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# PROVABLY $\Delta_1$ GAMES

J. P. AGUILERA AND D. W. BLUE

**Abstract.** We isolate two abstract determinacy theorems for games of length $\omega_1$ from work of Neeman and use them to conclude, from large-cardinal assumptions and an iterability hypothesis in the region of measurable Woodin cardinals that

(1) if the Continuum Hypothesis holds, then all games of length $\omega_1$ which are provably $\Delta_1$-definable from a universally Baire parameter (in first-order or $\Omega$-logic) are determined;
(2) all games of length $\omega_1$ with payoff constructible relative to the play are determined; and
(3) if the Continuum Hypothesis holds, then there is a model of ZFC containing all reals in which all games of length $\omega_1$ definable from real and ordinal parameters are determined.

**§1. Introduction.** The study of games of infinite length has proven to be fruitful, particularly in solving problems in descriptive set theory. Such games usually have the following form. Let $A \subset \mathbb{R}$. Two players, Player I and Player II, alternate playing natural numbers infinitely often, collaborating to produce an infinite sequence $x \in \omega^\omega$. Since $\omega^\omega$ is homeomorphic to the irrationals, there is no harm in identifying it with $\mathbb{R}$. Thus Player I and Player II produce a real number, and the game is won by Player I if that real is in $A$ and by Player II if not.

The statement that a class of these games is *determined* means that one of the players has a winning strategy, a function from initial moves in the game to further moves such that if the player plays according to the moves dictated by the function, they are guaranteed to defeat any play by their opponent. Many classes of games are known to be determined. It is a landmark result of Martin [6] that all games with Borel payoff are determined. If there exist infinitely many Woodin cardinals and $A$ is projective (i.e., obtained from a closed set by applying complements and continuous functions finitely many times), then $A$ is determined, as follows from work of Martin and Steel [7]; if, additionally, there is a measurable cardinal above the Woodin cardinals and $A \in L(\mathbb{R})$, then $A$ is determined, as follows from work of Woodin [17].

What about the determinacy of games in which players collaborate to play a sequence of natural numbers (or real numbers) of length greater than $\omega$? Games of length $\alpha$, for $\alpha$ countable, have been shown to be determined by Neeman [11], provided that the payoff set is reasonably definable and sufficiently large cardinals exist. Regarding games of uncountable length, we have the following limitative results, whose proofs we include for the reader's convenience.

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PROPOSITION 1.1. *Work in* ZF.

(1) (*Mycielski* [9]) *There is a nondetermined game of length* $\omega_1$.
(2) (*Folklore*) *There is a nondetermined game of length* $\omega_1 + \omega$ *with payoff set definable by a formula in the language of set theory.*

PROOF. For the first claim, consider the following game: Player I begins by playing a code of an ordinal $\alpha$ during the first $\omega$-many rounds. Then, Player II plays $\alpha$-many distinct reals. Clearly Player I cannot have a winning strategy, and from any winning strategy for Player II one can extract an $\omega_1$-sequence of distinct reals. Recall that determinacy of all games of length $\omega$ implies that there is no uncountable (wellordered) sequence of distinct reals. It follows that if this game is determined, then there is some other game of length $\omega_1$ which is not determined.

For the second claim, there are three cases. If there is no uncountable wellordered sequence of distinct reals, then the previous paragraph defines a nondetermined game of length $\omega_1$.

If there is an $\omega_1$-sequence of reals enumerating a perfect subset of $\mathbb{R}$, then there is one enumerating all of $\mathbb{R}$. It is well known that from an $\omega_1$-sequence of reals one can (uniformly) extract a nondetermined game, so we consider the following game: Player I begins by enumerating all reals during the first $\omega_1$-turns, after which the two players play the nondetermined game.

The final possibility is that there are $\omega_1$-sequences of reals, but all such sequences do not enumerate perfect sets. A well-known result of Davis [2] states that for every $A \subset \mathbb{R}$ there is a game (defined uniformly) whose determinacy implies that $A$ has the perfect set property. We consider the following game: Player I begins by enumerating $\omega_1$-many reals during the first $\omega_1$-turns, after which the two players play the perfect-set game for the enumerated set. ⊣

In view of this proposition, one is left with the question of whether *definable* games of length $\omega_1$ are determined.

Substantial work in the study of such games was carried out by Neeman, who showed the following. Identifying a game with the payoff set for Player I, a game $X$ of length $\omega_1$ is *open* if there is a formula $\varphi$ in the language of set theory such that

$$X = \{f \in 2^{\omega_1} : \exists \alpha < \omega_1 \, \varphi(f \restriction \alpha)\}.$$

THEOREM 1.2 (Neeman [12]). *Assume there is a countable, iterable inner model of* ZFC+ *"there is a measurable Woodin cardinal." Then all open games of length* $\omega_1$ *are determined. In fact, every game with payoff of the form*

$$\{f \in 2^{\omega_1} : (L_{\omega_1}[f], f) \models \varphi\}$$

*is determined, where* $\varphi$ *is a formula in the language of set theory with an additional predicate symbol.*

Neeman's result in fact extends to an even wider class of games (see Neeman [12]) employing nondefinable parameters. His hypothesis is also slightly weaker than the one we stated (in fact, the hypothesis can be replaced by the one used by Neeman throughout this article). Moreover, the techniques developed to prove Theorem 1.2 are very general. Using Neeman's theorem, Woodin proved from a Woodin cardinal which is a limit of Woodin cardinals that it is consistent that *all* games of length $\omega_1$

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definable from parameters in $\mathbb{R}$ and Ord are determined, see [11, Exercise 7F.15]. This establishes that one cannot refute determinacy at the level of length $\omega_1$ games. Yet the question of whether one can *prove* this determinacy remains open.

The following is the analogue of Martin's theorem on the determinacy of Borel games in the context of $\omega_1$.

THEOREM 1.3. *Assume that there is a countable, iterable inner model with a measurable Woodin cardinal. Suppose $X \subset 2^{\omega_1}$ is provably $\Delta_1$-definable from $\omega_1$. Then $X$ is determined.*

Theorem 1.3 can be proved by following Neeman's argument for Theorem 1.2, with a few changes. What we will do in this article is pursue that train of thought to isolate an abstract determinacy theorem (Theorem 3.4 below) from Neeman's work and use it to conclude Theorem 1.3.

Theorem 1.3 is close to being optimal, as Larson [5] has shown that Martin's Maximum implies that there is a nondetermined game which is $\Sigma_1$-definable. Nevertheless, using the abstract determinacy theorem, we will see that Theorem 1.3 admits generalizations from reasonable hypotheses. For instance:

THEOREM 1.4. *Assume the Continuum Hypothesis and that there is a countable, iterable inner model with a measurable Woodin cardinal. Suppose $X \subset 2^{\omega_1}$ is provably $\Delta_1$-definable from $\mathbb{R}$. Then $X$ is determined.*

THEOREM 1.5. *Suppose $\varphi$ is a formula in the language of set theory with an additional predicate constant. Let $M$ be an active, countable $\omega_1+1$-iterable extender model with a measurable Woodin cardinal $\theta$ below a measurable cardinal $\kappa$. Then, the set*

$$X = \left\{ f \in 2^{\omega_1} : (L[f], f) \models \varphi(f) \right\}.$$

*is determined.*

Although Theorem 1.5 subsumes Theorem 1.2 as stated, we emphasize that Neeman's work [12] extends to other classes of games that are not given in terms of usual notions of definability. It could be that there are other interesting classes of games that can be proved to be determined from the same hypothesis.

We mentioned earlier Woodin's theorem that it is consistent that all definable games of length $\omega_1$ are determined. By arguing similarly, we can obtain models of set theory satisfying this conclusion which additionally contain all reals.

THEOREM 1.6. *Suppose there is an active, countable $\omega_1+1$-iterable extender model with a measurable Woodin cardinal $\theta$ below a measurable cardinal $\kappa$ and that the Continuum Hypothesis holds. Then, there is a transitive $M \models \mathsf{ZFC}$ containing $\mathbb{R}$ such that if $X$ is a game ordinal-definable from real parameters, then*

$$M \models \text{“}X \text{ is determined.”}$$

We will also be able to weaken the assumption that the game is provably $\Delta_1$-definable. To do so, we need to consider universally Baire sets.

DEFINITION 1.7 (Feng, Magidor, Woodin [3]). A set $A \subseteq \mathbb{R}$ is *universally Baire* if for every compact Hausdorff space $X$ and every continuous function $f : X \to \mathbb{R}$,

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$f^{-1}[A]$ has the property of Baire in $X$. We denote by $\Gamma^\infty$ the set of all universally Baire sets of reals.

Universally Baire sets yield a generalization of provability called $\Omega$-*provability*. In $\Omega$-logic, proofs are not natural numbers, but members of $\Gamma^\infty$ (see Section 2).

DEFINITION 1.8. Suppose $A \in \Gamma^\infty$ and let $X \subset 2^{\omega_1}$. We say $X$ is $\Omega$-*provably* $\Delta_1(A)$-*definable* if there are $\Sigma_1$-formulae $\varphi$ and $\psi$ in the language of set theory with two extra predicates such that

(1) $X = \{f \in 2^{\omega_1} : \varphi(f, A)\}$, and
(2) $\mathsf{ZFC} \vdash_\Omega \forall f \in 2^{\omega_1}\, \varphi(f, A) \leftrightarrow \neg\psi(f, A)$.

In this definition, ZFC could be replaced by any theory, as long as it holds in both $V$ and in a suitable canonical inner model.

Determinacy for this class of games is shown under a large cardinal assumption postulating the existence of certain inner models called *A-models* (see Section 2). Roughly, if $A$ is a universally Baire set, an $A$-model is a countable model that has a measurable Woodin cardinal below an inaccessible and iterates in a way that respects $A$.

