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Transverse stability of plane solitons using the variational method

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We present stability results for plane soliton solutions of two versions of the two-dimensional KdV equation, namely the Zakharov–Kuznetsov (ZK) equation and the Kadomtsev–Petviashvili equation for positive dispersion (KP+ equation). To do this we use a linear variation-of-action method (VAM). Others have used this method, but with little success when applied to these two equations. The best results have given the correct instability range, but the predicted growth rates have significant errors. For the ZK equation we show, by paying more attention to the spatially asymptotic form of the trial function, how better estimates of the dispersion relation can be obtained. We go on to obtain the *exact* dispersion relation for perturbations of the plane soliton solution of the KP+ equation.

1. Introduction

The linear stability of many one-dimensional kink or soliton solutions has been studied, in the context of many different equations (e.g. the Zakharov–Kuznetsov (ZK) equation by Allen and Rowlands (1993) and the Cahn–Hilliard equation by Bettinson and Rowlands (1996)), using perturbative techniques. The dispersion relations obtained have been shown to be very accurate, but unfortunately this linear theory tells us nothing more than the short-time behaviour. On the other hand, the variation-of-action method (VAM) can be used to consider the non-linear behaviour of linearly unstable plane solitons. Clearly, for a solution accurate for all times, we must insist that the solution be valid for short times. Therefore in this paper we study the short-time evolution of plane solitons of two versions of the two-dimensional KdV equation, using a *linear* VAM. This should be seen as a necessary preliminary to a nonlinear theory, but further we insist that the linear method is relatively simple.

In one dimension the KdV equation admits fully stable plane soliton solutions (Benjamin 1972; Jeffrey and Kakutani 1970), and these theoretical predictions have been verified experimentally (Ikezi *et al.* 1970). To study the stability of such plane solitons to perpendicular perturbations, we shall consider the ZK equation and the Kadomtsev–Petviashvili equation for positive dispersion (KP+ equation), both of which reduce to the KdV equation in one dimension. We use a linear VAM where the quantity minimized (dependent on the unknown solution) is the action. To date, attempts at using this method in the linear regime, for both the ZK and KP+ equations, have at best given the correct instability region, but with significant error in the values of the growth rate of perturbations (for the ZK equation see

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Allen (1994), Infeld and Frycz (1987) and Makhankov (1978), and for the KP+ equation see Katyshev and Makhankov (1976) and Laedke and Spatschek (1979)).

Section 2 considers the stability of the plane soliton solution to the the ZK equation, and it is shown that by choosing a more suitable trial function, a good approximation to the true dispersion relation can be found. In Sec. 3 we show that this method is capable of obtaining the *exact* dispersion relation for soliton solutions to the KP+ equation. Some conclusions are drawn in Sec. 4.

2. Perturbing a plane soliton solution to the ZK equation

The two-dimensional ZK equation for ion-acoustic, nonlinear waves and solitons, propagating along a strongly magnetized, two component plasma is, in non-dimensional form (Zakharov and Kuznetsov 1974),

$$n_t + nn_x + n_{xxx} + n_{xyy} = 0, \tag{2.1}$$

where subscripts denote partial differentiation with respect to the subscript variable. The plane soliton solution of this is of the form $n = 12\eta^2 \operatorname{sech}^2 [\eta(x - x_0 - 4\eta^2 t)]$. It is convenient to transform the equation to a reduced-variables coordinate system moving with the unperturbed soliton. This is achieved by the transformation

$$(x, y, t) \mapsto \left(\frac{x + 4t}{\eta}, \frac{y}{\eta}, \frac{t}{\eta^3} \right), \qquad n \mapsto \eta^2 n, \tag{2.2}$$

which gives

$$n_t + (n - 4)n_x + n_{xxx} + n_{xyy} = 0. \tag{2.3}$$

To apply a small perturbation perpendicular to the direction of propagation, we write

$$n = n_0 + \varepsilon \Phi(x) e^{iky} e^{\gamma t}, \tag{2.4}$$

where $n_0 \equiv 12 \operatorname{sech}^2 x$ is the stationary plane soliton solution. Substituting (2.4) into (2.3), and neglecting products of ε^2, gives

$$\frac{d}{dx} L_0 \Phi = -\gamma \Phi + k^2 \frac{d\Phi}{dx}, \tag{2.5}$$

where L_0 is the operator $(d^2/dx^2) + n_0 - 4$. It has been shown that (2.5) has two marginally stable ($\gamma = 0$), bounded solutions (Laedke and Spatschek 1982). They are $\Phi = \operatorname{sech}^2 x \tanh x$ when $k = 0$, and $\Phi = \operatorname{sech}^3 x$ for $k = \sqrt{5}$.

