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FURTHER RESULTS ON STOCHASTIC ORDERINGS AND AGING CLASSES IN SYSTEMS WITH AGE REPLACEMENT

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Reliability properties associated with the classic models of systems with age replacement have been a usual topic of research. Most previous works have checked the aging properties of the lifetime of the working units using stochastic comparisons among the systems with age replacement at different times. However, from a practical point of view, it would also be interesting to deduce to which aging classes the lifetime of the system belongs, making use of the aging properties of the lifetime of its working units. The first part of this article deals with this problem. Further along, stochastic orderings are established between the systems with replacement at the same time using several stochastic comparisons among the lifetimes of their working units. In addition, the lifetimes of two systems with age replacement are compared as well. This is performed assuming stochastic orderings between the number of replacement until failure, and the lifetimes of their working units conditioned to be less or equal than the replacement time. Similar comparisons are accomplished considering two systems with age replacement where the replacements occur at a random time. Illustrative examples are presented throughout the paper.

Keywords: age replacement, aging classes, parallel systems, random time replacement, stationary pointwise availability, stochastic orders

1. INTRODUCTION

A natural way to improve the reliability of systems is implementing a replacement policy for a unit after it had been working for a period of time. Acting in this way it is possible to avoid system failures and periods of inactivity. Stochastic properties of lifetimes of systems with planned replacement policies have been widely studied and the age replacement policy comes up as one of the most studied kinds of replacement [29]. Under an age replacement policy, it is supposed that a single unit works upon failure or upon a specified age T,

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when a replacement by a new unit occurs, whichever comes first. The times of replacement are assumed as instantaneous. We assume that the lifetimes of all units to be placed in service are independent and equally distributed with the same distribution of a random variable X, which has finite mean, distribution function F, survival function $\overline{F} = 1 - F$ and density function f. Let us denote by $\tau_{X,T}$ the lifetime of the system with age replacement planned at $T > 0$. It is well known, see for example [4], that the survival function and the density function of $\tau_{X,T}$, denoted by $\overline{F}_{X,T}$ and $f_{X,T}$, respectively, satisfy

$$\begin{aligned} \overline{F}_{X,T}(t) &= [\overline{F}(T)]^{\lfloor t/T \rfloor} \overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right), \\ f_{X,T}(t) &= [\overline{F}(T)]^{\lfloor t/T \rfloor} f\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right), \end{aligned} \tag{1}$$

for all $t \geq 0$, where $\lfloor x \rfloor$ stands for the greater integer less or equal than x.

Systems with age replacement are important for practical and theoretical reasons. These systems have been used to characterize some aging classes. For example, Barlow and Proschan [4,5] prove that the lifetime of a system with age replacement, $\tau_{X,T}$, is stochastically decreasing (increasing) in T if and only if $X \in$ IFR ($X \in$ DFR). They also consider a model where the time until replacement is random and all lifetimes and times until replacement are assumed independent. A similar result for the discrete-time version of the age replacement model is proved in the recent work of Sudheesh *et al.* [40].

Marshall and Proschan [24] provide characterizations of NBU (NWU) and NBUE (NWUE) aging classes using the age replacement model. They show that $X \leq_{\text{st}} (\geq_{\text{st}}) \tau_{X,T}$ for all $T > 0$ if and only if $X \in$ NBU ($X \in$ NWU). Furthermore, $\mu = \mathbb{E}[X] \leq (\geq) \mathbb{E}[\tau_{X,T}]$ for all $T > 0$ if and only if $X \in$ NBUE ($X \in$ NWUE). See also [33] for these and other results associated with models with replacement policies. Similar characterizations are deduced by Belzunce *et al.* [6] for the increasing convex order and its associated *new better than used* class. In Theorem 2.6, we analyze the hazard rate and likelihood ratio orderings between X and $\tau_{X,T}$ when X belongs to IFR (DFR) and ILR (DLR) aging classes, respectively.

Those previous results characterize the aging properties of the random variable X using the random variable $\tau_{X,T}$, for $T > 0$. However, in practical situations, it would be interesting to deduce some properties of $\tau_{X,T}$ using the aging class to which X belongs. In this direction, we provide several results: in Theorem 2.1, we prove that $\tau_{X,T} \in$ NBU ($\tau_{X,T} \in$ NWU), whenever $X \in$ IFR ($X \in$ DFR). Moreover, in Theorem 2.2, we prove that $\tau_{X,T} \in$ DMRL, or $\tau_{X,T} \in$ IMRL, if and only if it has exponential distribution.

It is well known, see for example [4], that

$$\mathbb{E}[\tau_{X,T}] = \frac{\displaystyle\int_0^T \overline{F}(x)\,\mathrm{d}x}{F(T)}. \tag{2}$$

Marshall and Proschan [24] point out the lack of relation between the monotony of $\mathbb{E}[\tau_{X,T}]$ in T and the most common aging classes. As they noted, $X \in$ IFR implies that $\mathbb{E}[\tau_{X,T}]$ is decreasing in T, and that implies $X \in$ NBUE. Furthermore, both implications are false in the opposite sense. Due to this fact, some authors, as Klefsjö [18], Knopik [19,20], Kayid *et al.* [15] and Nair *et al.* [28], study the *Decreasing Mean Time to Failure* (DMTTF) aging class. See also the related work of Li and Xu [22] where the equivalent aging notion of *new better than renewal used in the reversed hazard rate order* (NBU_{rh}) is defined and studied. We say that X belongs to the DMTTF class, denoted by $X \in$ DMTTF, if $\mathbb{E}[\tau_{X,T}]$ given by (2) is decreasing in $T > 0$. It is well known that DMTTF $\subset$ NBUE [4,18] and

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IFR $\subset$ DMTTF [20]. Kayid *et al.* [16], Izadi *et al.* [12] and Ali Khan *et al.* [17] consider generalizations of this aging class.

The following characterization of the DMTTF and IMTTF aging classes can be obtained: $X \in$ DMTTF ($X \in$ IMTTF) if and only if $\mathbb{E}[\tau_{X,T}] \geq 1/\lambda(T)$ ($\mathbb{E}[\tau_{X,T}] \leq 1/\lambda(T)$), where $\lambda = f/\overline{F}$ is the hazard rate function of X. Besides, $\mathbb{E}[\tau_{X,T}]$ is constant in T only if X is exponentially distributed (cf. [19]). The equivalent results are obtained for the discrete-time age replacement model in [40] Thms. 3.1 and 3.2. In Corollary 2.4, we obtain another characterization of the DMTTF and IMTTF aging classes proving that $X \in$ DMTTF ($X \in$ IMTTF) if and only if $\tau_{X,T} \in$ NBUE ($\tau_{X,T} \in$ NWUE) for all $T > 0$.

