X$ (in which case U_j will have ω_2-many branches); or
- (b) For every $X \in [T]$ with $\sigma \prec X$, one of the following holds:
  - (i) Φ_e^X is not total;
  - (ii) $\Phi_e^X \notin [U_j]$; or
  - (iii) $\Phi_e^X = Y$ with $\Phi_i^Y \neq X$.

It should be clear that if we can ensure that (1) every path through T passes through the σ chosen by at least one $R_{e,i,j}$ strategy for each triple (e, i, j), and (2) every strategy satisfies its chosen requirement, then the desired properties will hold of $[T]$. Accordingly, we call such a σ the *satisfaction point* of the corresponding requirement.

Fix a standard effective enumeration $\psi : \omega_1 \to \omega_1^3$ of triples. We then order the requirements $R_{\psi(0)} < R_{\psi(1)} < \cdots$.

6.2.3. The preliminaries. At any stage s, countably many strategies will have been *initialized*. If a strategy η_σ with $|\sigma|$ odd (an *odd-level strategy*) has been initialized, it will have an *outcome*, which will be a member of $\Lambda \setminus \{0, 1\}$.

The *execution tree at stage* s, δ_s, is the subtree of $\mathfrak{S}$ consisting of those σ such that the following hold:
- (i) η_σ has been initialized by stage s; and
- (ii) for every $\tau \prec \sigma$ of odd length, $\sigma \succcurlyeq \tau \frown u$, where u is the stage-s outcome of η_τ.

The *true execution tree* δ is the subset of $\mathfrak{S}$ consisting of those σ such that $\sigma \in \delta_s$ for uncountably many s. Note that δ must be a tree since every δ_s will be a tree.

6.2.4. Strategy for requirement $R_{e,i,j}$. Let η_σ be a strategy assigned to requirement $R_{e,i,j}$. The behavior of η_σ varies depending on whether it is an even-level strategy or an odd-level strategy. Regardless, when initialized at stage s, η_σ is given a string $\tau \in T_s$ of length at least s; this will be the *anchor point* of η_σ.

6.2.5. Odd-level strategies. Let η_σ be an odd-level strategy assigned to $R_{e,i,j}$, and let τ be its anchor point. Again, for convenience, we drop the subscript σ. The strategy η will work directly to satisfy $R_{e,i,j}$.

At any stage s, if η is not in outcome Halt, then η *requires attention* if there exists $\rho \succ \tau$, $\rho \in T_s$ with either $\Phi_e^\rho \notin U_j$ or $\Phi_e^\rho = \theta$ and $(\exists t < s)\Phi_i^\theta \upharpoonright t \downarrow \notin T_s$. If permitted to act, η changes its anchor to a maximal extension of ρ on T_s and switches to outcome Halt.

Suppose that $\sigma \in \delta_s$, η currently has outcome Kurepa, and there exists a string $\rho \succ \sigma$ such that the following holds:
- (i) $\rho \in \delta_s$,
- (ii) η_ρ was initialized *before* stage s with an anchor τ_ρ, and
- (iii) For some $a \in \{0, 1\}$, $\tau_\rho \frown a$ has no σ-coherent extension in T_s.

Then η *requires attention* at stage s. If permitted to act, it re-anchors itself at a maximal extension θ of τ_ρ, switches to the first $\langle \mathrm{Div}, \alpha \rangle$ outcome it has never used

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before. θ is not σ-coherent. Since η has been permitted to act, it must be that θ is σ^*-coherent for every $\sigma^* \prec \sigma$ of odd length; otherwise η_{σ^*} would have acted instead; so either $\Phi_e^\theta \upharpoonright |\theta| \uparrow$ or $\Phi_e^\theta = \chi$ and $\Phi_i^\chi \upharpoonright |\theta| \uparrow$. Let α be least so that $\Phi_e^\theta(\alpha) \uparrow$, and let β be least so that $\Phi_i^\chi(\beta) \uparrow$. Then η *asserts* one of the following.

(a) If $\alpha < |\theta|$, then η asserts that for every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi_e^{\theta'}(\alpha) \uparrow$.

(b) Otherwise, η asserts that for every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi_i^{\Phi_e^{\theta'}}(\beta) \uparrow$.

We say assertion (a) has been *violated* if there exists $\theta' \succ \theta$, $\theta' \in T_s$, so that $\Phi_e^{\theta'}(\alpha) \downarrow$. Likewise, (b) has been *violated* if there exists θ' with the same conditions so that $\Phi_i^{\Phi_e^{\theta'}}(\beta) \downarrow$.

Suppose instead that $\sigma \in \delta_s$, η currently has outcome $\langle \mathrm{Div}, \alpha \rangle$ for some α, and the assertion η made when it entered that outcome has been violated. Then η *requires attention*. If permitted to act, η returns to its original anchor point, returns to the Kurepa outcome, and *posts* the tuple $\langle k, e, i, x \rangle$, where $k = 0$ and $x = \alpha$ if η was in Case (a) and $k = 1$ and $x = \beta$ otherwise.