THEOREM 1.9. *Suppose $A \in \Gamma^\infty$ and that $X \subset 2^{\omega_1}$ is $\Omega$-provably $\Delta_1(A)$-definable. Assume that*

(1) *the Continuum Hypothesis holds,*
(2) *there is a proper class of Woodin cardinals, and*
(3) *for every universally Baire set $B$, there is a $B$-model.*

*Then $X$ is determined.*

*Outline*. Our main technical determinacy theorem, Theorem 3.4, is proved in Section 3. In Section 4, we present applications of Theorem 3.4. These do not require the proof, but they do require the basic theory of $\Omega$-logic, which we recall in Section 2.

**§2. Ω-logic.** We recall some elements of $\Omega$-logic (see also Woodin [18, 19] and Bagaria–Castells–Larson [1]), beginning with a useful characterization of the collection of universally Baire sets. Recall that if $T$ is a tree on $\omega \times X$, then the projection $p[T]$ of $T$ is the set $\{x \in \mathbb{R} : \forall n \in \mathbb{N}\, \exists f \in X^n\, (x \restriction n, f) \in T\}$.

THEOREM 2.1 (Feng, Magidor, Woodin [3]). *Let $A \subset \mathbb{R}$. The following are equivalent*:

(1) *$A$ is universally Baire.*
(2) *For every cardinal $\kappa$, there exist some ordinal $\lambda$ and a pair of trees $T, S$ on $\omega \times \lambda$ such that $A = p[T]$ and*

$$p[T] = \mathbb{R} \setminus p[S]$$

*in every forcing extension by a partial order of size $< \kappa$.*
(3) *For every partial order $\mathbb{P}$, there is a pair of trees $T$, $S$ on $\omega \times 2^{|\mathbb{P}|}$ such that $A = p[T]$ and*

$$\Vdash^V_{\mathbb{P}} p[T] = \mathbb{R} \setminus p[S].$$

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The language of Ω-logic is the usual language of set theory. We associate to it a satisfaction relation.

DEFINITION 2.2 (Ω-satisfaction). Let $T$ be a theory in the language of set theory. We write $T \models_\Omega \varphi$ if for all set generic extensions $V[G]$ of $V$ and all ordinals $\alpha$ such that $(V_\alpha)^{V[G]} \models T$, we have

$$(V_\alpha)^{V[G]} \models \varphi.$$

Now let $\kappa$ be a cardinal, $A \subset \mathbb{R}$ be universally Baire, and let $\mathbb{P}$ be a partial order of cardinality $< \kappa$. The set $A$ has a canonical interpretation in any extension $V[g]$, where $g \subset \mathbb{P}$ is generic: letting $T$ be the tree given by Theorem 2.1 (for $\kappa$), we define

$$A_g = (p[T])^{V[g]}.$$

Alternatively, we define

$$A_g = \bigcup_T \{p[T]^{V[g]} : A = p[T]^V\}.$$

This latter definition has the possible advantage that $A_g$ is defined regardless of whether $A$ is universally Baire or not, although it might not be very meaningful in that case. We sometimes abuse notation and write $A$ instead of $A_g$ if this does not seem to cause confusion.

DEFINITION 2.3. Let $A \subseteq \mathbb{R}$ be universally Baire and let $M$ be a transitive model of ZFC.

(1) $M$ is *$A$-closed* if for every partial order $\mathbb{P} \in M$ and every $V$-generic $g \subset \mathbb{P}$, we have

$$V[g] \models M[g] \cap A_g \in M[g].$$

(2) $M$ is *strongly $A$-closed* if for any generic extension $N$ of $M$ by a filter in $V$, $A \cap N \in N$.

Both the notion of $A$-closure and that of strong $A$-closure remain unchanged if one only requires the corresponding defining property to hold of generic extensions by collapse algebras. By Theorem 2.37 of [1], if there is a proper class of Woodin cardinals and $A$ is a universally Baire set of reals, then there is a universally Baire set $B$ such that every countable, $B$-closed model of ZFC is strongly $A$-closed. By Proposition 2.11 of [1], every countable, strongly $A$-closed model is also $A$-closed.

DEFINITION 2.4 (Ω-provability). Suppose there is a proper class of Woodin cardinals and $T$ is a recursive theory in the language of set theory. We write $T \vdash_\Omega \varphi$ if there is a universally Baire $A \subseteq \mathbb{R}$ such that for all countable transitive sets $M$, if $M$ is $A$-closed, then

$$M \models \text{“}T \models_\Omega \varphi.\text{”}$$

By the comment preceding Definition 2.4, the definition of $\vdash_\Omega$ does not change if one replaces $A$-closure with strong $A$-closure. One can also ask, instead of $M \models$ “$T \models_\Omega \varphi$,” that $(V_\alpha)^M \models \varphi$ whenever $(V_\alpha)^M \models T$ and $M$ is $A$-closed.

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The *term relation* (extended to all partial orders) for $A$ is defined to be the collection

$$\tau_A^\infty = \{(\gamma, p, \sigma) : p \Vdash^V_{\mathrm{Coll}(\omega,\gamma)} \sigma \in A_g\}.$$

(Note that $\tau_A^\infty$ is defined even if one does not assume that $A$ is universally Baire.)

The term relation is, of course, a proper class. By a theorem of Woodin, a model $M$ is $A$-closed if and only if it contains all initial segments of the term relation for $A$, that is, if and only if $\tau_A^\infty \cap (V_\alpha)^M \in M$ for every $\alpha \in M$ (see Woodin [19, p. 128]). The term relation can also be defined in terms of all partial orders, instead of only collapse algebras; we will sometimes identify these two formulations.

DEFINITION 2.5 (Strong $A$-iterability). Let $A \subset \mathbb{R}$ be universally Baire and let $M$ be an $A$-closed countable transitive model of ZFC+ "there is a measurable Woodin cardinal." Then $M$ is *strongly $A$-iterable* if there is a universally Baire $(\omega_1+1)$-iteration strategy $\Sigma$ such that whenever $N$ is an iterate of $M$ by $\Sigma$ and $j$ is the embedding, the following hold:

(1) $N$ is $A$-closed, and
(2) for every ordinal $\alpha$, $j(\tau_A^\infty \cap (V_\alpha)^M) = \tau_A^\infty \cap (V_{j(\alpha)})^N$.

We can now state the large cardinal hypothesis we use in proving our abstract determinacy result.

DEFINITION 2.6 ($A$-model). Let $A \subset \mathbb{R}$ be universally Baire. An *$A$-model* is a strongly $A$-closed, strongly $A$-iterable countable transitive model $M$ of ZFC+ "there is a measurable Woodin cardinal below an inaccessible cardinal."

Rather than a true measurable Woodin cardinal below an inaccessible within each $A$-model, all we need is the weaker assumption of an *active* Woodin cardinal (as in Neeman [12]). This cardinal can be iterated to yield a proper class model with indiscernible Woodin cardinals and in particular gives us the inaccessible above for free. However, we shall stick with the more familiar hypothesis.

We extend the definitions of $\Omega$-satisfaction and $\Omega$-provability to sentences in the language of set theory with an added unary predicate for a universally Baire set. This is done in the natural way:

DEFINITION 2.7 (Relativized $\Omega$-satisfaction). Let $T$ be a theory in the language of set theory with an added unary predicate symbol. Suppose that $T$ extends ZFC and that $A$ is a set of reals. We write $T \models_\Omega \varphi(A)$ if for all generic extensions $V[G]$ of $V$ and all ordinals $\alpha$ such that $(V_\alpha)^{V[G]} \models T$, we have

$$(V_\alpha)^{V[G]} \models \varphi(A_G).$$

We will usually be interested in $\Omega$-satisfiability in the case that $A$ is a universally Baire set of reals. The definition makes sense regardless. Similarly, we define:

DEFINITION 2.8 (Relativized $\Omega$-provability). Let $T$ be a theory in the language of set theory with an added unary predicate symbol. Suppose there is a proper class of Woodin cardinals, $A$ is a universally Baire set of reals, and $T$ extends ZFC. We write $T \vdash_\Omega \varphi(A)$ if there is a universally Baire $B \subseteq \mathbb{R}$ such that $A$ continuously reduces to

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$B$ and for all countable transitive sets $M$ and all ordinals $\alpha$, if $M$ is $B$-closed, then

$$M \models \text{“}T \models_\Omega \varphi(A)\text{.”}$$

The following result is the soundness theorem for $\Omega$-logic.

THEOREM 2.9 (Woodin, see [19]). *Suppose that there is a proper class of Woodin cardinals. Then, for every $\varphi$ and every universally Baire set $A$ and every theory $T$ in the language of set theory with $\mathsf{ZFC} \subset T$, we have*

$$T \models_\Omega \varphi(A) \to T \vdash_\Omega \varphi(A).$$

Let us say something about our large-cardinal hypothesis. It is a widely believed conjecture that the existence of countable, fine-structural, iterable models with measurable Woodin cardinals is provable in the theory $\mathsf{ZFC}$ + “there is a measurable Woodin cardinal.” This and stronger forms of the conjecture past superstrong cardinals have been verified by Neeman–Steel [13] and Woodin [20] conditioned on iterability hypotheses. Neeman [10] has verified that the conjecture holds for Woodin cardinals which are limits of Woodin cardinals, unconditionally. Recent advances in Inner Model Theory (see, e.g., Sargsyan [14] and Sargsyan–Trang [15]) provide progress towards a positive answer. Empirically, one observes that proofs of the existence of fine-structural models relativize to allow universally Baire parameters and provide universally Baire iteration strategies.