Consider we have two systems with age replacement whose units have lifetimes X_1 and X_2, respectively, and with time until replacement $T > 0$ for both systems. It is natural to ask for the relation between the stochastic orderings that X_1, X_2 and $\tau_{X_1,T}$, $\tau_{X_2,T}$ satisfy. For example, Asha and Unnikrishnan Nair [3] and Kayid *et al.* [15] study the comparisons of the mean lifetimes of two systems with age replacement. Jain [13] Thm. 32 extends these results proving that $X_1 \succ X_2$ if and only if $\tau_{X_1,T} \succ \tau_{X_2,T}$, for all $T > 0$; where $\succ$ denotes the *usual stochastic order*, the *hazard rate order* or the *likelihood ratio order* (see also the work of Block *et al.* [8] related to the usual stochastic order). In Theorems 2.8 and 2.9, we obtain similar results for the *reversed hazard rate order*, the *mean residual lifetime order* and the *increasing convex and concave orders*. Possible applications of these results are also commented.

The existing results on stochastic orders for comparing parallel systems with constant and random number of units [9,32], allow us to establish stochastic orderings between the lifetimes of two parallel systems with age replacement. We consider a parallel system with age replacement with a constant number of units, and also the case where the number of units is random. These models have recently received considerable attention, and optimization studies have been carried out to determine the optimum replacement time that minimizes a defined cost function [10,29,30,44]. For the random-size parallel system with age replacement, we particularly focus on the case where the number of units is given by a *truncated Poisson distribution*, as it is considered, for example, by Nakagawa and Zhao [31] and Zhao *et al.* [44].

The last part of the present work deals with the system with replacement occurring at a random time. Similarly to the age replacement models, the optimal replacement time which minimizes a given cost function has been extensively studied on models with replacements at random time [30,44]. However, as far as we know, stochastic orderings between lifetimes of such systems have not been studied. For the sake of filling this gap, lifetimes of two systems with this kind of replacement are compared using the *usual stochastic order* and the *Laplace transform order*. In order to exemplify the applications of these results, we compare the lifetimes of two systems with *replacement first* or *replacement last*, that is, where the replacement time is given by the minimum or the maximum of n independent random times and a fixed time $T > 0$. Our study of both the random size parallel systems with age replacement, and the systems with replacements at first and last, is motivated by their possible application to real-life systems as it is reviewed in [44, Sect. 10.]

We must emphasize that the lifetimes of systems with replacement at random time are considerably more difficult to compare than the lifetimes of systems with age replacement previously considered. In general, it is necessary to assume stronger aging properties to get weaker order relations. For this reason, we also compare the mean lifetimes and the stationary pointwise availabilities of two systems as other measures of their reliability. The stationary pointwise availability of systems with replacement has been previously considered as a measure of the reliability of such systems. Moreover, the stationary pointwise availability of systems with replacement has been used to construct cost functions in studies about

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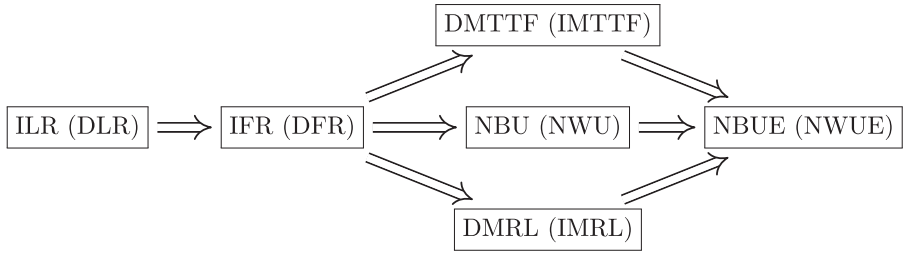


FIGURE **1.** Relations among the aging classes in Definition 1.1.

determining the optimal replacement time. See, for example, the works of Angus *et al.* [1], Zhao *et al.* [44] and Park *et al.* [34].

The rest of the paper is organized as follows: Section 1.1 provides the definitions of aging classes and stochastic orders we use in the sequel. Section 2 deals with the system with age replacement where the time until replacement is constant. Section 2.1 mainly deals with the results associated with aging classes and Section 2.2 is devoted to the results associated with stochastic orderings. Finally, Section 3 studies a system with replacement at a random time.

1.1. Background on Stochastic Orders and Aging Classes

During this work, we assume that all lifetimes and times until replacements are nonnegative absolutely continuous random variables with finite means. Throughout the article, the terms increasing and decreasing are used in the non-strict sense. The following definitions introduce some well-known concepts related to stochastic orders and aging classes.

DEFINITION 1.1 (Aging classes): *Let X be a nonnegative random variable with density function f, survival function $\overline{F}$ and finite mean μ. We say that X belongs to the aging class*

(a) New Better (Worse) than Used in Expectation, denoted by NBUE *(*NWUE*), if* $\int_t^\infty \overline{F}(x)\,\mathrm{d}x \leq\ (\geq)\ \mu\,\overline{F}(t)$ *for all* $t \geq 0$,

(b) Decreasing (Increasing) Mean Time to Failure, denoted by DMTTF *(*IMTTF*), if the function* $m_X(t) = \int_0^t \overline{F}(x)\,\mathrm{d}x/F(t)$ *is decreasing (increasing) in* $t > 0$,

(c) New Better (Worse) than Used, denoted by NBU *(*NWU*), if* $\overline{F}(x+t) \leq (\geq)\overline{F}(x)\overline{F}(t)$ *for all* $x, t \in \mathbb{R}_+$,

(d) Decreasing (Increasing) Mean Residual Life, denoted by DMRL *(*IMRL*), if*

$$\mu_X(t) = \int_t^\infty \overline{F}(x)\,\mathrm{d}x/\overline{F}(t)$$

is decreasing (increasing) in $\{t \geq 0 : \overline{F}(t) > 0\}$,

(e) Increasing (Decreasing) Failure Rate, denoted by IFR *(*DFR*), if* $\overline{F}(x+t)/\overline{F}(x)$ *is decreasing (increasing) in* $\{x \geq 0 : \overline{F}(x) > 0\}$ *for all* $t \geq 0$,

(f) Increasing (Decreasing) Likelihood Rate, denoted by ILR *(*DLR*), if* $f(x+t)/f(x)$ *is decreasing (increasing) in* $\{x \geq 0 : f(x) > 0\}$ *for all* $t \geq 0$.

Figure 1 shows the relations among the aging classes in Definition 1.1.