If η_σ has outcome i, no strategy η_τ requires attention for $\tau \preccurlyeq \sigma$, and $\eta_{\sigma^\frown i}$ has not yet been initialized, $\eta_{\sigma^\frown i}$ is initialized with anchor the $<_L$-least length-s extension of the anchor of η_σ.

6.2.6. Even-level strategies.

Definition 6.10. For any string σ, let $\mathrm{odd}(\sigma)$ be the string τ such that $\tau(\beta + n) = \sigma(\beta + 2n + 1)$ for all limit ordinals β and $n \in \omega$ so that $\beta + 2n + 1 < |\sigma|$.

Let $\mathrm{Kurepa}(\sigma) = \{\alpha < |\mathrm{odd}(\sigma)| \mid \mathrm{odd}(\sigma)(\alpha) = \mathrm{Kurepa}\}$; that is, this is the set of indices of strategies along σ that have the Kurepa outcome. A string θ is σ-*coherent* if the following holds for every $\alpha \in \mathrm{Kurepa}(\sigma)$:

(i) $\Phi_e^\theta \upharpoonright |\theta| \downarrow$, and
(ii) $\Phi_i^\rho \upharpoonright |\theta| \downarrow = \theta$ where $\rho = \Phi_e^\theta$,

where $\psi(\alpha) = \langle e, i, j \rangle$.

Let η_σ be an even-level strategy assigned to $R_{e,i,j}$, and let τ be its anchor point. For convenience, we drop the subscript σ. Let $S = \{a \mid \sigma^\frown a \in \mathfrak{S}\}$; η maintains a partial map $f : S \to 2^{<\omega_1}$ with range contained in T_s. When initialized, η immediately builds Aronszajn trees rooted at $\tau^\frown a$ for $a \in S$, and adds these to T_s. η *requires attention at stage* s if $\mathrm{dom} f = \emptyset$ and for every $a \in S$ there exists $\theta \succ \tau^\frown a$ with $\theta \in T_s$ so that θ is σ-coherent. If permitted to act, η takes $f(a)$ to be the $<_L$-least length-s extension of θ in T_s and initializes $\eta_{\sigma^\frown a}$ with anchor point $f(a)$.

While the required θ have not yet been found, η_σ maintains *bookmark points* θ_a in T_s for each $a \in S$. Initially, $\theta_a = \tau^\frown a$. At any stage s, if $\sigma \in \delta_s$ and η_σ does not require attention, then it adds $\theta_a^\frown 0$ to T_s for each a, moves each θ_a to $\theta_a^\frown 0$, and adds an Aronszajn tree rooted at the new θ_a to T_s.

Suppose σ leaves δ_s and reenters at some later stage t. Then σ reentered because some strategy along σ switched from a Div outcome to the Kurepa outcome. Let that strategy be η. When η returned to outcome Kurepa, it *posted* a tuple $\langle k, e, i, x \rangle \in 2 \times \omega_1^3$. η_σ chooses its new bookmarks θ_a depending on the value of k as follows; let θ_a' be the previous value of θ_a.

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$k = 0$: θ_a is the $<_L$-least length-s available extension of θ'_a such that if $(\exists \theta_a \prec \theta \in T_s)\Phi_e^{\theta}(x)\downarrow$, then $\Phi_e^{\theta_a}(x)\downarrow$.

$k = 1$: θ_a is the $<_L$-least length-s available extension of θ'_a such that if $(\exists \theta_a \prec \theta \in T_s)\Phi_i^{\Phi_e^{\theta}}(x)\downarrow$ then $\Phi_i^{\Phi_e^{\theta_a}}(x)\downarrow$.

6.2.7. Limit stages. Let s be a limit stage, and let η be an odd-level strategy. The outcome of η at stage s is taken to be the lim inf of its outcomes at previous stages. T_s is taken to be $\bigcup_{t<s} T_t \cup U$, where U is a (countable) collection of limit nodes consisting of exactly those strings which are limits of anchor points of strategies on δ_s along branches of δ_s with limit in $\mathfrak{S}$. Any limit-level strategy all of whose predecessors have been initialized and are in δ_s is then initialized, with anchor equal to the limit of the anchors of its predecessors.

6.2.8. Defining F. At any stage s, take $F_s(\sigma)$ to be the anchor point of η_ρ, where ρ is the (unique) member of δ_s such that $\text{even}(\rho) = \sigma$.

6.2.9. Construction. At any stage s, any strategy η_σ is permitted to act as long as (1) η_σ requires attention, and (2) η_τ does not require attention for any $\tau \prec \sigma$.