**§3. An abstract determinacy theorem.** In this section, we prove the following abstract determinacy theorem for games of length $\omega_1$. It is a variation of Neeman's Theorem 1.2 which gives us the extra information we will need; it asserts the existence of many strategies that are winning (in a certain sense). Here, a strategy is just a function

$$\sigma : \bigcup_{\alpha<\omega_1} 2^\alpha \to 2.$$

Given a strategy $\sigma$, we say a play $f \in 2^{\omega_1}$ results by Player I (Player II) following $\sigma$ if

$$\sigma(f \restriction \alpha) = f(\alpha) \text{ for every even (odd) } \alpha < \omega_1.$$

In our statement of the theorem, the generic filter $g$ is a subset of a version of the extender algebra which was introduced by Neeman [11].

THEOREM 3.1. *Suppose that $M$ is an active, countable extender model with a measurable Woodin cardinal $\theta$ and that $\Sigma$ is an $\omega_1+1$-iteration strategy for $M$. Let $\phi(x_0,\dots,x_n)$ be a formula in the language of set theory and $a_1,\dots,a_n$ be parameters in $M$. Then, there is a strategy $\sigma \in V$ and a formula $\psi$ evaluated with parameters in $M$ such that one of the following holds*:

(1) *$M \models \psi$ and whenever $f$ results from Player I following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $g \in V$ which is $N$-generic such that, letting $j : M \to N$ be the iteration embedding, the following hold*:
   (a) $\omega_1^{N[g]} = \omega_1 = j(\theta)$;
   (b) $f \in N[g]$;
   (c) $N[g] \models \phi(f, j(a_1),\dots,j(a_n))$.

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(2) *$M \not\models \psi$ and whenever $f$ results from Player II following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $g \in V$ which is $N$-generic such that, letting $j : M \to N$ be the iteration embedding, the following hold*:
(a) $\omega_1^{N[g]} = \omega_1 = j(\theta)$;
(b) $f \in N[g]$;
(c) $N[g] \not\models \phi(f, j(a_1), \ldots, j(a_n))$.

The theorem is proved by closely following the argument from Neeman [12]. We outline the proof, highlighting the differences (in particular, condition (P2) below), referring the reader to Neeman [11] and Neeman [12], and borrowing notation from those sources.

Proof Sketch of Theorem 3.1. First we describe the main difference from the proof in Neeman [12]. Recall the game $G_{\text{branch}}(M, t, \theta)(C)$ defined in [11, Section 6A]. Condition (P2) states that if $\delta_{\beta+1} = j_{0,\beta+1}(\theta)$, then $G_{\text{branch}}$ ends and Player I wins just in case there is some $M_{\beta+1}$-generic $G \subset j_{0,\beta+1}(\mathbb{W}_\theta)$ such that $t_{\beta+1} \in j_{0,\beta+1}(C)[G]$. We will modify this winning condition slightly. Recall that $\mathbb{W}_\theta$ is Neeman's variant of the extender algebra designed to add generic annotated positions to models with a Woodin cardinal which is also a limit of Woodin cardinals (see [11, Section 4B]). By [11, Lemma 4B.23], if $t$ is an annotated $M$-position in $V$ with relative domain $\theta$, then there is a canonical filter $G(t) \subset \mathbb{W}_\theta$ induced by $t$. Moreover, if $t$ satisfies the axioms of the extender algebra $\mathbb{W}_\theta$, then $t$ is $M$-generic. Then our winning condition is:

(P2) $G(t_{\beta+1}) \subset j_{0,\beta+1}(\mathbb{W}_\theta)$ is $M_{\beta+1}$-generic and $t_{\beta+1} \in j_{0,\beta+1}(C)[G(t_{\beta+1})]$.

The main result to be proved about $G_{\text{branch}}$ is [11, Theorem 6G.1], which is proved in [11, Chapter 6] with the original winning condition. This is a very general theorem in which $C$ is allowed to be an arbitrary name for a set of $\theta$-sequences. Our modification makes the game potentially more difficult for Player I and so it might not be immediately clear that the result still holds; however, one can see that it does by following Neeman's proof:

The winning strategy for $G_{\text{branch}}(M, t, \theta)(C)$ is constructed inductively by "piecing together" strategies for an auxiliary game $G_{\text{skip}}(M, t, \theta)(C)$. In constructing the winning strategy one needs in particular to consider limit steps $\beta$ such that $\delta_{\beta+1} = j_{0,\beta+1}(\theta)$. Here one proceeds according as to whether there exists a condition

$$[\sigma] \in j_{0,\beta+1}(\mathbb{W}_\theta) \cap (V_\alpha)^{M_{\beta+1}}$$

(where $\alpha$ is suitably chosen using the $j_{0,\beta+1}(\theta)$-c.c. of $j_{0,\beta+1}(\mathbb{W}_\theta)$) such that

$$[\sigma] \Vdash \dot{t} \in C$$

or not.

Following Neeman's proof, one proceeds to ensure that (i) if such a condition exists, then there is an $M_{\beta+1}$-generic filter $G \subset j_{0,\beta+1}(\mathbb{W}_\theta)$ such that $t_{\beta+1} \in j_{0,\beta+1}(C)[G]$. But Neeman's proof actually selects this filter to be $G(t_{\beta+1})$ (see [11, Claims 6D.6 and 6D.7]), which is what we desire. Otherwise, (ii) if no such condition exists, then Player I wins the game by getting Player II to select an illfounded branch at some stage.

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This is the only part of the proof of [11, Theorem 6G.1] in which condition (P2) is invoked. In particular, successor steps and the conversion of winning strategies of $G_{\text{skip}}$ into winning strategies of $G_{\text{branch}}$ work just as well under this modification.

The reason why this modification is needed is that in the argument of [12], one only considers $\mathbb{W}_\theta$-names $C$ for subsets of $2^{\omega_1}$ that are symmetric, in the sense that if $g, h \subset \mathbb{W}_\theta$ are $M$-generic and $x \in M[g] \cap M[h]$, then $x \in C[g]$ if and only if $x \in C[h]$ (see [12, Remark 3.10]). Since the formula $\phi$ in the statement of Theorem 3.1 is arbitrary, this symmetry need not hold.

To prove the theorem, one proceeds as in [12], with $k = 1$: let $\theta$ be a measurable Woodin cardinal (or only a Woodin cardinal which is indiscernible in $M$, as in [12]) and (within $M$) define $Y_1$ to be the canonical $\mathbb{W}_\theta$-name for the set of all $\theta$-sequences (in the sense of [11, Section 4D]) $t \in M[G]$ such that $\phi(\text{Re}(t), a_1, \dots, a_n)$ holds in $M[G]$, the extension of $M$ by a generic filter $G \subset \mathbb{W}_\theta$. Letting $\mu$ be the measure on $\theta$ (internal or external) and

$$\pi : M \to M^*$$

be the ultrapower embedding, we define $Y_0$ as the $(\theta, \pi(\theta))$-pullback of $\pi(Y_1)$, as computed in $M^*$. As in [12], one shows $Y_0 \in M$. We note that there is no condition in Theorem 3.1 on $\phi$ holding on a stationary set, which makes the situation here significantly simpler. One needs to show:

(1) If $M \models \varphi_{\text{ini}}(\theta, Y_0)$, then conclusion (1) in the statement of Theorem 3.1 holds.
(2) If $M \not\models \varphi_{\text{ini}}(\theta, Y_0)$, then conclusion (2) in the statement of Theorem 3.1 holds.

To show the first item, one considers an arbitrary run of the game and constructs a tree of trees (see [11, Section 7]) $\mathfrak{T}$ whose direct limit $N$ will witness (1). To do this, one defines for $\alpha < \omega_1$, among other things, models $M_\alpha$, annotated $M_\alpha$-positions $t_\alpha$, and embeddings $j_{0,\alpha} : M \to M_\alpha$. At each stage $\alpha$, one maintains inductively that Player I has a winning strategy in

$$G_{\text{branch}}(M_\alpha, t_\alpha, j_{0,\alpha}(\theta))(j_{0,\alpha}(Y_{i_\alpha})),$$

where $i_\alpha$ (which is either 0 or 1) is suitably chosen. More specifically, $i_\alpha$ will be set equal to 0 for all $\alpha$ with the property that for no $\beta \leq \alpha$ on the strand (see [11, Section 7B]) of $\mathfrak{T}$ from 0 to $\alpha$ is $t_\beta$ a terminal position in

$$G_{\text{branch}}(M, \varnothing, \theta)(Y_0);$$

and $i_\alpha$ will be 1 for all other $\alpha$. The base case is given by the assumption that $M \models \varphi_{\text{ini}}(\theta, Y_0)$ and [11, Corollary 6G.2], together with the remark above. The construction is like the one in [12]. Instead of distinguishing between internal, terminal, and external limit cases, one needs to distinguish between two cases here (this is because the game we consider here is not defined in terms of sequences of stationary sets). Letting $\text{rdm}(t)$ denote the relative domain of $t$, the two cases to be considered are:

(3) $\text{rdm}(t_\alpha) < j_{0,\alpha}(\theta)$ or there is no $\beta <_{\mathfrak{T}} \alpha$ such that $\text{rdm}(t_\beta) = j_{0,\beta}(\theta)$; and
(4) $\text{rdm}(t_\alpha) = j_{0,\alpha}(\theta)$ and there is $\beta <_{\mathfrak{T}} \alpha$ such that $\text{rdm}(t_\beta) = j_{0,\beta}(\theta)$.