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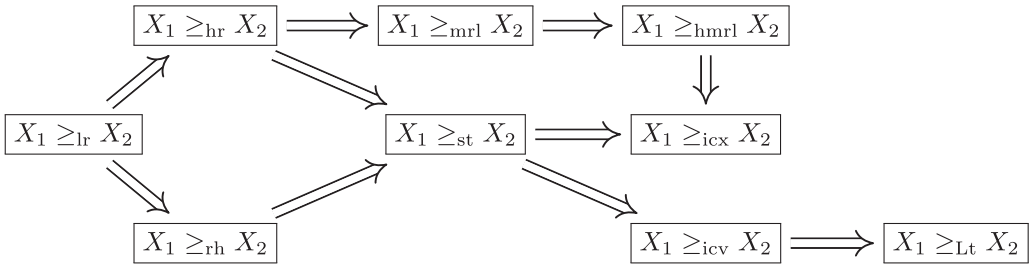


FIGURE **2.** Relations among the stochastic order in Definition 1.2.

Let us denote the hazard rate of X by $\lambda(t) = f(t)/\overline{F}(t)$, which is defined for all $t \geq 0$ such that $\overline{F}(t) > 0$. It is well known that $X \in \mathrm{IFR}$ ($X \in \mathrm{DFR}$) if and only if λ is increasing (decreasing) in its domain.

For a nonnegative integer-valued random variable N, these aging classes are analogously defined. For example, we say that N is IFR (DFR) if $\mathbb{P}[N = n]/\mathbb{P}[N > n]$ is increasing (decreasing) in $n \in \mathbb{N}$. Analogously, we say that N is ILR (DLR) if $\mathbb{P}[N = n + m]/\mathbb{P}[N = n]$ is decreasing (increasing) in $n \in \mathbb{N}$, for every $m \in \mathbb{N}$. Moreover, as in the continuous case, we have ILR $\subset$ IFR and DLR $\subset$ DFR. See Barlow and Proschan [5], Lai and Xie [21] and Unnikrishnan Nair *et al.* [28] for more details about these aging notions.

DEFINITION 1.2 (Stochastic orders): *Let X_1 and X_2 be two nonnegative random variables with finite means and with density functions f_1 and f_2, distribution functions F_1 and F_2 and survival functions $\overline{F}_1$ and $\overline{F}_2$, respectively. We say that X_1 is greater than X_2 in the*

(a) Laplace transform order, denoted by $X_1 \geq_{\mathrm{Lt}} X_2$, if $\int_0^\infty \overline{F}_1(t)\,\mathrm{e}^{-\mathrm{st}}\,\mathrm{d}t \geq \int_0^\infty \overline{F}_2(t)\,\mathrm{e}^{-\mathrm{st}}\,\mathrm{d}t$, for all $s \geq 0$,

(b) Increasing convex order, denoted by $X_1 \geq_{\mathrm{icx}} X_2$, if $\int_t^\infty \overline{F}_1(x)\,\mathrm{d}x \geq \int_t^\infty \overline{F}_2(x)\,\mathrm{d}x$, for all $t \geq 0$,

(c) Increasing concave order, denoted by $X_1 \geq_{\mathrm{icv}} X_2$, if $\int_0^t \overline{F}_1(x)\,\mathrm{d}x \geq \int_0^t \overline{F}_2(x)\,\mathrm{d}x$, for all $t \geq 0$,

(d) Harmonic mean residual lifetime order, denoted by $X_1 \geq_{\mathrm{hmrl}} X_2$, if

$$\int_t^\infty \frac{\overline{F}_1(x)}{\mathbb{E}[X_1]}\,\mathrm{d}x \geq \int_t^\infty \frac{\overline{F}_2(x)}{\mathbb{E}[X_2]}\,\mathrm{d}x,$$

for all $t \geq 0$,

(e) Mean residual lifetime order, denoted by $X_1 \geq_{\mathrm{mrl}} X_2$, if $\int_t^\infty \overline{F}_2(x)\,\mathrm{d}x/\int_t^\infty \overline{F}_1(x)\,\mathrm{d}x$ is decreasing for all $t \geq 0$ such that $\int_t^\infty \overline{F}_1(x)\,\mathrm{d}x > 0$,

(f) Usual stochastic order, denoted by $X_1 \geq_{\mathrm{st}} X_2$, if $\overline{F}_1(t) \geq \overline{F}_2(t)$, for all $t \geq 0$,

(g) Reversed hazard rate order, denoted by $X_1 \geq_{\mathrm{rh}} X_2$, if $F_1(t)/F_2(t)$ is decreasing for all $t \geq 0$,

(h) Hazard rate order, denoted by $X_1 \geq_{\mathrm{hr}} X_2$, if $\overline{F}_2(t)/\overline{F}_1(t)$ is decreasing for all $t \geq 0$ such that $\overline{F}_1(t) > 0$,

(i) Likelihood ratio order, denoted by $X_1 \geq_{\mathrm{lr}} X_2$, if $f_2(t)/f_1(t)$ is decreasing for all $t \geq 0$ such that $f_1(t) > 0$.

The relations among these stochastic orders are shown in Figure 2.

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For a nonnegative integer-valued random variable N, its *probability generating function*, denoted by ϕ_N, is defined as follows:

$$\phi_N : t \in (0,1) \mapsto \mathbb{E}[t^N].$$

A stochastic ordering between two nonnegative integer-valued random variables, N_1 and N_2, can be defined using the probability generating functions of these random variables, which turns to be equivalent to the Laplace transform ordering. Indeed, we say that N_1 is greater than N_2 in the sense of the probability generating function, denoted $N_1 \geq_{\text{pgf}} N_2$, if $\phi_{N_1}(t) \leq \phi_{N_2}(t)$, for all $t \in [0,1]$. $N_1 \geq_{\text{pgf}} N_2$ is thus equivalent to $\mathbb{E}[\mathrm{e}^{-sN_1}] \leq \mathbb{E}[\mathrm{e}^{-sN_2}]$, for all $s \geq 0$, which is equivalent to $N_1 \geq_{\text{Lt}} N_2$.

Denote by λ_1 and λ_2 the hazard rates of X_1 and X_2, respectively. The ordering $X_1 \geq_{\text{hr}} X_2$ holds if and only if $\lambda_1(t) \leq \lambda_2(t)$, for all $t \geq 0$. Similarly, $X_1 \geq_{\text{rh}} X_2$ if and only if $r_1(t) \geq r_2(t)$, for all $t \geq 0$, where $r_i = f_i/F_i$ is the reversed hazard rate of X_i, for $i = 1, 2$. A deeply treatment of these and other stochastic orders can be found in Müller and Stoyan [27], Shaked and Shanthikumar [37] and Belzunce *et al.* [7].

Consider the following sets of functions:

$$\mathcal{G}_{\text{rh}} = \{g : \mathbb{R}^2 \to \mathbb{R} \text{ such that } g(x,y) - g(y,x) \text{ is increasing in } x \text{ for all } x \leq y\},$$
$$\mathcal{G}_{\text{lr}} = \{g : \mathbb{R}^2 \to \mathbb{R} \text{ such that } g(x,y) \geq g(y,x) \text{ for all } x \geq y\}.$$

These sets of functions can be used to characterize the reversed hazard rate and likelihood ratio orders.

