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101.24 Generalisation and sharpening of Problem 100.C

The main goal of this Note is to formulate and prove a generalised and sharpened version of the inequality proposed in Problem 100.C [1]. On the one hand we extend the domain of the exponent from the special case $k = 2$ to all real numbers satisfying $k \geqslant 1$; on the other hand we sharpen it by inserting a simply-built symmetrical function of a, b, c between the left-hand side and right-hand side of the inequality of 100.C. We formulate our result in the following.

Theorem 1: Let a, b, c and r_a, r_b, r_c be the respective sides and ex-radii of the triangle ABC. Then for all $k \geqslant 1$, we have

$$\left(\frac{bc}{r_a}\right)^k + \left(\frac{ac}{r_b}\right)^k + \left(\frac{ab}{r_c}\right)^k \geqslant \left(\frac{2}{\sqrt{3}}\right)^k \left[(-a+b+c)^k + (a-b+c)^k + (a+b-c)^k\right]$$

$$\geqslant \left(\frac{2}{\sqrt{3}}\right)^k \left(a^k + b^k + c^k\right) \tag{1}$$

with equality in (1) if, and only if, $a = b = c$.

One of the main tools which plays a crucial role in the proof is the following lemma about the angles of ΔABC.

Lemma 1: Let α, β, γ be the angles of ΔABC. Then for all $k \geqslant 1$ we have

$$\frac{1}{\sin^k \alpha} + \frac{1}{\sin^k \beta} + \frac{1}{\sin^k \gamma} \geqslant \frac{2^k}{(\sqrt{3})^{k-2}} \tag{2}$$

with equality if, and only if, $\alpha = \beta = \gamma = \frac{\pi}{3}$.

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Proof: Since, for $k \geqslant 1$, $\dfrac{1}{\sin^k x}$ is convex on $0 < x < \pi$, by virtue of Jensen's inequality we have

$$\frac{\dfrac{1}{\sin^k \alpha} + \dfrac{1}{\sin^k \beta} + \dfrac{1}{\sin^k \gamma}}{3} \geqslant \frac{1}{\sin^k\left(\dfrac{\alpha + \beta + \gamma}{3}\right)} = \left(\frac{2}{\sqrt{3}}\right)^k,$$

where the inequality is strict unless $\alpha = \beta = \gamma$. Hence (2) is proved.

Now we need one more inequality which is equivalent to the second part of (1). With respect to its importance we formulate and interpret it separately. Since by virtue of the triangle inequality we have $a < b + c$, $b < a + c$, $c < a + b$ thus $-a + b + c > 0$, $a - b + c > 0$, $a + b - c > 0$; consequently all the kth powers involved in (1) are well-defined and we have the following.

Lemma 2: Let a, b, c be the sides of ΔABC Then for all $k \geqslant 1$ we have

$$(-a + b + c)^k + (a - b + c)^k + (a + b - c)^k \geqslant a^k + b^k + c^k \quad (3)$$

with equality for all a, b, c if $k = 1$, otherwise if, and only if, $a = b = c$.

Proof: If $k = 1$ then it is easy to see that (3) is an identity without setting any restrictions on a, b, c; thus we restrict our attention to the case $k > 1$. For $k > 1$ the function $f(u) = u^k$ is strictly convex for $u > 0$ and thus we can estimate the sum of powers of the sides of ΔABC above as follows:

$$\begin{aligned} a^k + b^k + c^k &= \left[\frac{(a - b + c) + (a + b - c)}{2}\right]^k + \left[\frac{(-a + b + c) + (a + b - c)}{2}\right]^k \\ &\quad + \left[\frac{(-a + b + c) + (a - b + c)}{2}\right]^k \\ &\leqslant \frac{(a - b + c)^k + (a + b - c)^k}{2} + \frac{(-a + b + c)^k + (a + b - c)^k}{2} \\ &\quad + \frac{(-a + b + c)^k + (a - b + c)^k}{2} \\ &= (-a + b + c)^k + (a - b + c)^k + (a + b - c)^k, \end{aligned}$$

as required. There is equality if, and only if, $-a + b + c = a - b + c = a + b - c$ which is equivalent to $a = b = c$.

We now need the following standard results about the geometry of triangles:

$$\Delta = r_a(s - a) = r_b(s - b) = r_c(s - c) \quad (4)$$

$$\Delta = \frac{bc \sin \alpha}{2} = \frac{ac \sin \beta}{2} = \frac{ab \sin \gamma}{2} \quad (5)$$

where, as usual, Δ refers to the area and s to the semi-perimeter of ΔABC.

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Now we are able to prove our main result, Theorem 1.