6.2.10. Verification.

Claim 6.11. *Let $\sigma \in \delta$ have even length. Then η_σ eventually initializes $\eta_{\sigma^\frown a}$ for each a such that $\sigma \frown a \in \mathfrak{S}$.*

Proof. Suppose otherwise. Then η_σ extends its bookmarks θ_a unboundedly often; so for each a, the limit X_a of θ_a is a branch of length ω_1. If η_σ never initializes its successors, then at least one of the X_a must have no σ-coherent initial segment. Fix one such a, and let $X = X_a$.

For each pair (e, i) so that η_σ has a predecessor attached to an $R_{e,i,j}$ requirement and η_σ is above that predecessor's Kurepa outcome, suppose that Φ_e^X is total and that $\Phi_i^{\Phi_e^X} = X$. For any j, Y, let $f_j^Y(\alpha)$ be the least s so that $\Phi_j^{Y\restriction s}(\beta)\downarrow$ for all $\beta < \alpha$. Note that, as long as Φ_j^Y is total, f_j^Y is continuous, and therefore has a club C_j^Y of fixed points. $C_e^X \cap C_i^{\Phi_e^X}$ is therefore a club.

Let $C = \bigcap \left(C_e^X \cap C_i^{\Phi_e^X} \right)$, where the intersection is taken over all pairs (e, i) meeting the aforementioned condition. Then C is a countable intersection of clubs, and hence is itself a club. Let $\alpha \in C$ be greater than the length of the anchor of η_σ. Then $X \restriction \alpha$ is σ-coherent.

It must therefore be the case that for some such pair (e, i), either Φ_e^X is not total or $\Phi_i^{\Phi_e^X} \neq X$. If the latter holds because there is some α with $\Phi_i^{\Phi_e^X} \restriction \alpha \downarrow \neq X \restriction \alpha$, then the first strategy corresponding to (e, i) should have used an initial segment of X to move to a Halt outcome. So either Φ_e^X is not total or $\Phi_i^{\Phi_e^X}$ is not total. Then there is an initial segment θ of X sufficiently long that this is realized; that is, for some x, either $\Phi_e^{\theta'}(x)\uparrow$ for every $\theta \prec \theta' \prec X$ or $\Phi_i^{\Phi_e^{\theta'}}(x)\uparrow$ for every $\theta \prec \theta' \prec X$.

At some stage, the first (e, i) strategy below σ will notice this θ and x, and switch to a divergence outcome. Call this strategy η_*. If σ is to be visited unboundedly often, then it must be that η_* eventually returns to its Kurepa outcome, because its assertion was violated by some extension of θ. But by construction, the bookmark

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η_σ now selects, which will be an initial segment of X, has the same convergence—contradicting our hypothesis on X. ⊣

CLAIM 6.12. *$F = \liminf_s F_s$ is a Δ^0_3 total embedding $K \to T$.*

PROOF. That F is Δ^0_3 is clear from definition.

To see that F is total, observe that it is sufficient to show that for every $\tau \in K$, some strategy η_σ for $\text{even}(\sigma) = \tau$ is eventually initialized. But this is evident by Claim 6.11.

Finally, to see that F is an embedding, observe that the successors of any given strategy always take anchors incomparable with one another. ⊣

CLAIM 6.13. *If $\sigma \in \delta$ has odd length, then $\sigma \frown i \in \delta$ for exactly one i, and the following holds, with τ the liminf of the anchor points of η_σ.*

(a) *If $i = \text{Halt}$, then either $\Phi^\tau_e \notin T_j$ or $\Phi^{\Phi^\tau_e}_i|\tau$,*
(b) *If $i = \langle \text{Div}, \alpha \rangle$, then there is an x so that for every $\rho \succ \tau$, $\rho \in T$, either $\Phi^\rho_e(x) \uparrow$ or $\Phi^{\Phi^\rho_e}_i(x) \uparrow$, and*
(c) *If $i = \text{Kurepa}$, then $F \circ \Phi^\cdot_e$ embeds the part of K above $\text{even}(\sigma)$ into T_j.*

PROOF. By construction, there are three possible cases for the limit behavior of η_σ, for $\sigma \in \delta$.

(1) η_σ eventually enters outcome Halt and does not leave.
(2) η_σ eventually enters outcome $\langle \text{Div}, \alpha \rangle$ for some α and does not leave.
(3) η_σ visits outcome Kurepa unboundedly often.

Observe that this list is complete, because whenever a $\langle \text{Div}, \alpha \rangle$ outcome is abandoned, η_σ returns to outcome Kurepa (however briefly).

In Case (1), η_σ moved to outcome Halt because it encountered a string ρ with either $\Phi^\rho_e \notin U_j$ or $\Phi^{\Phi^\rho_e}_i|\rho$, and set $\tau = \rho$. There is no occurrence that could cause η_σ to give up this definition, so η_σ falls into Case (a).

In Case (2), when η_σ entered outcome $\langle \text{Div}, \alpha \rangle$ it made one of the following assertions, for some choice of θ, α, and β:

(i) For every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi^{\theta'}_e(\alpha) \uparrow$.