PROPOSITION 1.1 (Characterization of rh and lr orders): *Let X_1 and X_2 be two independent absolutely continuous random variables. Then,*

(a) $X_1 \geq_{\text{rh}} X_2$ *if and only if* $\mathbb{E}[\phi(X_1, X_2)] \geq \mathbb{E}[\phi(X_2, X_1)]$, *for all* $\phi \in \mathcal{G}_{\text{rh}}$,
(b) $X_1 \geq_{\text{lr}} X_2$ *if and only if* $\mathbb{E}[\phi(X_1, X_2)] \geq \mathbb{E}[\phi(X_2, X_1)]$, *for all* $\phi \in \mathcal{G}_{\text{lr}}$.

See [38] Thm. 2.3 and [37] Thm. 1.B.48 for the proofs of these results.

A nonnegative function $h : \mathbb{R}^2 \to [0, \infty)$ is said to be *totally positive of order* 2 (TP_2) if $h(x_1, y_1)h(x_2, y_2) \geq h(x_2, y_1)h(x_1, y_2)$, for all $x_1 \leq x_2$ and $y_1 \leq y_2$, see [14]. The following characterizations of IFR and DFR aging classes can be found in Propositions B.8 and B.9 of [23].

PROPOSITION 1.2 (Characterization of IFR and DFR aging classes): *Let X be an absolutely continuous random variable. Then,*

(a) $X \in \text{IFR}$ *if and only if* $\overline{F}(y - x)$ *is a* TP_2 *function in* (x, y),
(b) $X \in \text{DFR}$ *if and only if* $\overline{F}(x + y)$ *is a* TP_2 *function in* (x, y).

The Gamma and the Weibull distributions are two of the more used distributions to model the lifetimes and repair times of components in reliability theory [25,29,30]. We say that a random variable X follows a Gamma distribution with parameters $\alpha, \beta \in (0, \infty)$, denoted $X \sim \text{Gamma}(\alpha, \beta)$, if it has density $f(x) = \beta(\beta x)^{\alpha-1}\mathrm{e}^{-\beta x}/\Gamma(\alpha)$, for all $x > 0$. Also, X follows a Weibull distribution with parameters $\alpha, \beta \in (0, \infty)$, denoted $X \sim \text{Weibull}(\alpha, \beta)$, if it has density $f(x) = \beta\alpha x^{\alpha-1}\mathrm{e}^{-\beta x^\alpha}$, for all $x > 0$. In Table 1, we show sufficient conditions to order two random variables with Gamma or Weibull distribution according to some of the stochastic orders in Definition 1.2. See Taylor [41], Müller and Stoyan [27] and Belzunce *et al.* [7] for the proofs of these and further results.

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TABLE 1. Sufficient conditions to order two Gamma or Weibull random variables

	$X_1 \geq_{\mathrm{lr}} X_2$	$X_1 \geq_{\mathrm{mrl}} X_2$	$X_1 \geq_{\mathrm{icv}} X_2$
$X_i \sim \mathrm{Gamma}(\alpha_i, \beta_i)$			
$f_i(x) = \beta_i(\beta_i x)^{\alpha_i - 1}\,\mathrm{e}^{-\beta_i x}/\Gamma(\alpha_i)$	$\alpha_1 \geq \alpha_2$	$\alpha_1 \leq \alpha_2$	$\beta_1 \geq \beta_2$
$\mathbb{E}[X_i] = \alpha_i/\beta_i$, $i = 1, 2$	$\beta_1 \leq \beta_2$	$\mathbb{E}[X_1] \geq \mathbb{E}[X_2]$	$\mathbb{E}[X_1] \geq \mathbb{E}[X_2]$
$X_i \sim \mathrm{Weibull}(\alpha_i, \beta_i)$			
$f_i(x) = \beta_i \alpha_i x^{\alpha_i - 1}\,\mathrm{e}^{-\beta_i x}$	$\alpha_1 = \alpha_2$	$\alpha_1 \leq \alpha_2$	$\alpha_1 \geq \alpha_2$
$\mathbb{E}[X_i] = (\beta_i)^{-1/\alpha_i}\Gamma(1 + 1/\alpha_i)$, $i = 1, 2$	$\beta_1 \leq \beta_2$	$\mathbb{E}[X_1] \geq \mathbb{E}[X_2]$	$\mathbb{E}[X_1] \geq \mathbb{E}[X_2]$

2. RESULTS ON SYSTEMS WITH AGE REPLACEMENT

This section is devoted to the study of the lifetime of systems with age replacement at age $T > 0$ using aging classes and stochastic orders. In Section 2.1, we study the relation between the aging classes to which X and $\tau_{X,T}$ belong. In Section 2.2, we establish stochastic orderings between the lifetimes of two systems with age replacement using stochastic orderings between the lifetimes of the working units.

2.1. Aging Classes

The relation between τ_{X,T_1} and τ_{X,T_2}, when X belongs to an aging class, has been widely studied. However, as far as we know, it has not been analyzed the aging classes to which $\tau_{X,T}$ belongs, using aging properties of X. The following results deal with this problem.

THEOREM 2.1: *If* $X \in$ IFR ($X \in$ DFR), *then* $\tau_{X,T} \in$ NBU ($\tau_{X,T} \in$ NWU), *for all* $T > 0$.

PROOF: To prove $\tau_{X,T} \in \mathrm{NBU}$, we have to check the inequality $\overline{F}_{X,T}(x)\overline{F}_{X,T}(t) \geq \overline{F}_{X,T}(x+t)$ for all nonnegative x and t, where $\overline{F}_{X,T}(t)$ is defined as in (1). This inequality is equivalent to

$$[\overline{F}(T)]^{\lfloor x/T \rfloor + \lfloor t/T \rfloor}\overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T\right)\overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right) \geq [\overline{F}(T)]^{\lfloor (x+t)/T \rfloor}\overline{F}\left(x + t - \left\lfloor \frac{x+t}{T} \right\rfloor T\right). \tag{3}$$

As $\lfloor (x+t)/T \rfloor = \lfloor t/T \rfloor + \lfloor x/T \rfloor$ or $\lfloor (x+t)/T \rfloor = \lfloor t/T \rfloor + \lfloor x/T \rfloor + 1$, we can focus our attention on the following two excluding cases:

Case 1: $\lfloor (x+t)/T \rfloor = \lfloor t/T \rfloor + \lfloor x/T \rfloor$
In this case, (3) can be written as

$$\overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T\right)\overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right) \geq \overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T + t - \left\lfloor \frac{t}{T} \right\rfloor T\right). \tag{4}$$

Then, for (4) to hold it is sufficient that $\overline{F}(w)\overline{F}(z) \geq \overline{F}(w+z)$ for all $w \geq 0$ and $z \geq 0$ satisfying $w + z \in [0, T]$. So, the inequality (3) holds because $X \in$ NBU.

