$2m+1-(h+1+m-h)$ equations (3) since we only assert here that y ranges over at most p points in R^{2m+2}. It is also clear that

$$\text{if } z = (0, \ldots, 0) \in f^{-1}(w), \quad \text{then} \quad f^{-1}(w) = (0, \ldots, 0).$$

Next, it follows easily from (2) and the choice of the integers q_j, r_j that the value of every summand in (3) remains unchanged when $z_0, \ldots, z_m$ are replaced by $z_0', \ldots, z_m'$, respectively. Thus, $g \circ T = g$.

This and the preceding remarks imply that, for any $w \in R^{4m+2}$, the set $g^{-1}(w)$ is either empty or an orbit under T. Therefore, g induces a univalent map

$$\phi\colon\ L(p; p_0, \ldots, p_m) \to R^{4m+2}.$$

Let ρ denote the radial projection of $R^{4m+2} - (0, \ldots, 0)$ onto S^{4m+1}. As noticed above, $g(S^{2m+1}) \subset R^{4m+2} - (0, \ldots, 0)$ so that the composition

$$\rho \circ \phi\colon\ L(p; p_0, \ldots, p_m) \to S^{4m+1}$$

is defined. To prove $\rho \circ \phi$ univalent it suffices to prove that each ray through the origin in R^{4m+2} meets the image of g at most once. Suppose $w = g(z)$ and $\lambda w = g(y)$ for some $\lambda > 0$ and $z, y \in S^{2m+1}$. It follows from (3) that $\lambda w = f(\lambda^{1/2d} z)$ and, by the above remarks, we obtain $\lambda^{1/2d} z = T^q(y)$ with $0 \leqslant q < p$. Since $T^q(y) \in S^{2m+1}$, by (1) we finally obtain $\lambda = 1$.

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SOME SELF-RECIPROCAL FUNCTIONS AND KERNELS

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The aim of this note is to find out some self-reciprocal functions and kernels for Fourier–Bessel integral transforms. Following Hardy and Titchmarsh (1), we shall denote by R_p the class of functions which satisfy the homogeneous integral equation

$$f(x) = \int_0^\infty J_p(xt)\,(xt)^{\frac{1}{2}} f(t)\,dt, \tag{1}$$

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where $J_p(x)$ is a Bessel function of order $p \geqslant -\frac{1}{2}$. For particular values of $p = \frac{1}{2}, -\frac{1}{2}$, we write R_s and R_c, respectively.

Consider the integral (6)

$$\int_0^\infty J_p(at)\, J_p(bt)\, J_p(ct)\, t^{-p+1} dt = \frac{2^{p-1}\Delta^{2p-1}}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})\,(abc)^p},$$

where a, b, c represent the sides of the triangle of area Δ. This reduces, if $p = 1$ and $a = x$, to the form

$$\int_0^\infty J_1(xt)\,(xt)^{\frac{1}{2}}\, J_1(bt)\, J_1(ct)\, t^{-\frac{1}{2}} dt = \frac{x^{-\frac{1}{2}}\Delta}{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})\,bc}, \tag{2}$$

where x, b, c represent the sides of the triangle of area Δ. Since $x^{-\frac{1}{2}}$ is also R_1, we deduce from (2) that the function

$$J_1(bx)\, J_1(cx)\, x^{-\frac{1}{2}} \tag{3}$$

is R_1. It is known (2) that the function

$$\prod_{i=1}^{n} J_{p_i}(a_i x)\, x^{-(\Sigma p_i + \frac{1}{2})} \tag{4}$$

is R_1 if $\Sigma a_i < x$. We shall find some R_1 functions using (4). The integral (6)

$$\int_0^\infty e^{-ct} J_{\alpha-\beta}(at)\, J_{p-1}(zt)\, t^{-p+\alpha+\beta} dt$$
$$= \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}z)^{p+2m-1} (\frac{1}{2}a)^{\alpha-\beta}\,\Gamma(2\alpha+2m)}{m!\,\Gamma(p+m)\,(a^2+c^2)^{\alpha+m}\,\Gamma(\alpha-\beta+1)}\, {}_2F_1\left[\begin{matrix}\alpha+m, \frac{1}{2}-\beta-m; \\ \alpha-\beta+1;\end{matrix} \frac{a^2}{a^2+c^2}\right] \tag{5}$$

provided $\Re(z) > 0$, $\Im(z) < c$, $|z| < (a^2+c^2)^{\frac{1}{2}} - c$. If we take $a = x$, $c = 0$, $\alpha = 1$, $\beta = 0$ and $z = 1$, the integral (5) reduces to the form,

$$\int_0^\infty J_1(xt)\,(xt)^{\frac{1}{2}} J_{p-1}(t)\, t^{-p+\frac{1}{2}} dt = \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2})^{p+2m}\,\Gamma(2+2m)}{m!\,\Gamma(p+m)\,\Gamma(2)}\, x^{-(2m+\frac{1}{2})}\, {}_2F_1[1+m, \tfrac{1}{2}-m; 2; 1]. \tag{6}$$

Since it is clear from (4) that $J_{p-1}(x)\, x^{-p+\frac{1}{2}}$ is R_1, it follows that the infinite series on the right-hand side of (6) is also R_1.