(ii) For every $\theta' \succ \theta$ with $\theta' \in T$, $\Phi^{\Phi^{\theta'}_e}_i(\beta) \uparrow$.

It also set $\tau = \theta$. If η_σ never leaves this outcome, then whichever assertion it made was never violated; thus Case (b) holds.

In Case (3), observe that by Claim 6.11, every even-level strategy on δ eventually initializes its successors. Odd-level strategies have no particular constraints on initializing their successors, so odd-level strategies always initialize their successors. Limit-level strategies are initialized when their predecessors are all initialized. So sufficiently many strategies will be initialized that the part of K above $\text{even}(\sigma)$ embeds into the part of δ above σ. Note also that for any even-level $\tau \succ \sigma$, η_τ chooses the anchor points of its successors to be τ-coherent and hence σ-coherent; in particular, if θ is one of these anchor points, then $\Phi^\theta_e \upharpoonright |\theta| \downarrow$ and $\Phi^{\Phi^\theta_e}_i \upharpoonright |\theta| \downarrow = \theta$. This latter clause entails that $\Phi^{\theta_0}_e|\Phi^{\theta_1}_e$ whenever $\theta_0|\theta_1$ are such strings, so $F \circ \Phi^\cdot_e$ is an embedding. ⊣

CLAIM 6.14. *For each $\sigma \in K$, there is exactly one $\tau \in \delta$ with $\text{even}(\tau) = \sigma$.*

PROOF. By Claim 6.13 and induction on $|\sigma|$. ⊣

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CLAIM 6.15. *Every requirement is eventually satisfied, and every path through T eventually passes through a satisfaction point of every requirement.*

PROOF. The latter part of the claim is clear from the construction: if X eventually avoids the anchor points of strategies that are visited unboundedly often, then X must be contained inside an Aronszajn tree built to bookmark the position of an inactive strategy; but then X cannot be a path. Moreover, the anchor point of a strategy that succeeds at satisfying its requirement will be a satisfaction point of that requirement.

By Claim 6.13, the requirements are satisfied. ⊣

CLAIM 6.16. *For any $X \in [K]$, $F[X]$ has Kurepa degree.*

PROOF. Fix $X \in [K]$, U a computable tree of countable width, and $F[X] \equiv_T Y \in [U]$. Take e, i such that $\Phi_e^{F[X]} = Y$ and $\Phi_i^Y = F[X]$. Fix j so that $U_j = U$.

By Claim 6.15, $F[X]$ passes through a satisfaction point τ of the requirement $R_{e,i,j}$. One of the following holds:

(a) For every $Z \succ \tau$ in $[T]$, there is a $W \in [U_j]$ so that $\Phi_e^Z = W$ and $\Phi_i^W = Z$;
(b) $\Phi_e^{F[X]}$ is not total;
(c) $\Phi_e^{F[X]} \notin [U_j]$; or
(d) $\Phi_i^Y \neq F[X]$.

Cases (b) through (d) directly contradict our hypothesis, so it must be that τ falls under Case (a). But F embeds the part of K above some initial segment of X into the part of T above τ, and K was a perfect Kurepa tree, so that part of K has ω_2-many branches. Therefore there are ω_2-many Z satisfying the hypothesis of (a), and hence $U = U_j$ must have ω_2-many branches. ⊣

CLAIM 6.17. *The image under F of $[K]$ is exactly $[T]$.*

PROOF. This is evident from Claim 6.15 and the fact that every path in T must pass through a satisfaction point of every (and hence unboundedly many) requirement. ⊣

By Claims 6.12, 6.16, and 6.17, T and F have the desired properties. This completes the proof of Theorem 6.9. ⊣

The construction above relied very little on the nature of K, and in fact could be used to show the following instead.

THEOREM 6.18. *Let K be a computable tree of countable width. Then there exists, uniformly in an index for K, a Δ^0_3 functional Γ and a computable tree T of countable width such that the following holds*:

(i) *For each $X \in [K]$, $\Gamma^X \in [T]$;*
(ii) *For each $Y \in [T]$, there is exactly one $X \in [K]$ so that $Y = \Gamma^X$; and*
(iii) *For every $Y \in [T]$ and every computable tree U, if there exists $Y \equiv_T Z \in [U]$ then there exists a $\sigma \in K$ so that $K(\sigma) = \{\tau \in K : \tau \succcurlyeq \sigma\}$ embeds in U. Furthermore, this embedding is Δ^0_3 uniformly in an index for U and the equivalence between Y and Z.*

As an example, we get the following:

COROLLARY 6.19. *There exists a thin Π^0_1 class of degrees $\mathbf{d}$ so that if $\mathbf{d}$ appears as a path in a computable tree T, then T has uncountably many Δ^0_3 branches.*

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PROOF. Take K to be any computable tree of countable width with uncountably many computable paths, and apply the theorem. ⊣

Theorem 6.9 can be combined with Theorem 5.11 to produce a Kurepa tree in which every path has minimal Turing degree.