2.1. The variation-of-action method

The ZK equation (2.3) is derivable from a Lagrangian density (Infield 1985)

$$\mathscr{L} = \tfrac{1}{2}\psi_t \psi_x + \tfrac{1}{6}\psi_x^3 - 2\psi_x^2 - \tfrac{1}{2}\psi_{xx}^2 - \tfrac{1}{2}\psi_{xy}^2, \qquad \psi_x \equiv n, \tag{2.6}$$

via the variational principle

$$\delta \int\int\int \mathscr{L} \, dx \, dy \, dt = 0. \tag{2.7}$$

The present VAM is based on the idea that one specifies the x dependence of ψ, but the unknown y and t dependence of ψ is introduced through m parameters $A_j(y, t)$ ($j = 1, 2, \ldots, m$). Thus (2.7) becomes known as the reduced variational principle,

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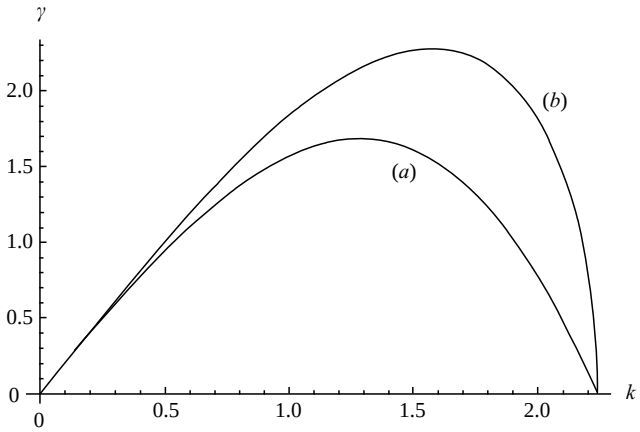


Figure 1. (*a*) The dispersion relation obtained using numerical methods. (*b*) The previous best dispersion relation obtained using the VAM.

namely

$$\delta \int \int \langle \mathscr{L}(A_j, A_{jy}, A_{jt}, \ldots) \rangle \, dy \, dt = 0, \tag{2.8}$$

where $\langle\ \rangle = \int_{-\infty}^{\infty} dx$. This is then minimized with respect to the parameters A_j, to give m coupled Euler–Lagrange equations to solve for the A_j.

To date, the best approximation to the true dispersion relation for the ZK equation, obtained using the VAM, is when the trial function used is of the form (Allen 1994)

$$n = \psi_x = A \operatorname{sech}^2 B(x - P) + H \operatorname{sech}^3 B(x - P). \tag{2.9}$$

Equation (2.9) corresponds to a combination of the equilibrium solution and the eigenfunctions of the linearized problem corresponding to the two marginally stable eigenvalues (note that expanding the $\operatorname{sech}^2 B(x-P)$ term gives rise to a term of the form $\operatorname{sech}^2 Bx \tanh Bx$). Once the Euler–Lagrange equations have been obtained for this trial function, the parameters A, H (amplitudes), B (width), and P (velocity of the soliton) are linearized about their unperturbed values; thus

$$\left.\begin{aligned} A &= 12 + \delta A(y,t), \\ B &= 1 + \delta B(y,t), \\ P &= \delta P(y,t), \\ H &= \delta H(y,t). \end{aligned}\right\} \tag{2.10}$$

The form of the linear perturbation is chosen so that $\delta A(y,t) = \bar{A} \exp(\gamma t) \exp(iky)$, $\delta B(y,t) = \bar{B} \exp(\gamma t) \exp(iky)$, etc., where $\bar{A}, \bar{B}$, etc. are constants.

Using (2.9) as the trial function, and applying the linear VAM yields the dispersion relation shown in Fig. 1(*b*). The predicted instability region is correct, as is the value of $d\gamma/dk$ at $k = 0$. Unfortunately the value of $d\gamma/dk$ at $k = \sqrt{5}$ and the values of γ within the instability region are substantially in error.

2.2. The new trial function

The trial function (2.9) is chosen because it contains a $\operatorname{sech}^2 x$ term (the plane soliton solution), a $\operatorname{sech}^2 x \tanh x$ term (the marginally stable eigenfunction of the linear problem (2.5) for $k = 0$) and a $\operatorname{sech}^3 x$ term (the marginally stable eigenfunction of

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the linear problem for $k = \sqrt{5}$). We note that integrating the linearized equation (2.5) over all space gives the consistency condition

$$\gamma \int_{-\infty}^{\infty} \Phi \, dx = 0, \tag{2.11}$$

when Φ and its low-order derivatives go to zero sufficiently fast at both infinities. Thus for non-zero γ the integral must vanish. This requirement is satisfied for $\Phi = \operatorname{sech}^2 x \tanh x$ but not for $\Phi = \operatorname{sech}^3 x$. Thus we need an eigenfunction Φ that satisfies (2.11) for k approaching $\sqrt{5}$, but is proportional to $\operatorname{sech}^3 x$ for $k = \sqrt{5}$. This is the first key change we make to our trial function when compared with those used by previous authors (Allen 1994; Infeld and Frycz 1987; Makhankov 1978).