The integral (6)

$$\int_0^\infty J_p(at)\, J_p(bt)\, t^{-q} dt$$
$$= \frac{(ab)^p\,\Gamma(p-\frac{1}{2}q+\frac{1}{2})}{2^q (a^2+b^2)^{p-\frac{1}{2}q+\frac{1}{2}}\,\Gamma(p+1)}\, {}_2F_1\left[\frac{2p+1-q}{4}, \frac{2p+3-q}{4}; p+1; \frac{4a^2b^2}{(a^2+b^2)^2}\right]$$

which is valid whenever $a > b$ or $a < b$, transforms into

$$\int_0^\infty J_1(xt)\,(xt)^{\frac{1}{2}} J_1(bt)\, t^{-\frac{3}{2}} dt = \frac{x^{\frac{3}{2}} b}{2(x^2+b^2)\,\Gamma(2)}\, {}_2F_1\left[\tfrac{1}{2}, 1; 2; \frac{4x^2b^2}{(x^2+b^2)^2}\right], \tag{7}$$

when $p = q = 1$ and $a = x$. Since it follows from (4) that, $J_1(bx)\, x^{-\frac{3}{2}}$ is R_1, we find from (7) that the function

$$\frac{x^{\frac{3}{2}}}{(x^2+b^2)}\, {}_2F_1\left[\tfrac{1}{2}, 1; 2; \frac{4x^2b^2}{(x^2+b^2)^2}\right] \tag{8}$$

is R_1.

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We shall use the following results proved elsewhere (3).

THEOREM. *If $h(x)$ is a kernel function which transforms $R_p(R_{p+1})$ into $R_{p+1}(R_p)$, then $f(x) = x^{\frac{1}{2}}h(\frac{1}{2}x^2)$ is R_{2p+1} and conversely.*

It is known (4) that

$$h(x) = x^{\frac{1}{2}}K_{\frac{1}{2}(p-q)}(\alpha x)\,J_{\frac{1}{2}(p+q)}(\beta x) \tag{9}$$

transforms R_p into R_q if $\alpha^2+\beta^2 = 1$. Putting $p = q+1$, we get the kernel

$$h(x) = x^{\frac{1}{2}}K_{\frac{1}{2}}(\alpha x)\,J_{q+\frac{1}{2}}(\beta x)$$

which transforms R_{p+1} into R_p. Hence the theorem shows that

$$f(x) = x^{\frac{3}{2}}K_{\frac{1}{2}}(\tfrac{1}{2}\alpha x^2)\,J_{q+\frac{1}{2}}(\tfrac{1}{2}\beta x^2) \tag{10}$$

is R_{2p+1} if $\alpha^2+\beta^2 = 1$. If $q = 0$, (10) reduces to $x^{\frac{3}{2}}K_{\frac{1}{2}}(\frac{1}{2}\alpha x^2)\,J_{\frac{1}{2}}(\frac{1}{2}\beta x^2)$ which is R_1. Further, putting $\alpha = \beta = 2^{-\frac{1}{2}}$, we find that the function $x^{\frac{3}{2}}K_{\frac{1}{2}}(2^{-\frac{3}{2}}x^2)\,J_{\frac{1}{2}}(2^{-\frac{3}{2}}x^2)$ is R_1.

Similarly from the known (4) kernel $h(x) = x^{\frac{1}{2}}J_{\frac{1}{2}(p+q)}(\alpha x)\,J_{\frac{1}{2}(p-q)}(\beta x)$ which transforms R_p into R_q, if $\alpha^2-\beta^2 = 1$, it follows that

$$f(x) = x^{\frac{3}{2}}J_{q+\frac{1}{2}}(\tfrac{1}{2}\alpha x^2)\,J_{\frac{1}{2}}(\tfrac{1}{2}\beta x^2) \tag{11}$$

is R_{2p+1}, if $\alpha^2-\beta^2 = 1$.

We know (4) that $f(x) = x^{\frac{1}{2}}e^{-\alpha x^2}J_{p+\frac{1}{2}}(\beta x^2)$ is R_{2p+1}, if $\alpha^2+\beta^2 = \frac{1}{4}$. It therefore follows from the converse of the theorem that

$$h(x) = e^{-2\alpha x}\,J_{p+\frac{1}{2}}(2\beta x) \tag{12}$$

is a kernel which transforms R_p into R_{p+1}, if $\alpha^2+\beta^2 = \frac{1}{4}$. Putting $p = 0$, we get the kernel $e^{-2\alpha x}J_{\frac{1}{2}}(2\beta x)$ which transforms R_0 into R_1 if $\alpha^2+\beta^2 = \frac{1}{4}$. Taking $\beta = \frac{1}{2}$, $\alpha = 0$, this gives us the well-known kernel $J_{\frac{1}{2}}(x)$ which transforms R_0 into R_1.

From the known (5) general R_{2p+1} function $T^{2m}_{2p+1}(x^2)\,x^{2p+\frac{3}{2}}e^{-\frac{1}{2}x^2}$ and from the stated result, we find that

$$h(x) = T^{2m}_{2p+1}(2x)\,x^{p+\frac{1}{2}}e^{-x} \tag{13}$$

transforms R_p into R_{p+1}. Taking $p = -\frac{1}{2}$, we get the well-known kernel e^{-x} which transforms R_c into R_s.

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