If $\text{rdm}(t_\alpha) < j_{0,\alpha}(\theta)$ as in case (i), then the construction is like the successor or internal limit case in [12]. If $\text{rdm}(t_\alpha) = j_{0,\alpha}(\theta)$, but there is no $\beta <_{\mathfrak{T}} \alpha$ such that

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$\mathrm{rdm}(t_\beta) = j_{0,\beta}(\theta)$, then, the play $t_\alpha$ is a terminal position in the game

$$G_{\mathrm{branch}}(M, \varnothing, \theta)(Y_0). \tag{3.1}$$

One shows as in [12] that it must be a winning play for Player I, so that by our modified winning condition, $G(t_\alpha) \subset j_{0,\alpha}(\mathbb{W}_\theta)$ is $M_\alpha$-generic and $t_\alpha \in j_{0,\alpha}(Y_0)[G(t_\alpha)]$. Letting

$$j_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$$

be the ultrapower embedding by the normal measure on $j_{0,\alpha}(\theta)$, we have that $j_{0,\alpha}(Y_0)$ is the $(j_{0,\alpha}(\theta), j_{0,\alpha+1}(\theta))$-pullback of $j_{0,\alpha+1}(Y_1)$, as computed in $M_{\alpha+1}$. By [11, Theorem 6G.1], Player I has a winning strategy in

$$G_{\mathrm{branch}}(M_{\alpha+1}, t_\alpha, j_{0,\alpha+1}(\theta))(j_{0,\alpha+1}(Y_1)),$$

which is the inductive assumption. In this case, we also note the following: as in [11, Definition 4B.39], one can define an annotated position $t(g)$ associated with a given filter $g \subset j_{0,\alpha}(\mathbb{W}_\theta)$ which is generic over $M_\alpha$ in such a way that

$$t(G(t_\alpha)) = t_\alpha$$

and a canonical $j_{0,\alpha}(\mathbb{W}_\theta)$-name $\dot{t}$ for $t(g)$. Let $[\sigma] \in G(t_\alpha)$ be a condition such that

$$[\sigma] \Vdash^{M_\alpha}_{j_{0,\alpha}(\mathbb{W}_\theta)} \dot{t} \in j_{0,\alpha}(Y_0)[g].$$

Since the critical point of $j_{\alpha,\alpha+1}$ is $\theta$ and $[\sigma]$ is smaller, we have

$$[\sigma] \Vdash^{M_{\alpha+1}}_{j_{0,\alpha+1}(\mathbb{W}_\theta)} \dot{t} \in j_{0,\alpha+1}(Y_0)[g].$$

Since $[\sigma] \in G(t_\alpha)$, it follows that $t_\alpha \models \sigma$.

In case (ii), in which $\mathrm{rdm}(t_\alpha) = j_{0,\alpha}(\theta)$ and there is $\alpha' <_{\mathfrak{T}} \alpha$ such that $\mathrm{rdm}(t'_\alpha) = j_{0,\alpha'}(\theta)$, the remarks just given yield a condition $\sigma$ as above such that $t_{\alpha'} \models \sigma$ and

$$[\sigma] \Vdash^{M_{\alpha'}}_{j_{0,\alpha'}(\mathbb{W}_\theta)} \dot{t} \in j_{0,\alpha'}(Y_0)[g].$$

As above, let

$$j_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$$

be the ultrapower embedding by the normal measure on $j_{0,\alpha}(\theta)$. Since $t_\alpha$ extends $t_{\alpha'}$, we have $t_\alpha \models \sigma$, so that $[\sigma] \in G(t_\alpha)$, which (since, as above $[\sigma]$ is not moved by the embedding $j_{\alpha',\alpha}$) implies that

$$M_\alpha[G(t_\alpha)] \models \dot{t}[G(t_\alpha)] \in j_{0,\alpha+1}(Y_0)[G(t_\alpha)].$$

Since $\dot{t}[G(t_\alpha)] = t_\alpha$, we have

$$t_\alpha \in j_{0,\alpha+1}(Y_0)[G(t_\alpha)].$$

By elementarity, $j_{0,\alpha}(Y_0)$ is the $(j_{0,\alpha}(\theta), j_{0,\alpha+1}(\theta))$-pullback of $j_{0,\alpha+1}(Y_1)$ as computed in $M_{\alpha+1}$, and so [11, Theorem 6G.1] implies that Player I has a winning strategy in

$$G_{\mathrm{branch}}(M_{\alpha+1}, t_\alpha, j_{0,\alpha+1}(\theta))(j_{0,\alpha+1}(Y_1)),$$

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which is the inductive assumption. In addition to this, the fact that $i_\beta$ is equal to 1 for all $\beta$ which have $\alpha'$ as a $\mathfrak{T}$-predecessor, together with an argument as before (the one in which we referred the reader to [12] after equation (3.1)) shows that $t_\alpha \in j_{0,\alpha}(Y_1)[G(t_\alpha)]$. By elementarity, $j_{0,\alpha}(Y_1)$ is the canonical $j_{0,\alpha}(\mathbb{W}_\theta)$-name for the set of all $j_{0,\alpha}(\theta)$-sequences $t \in M_\alpha[g]$ such that

$$M_\alpha[g] \models \phi(\mathrm{Re}(t), j_{0,\alpha}(a_1), \dots, j_{0,\alpha}(a_n)).$$

Hence,

$$M_\alpha[G(t_\alpha)] \models \phi(\mathrm{Re}(t_\alpha), j_{0,\alpha}(a_1), \dots, j_{0,\alpha}(a_n)).$$

The construction ends when $\alpha = \omega_1$. Once at this stage, $j_{0,\omega_1}(\theta) = \omega_1$. Letting $t = t_{\omega_1}$, $N = M_{\omega_1}$, $g = G(t_{\omega_1})$, and $j = j_{0,\omega_1}$, an argument as above shows that $t \in j(Y_1)[g]$, and thus

$$N[g] \models \phi(\mathrm{Re}(t), j(a_1), \dots, j(a_n)),$$

as desired.

If $M \not\models \varphi_{\mathrm{ini}}(\theta, Y_0)$ holds instead, then a similar argument yields (2) in the conclusion of the theorem (see the "mirrored" Lemma 3.37 in [12]). ⊣

There is a second, more general version of Theorem 3.1 which admits universally Baire parameters. Before stating it, we mention the following result, alluded to in the introduction, which will be needed for the proof. It makes use of the notion of a *scale* from descriptive set theory (see, e.g., Moschovakis [8]).

THEOREM 3.2 (Woodin). *Let $A$ be a universally Baire set of reals and suppose that $M$ is an $A$-closed countable, transitive model of* ZFC. *Let $B = \mathbb{R} \setminus A$. Let $\vec{\phi}$ be a universally Baire scale on $A$ and $\vec{\psi}$ be a universally Baire scale on $B$ and suppose that $M$ is $\vec{\phi} \times \vec{\psi}$-closed. Then, $M$ is strongly $A$-closed.*

A proof of the theorem can be found in Bagaria–Castells–Larson [1, Theorem 2.37] or in Steel [16]. It gives the additional information that if $M$ is as in the statement, then for every partial order $\mathbb{P}$ in $M$, there are trees $T, S \in M$ such that $p[T]^N = N \cap A$ and $p[S]^N = N \cap B$ for every generic extension $N$ of $M$ by a generic subset of $\mathbb{P}$. This has the following additional consequence:

COROLLARY 3.3. *Suppose that there is a proper class of Woodin cardinals and $A$ is universally Baire. Then, there is a universally Baire set $B$ such that whenever $M$ is a $B$-model and $\mathbb{P} \in M$ is a partial order, there is a tree $T \in M$ such that $p[T] = A \cap N$ in every generic extension $N$ of $M$ by $\mathbb{P}$.*

PROOF. Woodin has shown (see, e.g., Woodin [19]) that if there is a proper class of Woodin cardinals and $A$ is universally Baire, then

$$L(A, \mathbb{R}) \models \mathsf{AD}^+$$

and every set of reals in $L(A, \mathbb{R})$ is universally Baire. Woodin has also shown that it is a theorem of $\mathsf{AD}^+$ that the pointclass $\mathbf{\Sigma}^2_1$ has the scale property. In particular, there is a universally Baire scale on $A$. Letting $B$ be a universally Baire set coding all these sets and $M$ be a $B$-model, the previous theorem and the remark after it apply to $M$ and $B$, and thus, there is a tree $T$ as desired. ⊣

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THEOREM 3.4. *Suppose that there is a proper class of Woodin cardinals and for every universally Baire set $A$ there is an $A$-model.*

*Let $M$ be an $A$-model, as witnessed by an iteration strategy $\Sigma$. Let $\phi(x, x_0, \dots, x_n)$ be a formula in the language of set theory and $a_1, \dots, a_n$ be parameters in $M$. Then, there is a strategy $\sigma \in V$ and a formula $\psi$ with parameters in $M$ such that one of the following holds*:

(1) *$M \models \psi$ and whenever $f$ results from Player I following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $g \in V$ which is $N$-generic such that, letting $j : M \to N$ be the iteration embedding, the following hold*:
   (a) $\omega_1^{N[g]} = \omega_1 = j(\theta)$;
   (b) $f \in N[g]$;
   (c) $N[g] \models \phi(A \cap N[g], f, j(a_1), \dots, j(a_n))$.

(2) *$M \not\models \psi$ and whenever $f$ results from Player II following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $g \in V$ which is $N$-generic such that, letting $j : M \to N$ be the iteration embedding, the following hold*:
   (a) $\omega_1^{N[g]} = \omega_1 = j(\theta)$;
   (b) $f \in N[g]$;
   (c) $N[g] \not\models \phi(A \cap N[g], f, j(a_1), \dots, j(a_n))$.

PROOF SKETCH OF THEOREM 3.4. Let $M$ be an $A$-model, as witnessed by the iteration strategy $\Sigma$. Thus, if $N$ is a countable iterate of $M$ consistent with $\Sigma$, then $N$ is $A$-closed. The proof of the theorem is similar to the one of Theorem 3.1, the main difference being that the set $A \subset \mathbb{R}^V$ is involved. We need to rephrase questions about $A$ in $M$ in a way that is amenable to being shifted by elementary embeddings (by the iteration strategy $\Sigma$) and behaves properly when passing to generic extensions of $M$.