Case 2: $\lfloor (x+t)/T \rfloor = \lfloor t/T \rfloor + \lfloor x/T \rfloor + 1$

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Now, (3) is equivalent to

$$\overline{F}\left(x-\left\lfloor\frac{x}{T}\right\rfloor T\right)\overline{F}\left(t-\left\lfloor\frac{t}{T}\right\rfloor T\right)\geq\overline{F}(T)\overline{F}\left(x+t-\left\lfloor\frac{x}{T}\right\rfloor T-\left\lfloor\frac{t}{T}\right\rfloor T-T\right). \quad \textbf{(5)}$$

Let us suppose, without lost of generality, that $t-\lfloor t/T\rfloor T\leq x-\lfloor x/T\rfloor T\leq T$. Then, the following inequalities hold

$$T\geq x-\left\lfloor\frac{x}{T}\right\rfloor T\geq t-\left\lfloor\frac{t}{T}\right\rfloor T\geq x-\left\lfloor\frac{x}{T}\right\rfloor T+t-\left\lfloor\frac{t}{T}\right\rfloor T-T.$$

Hence, (5) is true when $\overline{F}(x_1)\overline{F}(x_4)\leq\overline{F}(x_2)\overline{F}(x_3)$ for $x_1\geq x_2\geq x_3\geq x_4$ such that $x_i\in[0,T]$, $i=1,2,3,4$. So, (5) is satisfied when $\overline{F}(y-x)$ is a TP_2 function in (x,y), or equivalently by part (a) of Proposition 1.2, when $X\in$ IFR.

The proof when $X\in$ DFR can be analogously obtained using part (b) of Proposition 1.2. ■

REMARK 2.1: *It is easy to see that if λ is increasing (decreasing) in $[0,T]$, then $\tau_{X,T}\in$* NBU *($\tau_{X,T}\in$* NWU*). Thus, $\tau_{X,T}$ could belong to* NBU *even if $X\notin$* IFR.

When X has probability density, so does $\tau_{X,T}$ and its hazard rate, denoted by $\lambda_{X,T}$, satisfies $\lambda_{X,T}(t)=\lambda(t-\lfloor t/T\rfloor T)$, for all $t\geq 0$. As $\lambda_{X,T}$ is a periodic function, it is monotonic if and only if it is constant. So, $\tau_{X,T}\in$ IFR (DFR) for all $T>0$ if and only if it is exponentially distributed, that is, $\tau_{X,T}\notin$ IFR(DFR) except when there exists a $\lambda>0$ such that $\overline{F}(t)=\mathrm{e}^{-\lambda t}$, for all $t\in[0,T]$, which implies $\overline{F}_{X,T}(t)=\mathrm{e}^{-\lambda t}$, for all $t\geq 0$. In Theorem 2.2, below we set a stronger result considering the DMRL and IMRL aging classes.

First, let us denote by $\mu_{X,T}$ the *mean residual lifetime* of $\tau_{X,T}$ and denote $x=nT+h$, where $h\in[0,T)$ and $n=\lfloor x/T\rfloor\geq 0$. Next, we will obtain an expression for $\mu_{X,T}$. Note that

$$\mu_{X,T}(x)=\int_0^\infty\frac{\overline{F}_{X,T}(x+t)}{\overline{F}_{X,T}(x)}\,\mathrm{d}t=\frac{1}{\overline{F}_{X,T}(x)}\int_x^\infty\overline{F}_{X,T}(t)\,\mathrm{d}t. \quad \textbf{(6)}$$

Furthermore,

$$\begin{aligned}\int_x^\infty\overline{F}_{X,T}(t)\,\mathrm{d}t&=\int_x^{(n+1)T}\overline{F}_{X,T}(t)\,\mathrm{d}t+\int_{(n+1)T}^\infty\overline{F}_{X,T}(t)\,\mathrm{d}t\\&=\int_x^{(n+1)T}\overline{F}_{X,T}(t)\,\mathrm{d}t+\sum_{k=n+1}^\infty\int_{kT}^{(k+1)T}\overline{F}_{X,T}(t)\,\mathrm{d}t\\&=[\overline{F}(T)]^n\int_{nT+h}^{(n+1)T}\overline{F}(t-nT)\,\mathrm{d}t+\sum_{k=n+1}^\infty[\overline{F}(T)]^k\int_{kT}^{(k+1)T}\overline{F}(t-kT)\,\mathrm{d}t.\end{aligned}$$

Then,

$$\int_x^\infty\overline{F}_{X,T}(t)\,\mathrm{d}t=[\overline{F}(T)]^n\int_h^T\overline{F}(u)\,\mathrm{d}u+\frac{[\overline{F}(T)]^{n+1}}{1-\overline{F}(T)}\int_0^T\overline{F}(u)\,\mathrm{d}u. \quad \textbf{(7)}$$

Moreover,

$$\overline{F}_{X,T}(x)=[\overline{F}(T)]^n\overline{F}(h). \quad \textbf{(8)}$$

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Thus, plugging (7) and (8) in (6), we get

$$\mu_{X,T}(x) = \frac{1}{\overline{F}(h)} \left(\int_h^T \overline{F}(u)\,\mathrm{d}u + \frac{\overline{F}(T)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\,\mathrm{d}u \right), \tag{9}$$

for $x = nT + h$, with $h \in [0, T)$, and $n = \lfloor x/T \rfloor \geq 0$.

THEOREM 2.2: $\tau_{X,T} \in$ DMRL (IMRL) *if and only if it is exponentially distributed.*

PROOF: Suppose $\mu_{X,T}$ is monotonic. As $\mu_{X,T}$ is a periodic function, it must be constant. Let us define $c = \mathbb{E}[\tau_{X,T}] = \mu_{X,T}(0)$. Using (9), we get $\mu_{X,T}$ is a constant function if and only if

$$\int_h^T \overline{F}(u)\,\mathrm{d}u = c[\overline{F}(h) - \overline{F}(T)], \quad \text{for } h \in [0, T].$$

The solution of the previous integral equation satisfies $\overline{F}(h) = \mathrm{e}^{-h/c}$, for $h \in [0, T]$. Consequently, using (1), we obtain that $\tau_{X,T}$ is exponentially distributed. The converse implication is trivially true. ■

THEOREM 2.3: $\tau_{X,T} \in$ NBUE ($\tau_{X,T} \in$ NWUE) *if and only if* $\mathbb{E}[\tau_{X,h}] \geq (\leq)\mathbb{E}[\tau_{X,T}]$ *for all* $h \in [0, T]$.