To arrive at a good approximation to the eigenfunction in the neighbourhood of $k = \sqrt{5}$, we use the asymptotic form of the linearized ZK equation. Writing $\bar{k}^2 \equiv 5 - k^2$ in (2.5), as $x \to \pm\infty$ this becomes

$$\frac{d^3\bar{\Phi}}{dx^3} - (9 - \bar{k}^2)\frac{d\bar{\Phi}}{dx} + \gamma\bar{\Phi} = 0, \tag{2.12}$$

where $\lim_{x\to\pm\infty} \Phi = \bar{\Phi}$. Let $\gamma = \gamma_1\bar{k} + \gamma_2\bar{k}^2 + \ldots$ and let the solution be of the form $\Phi = \exp(px)$. Substitution of this form into (2.12) gives, to first order in $\bar{k}$

$$\lim_{x\to+\infty} \Phi = c_1 e^{-(3+\gamma/18)x} \tag{2.13}$$

$$\lim_{x\to-\infty} \Phi = c_2 e^{(3-\gamma/18)x} + c_3 e^{(\gamma/9)x}, \tag{2.14}$$

where the c_i are arbitrary constants. We now impose the natural condition that our eigenfunction, to be used in the VAM, should have the same asymptotic form. Clearly, as $x \to +\infty$ there is a fast decay, but as $x \to -\infty$ a very slowly decaying 'tail' dominates the eigenfunction. Using this, we choose

$$\Phi_c = e^{-(\gamma/18)x} \operatorname{sech}^3 x + e^{(\gamma/9)x}(2 - 2\tanh x - \operatorname{sech}^2 x), \tag{2.15}$$

for which

$$\lim_{x\to+\infty} \Phi_c = c_4 e^{-(3+\gamma/18)x} + c_5 e^{-(4-\gamma/9)x} + O(e^{-5x}), \tag{2.16}$$

$$\lim_{x\to-\infty} \Phi_c = c_6 e^{(\gamma/9)x} + c_7 e^{(2+\gamma/9)x} + c_8 e^{(3-\gamma/18)x} + O(e^{5x}). \tag{2.17}$$

We see that the chosen function, given by (2.15), has the correct slowest decaying term for both $x \to \pm\infty$. Using this information, the form of our eigenfunction for k in the neighbourhood of $\sqrt{5}$ is

$$\Phi = e^{-\alpha x} \operatorname{sech}^3 x + W\alpha e^{2\alpha x}(2 - 2\tanh x - \operatorname{sech}^2 x), \tag{2.18}$$

where $\alpha = \gamma/18$. Since γ is of order $\bar{k}^2$ (it is found from perturbation analysis that $\gamma_1 = 0$; Allen and Rowlands 1993), we can treat α as small for small $\bar{k}$. Within the constant $W\alpha$, the W denotes that we are yet to make any assumptions on the relative sizes of each part, and the α ensures that as $\gamma \to 0$ the eigenfunction tends to $\operatorname{sech}^3 x$. Thus a new trial function is devised, namely

$$\begin{aligned} n = {} & A \operatorname{sech}^2 B(x - P) + He^{-\alpha B(x-P)} \operatorname{sech}^3 B(x - P) \\ & + W\alpha e^{2\alpha B(x-P)}[2 - 2\tanh B(x - P) - \operatorname{sech}^2 B(x - P)]. \end{aligned} \tag{2.19}$$

Here A, B, P, and H are as in (2.9), $W = \delta W$, and α is treated as a small constant.

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Thus the second key change that we have made from previous authors is to insist on the correct asymptotic behaviour of the trial function.

Previously, the determination of ψ, for use in the Lagrangian density (2.6), was just a matter of performing integrals of hyperbolic functions. Now the small constant α means that this is no longer the case, and so we write

$$\int_{-\infty}^{x} n(x')\,dx' = \psi = \frac{A}{B}\tanh z + \frac{H}{B}f_1(z) + \frac{W}{B}f_2(z) + J, \tag{2.20}$$

where $z = B(x - P)$, $J = J(y, t)$ and

$$f_1(z) = \int_{-\infty}^{z} e^{-\alpha z'}\,\mathrm{sech}^3 z' dz', \qquad f_2(z) = \alpha \int_{-\infty}^{z} e^{2\alpha z'}(1 - \tanh z')^2 dz'. \tag{2.21}$$

Note that, with the integration constant J in (2.20), we now have a six parameter problem. This integration constant has been neglected in previous attempts at applying the VAM to the ZK equation. As shown in Appendix A, the integration constant is essential for the conservation-of-mass law, as expressed by (2.11), to hold.

