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Finite dimensional submodules of G-modules for a compact group G

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We propose an exceptionally short and effortless approach to the fundamental facts of the representation theory of compact groups on Hilbert spaces and expand our methods to prove some generalizations concerning representations on arbitrary locally convex vector spaces.

1. DEFINITION. *Let G be a Hausdorff topological group and E a Hausdorff topological vector space, with the algebra $L(E)$ of continuous vector space endomorphisms. We call E a G-module if there is a function $(g,x)\to g.x$ from $G\times E$ to E such that the following conditions are satisfied with $\pi(g)(x) = g.x$:*

(*a*) *$\pi(g)\in L(E)$ for all $g\in G$;*

(*b*) *$1.x = x$ and $gh.x = g.(h.x)$ for all $x\in E$, $g,h\in G$;*

(*c*) *$(g,x)\to g.x$ is continuous.*

The proof of the following lemma is included for the convenience of the reader.

2. LEMMA. *Let E be a Hausdorff topological vector space over the reals or complexes, whose underlying topological space is a Baire space. Let G be a locally compact group and $(g,x)\to g.x$ a function satisfying sections* 1(*a*), (*b*) *and* (*c'*) *$g\to g.x$ is continuous for all $x\in E$ (i.e. $\pi: G\to L(E)$ is continuous relative to the topology of pointwise convergence on $L(E)$). Then section* 1(c) *holds, i.e. E is a G-module.*

Moreover, if K is any compact subset of G and U any neighbourhood of 0 *in E, then there is a neighbourhood V of* 0 *in E with K, $V\subset U$.*

Proof. Let K be a compact symmetric neighbourhood of 1 in G. We show that $\pi(K)$ is equicontinuous at 0; for then the continuity of $(g,x)\to g.x$ at $(1,0)$ is clear and follows at all other points from section 1(*a*), (*b*), and (*c'*) above. Now let U be an arbitrary closed neighbourhood of 0 in E. Define $C = \cap\{g^{-1}.U: g\in K\}$. Since $g\to g.x$ is continuous and K compact, $K.x$ is compact and therefore bounded in E. Hence there is a natural number n with $K.x\subset nU$. So $x\in nC$, and thus $E = \cup\{nC: n = 1,\ldots\}$. But C is closed and E is a Baire space, hence some nC, and therefore C has a non-trivial interior V. If $t\in V$, then $\pi(g)(V-t)\subset U-U$ for all $g\in K$. This proves that $\pi(K)$ is equicontinuous at 0. The final assertion now follows with the joint continuity of $(g,x)\to g.x$ and a standard covering argument.

Note that Lemma 2 applies to each complete metric, hence, in particular, to each Hilbert space by Baire's category theorem.

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3. DEFINITION. *A vector space endomorphism $T \in L(E)$ is called precompact if it maps some neighbourhood of 0 onto a precompact set. The set of all compact endomorphisms is called $LC(E)$. (For complete E a precompact endomorphism is compact in the usual sense.)*

4. LEMMA. *$LC(E)$ is an ideal in $L(E)$.*

5. LEMMA. *If E is a G-module and G is compact, then for each $T \in LC(E)$ there is a neighbourhood V of 0 in E such that $\pi(G)T\pi(G)V = G.(T(G.V))$ is precompact.*

For the formulation of the following main lemma, we recall that for the continuous functions from a compact space X into a Hilbert space E and for each Radon measure ν one defines an integral $\int f d\nu$ by observing, that for every $u \in E$ the integral