PROOF: Using (2) and (9), we know that $\tau_{X,T} \in$ NBUE if and only if

$$\int_h^T \overline{F}(u)\,\mathrm{d}u + \frac{\overline{F}(T)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\,\mathrm{d}u \leq \frac{\overline{F}(h)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\,\mathrm{d}u,$$

for $h \in [0, T)$. This inequality is equivalent to

$$\begin{aligned} \int_0^T \overline{F}(u)\,\mathrm{d}u - \int_0^h \overline{F}(u)\,\mathrm{d}u = \int_h^T \overline{F}(u)\,\mathrm{d}u &\leq \frac{\overline{F}(h) - \overline{F}(T)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\,\mathrm{d}u \\ &\leq \left[1 - \frac{1-\overline{F}(h)}{1-\overline{F}(T)}\right] \int_0^T \overline{F}(u)\,\mathrm{d}u. \end{aligned}$$

Rearranging conveniently we obtain the equivalent inequality

$$\mathbb{E}[\tau_{X,T}] = \frac{1}{F(T)} \int_0^T \overline{F}(u)\,\mathrm{d}u \leq \frac{1}{F(h)} \int_0^h \overline{F}(u)\,\mathrm{d}u = \mathbb{E}[\tau_{X,h}],$$

which proves the result. The case $\tau_{X,T} \in$ NWUE can be analogously obtained. ■

Suppose that a system with age replacement at time T is already functioning, and it must be decided if a sooner replacement policy should be implemented or not. Because of the definition of the DMTTF aging class, when $X \in$ DMTTF a smaller time between replacement always increases the expected lifetime of the system, but this is not necessarily true for other (not stronger) aging classes. Moreover, if the system is already working, the lifetime data we can collect come from the lifetime of the system with replacement, and not from the unconditioned lifetime of a working unit without a replacement policy implemented. Theorem 2.3 offers a solution in this case because it is sufficient to check that $\tau_{X,T} \in$ NBUE to ensure that $\mathbb{E}[\tau_{X,T}] \leq \mathbb{E}[\tau_{X,h}]$, for all $h \in (0, T]$. The hypothesis $\tau_{X,T} \in$ NBUE could be accepted using some of the existing statistical tests. We refer the interested readers on these kind of tests to [2,21,39], and the references therein.

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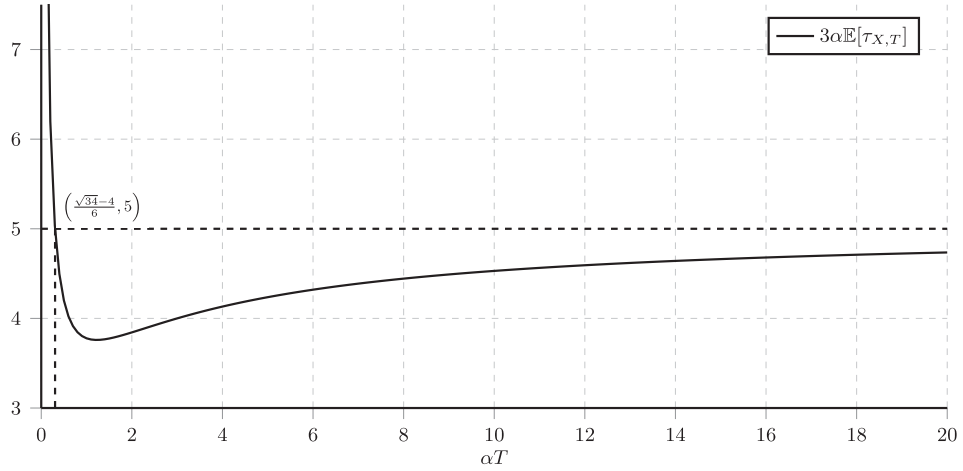


FIGURE **3.** Plot of $3\alpha\mathbb{E}[\tau_{X,T}]$ as a function of αT.

EXAMPLE 2.1: *Let us consider a random variable X with survival function*

$$\overline{F}_X(t) = 1 - \left[1 - \frac{1}{(1+\alpha t)^2}\right]^2,$$

where $\alpha > 0$. The expected value of $\tau_{X,T}$ can be computed using (2) and is equal to

$$\mathbb{E}[\tau_{X,T}] = \frac{1}{3\alpha}\frac{(\alpha T+1)(5\alpha^2T^2+9\alpha T+3)}{\alpha T(\alpha T+2)^2}, \quad \text{for } T > 0.$$

This example was considered first by Weiss [42]. In Figure 3, we can see that $3\alpha\mathbb{E}[\tau_{X,T}]$ is not monotonic as a function of αT. Also, as α is fixed, the global minimum of $\mathbb{E}[\tau_{X,T}]$ is attained when $\alpha T = \sqrt{6}/2$. Thus, by Theorem 2.3, we have that $\tau_{X,T} \in$ NBUE if and only if $T \leq \sqrt{6}/2\alpha$.

Note that since the graph of $3\alpha\mathbb{E}[\tau_{X,T}]$ crosses its limit value when T tends toward infinity, namely $3\alpha\mathbb{E}[X] = 5$, we get that X in not NBUE, *neither* NWUE. *However, it can be set up a replacement policy at age $T \in (0, \sqrt{6}/2\alpha)$ such that $\tau_{X,T}$ is* NBUE, *and equivalently $\mathbb{E}[\tau_{X,h}] \geq \mathbb{E}[\tau_{X,T}]$, for all $h \in (0, T]$. However, from a practical point of view, as Weiss [42] points out, it is inadvisable to set a replacement policy up unless*

$$\mathbb{E}[\tau_{X,T}] \geq \lim_{T\to\infty} \mathbb{E}[\tau_{X,T}] = \mathbb{E}[X] = 5/3\alpha,$$

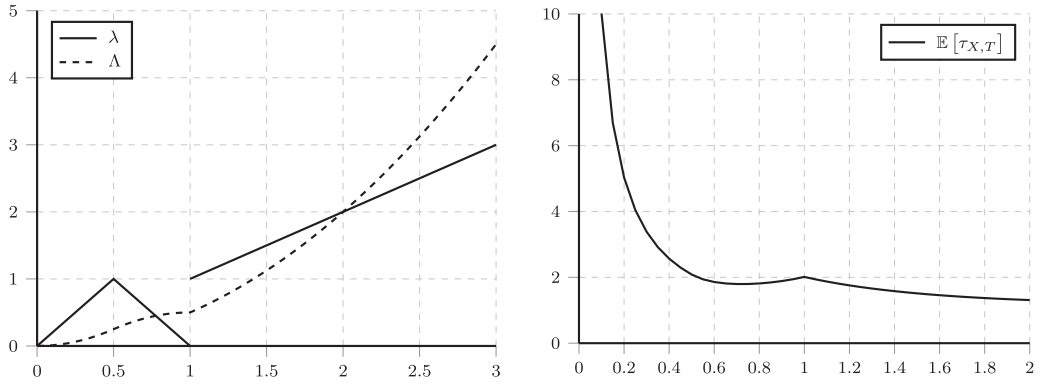
or equivalently $\alpha T \leq (\sqrt{34}-4)/6 \approx 0.3052$. Indeed, for a planned replacement at T such that $\mathbb{E}[\tau_{X,T}] \leq \mathbb{E}[X]$, it would be better not to implement a replacement in the sense of the expected value.