Let $\mathbb{P}$ be a partial order in $M$. By passing to a more complicated universally Baire set if necessary, we may assume by Corollary 3.3 that there are trees $T$ and $S$ in $M$ such that $p[T] = A \cap N$ and $p[S] = \mathbb{R}^N \setminus A$ for every generic extension $N$ of $M$ by $\mathbb{P}$. Let $\kappa$ be some cardinal in $M$ which is sufficiently large above $\theta$. Since $M$ is an $A$-model, $\tau_A^\infty \cap V_\kappa^M \in M$. Letting $A^*$ be the set of all reals $x$ in $M$ such that $(\theta, \varnothing, \check{x}) \in \tau_A^\infty \cap V_\kappa^M$, we have $A^* = A \cap M$ and

$$M \models p[T] = A^*. \tag{3.2}$$

Let $j : M \to N$ be a countable iteration embedding by $\Sigma$. Since $\Sigma$ witnesses strong $A$-iterability, by definition, we have

$$j(\tau_A^\infty \cap V_\kappa^M) = \tau_A^\infty \cap (V_{j(\kappa)})^N,$$

and thus $j(A^*) = A \cap N$. By elementarity applied to (3.2),

$$N \models p[j(T)] = j(A^*)$$

and $T$ and $S$ project to complements in all generic extensions of $N$ by $j(\mathbb{P})$, so that

$$p[T]^{N[g]} = A \cap N[g] \tag{3.3}$$

for every $N$-generic $g \subset j(\mathbb{P})$ with $g \in V$.

We claim that (3.3) holds for all $N$-generic $g \subset j(\mathbb{P})$ also if we assume that $j$ has length $\omega_1 + 1$. To see this, let $G \subset \mathrm{Coll}(\omega_1, \omega)$ be $V$-generic and work in $V[G]$. Recall that the iteration strategy $\Sigma$ is universally Baire. Letting $B$ be a universally Baire set

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coding $\Sigma$ and $A$, the statement that (3.3) holds for all countable $j$ and all $g$ is absolute to $L(B,\mathbb{R})$. Since there is a proper class of Woodin cardinals, there is an elementary embedding

$$i : L(B,\mathbb{R}) \to L(B_G,\mathbb{R}^{V[G]}),$$

and so it also holds in $L(B_G,\mathbb{R}^{V[G]})$, and thus in $V[G]$. In particular, it holds for all filters and embeddings in $V$ which are countable in $V[G]$; this implies the claim.

We have shown that for all partial orders $\mathbb{P} \in M$, there are trees $T, S \in M$ such that whenever $j$ is an iteration embedding of $M$ of length $\leq \omega_1 + 1$ by $\Sigma$ with target model $N$ and $g \subset j(\mathbb{P})$ is $N$-generic, then

$$p[T]^{N[g]} = A \cap N[g]$$

and

$$p[S]^{N[g]} = \mathbb{R}^{N[g]} \setminus A.$$

We apply this to $\mathbb{P} = \mathbb{W}_\theta$. In $M$, let $\dot{A}$ be the canonical $\mathbb{W}_\theta$-name for all reals $x \in M[g]$ in the projection of $T$.

Moving on with the proof as in Theorem 3.1, let $Y_1$ be the canonical $\mathbb{W}_\theta$-name for the set of all $\theta$ sequences $t \in M[g]$ such that

$$M[g] \models \phi(\dot{A}[g], \mathrm{Re}(t), a_1, \ldots, a_n)).$$

Continue by defining $Y_0$ as in the proof of Theorem 3.1 and act according as $M \models \varphi_{\mathrm{ini}}(\theta, Y_0)$ or not; for example, if $M \models \varphi_{\mathrm{ini}}(\theta, Y_0)$, then one proceeds by constructing the tree $\mathfrak{T}$ as in the proof of Theorem 3.1. This gives rise to an annotated position $t = t_{\omega_1}$, a model $N = M_{\omega+1}$, an $N$-generic filter $g = G(t_{\omega_1})$ with $g \in V$, and an embedding $j : M \to N$ such that

$$N[g] \models \phi(j(\dot{A})[g], \mathrm{Re}(t), j(a_1), \ldots, j(a_n)).$$

As before, $j(\theta) = \omega_1$. By choice of $T$, $j(\dot{A})$ is interpreted as $A \cap N[h]$ whenever $h \in \mathcal{P}(j(\mathbb{W}_\theta)) \cap V$ is $N$-generic; thus $j(\dot{A})[g] = A \cap N[g]$.

Therefore, we obtain

$$N[g] \models \phi(A \cap N[g], \mathrm{Re}(t), j(a_1), \ldots, j(a_n)),$$

as desired.

The case that $M \not\models \varphi_{\mathrm{ini}}(\theta, Y_0)$ is also taken care of as before. ⊣

Remark 3.5. It might be worth noting that the filters $g$ in the statements of Theorem 3.1 and Theorem 3.4 are generic for $j(\mathbb{W}_\theta)$, where $\mathbb{W}_\theta$ is Neeman's variant of the extender algebra that adds generic annotated positions. It might also be worth noting that if $\theta$ is a Woodin cardinal which is also a limit of Woodin cardinals, then $\mathbb{W}_\theta$ has the $\theta$-chain condition (see Neeman [10, 4B.19]).

**§4. Applications.** In this section, we present some applications of Theorem 3.4. Below, if $f$ is a function with domain an ordinal and $i \leq 3$, we write

$$f^i(\alpha) = f(4 \cdot \alpha + i),$$

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whenever the right-hand side is defined. If $f$ and $g$ are two functions with domain the same ordinal, we write

$$f \oplus g(2\cdot\alpha + i) = \begin{cases} f(\alpha), & \text{if } i = 0, \\ g(\alpha), & \text{if } i = 1. \end{cases}$$

The function $f^0 \oplus f^1 \oplus f^2 \oplus f^3$ is defined similarly. Thus, if $f$ is a function, then $f = f^0 \oplus f^1 \oplus f^2 \oplus f^3$. Moreover, if $f$ is a play in a game, then $f^0$ and $f^2$ make up Player I's moves, while $f^1$ and $f^3$ make up Player II's moves. Analogously, if $A$ is a set of reals, we denote by $A^i$ the set of all $x$ such that $(i, x) \in A$.

Another (abuse of) notation we shall employ is the following: suppose that $\tau$ is a strategy for a game of length $\omega_1$. Given a sequence of moves $f$ of length $\omega_1$ for a player, there is a sequence of responses to $f$ determined uniquely by $f$ and $\tau$, and we shall denote this by $\tau(f)$. This way, $f \oplus \tau(f)$ denotes the run of the game where Player I's moves are enumerated by $f$ and Player II's moves are obtained by responding according to $\tau$.

**4.1. $\Delta_1$ games.** The first application is Theorem 1.9 from the introduction, which we restate for convenience.

DEFINITION 4.1. Let $A \subset \mathbb{R}$ be universally Baire. A set $X \subset 2^{\omega_1}$ is $\Omega$-*provably* $\Delta_1(A)$ if there exist $\Sigma_1$-formulae $\varphi, \psi$ in the language of set theory, such that the following hold:

(1) $X = \{f \in 2^{\omega_1} : \varphi(f, A)\}$;
(2) $\mathsf{ZFC} \vdash_\Omega \forall f \in 2^{\omega_1}\, (\varphi(f, A) \leftrightarrow \neg\psi(f, A))$.

Clearly every generalized Borel set is $\Omega$-provably $\Delta_1(\varnothing)$.

THEOREM 4.2. *Suppose that*

(1) *There is a proper class of Woodin cardinals*;
(2) *for every universally Baire $A \subset \mathbb{R}$ there is an $A$-model*;
(3) *the Continuum Hypothesis holds.*

*Let $X \subset 2^{\omega_1}$ be $\Omega$-provably $\Delta_1(A)$ for some universally Baire $A \subset \mathbb{R}$. Then $X$ is determined.*

To prove the theorem, we assume hypotheses (1)–(3). For each universally Baire $A \subset \mathbb{R}$, we reduce all $\Omega$-provably $\Delta_1(A)$ games to those of a simple form.

DEFINITION 4.3. Let $\varphi$ be a $\Delta_0$ formula and $A \subset \mathbb{R}$. We define a game $Z^A_\varphi$. Players I and II take turns playing either 0 or 1:

| | | | | | |
|---|---|---|---|---|---|
| I | $f(0)$ | | $f(2)$ | ... | $(< \omega_1)$ |
| II | | $f(1)$ | | $f(3)$ ... | $(< \omega_1)$ |

Player I wins if and only if there is some uncountable ordinal number $\kappa$ and some $g \in 2^{\omega_1}$ such that, letting $N = L_\kappa(A, \mathbb{R})[f][g]$, the following hold:

(5) $N \models \mathsf{ZFC}$; and
(6) $N \models \varphi(f, g, A)$.

Otherwise, Player II wins.