The Lagrangian $\mathscr{L}$ is calculated using (2.6) and (2.20), and is then averaged by integrating over all x, namely

$$\langle \mathscr{L} \rangle = \int_{-\infty}^{\infty} \mathscr{L}\,dx. \tag{2.22}$$

Certain definite integrals required for this process are not easily calculated. In Appendix B, we show how to determine many of these non-trivial integrals by integrating in the complex plane. From the averaged Lagrangian, we derive the six Euler–Lagrange equations (one for each parameter), and the parameters within these equations are then linearized about their unperturbed values (as shown earlier). The Euler–Lagrange equation $\partial\langle \mathscr{L} \rangle/\partial J = 0$ actually gives the conservation-of-mass equation, namely (2.11) (see Appendix A). We are left with six coupled linear equations for the parameters $\bar{A}$, $\bar{B}$, $\bar{P}$, $\bar{H}$, $\bar{W}$ and $\bar{J}$. To solve, we create a 6×6 matrix consisting of the coefficients of each parameter from each of the six equations. The determinant of this matrix is equated to zero, leaving an implicit equation for the growth rate γ. This is solved to give the dispersion relation, namely the variation of γ with k.

2.3. The dispersion relation

To begin, we check our method by setting $\alpha = 0$. Clearly, since the new trial function (2.19) becomes (2.9) in this case, the dispersion relation obtained is that shown in Fig. 1(*b*). If we now set $\alpha = \gamma/18$, we must solve an implicit equation for γ. From this equation, a dispersion relation is obtained, and this is plotted in Fig. 2 along with the true dispersion relation obtained using numerical methods. As with the trial function (2.9), the new trial function gives a dispersion relation that gives the correct instability region, but it also has the correct slope at *both* of the marginally stable points. The maximum growth rate of instability is predicted to be when $k = 1.316$, which is only 2.5% greater than the true value of $k = 1.284$. The error in our results is plotted in Fig. 3, and has a maximum of 9.9%. This degree of accuracy is something never before achieved using a Lagrangian based variational method.

We see that the key to obtaining a good dispersion relation is not only to include

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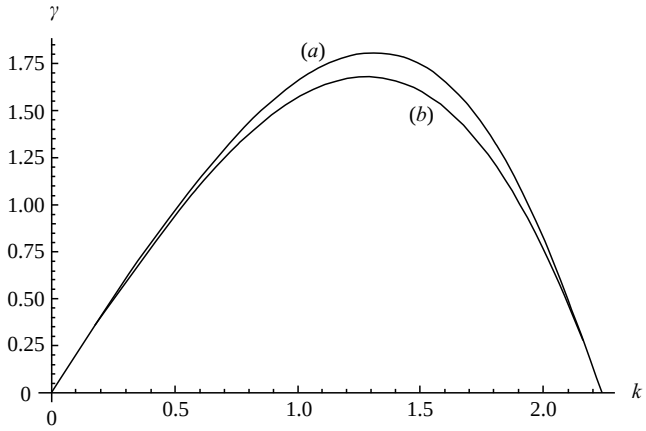


Figure 2. (*a*) The dispersion relation obtained using the VAM with our new trial function given by (2.19). (*b*) The dispersion relation obtained using numerical methods.

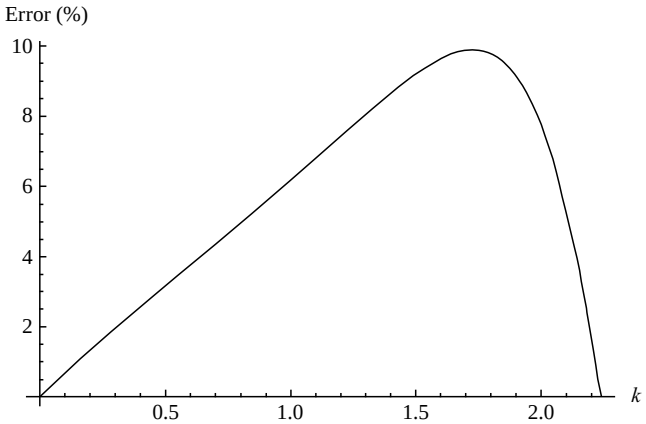


Figure 3. The percentage error of the new dispersion relation.

the stationary solution and its marginally stable eigenfunctions in the trial function, but also to ensure that the slowest decaying part of the asymptotic form of the marginally stable eigenfunction(s) is (are) correct. Put simply, this means that the chosen eigenfunction must have the correct asymptotic form around the marginally stable point(s).