$$\int \langle f(x), u\rangle \, d\nu(x)$$

is well defined in the usual sense and that $u \to \int \langle f(x), u\rangle d\nu(x)$ defines a continuous linear functional on E and therefore an element $\int f d\nu$ of E. Now let G be a compact group and μ the normalized Haar measure on G. Since $\|\pi(g)\|$ is bounded for $g \in G$ by Lemma 2, for each $T \in L(E)$ we define an operator $\tilde{T} \in L(E)$ by $\tilde{T}x = \int \pi(g)^{-1}T\pi(g)\,(x)\, d\mu(g)$. Note that $\|\tilde{T}\| \leqslant M^2\|T\|$ with $M = \sup\{\|\pi(g)\| : g \in G\}$. As usual, we denote with A' for $A \subset L(E)$ the commutant $\{T \in L(E) : TT' = T'T \text{ for } T' \in A\}$. The cone of all positive elements of $L(E)$ will be called $LP(E)$. A G-module will be called *unitary* if E is a Hilbert space and all $\pi(g)$ are unitary. It is well known that the inner product on a Hilbert space, which at the same time is a G-module over a compact group G, may be replaced by a new inner product $(x, y) = \int \langle g.x, g.y\rangle \, d\mu(g)$ which is equivalent to the old one and relative to which all $\pi(g)$ are unitary ('Weyl's trick'; see e.g. (11), § 32, 2).

6. MAIN LEMMA. *Let G be a compact group and E a unitary G-module. Then the function $T \to \tilde{T}$: $L(E) \to L(E)$ is a linear operator with norm $\leqslant M^2$ mapping $LC(E)$ into itself and $LP(E)$ into itself and $L(E)$ into $\pi(G)'$. If $T \in LP(E)$ and $\tilde{T} = 0$, then $T = 0$.*

Proof. The linearity is clear and the estimate for the norm had been observed above. If T is compact then the image of the unit ball B under $\tilde{T}$ is contained in the closed convex hull of $\pi(G)T\pi(G)B$ which is compact in view of section 5. Hence $\tilde{T}$ is compact. If $T \in LP(E)$, then

$$\langle \tilde{T}x, x\rangle = \int \langle \pi(g)^{-1}T\pi(g)\,(x), x\rangle \, d\mu(g) = \int \langle T\pi(g)\,(x), \pi(g)\,(x)\rangle \, d\mu(g) \geqslant 0;$$

and if this integral is zero, then the continuous non-negative function

$$g \to \langle T\pi(g)\,(x), \pi(g)\,(x)\rangle$$

must be zero, which implies $T = 0$. Since

$$h^{-1}.(\tilde{T}h.x) = \int \pi(gh)^{-1}T\pi(gh)\,(x)\, d\mu(g) = \tilde{T}x$$

by the invariance of μ, we have $\tilde{T} \in \pi(G)'$.

7. THEOREM. *Let E be a Hilbert space and a G-module for a compact group G. Then E contains a finite dimensional non-zero submodule.*

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Proof. We may assume that E is unitary. Let T be an orthogonal projection onto a finite dimensional subspace. Then $T \in LC(E) \cap LP(E)$, whence

$$\tilde{T} \in LC(E) \cap LP(E) \cap \pi(G)' \quad \text{and} \quad \tilde{T} \neq 0$$

by section 6. Let S be the orthogonal projection onto one of the non-zero eigenspaces of $\tilde{T}$. Then $S \in \pi(G)'$ and the range of S is finite dimensional. It is the required submodule.

8. Corollary. *Let E be a Hilbert space over the reals or complexes. Let $\pi: G \to L(E)$ be a function from a compact group into the bounded operators of E which is continuous in the strong operator topology and satisfies $\pi(gh) = \pi(g)\pi(h)$ and $\pi(1) = 1_E$. Then E contains a finite dimensional subspace which is invariant under $\pi(G)$.*

Proof. This follows immediately from Theorem 7 and Lemma 2.

9. Corollary. *Let the hypotheses be as in Corollary 8 and assume in addition that all $\pi(g)$ are unitary. Then E is the Hilbert space direct sum of mutually orthogonal finite dimensional submodules (i.e. invariant subspaces).*

Proof. Since there is at least one non-zero finite dimensional submodule, by Zorn's lemma we may find a maximal set of mutually orthogonal, finite dimensional submodules. Their sum is a closed submodule, and since all $\pi(g)$ are unitary, its orthogonal complement is a submodule. If it is not zero, then it would contain a finite dimensional submodule by 8, contradicting the maximality of the set we chose.

10. Corollary. *The orthogonal and the unitary finite dimensional irreducible representations separate the points of a compact group.*

Proof. Apply Corollary 9 to $E = L^2(G)$ with $(\pi(g)f)(h) = f(g^{-1}h)$ for $f \in L^2(G)$. Then π is an injective unitary strongly continuous representation. Thus, if $1 \neq g \in G$, then there must be at least one finite dimensional submodule F of E such that $\pi(g) \mid F \neq 1_F$. Since F splits into a direct sum of simple (irreducible) submodules, the assertion follows.