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LEMMA 4.4. *The game $Z^A_\varphi$ is $\Sigma_1$-definable from $A$ and $\mathbb{R}$.*

PROOF. Whether $f$ belongs to the payoff set of the game depends on the existence of some $\kappa$ and some $g$ such that $g \in 2^{\omega_1}$ and a first-order theory is satisfied by

$$N = L_\kappa(A, \mathbb{R})[f][g].$$

Such an $N$ is $\Delta_1$-definable from $\kappa$, $A$, $\mathbb{R}$, $f$, and $g$. Moreover, $\omega_1$ is $\Delta_1$-definable from $\mathbb{R}$, as it is the least ordinal not coded by an element of WO, so whether $g \in 2^{\omega_1}$ is $\Delta_1$ over $\mathbb{R}$. ⊣

LEMMA 4.5. *Under the assumptions of the theorem, suppose that $X \subset 2^{\omega_1}$ is determined whenever there exist formulae $\varphi$ and $\psi$ for which*

(1) $X = Z^A_\varphi$, *and*
(2) $\mathsf{ZFC} + \mathsf{CH} \vdash_\Omega Z^A_\varphi = 2^{\omega_1} \setminus Z^A_\psi$,

*Then, every $\Omega$-provably $\Delta_1(A)$ subset of $2^{\omega_1}$ is determined.*

PROOF. Let us assume without loss of generality that $\mathbb{R}$ (and thus $\omega_1$) are $\Delta_1$-definable from $A$. Otherwise, we could replace $A$ by the set

$$A' = \{(0, x) : x \in A\} \cup \{(1, x) : x \in \mathbb{R}\}.$$

This is possible because such an $A'$ is universally Baire and is Wadge-equivalent to $A$, so that on the one hand, the $\Omega$-provably $\Delta_1(A)$ sets are precisely the $\Omega$-provably $\Delta_1(A')$ sets and, on the other, if

$$\mathsf{ZFC} + \mathsf{CH} \vdash_\Omega Z^A_\varphi = 2^{\omega_1} \setminus Z^A_\psi,$$

then

$$\mathsf{ZFC} + \mathsf{CH} \vdash_\Omega Z^{A'}_{\varphi'} = 2^{\omega_1} \setminus Z^{A'}_{\psi'},$$

where $\varphi'(f, g, Z)$ is the formula $\varphi(f, g, Z^0)$ and $\psi'(f, g, Z)$ is the formula $\psi(f, g, Z^0)$.

Let $X$ be an $\Omega$-provably $\Delta_1(A)$ game. Thus, there are $\Sigma_0$-formulae $\varphi$ and $\psi$ in the language of set theory such that, provably in $\Omega$-logic,

$$f \in X \text{ if and only if } \exists x \varphi(x, f, A) \text{ if and only if } \neg \exists x \psi(x, f, A). \tag{4.1}$$

Suppose $\varphi(x, f, A)$ holds. Let $\kappa$ be large and such that $x \in V_\kappa$. Let $N$ be the transitive collapse of an elementary substructure of $V_\kappa$ containing $x$, $A$, $f$, every countable ordinal, and every real, and such that $|N| = \aleph_1$. Such a structure exists, since the Continuum Hypothesis holds. Since $N$ contains every real, it contains $A$, so $N \models \varphi(\bar{x}, f, A)$ by elementarity, where $\bar{x}$ is the preimage of $x$ under the collapse embedding. By upwards absoluteness, $\varphi(\bar{x}, f, A)$ holds in $V$. Since $\bar{x} \in N$ and $N$ is transitive and of cardinality $\aleph_1$, there is some $g \in 2^{\omega_1}$ coding $\bar{x}$ in some simple way. For example, $g$ can be identified with an element of $2^{\omega_1 \times \omega_1}$ and seen as the membership relation of a structure of cardinality $\omega_1$ which is isomorphic to $(\bar{x}, \in)$.

Define $\theta_0(g, f, A)$ to hold if and only if $g \in 2^{\omega_1}$ codes an $x$ such that $\varphi(x, f, A)$ and define $\theta_1(g, f, A)$ to hold if and only if $g \in 2^{\omega_1}$ codes an $x$ such that $\psi(x, f, A)$. These formulae are easily seen to be $\Delta_1$. By the argument just given,

(7) $\theta_0(g, f, A)$ holds of some $g$ if and only if $\varphi(x, f, A)$ holds of some $x$;
(8) $\theta_1(g, f, A)$ holds of some $g$ if and only if $\psi(x, f, A)$ holds of some $x$.

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Since this argument only depends on the Continuum Hypothesis, it follows from (4.1)

$$\mathsf{ZFC}+\mathsf{CH}\vdash_\Omega \forall f\in 2^{\omega_1}\left(\exists g\in 2^{\omega_1}\,\theta_0(g,f,A)\longleftrightarrow\neg\exists g\in 2^{\omega_1}\,\theta_1(g,f,A)\right). \tag{4.2}$$

CLAIM 4.6. $\mathsf{ZFC}+\mathsf{CH}\vdash_\Omega Z^A_{\theta_0}=2^{\omega_1}\setminus Z^A_{\theta_1}$.

PROOF. Let $B$ be a universally Baire set of reals such that

(9) $A$ is continuously reducible to $B$;
(10) There is some $A_0$ such that $A_0$ witnesses the $\Omega$-provability of (4.2) and $A_0$ is continuously reducible to $B$.

Let $P$ be a strongly $B$-closed countable transitive model of ZFC and $Q$ be rank-initial segment of $P$ such that

$$Q\models\mathsf{ZFC}+\mathsf{CH}.$$

We need to show that

$$Q\models Z^{A\cap Q}_{\theta_0}=2^{\omega_1}\setminus Z^{A\cap Q}_{\theta_1}.$$

SUBCLAIM 4.7. *Work in $Q$ and suppose $f\in Z^{A\cap Q}_{\theta_0}$. Then $f\notin Z^{A\cap Q}_{\theta_1}$* ⊣

PROOF. Since $f\in Z^{A\cap Q}_{\theta_0}$, there is an ordinal $\kappa_0$ with $\omega_1^Q<\kappa_0$ and a function $g_0\in(2^{\omega_1})^Q$ such that, letting

$$N_0=L_{\kappa_0}(A\cap Q,\mathbb{R}^Q)[f][g_0],$$

we have

(11) $N_0\models\mathsf{ZFC}$; and
(12) $N_0\models\theta_0(g_0,f,A\cap Q)$.

Supposing towards a contradiction that $f\in Z^{A\cap Q}_{\theta_1}$, there is a $Q$-uncountable ordinal $\kappa_1$ and a function $g_1\in(2^{\omega_1})^Q$ such that, letting $\kappa=\sup\{\kappa_0,\kappa_1\}$ and

$$N=L_\kappa(A\cap Q,\mathbb{R}^Q)[f][g_0][g_1],$$

we have

(13) $N\models\mathsf{ZFC}$; and
(14) $N\models\theta_0(g_0,f,A\cap Q)$;
(15) $N\models\theta_1(g_1,f,A\cap Q)$.

(This follows from the fact that $\theta_0$ and $\theta_1$ are $\Delta_1$.) By assumption, $B$ witnesses (4.2) (in $V$) and $P$ is strongly $B$-closed, so, by the choice of $Q$,

$$Q\models\exists g\in 2^{\omega_1}\,\theta_0(g,f,A\cap Q)\longleftrightarrow\neg\exists g\in 2^{\omega_1}\,\theta_1(g,f,A\cap Q). \tag{4.3}$$

However, the functions $g_0$ and $g_1$ belong to $Q$ and are witnesses for $\theta_0$ and $\theta_1$ (since they are $\Delta_1$), respectively, which is a contradiction. ⊣

Conversely, and again working in $Q$, suppose $f\notin Z^{A\cap Q}_{\theta_0}$.

SUBCLAIM 4.8. *Work in $Q$ and suppose $f\notin Z^{A\cap Q}_{\theta_0}$. Then $f\in Z^{A\cap Q}_{\theta_1}$.*

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PROOF. Otherwise we would have that for all $g_0, g_1 \in (2^{\omega_1})^Q$ and all uncountable $\kappa_0, \kappa_1$ such that, letting $\kappa = \min\{\kappa_0, \kappa_1\}$ and

$$N = L_\kappa(A \cap Q, \mathbb{R}^Q)[f][g_0][g_1],$$

$N \models \mathsf{ZFC}$ implies that

(16) $N \not\models \theta_0(g_0, f, A \cap Q)$;
(17) $N \not\models \theta_1(g_1, f, A \cap Q)$.

(As before, this follows from the fact that $\theta_0$ and $\theta_1$ are $\Delta_1$.) Again, equation (4.3) holds by choice of $Q$. Assume without loss of generality that $\theta_0(g, f, A \cap Q)$ holds for some $g \in (2^{\omega_1})^Q$. Since $\theta_0$ is $\Delta_1$, we have

$$N^* \models \theta_0(g, f, A \cap Q),$$

where

$$N^* = L_\kappa(A \cap Q, \mathbb{R}^Q)[f][g],$$

and $\kappa$ is any uncountable ordinal in $Q$ ensuring that $N^* \models \mathsf{ZFC}$. This is again a contradiction. ⊣

The previous two subclaims together show that

$$Q \models Z_{\theta_0}^{A \cap Q} = 2^{\omega_1} \setminus Z_{\theta_1}^{A \cap Q}.$$

Since $P$ and $Q$ were arbitrary, it follows that

$$\mathsf{ZFC} + \mathsf{CH} \vdash_\Omega Z_{\theta_0}^A = 2^{\omega_1} \setminus Z_{\theta_1}^A. \tag{4.4}$$

This proves the claim. ⊣

Now, by hypothesis, $Z_{\theta_0}^A$ is determined. Suppose that Player II has a winning strategy, $\tau$; the other case is similar. We claim that $\sigma$ is a winning strategy for Player II also for the game with payoff $X$.

To see this, let $f$ be a run of the game $X$ by $\sigma$. Since $\sigma$ is a winning strategy for Player II in $Z_{\theta_0}^A$, it follows that $f \notin Z_{\theta_0}^A$, and so $f \in Z_{\theta_1}^A$, by Claim 4.6 and $\Omega$-soundness. Thus, there is an uncountable ordinal $\kappa$ and a function $g \in 2^{\omega_1}$ such that, letting

$$N = L_\kappa(A, \mathbb{R})[f][g],$$

then $N \models \mathsf{ZFC}$ and

$$N \models \theta_1(g, f, A).$$

Since $\theta_1$ is absolute, $\theta_1(g, f, A)$ holds in $V$ as well. Thus, $\psi(x, f, A)$ holds in $V$ for some $x$. This implies by (4.1) that $\varphi(x, f, A)$ cannot hold for any $x$ and therefore that $f \notin X$, as desired. This proves the lemma. ⊣

We now prove the theorem from the lemma. It suffices to show that $X$ is determined whenever

(18) $A$ is universally Baire;
(19) $X = Z_\varphi^A$; and
(20) $\mathsf{ZFC} + \mathsf{CH} \vdash_\Omega Z_\varphi^A = 2^{\omega_1} \setminus Z_\psi^A$.

