α	β	a	b	c	a', b', c'	$\Delta A'B'C' = \frac{1}{2}a'b' \sin C$
$\frac{9}{5}$	$\frac{2}{1}$	20	19	37	20, 19, 37	114
$\frac{6}{5}$	$\frac{3}{1}$	25	16	39	25, 16, 39	120
$\frac{1}{1}$	$\frac{18}{5}$	28	15	41	28, 15, 41	126
$\frac{3}{5}$	$\frac{6}{1}$	40	13	51	40, 13, 51	156
$\frac{1}{6}$	$\frac{108}{5}$	708	65	761	708, 65, 761	13 806
$\frac{7}{8}$	$\frac{144}{35}$	1712	805	2405	1712, 805, 2405	413 448

TABLE 4: $\cos C = -\frac{4}{5}$, $\sin C = \frac{3}{5}$, $m = -4$, $n = 5$

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100.02 Square pyramidal numbers and the multiplication table

A square pyramidal number represents the number of stacked spheres in a pyramid with a square base. The number of spheres in each layer of the pyramid is just the square of the number of spheres of the layer edge, and the square pyramidal number is the sum of the numbers of spheres in each layer. The first few square pyramidal numbers are: 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240.

In this Note we derive the summation formula for the first N squared natural numbers using the properties of a matrix with elements i times j in the ith row and jth column. This matrix is also known as the multiplication table. Our derivation of the square pyramidal number formula takes advantage of the multiplication table properties.

Let us imagine an $N \times N$ matrix $c_{i,j}$ with $i, j = 1, \ldots, N$ in which the elements $c_{i,j}$ are equal to the products of i and j: $c_{i,j} = ij$. We will use this

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matrix to compute $\sum_{k=1}^{N} k^2$. We will apply the fact that $\sum_{i,j=1}^{N} c_{i,j} = \sum_{k=1}^{N} k^3$ and that, on the other hand, $\sum_{i,j=1}^{N} c_{i,j} = A + 2B$ where A is the sum of the diagonal elements $c_{i,i}$: $A = \sum_{i=1}^{N} c_{i,i} = \sum_{k=1}^{N} k^2$, and B is the sum of numbers in the upper or lower triangle of the matrix $c_{i,j}$. The sum of numbers in the upper triangle is equal to the sum of numbers in the lower triangle because the matrix $c_{i,j}$ is symmetric. First we will compute the formula for B.

Formula for the sum of numbers in the upper or lower triangle: For B and $N > 1$ we have $N - 1$ columnar areas in the matrix upper triangle where we can add the numbers as follows:

$$\begin{aligned}
&1 : 2 && = 2 \times 1 \\
&2 : 3 + 6 && = 3 \times (1 + 2) \\
&3 : 4 + 8 + 12 && = 4 \times (1 + 2 + 3) \\
&4 : 5 + 10 + 15 + 20 && = 5 \times (1 + 2 + 3 + 4) \\
&\dots \\
&k : (k+1) + 2(k+1) + \dots + k(k+1) && \\
&\quad = (k+1)(1 + 2 + \dots + k) && = (k+1)k(k+1)/2 \\
&\dots \\
&N-1 : N + 2N + \dots + (N-1)N && \\
&\quad = N(1 + 2 + \dots + (N-1)) && = N(N-1)N/2.
\end{aligned} \tag{1}$$

We have used the following summation formula to compute the sum in the k th columnar area for the upper triangle:

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}. \tag{2}$$

Our conclusion is that

$$B = \sum_{k=1}^{N-1} \frac{k(k+1)^2}{2} \tag{3}$$

and that $2B$ can be expressed as

$$2B = \sum_{k=1}^{N-1} k(k+1)^2. \tag{4}$$

In the k th columnar and row area (e.g. for the $k = 4$ area highlighted in Figure 1) the sum of all numbers can be expressed as

$$k(k+1)^2 + (k+1)^2 = (k+1)^2(k+1) = (k+1)^3 \tag{5}$$

which proves that the sum of all numbers in the matrix $\sum_{i,j=1}^{N} c_{i,j}$ is equal to $\sum_{k=1}^{N} k^3$.

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1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

FIGURE 1: The k th columnar and row area with the diagonal element for $k = 4$ is highlighted in the example 10 by 10 matrix $c_{i,j}$. The sum of numbers in this area is equal to $5^3 = 125$.

Intermediate computations: We have then established that

$$A + 2B = \sum_{k=1}^{N} k^2 + \sum_{k=1}^{N-1} k(k+1)^2 = \sum_{k=1}^{N} k^3 \tag{6}$$

which can be rewritten as

$$\sum_{k=1}^{N-1} k^2 + N^2 + \sum_{k=1}^{N-1} k^3 + 2\sum_{k=1}^{N-1} k^2 + \sum_{k=1}^{N-1} k = \sum_{k=1}^{N-1} k^3 + N^3. \tag{7}$$

After simple computations (7) gives

$$3\sum_{k=1}^{N-1} k^2 + N^2 + \frac{N(N-1)}{2} = N^3 \tag{8}$$

and

$$6\sum_{k=1}^{N-1} k^2 = N\left(2N^2 - 3N + 1\right) = N(2N-1)(N-1). \tag{9}$$

Conclusions: Now in (9) we can substitute $N + 1$ for every N to obtain

$$6\sum_{k=1}^{N} k^2 = (N+1)(2N+1)N \tag{10}$$

and we conclude that the sum of the first N squared natural numbers is equal

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to

$$\sum_{k=1}^{N} k^2 = \frac{(N+1)(2N+1)N}{6}. \tag{11}$$

The proof presented in our Note is an alternative to that found, for example, in [1] and it uses the properties of the multiplication table.

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100.03 Evaluating the probability integral by annular partition

The evaluation of the probability integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{1}$$

can be achieved in numerous ways including the methods based on double integration and solids of revolution (for example, see [1, 2, 3, 4]). A well-known approach of double integral in numerous textbooks is to compute

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \pi$$

using Fubini's theorem and the change of variable theorem to transform the double integral from rectangular coordinates to polar coordinates. In this note we use double integration with annular partition to evaluate the probability integral.

Consider the double integral of $z = f(x, y) = e^{-(x^2+y^2)}$ over three different regions respectively. Region 1 is the disk, denoted by D_a, with radius a and centre at the origin. Region 2 is the square, denoted by S_a, with $-a \leqslant x \leqslant a$ and $-a \leqslant y \leqslant a$. Region 3 is the disk, denoted by $D_{\sqrt{2}a}$, with radius $\sqrt{2}a$ and centre at the origin. The construction of these three regions is offered in [2]. Since $f(x, y)$ is positive and the square S_a lies between the disk D_a and the disk $D_{\sqrt{2}a}$, we obtain

$$V_1 = \iint_{D_a} f(x,y)dA < V_2 = \iint_{S_a} f(x,y)dA < V_3 = \iint_{D_{\sqrt{2}a}} f(x,y)dA. \tag{2}$$

Each double integral in (2) represents the volume, V_i, of the solid over its respective region i between the surfaces $z = 0$ and $z = f(x, y)$.

In particular, V_2 has two properties. On the one hand, Fubini's theorem gives

$$V_2 = \iint_{S_a} f(x, y)\, dA = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)} dx\, dy = \left[\int_{-a}^{a} e^{-x^2} dx\right]^2.$$

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