THEOREM 6.20. *Let K be a computable perfect Kurepa tree. Then there exists a computable tree T of countable width together with a Δ^0_3 embedding $f : K \to T$ (uniformly in an index for K) so that the following holds*:

- *Every branch of T is the f-image of a branch of K, and*
- *For each $X \in [K]$, $f[X] \in T$ is of minimal Turing degree.*

An immediate corollary is the following:

COROLLARY 6.21. *There is a thin Π^0_1-class of cardinality $\aleph_2$ consisting entirely of sets of minimal Turing degree.*

PROOF OF THEOREM 6.20. We mimic the previous proofs involving Kurepa trees and minimal degrees, respectively. The argument will be by a modified form of unbounded injury, merged with a worker argument.

Let A be a fixed computable Aronszajn tree.

Let $\Lambda = \{S\} \cup \omega_1$. Let $\mathscr{T} = \Lambda^{<\omega_1}$. To each string $\sigma \in \mathscr{T}$ we attach a *worker* η_σ. At any stage s, each worker which has been initialized has one or two outcomes from Λ; let $\delta(s)$ be the maximal partial embedding $2^{<\omega_1} \to \mathscr{T}$ so that $\delta(s)(\sigma \frown 0)$ and $\delta(s)(\sigma \frown 1)$ are the leftmost and rightmost immediate extensions of $\delta(s)(\sigma)$ by its stage-s outcomes.

Each worker maintains one or more *scaffolds*, each one corresponding to an outcome of that worker. The *active* scaffold at stage s is the one corresponding to the worker's outcome at stage s.

DEFINITION 6.22. A *scaffold* is a partial function $f : 2^{<\omega_1} \to 2^{<\omega_1}$ together with a set $S \subseteq 2^{<\omega_1}$ such that the following conditions hold.

(i) $\mathrm{dom}(f) \subseteq S$;
(ii) For $\sigma, \tau \in \mathrm{dom}(f)$, $f(\sigma) \preccurlyeq f(\tau)$ iff $\sigma \preccurlyeq \tau$;
(iii) If $\sigma \frown i \in \mathrm{dom}(f)$, then $f(\sigma \frown i)(|f(\sigma)|) = i$; and
(iv) If $\sigma_0 \prec \sigma_1 \prec \cdots$ is a sequence in $\mathrm{dom}(f)$ and $\sigma = \lim_i \sigma_i \in \mathrm{dom}(f)$, then $f(\sigma) = \lim_i f(\sigma_i)$.

We denote the scaffold by (S, f).

For scaffolds (S_0, f_0) and (S_1, f_1), say (S_0, f_0) is a *subscaffold* of (S_1, f_1) if $\mathrm{ran}(f_0) \subseteq \mathrm{ran}(f_1)$. Observe that there is no restriction on the relationship between S_0 and S_1.

It will be the case that every scaffold maintained by η_σ is a subscaffold of η_τ for each $\tau \prec \sigma$. The root worker $\eta_{\langle\rangle}$ maintains only one scaffold, and the corresponding function is the identity; the underlying function of this scaffold will be the final tree T.

At any stage s, a worker η_σ may issue a *request* for any $\theta \in 2^{<\omega_1}$ every initial segment of which is currently in the range of the active scaffold of every η_τ for $\tau \prec \sigma$. Provided that $|\theta| \geq s$, this request will be *honored* by every such η_τ: η_τ will add an assignment to its active scaffold (possibly adding to the domain of the scaffold) so

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that θ will be in the range, and then add to the domain of the scaffold so that the preimage of θ has a copy of A above it.

Successor workers pursue two sorts of requirements:

D_e: For any $X \in [T]$, $\Phi_e \neq X$.
M_e: For any $X \in [T]$, one of the following holds:
(i) $(\exists x)\Phi_e^X(x)\uparrow$,
(ii) Φ_e^X is computable, or
(iii) $\Phi_e^X \geq_T X$.

Divide the countable successor ordinals into three computable unbounded sets; name these $\mathscr{D}$, $\mathscr{M}$, and $\mathscr{K}$. Workers η_σ for $|\sigma|$ the eth member of $\mathscr{D}$ pursue the diagonalization requirement D_e; workers η_σ for $|\sigma|$ the eth member of $\mathscr{M}$ pursue the minimality requirement M_e. Workers η_σ for $|\sigma|$ the eth member of $\mathscr{K}$ build an embedding from K into the tree. Workers $\eta_{\langle\rangle}$ and η_σ for σ of limit length behave differently and pursue no requirement.

Given σ, let $k(\sigma)$ be the string obtained from σ by deleting all entries with indices not in $\mathscr{K}$; that is, by considering only outcomes of $\mathscr{K}$ workers.