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Let $M$ be a $B$-model, for some $B$ such that both $A$ and the universally Baire set witnessing

$$\mathsf{ZFC}+\mathsf{CH}\vdash_\Omega Z^A_\varphi = 2^{\omega_1}\setminus Z^A_\psi$$

are continuously reducible to $B$, and let $\Sigma$ be the iteration strategy witnessing this. Thus, $M$ is also an $A$-model.

By definition, $M$ has a measurable Woodin cardinal $\theta$ below an inaccessible $\kappa_0$. Fix some large limit cardinal $\eta$ of $M$ with $\kappa_0<\eta$. Let $\phi(Y,v,v_0,v_1)$ be the formula in the language of set theory (with the indicated free variables) asserting that $v\in 2^{v_0}$ and there is an ordinal $\kappa$ with $v_0<\kappa<v_1$ such that for some $g\in 2^{v_0}$, $L_\kappa(Y,\mathbb{R})[v^0\oplus v^1][g]\models\mathsf{ZFC}$ and $L_\kappa(Y,\mathbb{R})[v^0\oplus v^1][g]\models\varphi(v^0\oplus v^1,g,Y)$.

We apply Theorem 3.4 to $M$ and to the formula $\phi(Y,v,v_0,v_1)$, with the parameters $\theta$ and $\eta$ (in the positions of $v_0$ and $v_1$, respectively). Thus, there is a strategy $\sigma$ such that one of the following holds:

(1) whenever $f$ results from Player I following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $G\in V$ which is $N$-generic such that, letting $j:M\to N$ be the iteration embedding, the following hold:
   (a) $\omega_1^{N[G]}=\omega_1=j(\theta)$;
   (b) $f\in N[G]$;
   (c) $N[G]\models\phi(A\cap N[G],f,j(\theta),j(\eta))$.

(2) whenever $f$ results from Player II following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $G\in V$ which is $N$-generic such that, letting $j:M\to N$ be the iteration embedding, the following hold:
   (a) $\omega_1^{N[G]}=\omega_1=j(\theta)$;
   (b) $f\in N[G]$;
   (c) $N[G]\not\models\phi(A\cap N[G],f,j(\theta),j(\eta))$.

Suppose that it is the first alternative that holds. Then $\sigma$ naturally induces a strategy $\tau$ for $X$ (i.e., for the game with payoff $X$) given by

$$\tau(f^1)=f^0 \text{ if and only if } \exists f^2\,\sigma(f^1\oplus h)=f^0\oplus f^2,$$

where $h$ is some fixed enumeration of $\mathbb{R}$ in order-type $\omega_1$. We claim that $\tau$ is a winning strategy for Player I for the game with payoff $X$ in $V$. Let $f^0\oplus f^1$ be a play by $\tau$ and let $h=f^3$ be as above. By the choice of $\tau$, there is $f^2\in 2^{\omega_1}$, an iteration elementary embedding

$$j:M\to N,$$

and an $N$-generic $G\in V$ such that $f\in N[G]$ and

$$N[G]\models\phi(A\cap N[G],f,\omega_1^V,j(\eta)).$$

Since $f\in N[G]$ and $f^3$ enumerates $\mathbb{R}$, we have $\mathbb{R}^{N[G]}=\mathbb{R}$ and $A\cap N[G]=A$.

By the definition of $\phi$, there is an ordinal $\kappa$ with $\omega_1<\kappa<j(\eta)$ such that for some $g\in 2^{\omega_1}\cap N[G]$,

$$L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\mathsf{ZFC},$$

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and

$$L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\varphi(f^0\oplus f^1,g,A),$$

as desired.

Suppose now that the second conclusion of the theorem holds and let $\sigma$ be the strategy. As before, define a strategy $\tau$ for Player II for the game with payoff $X$ by setting

$$\tau(f^0)=f^1 \text{ if and only if } \exists f^3\,\sigma(f^0\oplus h)=f^1\oplus f^3,$$

where $h$ is again a fixed enumeration of $\mathbb{R}$ in order-type $\omega_1$. We argue that $\tau$ is a winning strategy for Player II for the game with payoff $X$ in $V$. Let $f^0\oplus f^1$ be a play by $\tau$ and let $h=f^2$ be as above, so that $f$ is a play consistent with $\sigma$. Hence there is an iteration embedding

$$j:M\to N$$

and an $N$-generic $G\in V$ such that $f\in N[G]$ and

$$N[G]\not\models\phi(A\cap N[G],f,\omega_1^V,j(\eta)). \tag{4.5}$$

Since $f\in N[G]$ and $f^2$ enumerates all reals in $V$, $A\cap N[G]=A$. Unravelling the definition of $\phi$, we get

$$\begin{aligned}N[G]\models\text{“for all }\kappa\text{ with }\omega_1<\kappa<j(\eta)\text{ and for all }g\in 2^{\omega_1}\\ \text{if } L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\mathsf{ZFC},\text{ then}\\ L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\varphi(f^0\oplus f^1,g,A).\text{”}\end{aligned}$$

Since $A\in N[G]$, $L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]$ is defined correctly in $N[G]$, so we can conclude that for all $\kappa$ with $\omega_1<\kappa<j(\eta)$ and all $g\in 2^{\omega_1}\cap N[G]$, if

$$L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\mathsf{ZFC},$$

then

$$L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\not\models\varphi(f^0\oplus f^1,g,A).$$

Claim 4.9. $N[G]\not\models f^0\oplus f^1\in Z^A_\varphi$.

Proof. This is mostly a matter of unravelling the definition of $Z^A_\varphi$. If we had $N[G]\models f^0\oplus f^1\in Z^A_\varphi$, there would be some $\kappa\in N[G]$ and some $g\in 2^{\omega_1}\cap N[G]$ such that

$$L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\mathsf{ZFC}+\varphi(f^0\oplus f^1,g,A).$$

$f$ contains an enumeration of all reals of order-type $\omega_1$, so $N[G]\models\mathsf{CH}$. Thus, by Löwenheim–Skolem, we would have a $\kappa$ as above which additionally satisfies $\kappa<\omega_2^{N[G]}$. We chose $\eta$ to be a large limit cardinal of $M$, so $j(\eta)$ is a large limit cardinal of $N$. By Remark 3.5, $G$ is generic for a partial order that has the $\omega_1^V$-chain condition, so $j(\eta)$ is a large limit cardinal in $N[G]$. We have shown that such a $\kappa$ cannot exist, so this is a contradiction. ⊣

Claim 4.10. $N[G]\models Z^A_\varphi=2^{\omega_1}\setminus Z^A_\psi$.

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PROOF. By hypothesis, $N$ is $B$-closed and $B$ witnesses that

$$\mathsf{ZFC}+\mathsf{CH}\vdash_\Omega Z^A_\varphi = 2^{\omega_1}\setminus Z^A_\psi,$$

so

$$N\models \text{“}\mathsf{ZFC}+\mathsf{CH}\models_\Omega Z^A_\varphi = 2^{\omega_1}\setminus Z^A_\psi.\text{”}$$

As we argued in the previous claim, $N[G]\models\mathsf{CH}$. We had chosen $\kappa_0$ to be an inaccessible cardinal of $M$, so $j(\kappa_0)$ is an inaccessible cardinal of $N$ and thus of $N[G]$, because $G$ is generic for a partial order that has the $\omega_1^V$-chain condition (Remark 3.5). Thus, $(V_{j(\kappa_0)})^{N[G]}\models\mathsf{ZFC}+\mathsf{CH}$, so

$$(V_{j(\kappa_0)})^{N[G]}\models Z^A_\varphi = 2^{\omega_1}\setminus Z^A_\psi.$$

This statement depends only on which subsets of $\omega_1$ exist and is thus absolute to all of $N[G]$, so we have

$$N[G]\models Z^A_\varphi = 2^{\omega_1}\setminus Z^A_\psi,$$

as claimed. ⊣

Combining the previous two claims, we get

$$N[G]\models f^0\oplus f^1\in Z^A_\psi,$$

so there is an ordinal $\kappa\in N[G]$ with $\omega_1<\kappa$ and some $g\in 2^{\omega_1}\cap N[G]$ such that

$$L_\kappa(A,\mathbb{R})[f^0\oplus f^1][g]\models\psi(f^0\oplus f^1,g,A).$$

Hence, we really have $f^0\oplus f^1\in Z^A_\psi$. By $\Omega$-soundness, $f^0\oplus f^1\notin Z^A_\varphi$, as desired. ⊣

**4.2. Definable games.** Another application of Theorem 3.4 is determinacy for arbitrary games in various minimal models. For simplicity, we shall restrict ourselves to the lightface cases.

THEOREM 4.11. *Suppose $\varphi$ is a formula in the language of set theory with an additional predicate constant. Let $M$ be an active, countable $\omega_1+1$-iterable extender model with a measurable Woodin cardinal $\theta$ below a measurable cardinal $\kappa$. Then, the set*

$$X=\Big\{f\in 2^{\omega_1}:(L[f],f)\models\varphi(f)\Big\}.$$

*is determined.*

REMARK 4.12. The large-cardinal hypothesis in the statement of Theorem 4.11 can be weakened in the obvious way. The determinacy of these sets is not immediate from the statement of Theorem 4.2, although one can adapt the proof. A proof directly from Theorem 3.1, however, allows us to dispose of the assumption on CH. This form of determinacy was previously proved by Woodin (see [20]) from $\Sigma^2_2$-absoluteness conditioned on generic-diamond.