In Allen and Rowlands (1993) it is shown that a very good approximation for $\gamma(k)$ can be obtained using a Padé approximation. Here information on the growth rate about $k = 0$ and $k = \sqrt{5}$, obtained using multiscale perturbation analysis, is used to determine the coefficients of the Padé approximation. This approximation has a maximum error of 0.42%, whereas our solution has an error of 9.9%. Clearly the use of multiscale perturbation analysis and a Padé approximation is the more accurate method for determining the dispersion relation. But when using the VAM, the Euler–Lagrange equations do not have to be linearized, and so – in theory at least – a full nonlinear solution can be sought.

All the detailed working has been done using the mathematical computer package

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Mathematica (Wolfram 1991). The true dispersion relation is taken as the numerical solution to the linear equation (2.5), given in Allen and Rowlands (1993).

3. Perturbing a plane soliton solution to the KP+ equation

The two-dimensional KP+ equation (Kadomstev and Petviashvili 1970) is

$$(n_t + nn_x + n_{xxx})_x - n_{yy} = 0. \tag{3.1}$$

Since this equation reduces to the KdV equation in one dimension, the stationary soliton solution is $n = 12\eta^2 \operatorname{sech}^2[\eta(x - x_0 - 4\eta^2 t)]$. As for the ZK equation we transform the equation to a reduced-variables coordinate system moving with the unperturbed soliton. This is achieved by a similar transformation to that used for the ZK equation, namely

$$(x, y, t) \mapsto \left(\frac{x + 4t}{\eta}, \frac{y}{\eta^2}, \frac{t}{\eta^3}\right), \qquad n \mapsto \eta^2 n, \tag{3.2}$$

which gives

$$[n_t + (n - 4)n_x + n_{xxx}]_x - n_{yy} = 0, \tag{3.3}$$

which has the stationary solution $n_0 = 12 \operatorname{sech}^2 x$. Equation (3.3) is derivable from a Lagrangian density, namely

$$\mathscr{L} = \tfrac{1}{2}\psi_t\psi_x + \tfrac{1}{6}\psi_x^3 - \tfrac{1}{2}\psi_{xx}^2 - \tfrac{1}{2}\psi_y^2 - 2\psi_x^2, \tag{3.4}$$

via the VAM, where $\psi_x = n$. Earlier attempts at using the linear VAM to study perpendicular perturbations to the soliton solutions of the KP+ equation failed (Katyshev and Makhankov 1976; Laedke and Spatschek 1979). In Katyshev and Makhankov (1976), the trial function used was

$$n = 12B^2 \operatorname{sech}^2 B(x - M), \tag{3.5}$$

where $B = B(y, t)$ and $M = M(y, t)$ are the variable parameters. From such a trial function, the dispersion relation obtained is

$$\gamma = \frac{4k}{\sqrt{3}}\left(1 - \frac{3 + \pi^2}{144}k^2\right)^{1/2}. \tag{3.6}$$

This expression has the correct value and slope for $k = 0$ but gives the region of instability as $0 < k < (3 + \pi^2)/144$, which, as proved in Laedke and Spatschek (1979), is larger than the true region of instability, namely $0 < k < \sqrt{3}$.

When linearized about the soliton solution n_0, (3.3) becomes

$$\frac{d^2}{dx^2}L_0\Phi = -\gamma\frac{d\Phi}{dx} - k^2\Phi, \tag{3.7}$$

where $L_0 = (d^2/dx^2) + n_0 - 4$ as before. It has been shown (Allen 1994) that this linearized equation has the solution

$$\Phi = e^{-\alpha x}[\operatorname{sech}^2 x \tanh x + \alpha \operatorname{sech}^2 x - \tfrac{1}{2}\alpha^2(1 + \tanh x)], \tag{3.8}$$

where $\alpha = 1 - (1 - k/\sqrt{3})^{1/2}$. In this section we use (3.8) to obtain a trial function that gives the exact dispersion relation by only varying two parameters.

As a trial function we use

$$n = 12 \operatorname{sech}^2 z + He^{-\alpha z}[\operatorname{sech}^2 z \tanh z + \alpha \operatorname{sech}^2 z - \tfrac{1}{2}\alpha^2(1 + \tanh z)]. \tag{3.9}$$

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Clearly, since we have used the *exact* eigenfunction (3.8) in the trial function (3.9), the requirement that the trial function have the correct asymptotic form is automatically satisfied. When integrated, (3.9) becomes

$$\int_{-\infty}^{x} n(x')dx' = \psi$$
$$= 12\tanh z + \tfrac{1}{2}H\,e^{-\alpha z}\,\mathrm{sech}^2 z\,(\cosh z + \sinh z)[(\alpha-1)\cosh z + \sinh z], \quad (3.10)$$

where $z = x - P$, $\alpha = 1-(1-k/\sqrt{3})^{1/2}$, and H and P are parameters dependent on y and t (because the KP+ equation does not conserve mass, we have no parameter J as before). Again using definite integrals such as those in Appendix B, we obtain an averaged Lagrangian, and from this two Euler equations. These are linearized and solved to give

$$\gamma = \frac{4k}{\sqrt{3}}\left(1-\frac{k}{\sqrt{3}}\right)^{1/2}, \quad (3.11)$$

which is shown in Allen (1994) to be the exact dispersion relation.