For ease of notation, when σ is a string of successor length, denote by σ^- its immediate predecessor.

For any worker η_σ, let (S^σ, f^σ) be the currently active scaffold.

6.2.11. Strategy for limit workers. $\eta_{\langle\rangle}$ does nothing except respond to requests; we therefore concentrate on η_σ for $|\sigma|$ of limit length.

A limit worker η_σ is initialized at stage s if and only if it isn't yet initialized, every direct predecessor is active, and $k(\sigma) \in K$.

When initialized at stage s, η_σ takes S^σ to be the usual computable Aronszajn tree A and sets $f^\sigma(\langle\rangle)$ to be the limit of $f^\tau(\langle\rangle)$ for $\tau \prec \sigma$. It enters outcome 0, and remains there permanently.

At later stages, η_σ works to define f^σ on progressively higher and higher levels of S^σ. Limit levels are straightforward: η_σ simply defines the function as necessary to preserve continuity, requesting any strings not already present in the ranges of the previous scaffolds. At successor levels, suppose η_σ has defined $f^\sigma(\tau)$ and needs to define $f^\sigma(\tau \frown i)$. For each $\rho \prec \sigma$, let $\theta(\rho)$ be the element in the domain of f^ρ so that $f^\rho(\theta(\rho)) = f^\sigma(\tau)$. Let $f^\sigma(\tau \frown i) = \lim f^\rho(\theta(\rho) \frown i)$, requesting the string if necessary. This limit exists and has the property that every f^ρ has a sequence cofinal along it; so the request is a valid one.

6.2.12. Strategy for diagonalization requirements. Let η_σ be a successor worker pursuing requirement D_e. When initialized at stage s, η_σ looks for θ_0 and θ_1 in the range of f^{σ^-} with length at least s differing at some position $\alpha < s$. η_σ sets $S^\sigma = A$, puts $f^\sigma(\langle\rangle) = \theta_0$, and enters outcome 0.

If at some stage $t > s$, $\Phi_{e,t}(\alpha)\downarrow = \theta_0(\alpha)$, then η_σ requires attention. If it receives attention, it deletes its existing scaffold, sets $S^\sigma = A$ and $f^\sigma(\langle\rangle) = \theta_1$, and enters outcome 1.

At any successor stage, η_σ considers each leaf node τ of $\mathrm{dom}(f^\sigma)$. For each such τ, by induction there exists $\theta \in \mathrm{dom}(f^{\sigma^-})$ with $f^{\sigma^-}(\theta) = f^\sigma(\tau)$; if $\theta \frown 0, \theta \frown 1 \in \mathrm{dom}(f^{\sigma^-})$, then η_σ takes $f^\sigma(\tau \frown i) = f^{\sigma^-}(\theta \frown i)$ for each i. Otherwise, it simply waits until the necessary assignments have been made.

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At a limit stage, η_σ considers each limit node $\tau \in S^\sigma$ so that f^σ has been defined on every initial segment of τ. The partial function f^σ induces an obvious choice of $f^\sigma(\tau)$; if this string is not already in the range of f^{σ^-}, then η_σ requests it. Regardless, it takes this as the assignment of $f^\sigma(\tau)$.

6.2.13. Strategy for $\mathscr{K}$ workers. Let η_σ be a successor worker with $|\sigma|$ the eth member of $\mathscr{K}$. Let τ be the length-e subsequence of σ given by $k(\sigma)$. For each $i \in \{0, 1\}$ so that $\tau \frown i \in K$, η_σ activates outcome i. It constructs scaffolds S_0 and S_1 (the left and right subscaffolds of its parent scaffold, respectively) and assigns S_i to outcome i.

6.2.14. Strategy for minimality requirements. Let η_σ be a successor worker pursuing requirement M_e. When initialized at stage s, η_σ sets $S^\sigma = A$ and puts $f^\sigma(\langle\rangle)$ some element of $\mathrm{ran}(f^{\sigma^-})$ with length at least s (if none exists, η_σ waits until it does). This scaffold is designated the *splitting scaffold*; it will be the active scaffold whenever η_σ is in outcome S. η_σ begins in outcome S.

η_σ will maintain the condition that if (S, f) is its splitting scaffold and $\tau_0, \tau_1 \in S$ are incomparable, then $\Phi_e^{f(\tau_0)}$ and $\Phi_e^{f(\tau_1)}$ are incomparable. Call this condition (*).

