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# NOTES AND PROBLEMS

# PERMANENT-TRANSITORY DECOMPOSITIONS UNDER WEAK EXOGENEITY

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This note shows that the orthogonal shocks obtained by imposing a recursive structure on the contemporaneous impacts of the errors in a vector-error correction model (VECM) are related to the orthogonal permanent-transitory shocks in a Gonzalo and Ng (2001, *Journal of Economic Dynamics and Control* 25, 1527–1546) decomposition provided some of the variables in the system are weakly exogenous. Specifically, when there are as many weakly exogenous variables as common trends and the weakly exogenous variables are ordered first, the orthogonal shocks obtained directly from the VECM constitute a permanent-transitory decomposition, and the permanent shocks are identical to those in the Gonzalo and Ng decomposition.

## 1. MOTIVATION AND RESULTS

In the context of a cointegrated system, Gonzalo and Ng (2001) identify the underlying structural shocks as either permanent or transitory. In their procedure, the errors from a vector-error correction model (VECM) are transformed into a set of shocks that have permanent effects on the levels of the series and those that have only transitory effects. The permanent-transitory shocks are made mutually orthogonal by imposing a recursive structure on their contemporaneous impacts as in Sims (1980).

This note shows that when some of the variables in the system are weakly exogenous, the orthogonal shocks obtained directly from the VECM are related

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to the orthogonal permanent-transitory shocks obtained from the Gonzalo and Ng (2001) procedure. Specifically, when (i) there are as many weakly exogenous variables as common trends in the system and (ii) the weakly exogenous variables are ordered first, the orthogonal shocks obtained by imposing a contemporaneous recursive structure on the errors directly from the VECM constitute a permanent-transitory decomposition. Further, the orthogonal shocks with permanent effects are identical to those in the Gonzalo and Ng (2001) decomposition, and, in the case of only one cointegrating relation, all of the orthogonal shocks are identical to those in the Gonzalo and Ng (2001) decomposition. When the number of weakly exogenous variables is less than the number of common trends, the orthogonal shocks obtained directly from the VECM do not constitute a permanent-transitory decomposition, though some of the shocks are identical to those with permanent effects from the Gonzalo and Ng (2001) procedure.

In related literature, Ribba (1997) considered the case of a two-variable system with one cointegrating relation and one weakly exogenous variable. He shows that the orthogonal shocks obtained by imposing a contemporaneous recursive structure on the errors directly from the VECM form a permanent-transitory decomposition when the weakly exogenous variable is ordered first. Here the orthogonal shocks are identical to those from the Gonzalo and Ng (2001) decomposition. As a further example, Lettau and Ludvigson (2004) find evidence for one long-run relationship among U.S. consumption, labor income, and household wealth and also that consumption and labor income are weakly exogenous. Thus, in their system, there are two common stochastic trends, two weakly exogenous variables, and one cointegrating relationship. For consumption and labor income ordered before household wealth, they identify the orthogonal shocks under both weak exogeneity restrictions using the Gonzalo and Ng (2001) procedure. These orthogonal shocks are identical to those obtained by imposing a recursive structure on the contemporaneous impacts of the errors directly from the VECM.

## 2. PROOFS AND DISCUSSION

Let $X_t$ be a $p$-dimensional vector of $I(1)$ series and let the number of cointegrating relations among them be $r$, where $r < p$, so that the series share $m = p - r$ common trends. In that case, $X_t$ has a VECM representation of infinite order that in practice is approximated by the finite-order VECM

$$\pi(L)\Delta X_t = \alpha\beta' X_{t-1} + e_t, \tag{1}$$

where $\pi(L) = I_p - \pi_1 L - \pi_2 L^2 - \cdots - \pi_{q-1} L^{q-1}$, $L$ is the lag operator, $\Delta$ is the first difference operator, and $e_t$ satisfies $E(e_t) = 0$, $E(e_t e_t') = \Omega$, and $E(e_t e_s') = 0$, $t \neq s$. Both $\alpha$ and $\beta$ are $(p \times r)$ matrices where the columns of $\beta$ are the coefficients in the cointegrating relations and the rows of $\alpha$ are the

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loadings on the error-correction terms $\beta' X_{t-1}$, respectively. In (1), under the condition that $e_t$ is a vector of independent Gaussian innovations, variable $x_{i,t}$ can be shown to be weakly exogenous with respect to the matrix of cointegrating vectors if the $r$ error-correction terms do not appear in the equation for $\Delta x_{i,t}$ so that $\alpha_{ik} = 0$ for $k = 1, \ldots, r$; see Johansen (1992, Thm. 1, p. 392) and Urbain (1992, Prop. 1, p. 193).