We now ask whether this exact result can guide us in our choice of trial function for the ZK equation. From the result obtained in this section, we should expect to obtain an accurate dispersion relation by constructing the trial function from the plane soliton and a good approximation to the eigenfunction (for all k), by varying only the phase and the relative amplitude of the eigenfunction. Unfortunately we were unable to determine a suitable eigenfunction. It seems that to have an eigenfunction with the correct asymptotic behaviour at both $k = 0$ and $k = \sqrt{5}$, one requires a function implicitly dependent upon k and γ. Since the trial function is then integrated over all space, we obtain a relationship that is extremely difficult to solve, which is counter to our desire to obtain a simple but accurate VAM.

4. Conclusions

In this paper we have used the linear VAM to determine the initial growth rate of perpendicular perturbations to plane soliton solutions of the ZK and KP+ equations. For the ZK equation, a new form of trial function is chosen to include a good approximation to the eigenfunction of the linearized ZK equation around both $k = 0$ and $k = \sqrt{5}$, and also to satisfy mass conservation. Further, our trial function differs significantly from previous ones because we include the very slow decay of the linearized solution as $x \to -\infty$, and it is this inclusion that leads to the substantial improvement in the results.

The linear VAM is used to give an implicit equation for the growth rate γ. We obtain a solution that has the correct instability region, has the correct slopes at $k = 0$ and $k = \sqrt{5}$, and has a maximum error of less than 9.9%. This is a great improvement on the best result of Allen (1994), although is not as accurate as the approximation obtained using a perturbative method in Allen and Rowlands (1993). Also, by including a constant of integration as a parameter dependent on y and t, our new trial function *automatically* conserves mass. Though better agreement could undoubtably be obtained by using better eigenfunctions containing more variational constants, the complexity of the analysis grows quickly with such changes and the simplicity of the method is lost.

We go on to consider the stability of a plane soliton solution to the KP+ equation. By using knowledge of the exact eigenfunction solution to the linear equation,

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we chose a trial function in which only the phase and relative amplitude of the eigenfunction are variable parameters. From this, the *exact* dispersion relation is obtained.

Finally, in Laedke and Spatschek (1979) it is stated that for the VAM, 'the proper choice of the trial function is somehow a game of chance'. We disagree with this statement, at least when applied to the ZK and KP+ equations, and feel that the analysis and results of this paper verify our opinion.

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Appendix A. Conservation law of the Zakharov–Kuznetsov equation

If we can write an evolution equation in the form

$$T_t + X_x = 0, \tag{A 1}$$

where T is known as the density, X the flux, and $X \to$ constant as $|x| \to \infty$, then

$$\int_{-\infty}^{\infty} T\,dx = C, \tag{A 2}$$

where C is a constant of the motion.

Only five conservation laws have been found for the ZK equation (note that this is one reason why this equation is thought to be non-integrable, since integrable equations have an infinite number of conservation laws). The simplest of the five laws is the conservation of mass along the direction parallel to the field. This is found by rewriting the ZK equation (2.3) as

$$n_t + [\tfrac{1}{2}(n-4)^2 + n_{xx} + n_{yy}]_x = 0. \tag{A 3}$$

Since $\frac{1}{2}(n-4)^2 + n_{xx} + n_{yy} \to$ constant as $|x| \to \infty$, we have

$$\int_{-\infty}^{\infty} n\,dx = C, \tag{A 4}$$

where C is some constant. Now consider the plane soliton solution of (2.3), namely $n_0 = 12\,\mathrm{sech}^2 x$, which, when substituted into (A 4), gives

$$\int_{-\infty}^{\infty} 12\,\mathrm{sech}^2 x\,dx = 24. \tag{A 5}$$

Having determined $C = 24$, we substitute our trial function (2.19) into (A 4) to determine a relationship between the parameters so that mass is conserved. It is found that

$$4A - 48B + (1-\alpha^2)\pi\,\mathrm{sec}(\tfrac{1}{2}\alpha\pi)H + 4(1-\alpha)\pi\,\mathrm{cosec}(\alpha\pi)W = 0. \tag{A 6}$$

Note that if an integration constant, call it $J(y,t)$, is included as an extra parameter in ψ (see (2.20)) then this parameter will only appear in the Lagrangian (2.6) as

$$\mathscr{L} \propto (J_t \times n) + \ldots, \tag{A 7}$$

which, when integrated over all x gives a contribution

$$\langle \mathscr{L} \rangle \propto (J_t \times \langle n \rangle) + \ldots, \tag{A 8}$$