11. Corollary. *If G is a compact group, then the algebra $R(G) \subset C(G)$ of real (resp. complex) coefficients of all finite dimensional orthogonal (resp. unitary) representations of G is uniformly dense in $C(G)$, the algebra of all real valued (resp. complex valued) continuous functions on G.*

Proof. The assertion follows from Corollary 10 together with the Stone–Weierstrass theorem.

Note that the use of a few straightforward facts concerning projective limits of compact groups as a consequence of section 10 immediately yields the fact that any compact group is a projective limit of finite dimensional orthogonal (resp. unitary) groups. Likewise it follows that the category of compact groups has a cogenerator, namely the product of the unitary groups $U(n)$ $(n = 1, 2, \ldots)$ on the standard finite dimensional Hilbert spaces $\mathbf{C}^n$ $(n = 1, 2, \ldots)$.

After we have presented the core of the representation theory for compact groups a few comments are in place. Practically all textbooks and articles concerned with the representation and structure theory of compact groups present these facts in one way

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or another (3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 20). Most of them use the analytic geometry of Hilbert space including Hilbert–Schmidt or compact operators; all of them, of course, use invariant integration in some form, and in some presentations the work is done by highly elegant, but frequently quite elaborate, machinery like H^*-algebras or von Neumann algebras (4, 9, 11). A preferred approach is to first prove the theorem of Gelfand and Raikov that the irreducible unitary representations of a locally compact group separate points (which requires a good deal of the representation theory of involutive Banach algebras and the Krein–Milman theorem) and then to show that an irreducible representation of a compact group of unitary operators is finite dimensional. The approach we have suggested seems not to have been used for this purpose, although the operator $\tilde{T}$ was used by Rudin for Banach spaces (19) and by the author for topological groups under suitable circumstances ((18), p. 30). Parts of Lemma 6 were proved in both of these places. This approach uses very little background material, in fact nothing beyond the rudiments of Hilbert space theory and their compact operators, and elementary integration theory of continuous functions. It also has the advantage that it yields a number of useful generalizations concerning linear actions of compact groups on arbitrary locally convex spaces which we discuss in the remainder of this note.

One recalls that the theory of integration of vector valued functions which we used before generalizes in the following fashion to locally convex spaces: If X is a compact space and ν a Radon measure on X, then for any continuous function f from X into a locally convex topological vector space E the function $u \to \int \langle f(x), u\rangle \, d\nu(x)$ is a linaer functional on the topological dual E' of E. Suppose that E satisfies the following condition

(K) *The closed convex hull of a compact subset is compact in the weak topology of* E.

Then this functional is in fact an element of E (under the natural embedding of E into the algebraic dual of E!) ((2), VI, 1, no. 2). It is denoted with $\int f d\nu$, and is characterized by $\langle \int f d\nu, u\rangle = \int \langle f(x), u\rangle \, d\nu(x)$ for all $u \in E'$. If $\nu(X) = 1$, then $\int f d\nu$ is in the closed convex hull of $f(X)$ (2). The integral and continuous vector space endomorphisms commute. Note that (K) is certainly satisfied if E is quasicomplete, hence in particular if it is a Banach space. Now suppose that E is a locally convex space satisfying condition (K) and being a G-module for a compact group G. Then for every $T \in L(E)$ we can again define $\tilde{T}x = \int \pi(g)^{-1} T \pi(g)(x) \, d\mu(g)$. By the joint continuity of $(g, x) \to g.x$, and the compactness of G, by a standard covering argument it follows that for any neighbourhood U of 0 in E there is a neighbourhood V of 0 such that

$$\pi(G) \, T \pi(G)(V) \subset U.$$

If U was chosen convex and closed, then for each $v \in V$ we have $\tilde{T}v \in U$, which shows that also in this more general case $\tilde{T}$ is continuous. Thus the linear map

$$T \to \tilde{T} \colon L(E) \to L(E)$$

is well defined. The fact that it maps $LC(E)$ into $LC(E)$ remains true, and so does the fact that it is continuous relative to the topology of uniform convergence on bounded sets on $L(E)$. The following theorem is a generalization of Theorem 7.