While still in outcome S, η_σ attempts to extend f^σ while preserving (*). At a limit stage, it simply fills in any limit nodes in S^σ, requesting whatever is necessary. At a successor stage, η_σ will attempt to extend each leaf node of $\mathrm{dom}(f^\sigma)$ in both directions. For each leaf $\tau \in \mathrm{dom}(f^\sigma)$, if there exist $\rho_0, \rho_1 \in \mathrm{ran}(f^{\sigma^-})$ with length at least s and $x < s$ so that $\rho_i \succcurlyeq f^\sigma(\tau) \frown i$ and $\Phi_e^{\rho_0}(x) \downarrow \neq \Phi_e^{\rho_1}(x) \downarrow$, then η_σ takes $f^\sigma(\tau \frown i) = \rho_i$. If none exists, then let $\epsilon = f^\sigma(\tau)$. η_σ switches to a new ordinal outcome and associates with it a new scaffold as follows: $f^\sigma(\langle\rangle) = \theta$, where θ is the first extension of $\epsilon \frown 0$ in the range of f^{σ^-} (as usual, if none exists, wait until it does); S^σ is the translation of the part of S^{σ^-} extending that point.

While in an ordinal outcome, η_σ copies the scaffold of η_{σ^-}, just like the diagonalization workers do.

If a splitting pair is found within this new scaffold (that is, $\rho_0, \rho_1 \in S^\sigma$ with $\Phi_e^{\rho_0}$ and $\Phi_e^{\rho_1}$ incomparable) then η_σ again switches to a new ordinal outcome and replaces the current scaffold with a new scaffold, defined the same way as before but rooting at an extension of $\epsilon \frown 1$.

If a spitting pair is found within this scaffold as well, then between the two sets of splitting pairs is a pair appropriate for extending the splitting scaffold at τ; η_σ returns to the S outcome, re-activating the splitting scaffold, and adds those elements.

To ensure that the search for strings of sufficient length is successful (that is, that higher priority scaffolds are not unexpectedly short) at every stage we allow the active workers to act *in order*, beginning from the root. Responses to requests are immediate, but propagate in the same manner.

This completes the construction.

Claim 6.23. *For any α, only countably many requests are made for strings of length $<\alpha$, and none are requested past stage α.*

Proof. Any work after stage α will, by construction, place its root at a string of length at least α. Again by construction, no worker ever makes requests that do not extend its root; therefore, only the workers initialized before stage α will ever request strings of length less than α.

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Let η be such a worker, and suppose that it requests a limit string θ at stage s. In every case, this request is made as soon as the initial segments are available; that is, it must be that there was a sequence $s_0 < s_1 < \cdots$ cofinal in s at which η assigned strings in the domain of its scaffold to initial segments of θ. However, by construction a worker can only make stage-t assignments of length at least t; so these initial segments of θ have lengths at least $s_0, s_1, \ldots$. Thus θ has length at least s.

Clearly no worker requests uncountably many strings in a single stage; therefore no individual worker requests uncountably many strings of length $<\alpha$. Since only countably many workers can request these strings at all, only countably many such requests are made. ⊣

Claim 6.24. *The tree T is computable and of countable width.*

Proof. This is an immediate consequence of Claim 6.23. ⊣

Claim 6.25. *All requests are honored.*

Proof. Again, this is an immediate consequence of Claim 6.23; any request made at stage s for a string with length at least s is honored, and by the claim no requests not satisfying this condition are made. ⊣

Claim 6.26. *The set of strings in $\mathscr{T}$ that lie on the execution tree unboundedly often forms a tree.*

Proof. Observe that limit workers (including the root worker) never change outcomes; diagonalization workers change outcome at most once; and while minimalization workers may change outcome unboundedly often, the only outcome they may visit unboundedly often is S. So every non-$\mathscr{K}$ worker visits exactly one outcome unboundedly often; since the $\mathscr{K}$ workers always activate a fixed set of outcomes simultaneously, the inductive argument is straightforward. ⊣

We call this the *true tree*, and we call the workers along this tree *true workers*; their outcomes that lie along the true tree are *true outcomes*.

Note that the set of strings $f^\sigma(\langle\rangle)$ evaluated for σ with $k(\sigma) \prec Y$, during stages at which η_σ has its true outcome exactly defines a path through the tree T; we call this path $f[Y]$.

Claim 6.27. *The images $f[Y]$ of paths Y through K are the only paths through T.*

Proof. Suppose for contradiction that Z is a path through T and is not $f[Y]$ for any path Y through K. Z cannot be "native"—that is, it must be obtained through uncountably many requests, because otherwise it would be part of an Aronszajn tree.

No worker makes requests incompatible with its root; it therefore cannot be that the requests for initial segments of Z are unbounded along any path in the true tree. However, note that by construction, whenever $k(\sigma)$ and $k(\tau)$ are incompatible, the workers σ and τ maintain scaffolds consisting of strings incomparable with one another; in particular, for every $\alpha < \omega_1$ there is at most one σ of minimal length so that workers extending σ may add elements to the tree extending $Z \upharpoonright \alpha$. Furthermore, these σ (provided they exist) must be pairwise comparable. We therefore define the *true path for Z* to be the subset of the true tree consisting of the

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initial segments of these σ. Observe that this is certainly linearly ordered, but may have strictly countable length.