The VECM can be inverted to obtain the moving average representation for $\Delta X_t$,

$$\Delta X_t = C(L)e_t, \tag{2}$$

where $C(L) = I_p + C_1 L + C_2 L^2 + \cdots$. The long-run impacts of the reduced-form shocks on the levels of the series in $X_t$ are given by $C(1)$, which is a $(p \times p)$ matrix of rank $m$. Specifically, as shown in Johansen (1991),

$$C(1) = \beta_\perp \gamma \alpha_\perp', \tag{3}$$

where $\alpha_\perp$ and $\beta_\perp$ are $(p \times m)$ matrices of rank $m$ such that $\alpha' \alpha_\perp = 0$ and $\beta' \beta_\perp = 0$ and the $(m \times m)$ full rank matrix $\gamma$ is given by $(\alpha_\perp' \pi(1) \beta_\perp)^{-1}$.

Gonzalo and Ng (2001) define the $(p \times 1)$ vector $u_t$, whose first $m$ elements are the permanent shocks and remaining $r$ elements are the transitory shocks, as $u_t = Ge_t$ where

$$G = \begin{bmatrix} \alpha_\perp' \\ \beta' \end{bmatrix} \tag{4}$$

and $E(u_t u_t') = G\Omega G'$. The $(p \times p)$ matrix $G$ is assumed to be of full rank.[1] The long-run impact matrix of the permanent-transitory shocks $u_t$ is $C(1)G^{-1}$. It is easy to show that the last $r$ column vectors of this long-run impact matrix are all zero as required for a permanent-transitory decomposition after noting that

$$G^{-1} = [\beta_\perp (\alpha_\perp' \beta_\perp)^{-1} \quad \alpha(\beta' \alpha)^{-1}]. \tag{5}$$

This expression for $G^{-1}$ appears in Gonzalo and Granger (1995) and also in Levtchenkova, Pagan, and Robertson (1996).

Gonzalo and Ng (2001) apply the following transformation to orthogonalize the permanent-transitory shocks:

$$v_t = H^{-1} u_t = H^{-1} G e_t, \tag{6}$$

where $H$ is the unique lower triangular matrix such that

$$HH' = G\Omega G' \tag{7}$$

and by construction $E(v_t v_t') = I_p$. The long-run impact matrix of the orthogonal permanent-transitory shocks is $C(1)G^{-1}H$. The permanent-transitory decomposition is preserved because postmultiplication of $C(1)G^{-1}$ by the lower

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triangular matrix $H$ results in the matrix where the last $r$ column vectors are all zero.

The orthogonal shocks obtained from the Gonzalo and Ng (2001) procedure and those obtained directly from the VECM are related when some of the variables in the system are weakly exogenous, as the following proposition and corollaries show.

PROPOSITION. *Let $X_t$ be a p-dimensional vector of $I(1)$ variables with r cointegrating vectors and $m = p - r$ common trends. Suppose there are m weakly exogenous variables with respect to the cointegrating vectors and that they are ordered first in the vector $X_t$. Then the orthogonal shocks obtained directly from the VECM by imposing a recursive structure on the contemporaneous impacts of the errors constitute a permanent-transitory decomposition where the permanent shocks are identical to those from the Gonzalo and Ng (2001) permanent-transitory decomposition.*

Proof. Partition the $(p \times r)$ matrix $\alpha$ as

$$\alpha = \begin{bmatrix} \alpha_m \\ \alpha_r \end{bmatrix},$$

where $\alpha_m$ and $\alpha_r$ are, respectively, the first $m$ and last $r$ $(= p - m)$ rows of $\alpha$. Using Gaussian elimination with complete pivoting, the rows of $\alpha$ can be uniquely rearranged such that the first $r$ rows are independent. This is achieved by premultiplying $\alpha$ by the unique $(p \times p)$ matrix

$$F = \begin{bmatrix} 0_{rm} & I_{rr} \\ I_{mm} & 0_{mr} \end{bmatrix},$$

where $I_{rr}$ and $I_{mm}$ are $(r \times r)$ and $(m \times m)$ identity matrices and $0_{rm}$ and $0_{mr}$ are $(r \times m)$ and $(m \times r)$ null matrices, respectively. Then

$$F\alpha = \begin{bmatrix} \alpha_r \\ \alpha_m \end{bmatrix},$$

where $\alpha_r$ is an $(r \times r)$ matrix of full rank. The orthogonal complement of $\alpha$ is chosen as $\alpha_\perp' = [-\alpha_m \alpha_r^{-1} \quad I_{mm}]\, F$. Substituting for $F$, this becomes