Proof: Using (4) and (5) lead us to the following formulae for the ex-radii:

$$r_a = \frac{bc\,\sin\alpha}{2(s-a)}, \qquad r_b = \frac{ac\,\sin\beta}{2(s-b)}, \qquad r_c = \frac{ab\,\sin\gamma}{2(s-c)}.$$

Plugging these expressions into the left-hand side of (1) gives

$$\left(\frac{bc}{r_a}\right)^k + \left(\frac{ac}{r_b}\right)^k + \left(\frac{ab}{r_c}\right)^k = 2^k\left[(s-a)^k\frac{1}{\sin^k\alpha} + (s-b)^k\frac{1}{\sin^k\beta} + (s-c)^k\frac{1}{\sin^k\gamma}\right].$$

Henceforth we assume without loss of generality that $a \leqslant b \leqslant c$, and consequently

$$(s-a)^k \geqslant (s-b)^k \geqslant (s-c)^k \text{ and } \frac{1}{\sin^k\alpha} \geqslant \frac{1}{\sin^k\beta} \geqslant \frac{1}{\sin^k\gamma}.$$

Applying Chebyshev's sum inequality [2] to the sum in the square brackets, then Lemma 1 and finally using the definition of s completes the proof of the first inequality of (1):

$$\begin{aligned}
&\left(\frac{bc}{r_a}\right)^k + \left(\frac{ac}{r_b}\right)^k + \left(\frac{ab}{r_c}\right)^k \\
&\geqslant 2^k\cdot\frac{1}{3}\left[(s-a)^k + (s-b)^k + (s-c)^k\right]\left(\frac{1}{\sin^k\alpha} + \frac{1}{\sin^k\beta} + \frac{1}{\sin^k\gamma}\right) \\
&\geqslant \frac{2^{2k}}{(\sqrt{3})^k}\left[(s-a)^k + (s-b)^k + (s-c)^k\right] \\
&= \left(\frac{2}{\sqrt{3}}\right)^k\left[(-a+b+c)^k + (a-b+c)^k + (a+b-c)^k\right].
\end{aligned}$$

Lemma 2 proves the second inequality of (1). There is equality in the inequalities if, and only if, ΔABC is equilateral.

As an application of Theorem 1 we analyse two special cases. Detailed results will be given for $k = 2$ and following the lines of this part we provide some ideas about $k = 3$ and encourage the interested readers to work out the detailed analysis of other special cases.

Example: Setting $k = 2$ and doing some easy algebra one has

$$\begin{aligned}
&(-a+b+c)^2 + (a-b+c)^2 + (a+b-c)^2 \\
&= a^2 + b^2 + c^2 + (a-b)^2 + (b-c)^2 + (c-a)^2.
\end{aligned}$$

Thus we get more detailed information about the nature of the difference between the originally stated lower bound and the newly inserted estimator; more precisely

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$$\left(\frac{bc}{r_a}\right)^2 + \left(\frac{ac}{r_b}\right)^2 + \left(\frac{ab}{r_c}\right)^2$$

$$\geqslant \left(\frac{2}{\sqrt{3}}\right)^2 \left[(-a + b + c)^2 + (a - b + c)^2 + (a + b - c)^2\right]$$

$$= \frac{4}{3}\left(a^2 + b^2 + c^2\right) + \frac{4}{3}\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right]$$

$$\geqslant \frac{4}{3}\left(a^2 + b^2 + c^2\right).$$

It is now clear that the difference in this case is

$$\Delta_2(a, b, c) = \frac{4}{3}\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right]$$

which we call the deviation function of order 2 of ΔABC. Clearly $\Delta_2(a, b, c) \geqslant 0$ and $\Delta_2(a, b, c) = 0$ if, and only if, $a = b = c$. Thus this feature consistently supports the name, as $\Delta_2(a, b, c)$ measures in some sense the deviation of the triangle from being equilateral.

Avoiding the tedious details of computation for $k = 3$ we have the inequality

$$\left(\frac{bc}{r_a}\right)^3 + \left(\frac{ac}{r_b}\right)^3 + \left(\frac{ab}{r_c}\right)^3$$

$$\geqslant \left(\frac{2}{\sqrt{3}}\right)^3 \left[(-a + b + c)^3 + (a - b + c)^3 + (a + b - c)^3\right]$$

$$= \frac{8}{3\sqrt{3}}\left(a^3 + b^3 + c^3\right) + \frac{8}{\sqrt{3}}\left[a(b - c)^2 + b(c - a)^2 + c(a - b)^2\right].$$

Thus the deviation function of order 3 is

$$\Delta_3(a, b, c) = \frac{8}{\sqrt{3}}\left[a(b - c)^2 + b(c - a)^2 + c(a - b)^2\right].$$

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