Since Z is not $f[Y]$ for any Y, it cannot be that requests for initial segments of Z are unbounded along the true path for Z (otherwise Z would be the image of the corresponding path in K). There is therefore a minimal worker η along the true path for Z so that only countably many requests for initial segments of Z are made above the true outcome of η. Note that η is not a limit worker or the root worker, because neither class of worker changes outcomes. Note also that η is not a diagonalization worker; at some stage, a worker pursuing a diagonalization requirement will enter its true outcome and never leave, so only countably many requests for anything can be made above the other outcome. So η is a minimalization worker. Furthermore, η's true outcome must be S—otherwise, the same argument holds as for the diagonalization case.

Since η's true outcome is not one of the ordinal outcomes, it cannot be that uncountably many requests for initial segments of Z are made above only one of them; so it must be that an uncountable sequence of ordinal outcomes have at least one request each. The scaffolds for these outcomes must be rooted at points along Z; but all of them come from points at which the attempt to split in the splitting outcome temporarily failed. So all of these initial segments of Z are in the splitting scaffold—which means uncountably many requests for initial segments of Z must have been made above the S outcome, contradicting our assumption on η. ⊣

Claim 6.28. *Each image $f[Y]$ of paths Y through K is not computable.*

Proof. Suppose $\Phi_e = f[Y]$. Let η be the (unique) worker along the true path for Y pursuing requirement D_e. When initialized, η selected some pair of strings and an α at which they differed. At some stage t, $\Phi_{e,t}(\alpha)\downarrow$; at that stage, η switched outcomes if necessary and directed construction along the string of the pair that did not agree with Φ_e at α. This is a contradiction. ⊣

Claim 6.29. *Each image $f[Y]$ of paths Y through K has minimal Turing degree.*

Proof. Let $X = f[Y]$.

Suppose that Φ_e^X is total but not computable. Let η be the (unique) worker along the true path for Y pursuing requirement M_e. Observe that if the true outcome of η were an ordinal, Φ_e^X would be either partial or computable, because a split would never be found; $\Phi_e^X(x)$ could be computed by taking the first computation $\Phi_e^\tau(x)$ to converge for τ in the range of the scaffold of η. So the true outcome of η is S, and therefore in the limit η constructs a scaffold (U, f) so that if σ and τ are incomparable members of U then $\Phi_e^{f(\sigma)}$ and $\Phi_e^{f(\tau)}$ disagree somewhere. But then given Φ_e^X we can determine a unique branch Y of U so that $\Phi_e^{f(Y)} = \Phi_e^X$; this branch must be the preimage of Y. Thus $\Phi_e^X \geq_T X$.

So for every e we have that Φ_e^X is either partial, computable, or above X; since we already have that $X >_T \emptyset$, X is of minimal Turing degree. ⊣

This completes the proof of Theorem 6.20. ⊣

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§7. **Future directions.** The results of this chapter leave one significant question unanswered.

QUESTION 7.1. *Is there a degree-theoretic characterization of* $\mathcal{P}_{\text{thin}}$? *Failing that, is there a characterization of* $\mathcal{P}_{\text{thin}}$ *that is simpler than the definition in terms of computable trees of countable width*?

An appealing approach might be a positive answer to the following:

QUESTION 7.2. *Is there a natural extension of the notion of strong club-approximation that characterizes exactly the members of* $\mathcal{P}_{\text{thin}}$?

Such an extension would have to be somehow intermediate between strong and weak club-approximation, but it is not evident what that intermediate might be.

The work in Section 6 also suggests a sweeping question:

QUESTION 7.3. *For which classes of degrees* $\mathcal{C}$ *is there a computable tree of countable width whose branches realize exactly the members of* $\mathcal{C}$?

Certainly such a $\mathcal{C}$ must include a Δ^1_1 member, but the results of Section 2 demonstrate that even if every member of $\mathcal{C}$ is Δ^0_2 there may be no corresponding tree. At the same time, the results of Section 6 show that $\mathcal{C}$ need not consist entirely or even "mostly" of Δ^1_1 degrees; Theorem 6.20 in particular demonstrates that $\mathcal{C}$ may be large with very little in its lower cone.

Of further interest to computable structure theorists will be the relationship to the notion of α-true stages introduced by Montalbán [17] as an elaboration of the priority system introduced previously by Ash [1]: the results of Sections 3 and 4 illustrate that there are strong club-approximations to sets of degree $\mathbf{0}^{(\alpha)}$ for every hyperarithmetic α, which serve the same function as the approximations ∇^ξ in the notation of [17]. As a result, the development of Montalbán's α-true relations and α-true stages will be considerably simpler in the setting of ω_1. Furthermore, since these strong club-approximations extend considerably further than the hyperarithmetic hierarchy, it is reasonable to suppose that the techniques of [17] could be adapted to perform priority arguments of very large degree.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN-MADISON
480 LINCOLN DRIVE
MADISON, WI 53706, USA
E-mail: rwjohnston@math.wisc.edu

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