$$\alpha_\perp' = [I_{mm} \quad -\alpha_m \alpha_r^{-1}]. \tag{8}$$

When the weakly exogenous variables are ordered first and are equal in number to $m$, $\alpha_m$ is an $(m \times r)$ null matrix. Thus (8) becomes

$$\alpha_\perp' = [I_{mm} \quad 0_{mr}]. \tag{9}$$

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It then follows from (9) that (3) may be written as

$$C(1) = [\beta_\perp \gamma \quad 0_{pr}]. \tag{10}$$

Thus, the reduced-form errors constitute a permanent-transitory decomposition. Following Sims (1980), orthogonalize the errors from the VECM by imposing a recursive structure on their contemporaneous impacts by applying the transformation

$$v_t = A^{-1} e_t, \tag{11}$$

where $E(v_t v_t') = I_p$ and $A$ is the unique lower triangular matrix such that $AA' = \Omega$. From (2), (10), and (11), the long-run impact matrix of the orthogonalized shocks is

$$C(1)A = [\beta_\perp \gamma \quad 0_{pr}]A. \tag{12}$$

Because $A$ is lower triangular, the orthogonal shocks constitute a permanent-transitory decomposition because the last $r$ column vectors of $C(1)A$ are all zero.

Now partition the matrices $H$, $A$, and $G$ as follows:

$$H = \begin{bmatrix} H_{mm} & 0_{mr} \\ H_{rm} & H_{rr} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{mm} & 0_{mr} \\ A_{rm} & A_{rr} \end{bmatrix}, \qquad G = \begin{bmatrix} I_{mm} & 0_{mr} \\ \beta_{rm}' & \beta_{rr}' \end{bmatrix},$$

where $H_{mm}$, $H_{rr}$, $A_{mm}$, and $A_{rr}$ are lower triangular, because $H$ and $A$ are lower triangular, and where $G$ is obtained from (4) and (9). Partitioning $\Omega$ conformably with $H$ and $A$, (7) shows that

$$H_{mm} H_{mm}' = \Omega_{mm} \tag{13}$$

for the case under consideration. The lower triangular matrix $H_{mm}$ is the unique Cholesky factor of $\Omega_{mm}$. But by construction

$$A_{mm} A_{mm}' = \Omega_{mm}, \tag{14}$$

so that the lower triangular matrix $A_{mm}$ is also the unique Cholesky factor of $\Omega_{mm}$. Thus $H_{mm}^{-1}$ and $A_{mm}^{-1}$ are equal to the inverse of the unique Cholesky factor of $\Omega_{mm}$ and therefore equal to each other. Using the formula for the inverse of a partitioned matrix (Dhrymes, 2000, Prop. 2.31, p. 43), the first $m$ orthogonal shocks are $v_{mt} = A_{mm}^{-1} e_{mt}$, where $e_{mt}$ is the vector of the first $m$ elements of $e_t$ and, for the Gonzalo and Ng (2001) decomposition, the permanent shocks are $v_{mt} = H_{mm}^{-1} e_{mt}$. Because $H_{mm}^{-1} = A_{mm}^{-1}$, it immediately follows that $v_{mt} = H_{mm}^{-1} e_{mt} = A_{mm}^{-1} e_{mt}$. ■

COROLLARY 1. *In addition, when there is only one cointegrating relation* ($r = 1$), *the orthogonal shocks obtained by imposing a contemporaneous recur-*

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*sive structure on the VECM errors are identical to those in a Gonzalo and Ng (2001) orthogonal permanent-transitory decomposition.*

Proof. In this case, $G$ is lower triangular. It follows that $H^{-1}G$ is also lower triangular and $G^{-1}H$ is the unique Cholesky factor of $\Omega$. Because $A$ is also the unique Cholesky factor of $\Omega$, $A^{-1} = H^{-1}G$ so that this is an exact special case of the Gonzalo and Ng (2001) procedure. ■

COROLLARY 2. *Suppose there are* $0 < j < m$ *weakly exogenous variables with respect to the cointegrating vectors and they are ordered first in* $X_t$. *Then the orthogonal shocks obtained by imposing a contemporaneous recursive structure on the VECM errors do not constitute a permanent-transitory decomposition; however, the first* $j$ *of these shocks are identical to the first* $j$ *permanent shocks in the Gonzalo and Ng (2001) orthogonal permanent-transitory decomposition.*