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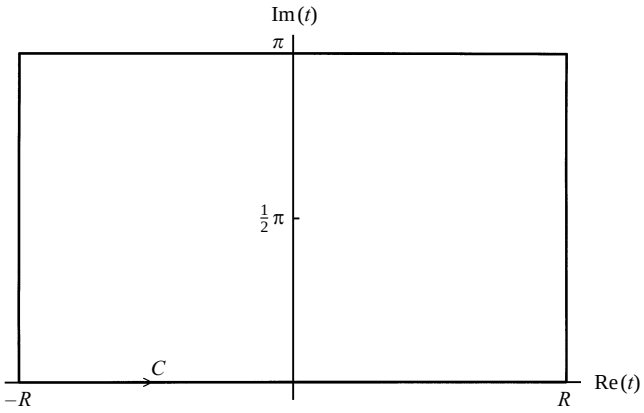


Figure 4. Path of integration in the complex plane.

and so the Euler equation $\partial\langle\mathscr{L}\rangle/\partial J = 0$ gives

$$\frac{\partial\langle n\rangle}{\partial t} = 0. \tag{A 9}$$

Thus (A 6) shows that the mass is automatically conserved in our analysis, whereas this is not the case in previous attempts at applying the VAM to the ZK equation, since the constant of integration J has not been included.

Appendix B. Definite integrals in the complex plane

The averaged Lagrangian involves performing non-trivial definite integrals, namely

$$I_1 = \int_{-\infty}^{\infty} f_1(z)\,dz = \int_{-\infty}^{\infty} e^{mz}\,\mathrm{sech}^n z\,dz, \tag{B 1a}$$

$$I_2 = \int_{-\infty}^{\infty} f_2(z)\,dz = \int_{-\infty}^{\infty} e^{mz}\,\mathrm{sech}^n z\tanh z\,dz, \tag{B 1b}$$

$$I_3 = \int_{-\infty}^{\infty} f_3(z)\,dz = \int_{-\infty}^{\infty} ze^{mz}\,\mathrm{sech}^n z\,dz, \tag{B 1c}$$

$$I_4 = \int_{-\infty}^{\infty} f_4(z)\,dz = \int_{-\infty}^{\infty} ze^{mz}\,\mathrm{sech}^n z\tanh z\,dz, \tag{B 1d}$$

$$I_5 = \int_{-\infty}^{\infty} f_5(z)\,dz = \int_{-\infty}^{\infty} z^2e^{mz}\,\mathrm{sech}^n z\,dz, \tag{B 1e}$$

$$I_6 = \int_{-\infty}^{\infty} f_6(z)\,dz = \int_{-\infty}^{\infty} z^2e^{mz}\,\mathrm{sech}^n z\tanh z\,dz. \tag{B 1f}$$

Such definite integrals are best evaluated using integration in the complex plane. We consider I_1, although the method for determining the other integrals is analogous. The function integrated to give I_1 has a pole at $z = \frac{1}{2}\pi i$. To find the value of I_1, we find the residue at this pole, and, integrating along the rectangular path shown in Fig. 4, appeal to Cauchy's residue theorem (Priestley 1990), p. 106), namely

$$\oint_c f(z)\,dz = 2\pi i\sum_{k=1}^{m}\mathrm{res}\{f(z); z_k\}, \tag{B 2}$$

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where z_k are the poles of the function f. Thus we write

$$\int_{C_1} f_1(z)\,dz + \int_{C_2} f_1(z)\,dz + \int_{C_3} f_1(z)\,dz + \int_{C_4} f_1(z)\,dz = 2\pi i \operatorname{res}\{f_1(z); \tfrac{1}{2}\pi i\}, \quad \text{(B 3)}$$

where C_1, C_2, C_3 and C_4 are the four sides of the rectangular path. The path chosen contains one pole at $z = \frac{1}{2}i\pi = z_0$. To find the residue of the function at this pole, we let $z = z_0 + \delta z$ and find the coefficient of δz^{-1}. First we note that

$$f_1(z) = e^{mz}\operatorname{sech}^n z = \frac{2^n e^{(m+n)z}}{(1+e^{2z})^n}, \quad \text{(B 4)}$$

and so

$$\operatorname{res}\{f_1(z); \tfrac{1}{2}\pi i\} = 2^n e^{(m+n)\pi i/2} \operatorname{res}\left[\frac{e^{(m+n)\delta z}}{(1-e^{2\delta z})^n}\right], \quad \text{(B 5)}$$

where $\operatorname{res}[g(\delta z)]$ denotes the coefficient of the $1/\delta$ term in g. We now consider the integral around C. The integral along the lower horizontal is

$$\lim_{R\to\infty} \int f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \equiv I_1, \quad \text{(B 6)}$$

whilst the integral along the upper horizontal is

$$\begin{aligned}\lim_{R\to\infty} \int f(z)\,dz &= \int_{\infty}^{-\infty} e^{(m+n)i\pi} f(x)\,dx, \\ &= -e^{(m+n)i\pi} I_1. \end{aligned} \quad \text{(B 7)}$$

The integrals along the vertical contours do not contribute as $R \to \infty$.