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12. THEOREM. *Let G be a compact group and E a locally convex G-module satisfying condition (K). Then*

(1) *the subspace generated algebraically by all finite dimensional simple modules of E is dense in E;*

(2) *if E splits into a direct sum of topological vector spaces with a submodule F as direct summand, then E is the direct sum of F and a module complement (i.e. E is then isomorphic to the direct sum of the modules in F and E/F).*

In particular each finite dimensional or cofinite dimensional submodule has a module complement.

Proof. (1) Let $R(G) \subset C(G)$ be as in Corollary 11. For $f \in C(G)$ and $x \in E$ define $f.x = \int f(g)\, g.x\, d\mu(g)$. If $C(G)$ is considered as an algebra with convolution, then algebraically E is a $C(G)$-module relative to the operation $(f, x) \to f.x$, and if we define ${}_gf(h) = f(g^{-1}h)$ for $f \in C(G)$, then we have $g.(f.x) = {}_gf.x$. This implies that every element of $R(G).E$ lies in a finite dimensional submodule. Thus it suffices to show that $R(G).E$ is dense in E. We first show that $f \to f.x$ is a continuous function relative to the sup-norm on $C(G)$: Let $f \in C(G)$ and let B be the closed convex circled hull of the compact set $G.x$. Then B is bounded and we have

$$|\langle f.x, u\rangle| \leqslant \int |f(g)|\, d\mu(g) \leqslant \|f\| \quad \text{if} \quad |\langle B, u\rangle| \leqslant 1.$$

Thus $f.x \in \|f\|\, B$, which shows the continuity of $f \to f.x$. Since $R(G)$ is dense in $C(G)$ by 11, it now suffices to show that $C(G).E$ is dense in E. Let U be a closed convex neighbourhood of O in E and W a neighbourhood of 1 in G such that $g.x - x \in U$ for $g \in W$. Then for any non-negative continuous function f on G vanishing outside W with $\int f\, d\mu = 1$, and for each functional $u \in E'$ with $\langle U, u\rangle \leqslant 1$ we have

$$\langle f.x - x, u\rangle = \int \langle f(g)\,(g.x - x), u\rangle\, d\mu(g) \leqslant \int f(g)\, d\mu(g) = 1;$$

so $f.x - x \in U$. Hence $C(G).E$ is dense in E.

(2) Let F be a submodule of E which is a direct summand of topological vector spaces. Then there is a continuous projection $T: E \to E$ onto F. Now $\tilde{T} \in \pi(G')$. If $x \in F$, then $\pi(G)(x) \subset F$, so $\tilde{T}x = \int \pi(g)^{-1}\pi(g)\, d\mu(g) = x$; hence $\tilde{T}|F = 1_F$, and $F = TE \subset \tilde{T}E$. But for all $x \in E$ we always have $\pi(G)\, T\pi(G)(x) \subset \pi(G)\, F \subset F$, so $\tilde{T}E \subset F$. Hence $TE = \tilde{T}E$, and therefore $\tilde{T}^2 = \tilde{T}$. Then E is the direct sum of $F = \tilde{T}E$ and $\ker \tilde{T}$, and both of these spaces are submodules, since $\tilde{T} \in \pi(G)'$.

For Banach spaces 12·2 is due to Rudin (19); a similar theorem for topological groups is to be found in (18). For Banach spaces 12·1 has to be compared with results in (20). A proof resembling the one given is found in (16). It should be pointed out that in case E is a Banach space, the hypotheses of Theorem 12 can be relaxed considerably: *If E is a Banach space, G a locally compact group, and $(g, x) \to g.x$ a function from $G \times E$ into E satisfying section* 1 *(a), (b) and*

(c″) $g \to g.x$ is continuous for all $g \in G$ relative to the weak topology on E. (That is, π is a weakly continuous representation of G.) Then section 1 *(c) holds, i.e. E is a G-module and Theorem* 12 *applies.* For a proof see e.g. ((17), 2.8).

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