The following result is a consequence of Theorem 2.3.

COROLLARY 2.4: *$X \in$* DMTTF (IMTTF) *if and only if $\tau_{X,T} \in$* NBUE (NWUE) *for all $T \geq 0$.*

In Theorem 2.1, we proved that $X \in$ IFR is a sufficient condition for $\tau_{X,T} \in$ NBU to hold. One could ask if $X \in$ NBU is a sufficient condition for $\tau_{X,T} \in$ NBU. The following

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(a) Plots of functions λ and Λ.

(b) Plot of $\mathbb{E}[\tau_{X,T}]$.

FIGURE **4.** Plots of (a) functions λ and Λ, and (b) the expectation $\mathbb{E}[\tau_{X,T}]$ as a function of T.

example gives us a negative answer to this question showing that $X \in \text{NBU}$ is not even a sufficient condition even for $\tau_{X,T} \in \text{NBUE}$ to hold.

EXAMPLE 2.2 ($X \in \text{NBU}$ and $\tau_{X,T} \notin \text{NBUE}$): *Let X be a random variable with hazard rate function λ and cumulative hazard rate function Λ satisfying*

$$\lambda(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2], \\ -2(x-1) & \text{if } x \in (1/2, 1], \\ x & \text{if } x \in (1, \infty). \end{cases}$$

$$\Lambda(x) = \int_0^x \lambda(t)\,\mathrm{d}t = \begin{cases} x^2 & \text{if } x \in [0, 1/2], \\ -(x-1)^2 + 1/2 & \text{if } x \in (1/2, 1], \\ x^2/2 & \text{if } x \in (1, \infty). \end{cases}$$

Figure 4(a) shows a plot of these functions.

Then, X has survival function $\overline{F}(x) = \mathrm{e}^{-\Lambda(x)}$. It can be shown that $\Lambda(x+t) \geq \Lambda(x) + \Lambda(t)$ for all x and t in $\mathbb{R}_+$, thus $X \in \text{NBU}$. However, λ is not monotonic, so $X \notin \text{IFR}$. Also, as it is showed in Figure 4(b), $\mathbb{E}[\tau_{X,T}]$ is not decreasing in T. Then, from Theorem 2.3, there are values of T such that $\tau_{X,T} \notin \text{NBUE}$. For example, taking $T_1 = 3/4$ and $T = 1$, we get $\mathbb{E}[\tau_{X,T_1}] \approx 1.7976$ and $\mathbb{E}[\tau_{X,T}] \approx 2.012$. Thus, even if $X \in \text{NBU}$ and $T_1 < T$, the inequality $\mathbb{E}[\tau_{X,T_1}] \leq \mathbb{E}[\tau_{X,T}]$ holds, and consequently $\tau_{X,T} \notin \text{NBUE}$.

Marshall and Proschan [24] prove that $X \leq_{\text{st}} \tau_{X,T}$ ($X \geq_{\text{st}} \tau_{X,T}$) for all $T > 0$ if and only if $X \in \text{NBU}$ ($X \in \text{NWU}$). In a similar way, the following lemma provides necessary and sufficient conditions for stochastic orderings in the hazard rate and the likelihood ratio senses between the lifetime of a system with replacement and the lifetime of its units.

LEMMA 2.5: *Let X be an absolutely continuous random variable. Then,*

(a) $\tau_{X,T} \geq_{\text{hr}} X$ ($\tau_{X,T} \leq_{\text{hr}} X$), for all $T \geq 0$, if and only if $(\overline{F}(x+y))/\overline{F}(x)$ is decreasing (increasing) for $x \in [0, y]$ and for all $y \geq 0$.

(b) $\tau_{X,T} \geq_{\text{lr}} X$ ($\tau_{X,T} \leq_{\text{lr}} X$), for all $T \geq 0$, if and only if $(f(x+y))/f(x)$ is decreasing (increasing) for $x \in [0, y]$ and for all $y \geq 0$.

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PROOF: (a) We will prove the result for $\tau_{X,T} \geq_{\text{hr}} X$. The proof for $\tau_{X,T} \leq_{\text{hr}} X$ is analogous.

Due to the continuity of $\overline{F}(t)$, the function

$$\frac{\overline{F}_{X,T}(t)}{\overline{F}(t)} = \frac{[\overline{F}(T)]^{\lfloor t/T \rfloor}\overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right)}{\overline{F}(t)} \tag{10}$$

is also continuous. Then, to prove that (10) is increasing is equivalent to check the function $\overline{F}(t-(n-1)T)/\overline{F}(t)$ is increasing in $t \in [(n-1)T, nT)$, for all $n \in \mathbb{N}$. Let us denote $x = t-(n-1)T$. Thus, $\tau_{X,T} \geq_{\text{hr}} X$ if and only if $\overline{F}(x)/\overline{F}(x+(n-1)T)$ is increasing in $x \in [0,T]$ for all $T > 0$ and $n \geq 1$. Setting $n = 2$ and $T = y$, it is clear that $\overline{F}(x)/\overline{F}(x+y)$ is increasing for $x \in [0,y]$ and for all $y \geq 0$.

Now, we will prove the converse implication. Assume $\overline{F}(x)/\overline{F}(x+y)$ is increasing in $x \in [0,y]$, for all $y \geq 0$, and let m be a natural number. Then, $\overline{F}(x)/\overline{F}(x+mT)$ is increasing for $x \in [0,T]$ because it is for $x \in [0,mT] \supset [0,T]$.

(b) The proof is very similar to the previous case. ■

The next result follows straightforward from Lemma 2.5.

THEOREM 2.6: *(a) If $X \in$ IFR, then $\tau_{X,T} \geq_{\text{hr}} X$, for all $T > 0$.*

(b) If $X \in$ ILR, then $\tau_{X,T} \geq_{\text{lr}} X$, for all $T > 0$.

Moreover, using that the hazard rate function of $\tau_{X,T}$ satisfies $\lambda_{X,T}(t) = \lambda(t - \lfloor t/T \rfloor T)$, for all $t \geq 0$, we get the following result.

PROPOSITION 2.1: *If $\sup_{t\in[0,T)} \lambda(t) \leq \inf_{t\in[T,\infty)} \lambda(t)$, then $\tau_{X,T} \geq_{\text{hr}} X$.*

Proposition 2.7 shows that the ordering $\tau_{X,T} \geq_{\text{hr}} X$ can hold even when $X \notin$ IFR. For instance, consider a random variable X with hazard rate λ as in Example 2.2. It can be proved that $\tau_{X,T} \geq_{\text{hr}} X$ for all $T \geq 1$.