Using this information and Cauchy's residue theorem, we write

$$I_1 - e^{(m+n)i\pi} I_1 = 2^{n+1}\pi i e^{(m+n)\pi i/2} \operatorname{res}\left[\frac{e^{(m+n)\delta z}}{(1-e^{2\delta z})^n}\right], \quad \text{(B 8)}$$

which can be simplified to

$$I_1 = -2^n \pi \operatorname{cosec}\left[\frac{(m+n)\pi}{2}\right] \operatorname{res}\left[\frac{e^{(m+n)\delta z}}{(1-e^{2\delta z})^n}\right]. \quad \text{(B 9)}$$

Again using Cauchy's residue theorem, we find that

$$I_2 = 2^n \pi \operatorname{cosec}\left[\frac{(m+n)\pi}{2}\right] \operatorname{res}\left[\frac{e^{(m+n)\delta z}(1+e^{2\delta z})}{(1-e^{2\delta z})^{n+1}}\right], \quad \text{(B 10}a\text{)}$$

$$I_3 = -\pi \operatorname{cosec}\left[\frac{(m+n)\pi}{2}\right] \left\{2^n \operatorname{res}\left[\frac{\delta z e^{(m+n)\delta z}}{(1-e^{2\delta z})^n}\right] \frac{I_1}{2} \cos\left[\frac{(m+n)\pi}{2}\right]\right\}, \quad \text{(B 10}b\text{)}$$

$$I_4 = \pi \operatorname{cosec}\left[\frac{(m+n)\pi}{2}\right] \left\{2^n \operatorname{res}\left[\frac{\delta z(1+e^{2\delta z})e^{(m+n)\delta z}}{(1-e^{2\delta z})^{n+1}}\right] - \frac{I_2}{2}\cos\left[\frac{(m+n)\pi}{2}\right]\right\}, \quad \text{(B 10}c\text{)}$$

$$I_5 = \pi \operatorname{cosec}\left[\frac{(m+n)\pi}{2}\right] \left\{2^n \operatorname{res}\left[\frac{\delta z^2 e^{(m+n)\delta z}}{(1-e^{2\delta z})^n}\right] - I_3 \cos\left[\frac{(m+n)\pi}{2}\right] + \frac{\pi I_1}{4}\sin\left[\frac{(m+n)\pi}{2}\right]\right\}, \quad \text{(B 10}d\text{)}$$

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$$I_6 = \pi \operatorname{cosec}\left[\frac{(m+n)\pi}{2}\right]\left\{2^n \operatorname{res}\left[\frac{\delta z^2(1+e^{2\delta z})e^{(m+n)\delta z}}{(1-e^{2\delta z})^{n+1}}\right]\right.$$
$$\left. -I_4 \cos\left[\frac{(m+n)\pi}{2}\right] + \frac{\pi I_2}{4}\sin\left[\frac{(m+n)\pi}{2}\right]\right\}. \quad \text{(B 10}e\text{)}$$

We also have to determine the integral

$$I_s = \int_{-\infty}^{\infty} f_2\, e^{-\alpha z}\operatorname{sech}^3 z\, dz, \quad \text{(B 11)}$$

where f_2 is given by (2.21), namely

$$f_2 = \int_{-\infty}^{z} \alpha e^{2\alpha z'}(2 - \operatorname{sech}^2 z' - 2\tanh z')\, dz',$$
$$= \int_{-\infty}^{z} \frac{4\alpha e^{2\alpha z'}}{(1+e^{2z'})^2}\, dz'. \quad \text{(B 12)}$$

Using the mathematical computer package *Mathematica* (Wolfram 1991), we find that

$$\int_{-\infty}^{z} \frac{e^{2\alpha z'}}{(1+e^{2z'})^2}\, dz' = \frac{e^{2\alpha z}}{2(1+e^{2z})} + \frac{1}{2\alpha}(1-\alpha)e^{2\alpha z}\, {}_2F_1[\alpha, 1; 1+\alpha; -e^{2z}], \quad \text{(B 13)}$$

where ${}_2F_1$ denotes the hypergeometric function (see Abramowitz and Stegun 1965, p. 556). Equation (B 13) is then substituted into (B 12), which is substituted into the integral I_s. For a particular value of α ($= \gamma/18$), I_s is determined using the numerical integrate procedure of *Mathematica*.

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