PROOF OF THE THEOREM. Let $M$ be as in the statement. Let $\phi(v,v_0)$ be the formula in the language of set theory (with the indicated free variables) asserting that $v\in 2^{v_0}$ and that, letting $\eta$ be the least measurable cardinal greater than $v_0$, we have

$$L_\eta[v]\models\varphi(v).$$

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Let $\theta$ be the measurable Woodin cardinal of $M$. We apply Theorem 3.1 to $M$ and the formula $\phi(v, v_0)$ and the parameter $\theta$ (in the position of $v_0$), so that $\eta = \kappa$. Thus, there is a strategy $\sigma$ such that one of the following holds:

(1) whenever $f$ results from Player I following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $g \in V$ which is $N$-generic such that, letting $j : M \to N$ be the iteration embedding, the following hold:
   (a) $\omega_1^{N[g]} = \omega_1 = j(\theta)$;
   (b) $f \in N[g]$;
   (c) $N[g] \models \phi(f, \omega_1^V)$.

(2) whenever $f$ results from Player II following $\sigma$, there is an iterate $N$ of $M$ by $\Sigma$ and a filter $g \in V$ which is $N$-generic such that, letting $j : M \to N$ be the iteration embedding, the following hold:
   (a) $\omega_1^{N[g]} = \omega_1 = j(\theta)$;
   (b) $f \in N[g]$;
   (c) $N[g] \not\models \phi(f, \omega_1^V)$.

In any such model $N$, $j(\kappa)$ is the least measurable cardinal in $N$ above $\omega_1^V$, and thus the least measurable cardinal in $N[g]$. Thus, if

$$N[g] \models \phi(f, \omega_1^V),$$

then clearly we have

$$L[f] \models \varphi(f),$$

and similarly in the other case. ⊣

The previous theorem implies the existence of a model containing all reals and in which all games of length $\omega_1$ are determined:

THEOREM 4.13. *Suppose there is an active, countable $\omega_1 + 1$-iterable extender model with a measurable Woodin cardinal $\theta$ below a measurable cardinal $\kappa$ and that the Continuum Hypothesis holds. Then, there is a transitive $M \models \mathsf{ZFC}$ containing $\mathbb{R}$ such that if $X$ is a game ordinal-definable from real parameters, then*

$$M \models \text{“}X \text{ is determined.”}$$

PROOF. The argument follows Woodin's proof of the fact that it is consistent that all definable games of length $\omega_1$ are determined and traces back to the arguments of Martin (unpublished) and Kechris and Solovay [4]. We construct a transitive $M^* \models \mathsf{ZFC}$ containing the reals in which all definable games are determined. Using standard techniques, one can extend the result to allow for ordinal or real parameters; see, for example, Neeman [11].

Let $M$ be a countable, active model of ZFC with a measurable Woodin cardinal below a measurable cardinal and let $\Gamma$ be an $(\omega_1 + 1)$-iteration strategy for $M$. Let $\{s_\alpha : \alpha < \omega_1\}$ be an enumeration of $\mathbb{R}$. Recursively define a sequence of reals $\vec{r} = \langle r_\alpha : \alpha < \omega_1 \rangle$ such that

(3) $r_0$ is a real coding $M$ and $s_0$; and
(4) if $0 < \alpha$, then $r_\alpha$ is a real coding $s_\alpha$ and $\Gamma$ restricted to countable iteration trees in

$$L[\langle r_\beta : \beta < \alpha \rangle].$$

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Let $N = L[\vec{r}]$. We have that $M \in H(\omega_1)^N$ moreover,

(5) in $N$, there is a countable, active $\omega_1 + 1$-iterable model $M$ of ZFC with a measurable Woodin cardinal below a measurable cardinal;
(6) the Continuum Hypothesis holds in $N$;
(7) $\mathbb{R} \subset N$.

It remains to show that all definable games are determined in $N$. Let $X \subset 2^{\omega_1}$ be such that for some formula $\varphi$,

$$X = \{f \in 2^{\omega_1} \cap N : N \models \varphi(\mathbb{R}, f)\}.$$

We claim that $X$ is determined in $N$. Writing

$$f = f_1 \oplus f_2 \oplus f_3 \oplus f_4,$$

let

$$Y = \{f \in 2^{\omega_1} : (L[f], f) \models \varphi(f_1 \oplus f_2, \mathbb{R})\},$$

that is, $Y$ is the game that results from two players playing a run of $X$ with dummy plays and then asking which player wins the game as computed in $L$ of the play.

By Theorem 4.11 applied inside $N$,

$$N \models \text{“}Y \text{ is determined.”}$$

Suppose Player I has a winning strategy $\sigma$ and assume that $\sigma$ is coded as a subset of $\omega_1$. We claim that Player I has a winning strategy for $X$ in $N$. The strategy, $\tau$, is as follows: a play $g$ is by $\tau$ if and only if $g = g_1 \oplus g_2$ and there exists some $h \in 2^{\omega_1} \cap N$ such that the play

$$f = g_1 \oplus g_2 \oplus h \oplus \vec{r}$$

is by $\sigma$, that is, Player I responds as she would in a game of $Y$ where, using their dummy plays, Player II codes $\vec{r}$. Any play by $\tau$ is winning for Player I. Similarly, if Player II has a winning strategy $\sigma$ in $Y$ in $N$, then Player II has a winning strategy for $X$ in $N$. ⊣

We finish with some questions. We do not know whether the Continuum Hypothesis is necessary for the determinacy of provably $\Delta_1$ games:

QUESTION 4.14. *Suppose that all games of length $\omega_1$ which are provably $\Delta_1$-definable from $\mathbb{R}$ are determined. Must the Continuum Hypothesis hold*?

The second question concerns the relation between provably $\Delta_1$-determinacy and open determinacy for games of length $\omega_1$.

QUESTION 4.15. *Suppose that there is a proper class of Woodin cardinals which are limits of Woodin cardinals and that open determinacy for games of length $\omega_1$ holds in every generic extension. Must all generic extensions that satisfy the Continuum Hypothesis satisfy that every game ($\Omega$-)provably $\Delta_1$-definable from $\mathbb{R}$ is determined*?

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## REFERENCES

[1] J. Bagaria, N. Castells, and P. Larson, *An Ω-logic primer*, ***Set Theory*** (J. Bagaria and S. Todorcevic, editors), Birkhäuser, Basel, 2006, pp. 203–242.

[2] M. Davis, *Infinite games of perfect information*, ***Advances in Game Theory*** (M. Dresher, L. S. Shapley, and A. W. Tucker, editors), Birkhäuser, Basel, 1964.

[3] Q. Feng, M. Magidor, and H. Woodin, *Universally Baire sets of reals*, ***Set Theory of the Continuum*** (H. Judah, W. Just, and H. Woodin, editors), Springer, New York, NY, 1992, pp. 203–242.

[4] A. S. Kechris and R. M. Solovay, *On the relative consistency strength of determinacy hypotheses*. ***Transactions of the American Mathematical Society***, vol. 290 (1985), pp 179–211.

[5] P. B. Larson, *The canonical function game*. ***Archive for Mathematical Logic***, vol. 44 (2005), no. 7, pp. 817–827.

[6] D. A. Martin, *Borel determinacy*. ***Annals of Mathematics***, vol. 102 (1975), pp. 363–371.

[7] D. A. Martin and J. R. Steel, *A proof of projective deterinacy*. ***Journal of the American Mathematical Society***, vol. 2 (1989), pp. 75–125.

[8] Y. N. Moschovakis, ***Descriptive Set Theory***, American Mathematical Society, Providence, RI, 2009.

[9] J. Mycielski, *On the axiom of determinacy*. ***Fundamenta Mathematicae***, vol. 53 (1964), pp. 205–224II.

[10] I. Neeman, *Inner models in the Region of a Woodin limit of Woodin cardinals*. ***Annals of Pure Applied Logic***, vol. 116 (2002), pp. 67–155.

[11] ———, ***The Determinacy of Long Games***, Walter de Gruyter, Berlin, 2004.

[12] ———, *Games of length* $\omega_1$. ***Journal of Mathematical Logic***, vol. 7 (2007), no. 1, pp. 83–124.

[13] I. Neeman and J. R. Steel, *Plus-one premice, parts I and II*. Handwritten notes, 2014. 94 + 64 pp.

[14] G. Sargsyan, *Hod Mice and the mouse set conjecture*. ***Memoirs of the American Mathematical Society*** (2014), p. 185.

[15] G. Sargsyan and N. D. Trang, *The Largest Suslin Axiom*, 2016. 307 pp.

[16] J. R. Steel, *A theorem of Woodin on mouse sets*, ***Ordinal Definability and Recursion Theory: The Cabal Seminar, vol. III*** (A. S. Kechris, B. Löwe, and J. R. Steel, editors), Lecture Notes in Logic 43, Association for Symbolic Logic and Cambridge University Press, Providence, RI, 2016, pp. 243–256.

[17] W. H. Woodin, *Supercompact cardinals, sets of reals, and weakly homogeneous trees*. ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 85 (1988), p. 6587.

[18] ———, *The continuum hypothesis, part II*. ***Notices of the American Mathematical Society***, vol. 48 (2001), pp. 681–690.

[19] ———, *Suitable extender models, I*. ***Journal of Mathematical Logic***, vol. 10 (2010), pp. 101–339.

[20] ———, *Fine structure at the finite levels of supercompactness*. Circulated manuscript, 2016. 710 pages.

INSTITUTE OF DISCRETE MATHEMATICS AND GEOMETRY
VIENNA UNIVERSITY OF TECHNOLOGY
WIEDNER HAUPTSTRAßE 8-10, 1040 VIENNA, AUSTRIA
and
DEPARTMENT OF MATHEMATICS, GHENT UNIVERSITY
KRIJGSLAAN 281-S8, 9000 GHENT, BELGIUM
*E-mail*: aguilera@logic.at

HARVARD LOGIC CENTER, HARVARD UNIVERSITY
2 ARROW STREET, CAMBRIDGE, MA 02138, USA
*E-mail*: blue@g.harvard.edu

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