Proof. Partition the $(m \times r)$ matrix $\alpha_m$ as

$$\alpha_m = \begin{bmatrix} \alpha_j \\ \alpha_{m-j} \end{bmatrix},$$

where $\alpha_j$ and $\alpha_{m-j}$ are, respectively, the first $j$ and last $(m-j)$ rows of $\alpha_m$. Given that there are $j < m$ weakly exogenous variables and that they are ordered first, $\alpha_j$ is the $(j \times r)$ null matrix. Then from (8), the matrix $G$ can be written as

$$G = \begin{bmatrix} \alpha_\perp' \\ \beta' \end{bmatrix} = \begin{bmatrix} I_{jj} & 0_{j(m-j)} & 0_{jr} \\ 0_{(m-j)j} & I_{(m-j)(m-j)} & -\alpha_{m-j}\alpha_r^{-1} \\ \beta_{rj}' & \beta_{r(m-j)}' & \beta_{rr}' \end{bmatrix}. \tag{15}$$

Each row of the matrix $\alpha_{m-j}$ has at least one nonzero element; otherwise there would be more than $j$ weakly exogenously variables. Thus, the $((m-j) \times r)$ matrix $-\alpha_{m-j}\alpha_r^{-1}$ is not null, and therefore the $(m \times r)$ matrix $-\alpha_m\alpha_r^{-1}$ is not null either. Using an expression analogous to (8) for $\beta_\perp$, it is straightforward to show from (3) that the upper right-hand block of $C(1)$ is the $(m \times r)$ matrix $-\gamma\alpha_m\alpha_r^{-1}$, which is not null as $\gamma$ is of full rank. Thus, the last $r$ column vectors of $C(1)$ are not null so that a permanent-transitory decomposition does not pertain in this case.

Now partition $H$ and $A$ conformably with $G$ as follows:

$$H = \begin{bmatrix} H_{jj} & 0_{j(m-j)} & 0_{jr} \\ H_{(m-j)j} & H_{(m-j)(m-j)} & 0_{(m-j)r} \\ H_{rj} & H_{r(m-j)} & H_{rr} \end{bmatrix},$$

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$$A = \begin{bmatrix} A_{jj} & 0_{j(m-j)} & 0_{jr} \\ A_{(m-j)j} & A_{(m-j)(m-j)} & 0_{(m-j)r} \\ A_{rj} & A_{r(m-j)} & A_{rr} \end{bmatrix},$$

where all the matrices along the principal diagonal of $H$ and $A$ are, respectively, lower triangular because $H$ and $A$ are both lower triangular. Also partition $\Omega$ conformably with $H$, $A$, and $G$. Then it follows from (7) and (15) that $H_{jj}H_{jj}' = \Omega_{jj}$ so that $H_{jj}$ is the unique Cholesky factor of $\Omega_{jj}$. But by construction $A_{jj}A_{jj}' = \Omega_{jj}$ so that $A_{jj}$ is also the unique Cholesky factor of $\Omega_{jj}$. It immediately follows that $v_{jt} = H_{jj}^{-1}e_{jt} = A_{jj}^{-1}e_{jt}$. ■

## NOTE

1. The matrix $G$ is not necessarily of full rank. Levtchenkova et al. (1996) present a simple example in a two-variable system that yields a singular $G$ matrix. It is assumed here that $G$ is nonsingular because that is a required assumption of the Gonzalo and Ng (2001) procedure as is evident from (5).

## REFERENCES

Dhrymes, P.J. (2000) *Mathematics for Econometrics*, 3rd ed. Springer-Verlag.

Gonzalo, J. & C.W.J. Granger (1995) Estimation of common long-memory components in cointegrated systems. *Journal of Business & Economic Statistics* 13, 27–35.

Gonzalo, J. & S. Ng (2001) A systematic framework for analyzing the dynamic effects of permanent and transitory shocks. *Journal of Economic Dynamics and Control* 25, 1527–1546.

Johansen, S. (1991) Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models. *Econometrica* 59, 1551–1580.

Johansen, S. (1992) Cointegration in partial systems and the efficiency of single-equation analysis. *Journal of Econometrics* 52, 389–402.

Lettau, M. & S.C. Ludvigson (2004) Understanding trend and cycle in asset values: Reevaluating the wealth effect on consumption. *American Economic Review* 94, 276–299.

Levtchenkova, S., A.R. Pagan, & J.C. Robertson (1996) Shocking stories. *Journal of Economic Surveys* 12, 507–532.

Ribba, A. (1997) A note on the equivalence of long-run and short-run identifying restrictions in cointegrated systems. *Economics Letters* 56, 273–276.

Sims, C.A. (1980) Macroeconomics and reality. *Econometrica* 48, 1–48.

Urbain, J.P. (1992) On weak exogeneity in error correction models. *Oxford Bulletin of Economics and Statistics* 54, 187–207.

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