2.2. Stochastic Orderings

Let us consider two systems with age replacement whose units have lifetimes with the same distribution of the random variables X_1 and X_2, respectively. Jain [13] Thm. 32 proves the existence of stochastic orderings between $\tau_{X_1,T}$ and $\tau_{X_2,T}$ when the same ordering hold between X_1 and X_2. Jain [13] considers the usual stochastic order, the hazard rate order and the likelihood ratio order. The next theorem expands these results considering other stochastic orders.

THEOREM 2.7: *The following statements hold:*

(a) The ordering $X_1 \geq_{\text{rh}} X_2$ holds, if and only if $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$, for all $T \geq 0$.

(b) If $\int_h^T \overline{F}_1(u)\,\mathrm{d}u / \int_h^T \overline{F}_2(u)\,\mathrm{d}u$ is increasing in $h \in (0,T)$, then $\tau_{X_1,T} \geq_{\text{mrl}} \tau_{X_2,T}$.

(c) If $\overline{F}_1(T) \geq \overline{F}_2(T)$ and

$$\int_0^t \overline{F}_1(x)\,\mathrm{d}x \geq \int_0^t \overline{F}_2(x)\,\mathrm{d}x, \tag{11}$$

for all $t \in [0,T]$, then $\tau_{X_1,T} \geq_{\text{icv}} \tau_{X_2,T}$.

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PROOF: (a) The reversed hazard rate of $\tau_{X_i,T}$, denoted by $r_{X_i,T}$, satisfies

$$r_{X_i,T}(t) = \frac{[\overline{F}_i(T)]^{\lfloor t/T \rfloor} f_i(t - \lfloor t/T \rfloor T)}{1 - [\overline{F}_i(T)]^{\lfloor t/T \rfloor} + [\overline{F}_i(T)]^{\lfloor t/T \rfloor} F_i(t - \lfloor t/T \rfloor T)}$$
$$= \frac{r_i(t - \lfloor t/T \rfloor T)}{((1 - [\overline{F}_i(T)]^{\lfloor t/T \rfloor})/([\overline{F}_i(T)]^{\lfloor t/T \rfloor} F_i(t - \lfloor t/T \rfloor T))) + 1}, \tag{12}$$

where r_i is the reversed hazard rate of X_i, for $i = 1, 2$.

Note that for $t \in [0, T]$, we get $r_{X_i,T}(t) = r_i(t)$, for $i = 1, 2$. So, if $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$, for all $T \geq 0$, then $r_1(t) \geq r_2(t)$ for all $t \geq 0$, and consequently $X_1 \geq_{\text{rh}} X_2$.

Now, let us prove that if $X_1 \geq_{\text{rh}} X_2$, then $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$. The ordering $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$ is equivalent to $r_{X_1,T}(t) \geq r_{X_2,T}(t)$, for $t \geq 0$, which using (12) becomes equivalent to

$$r_1(t - \lfloor t/T \rfloor T) \left[\frac{1 - [\overline{F}_2(T)]^{\lfloor t/T \rfloor}}{[\overline{F}_2(T)]^{\lfloor t/T \rfloor} F_2(t - \lfloor t/T \rfloor T)} + 1 \right]$$
$$\geq r_2(t - \lfloor t/T \rfloor T) \left[\frac{1 - [\overline{F}_1(T)]^{\lfloor t/T \rfloor}}{[\overline{F}_1(T)]^{\lfloor t/T \rfloor} F_1(t - \lfloor t/T \rfloor T)} + 1 \right].$$

As $r_1(t) \geq r_2(t)$, for all $t \geq 0$, it is sufficient to check the inequality

$$\frac{1 - [\overline{F}_2(T)]^{\lfloor t/T \rfloor}}{[\overline{F}_2(T)]^{\lfloor t/T \rfloor} F_2(t - \lfloor t/T \rfloor T)} \geq \frac{1 - [\overline{F}_1(T)]^{\lfloor t/T \rfloor}}{[\overline{F}_1(T)]^{\lfloor t/T \rfloor} F_1(t - \lfloor t/T \rfloor T)},$$

which can be written in the way

$$F_2(T) \frac{[1 + \overline{F}_2(T) + \cdots + [\overline{F}_2(T)]^{\lfloor t/T \rfloor - 1}]}{F_2(t - \lfloor t/T \rfloor T)[\overline{F}_2(T)]^{\lfloor t/T \rfloor}}$$
$$\geq F_1(T) \frac{[1 + \overline{F}_1(T) + \cdots + [\overline{F}_1(T)]^{\lfloor t/T \rfloor - 1}]}{F_1(t - \lfloor t/T \rfloor T)[\overline{F}_1(T)]^{\lfloor t/T \rfloor}}. \tag{13}$$

Since $\overline{F}_1(T) \geq \overline{F}_2(T)$, a sufficient condition for the inequality (13) to hold is

$$\frac{F_2(T)}{F_2(t - \lfloor t/T \rfloor T)} \geq \frac{F_1(T)}{F_1(t - \lfloor t/T \rfloor T)}.$$

But this last inequality is equivalent to

$$\frac{F_2(T)}{F_1(T)} \geq \frac{F_2(t - \lfloor t/T \rfloor T)}{F_1(t - \lfloor t/T \rfloor T)},$$

and it is true because, as $X_1 \geq_{\text{rh}} X_2$, the function $F_2(t)/F_1(t)$ is increasing for all $t \geq 0$.

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(b) Using (7), it can be seen that the ordering $\tau_{X_1,T} \geq_{\text{mrl}} \tau_{X_2,T}$ is equivalent to

$$\left[\frac{\overline{F}_1(T)}{\overline{F}_2(T)}\right]^n l(h)$$

being increasing in $h \in [0,T]$ and in $n \in \mathbb{N}$, where

$$l(h) = \left(\frac{\displaystyle\int_h^T \frac{\overline{F}_1(u)}{\overline{F}_1(T)}\,\mathrm{d}u + \mathbb{E}[\tau_{X_1,T}]}{\displaystyle\int_h^T \frac{\overline{F}_2(u)}{\overline{F}_2(T)}\,\mathrm{d}u + \mathbb{E}[\tau_{X_2,T}]} \right).$$

Observe that

$$\frac{\overline{F}_1(T)}{\overline{F}_2(T)} \geq \frac{\displaystyle\int_h^T \overline{F}_1(u)\,\mathrm{d}u}{\displaystyle\int_h^T \overline{F}_2(u)\,\mathrm{d}u} \geq \frac{\overline{F}_1(h)}{\overline{F}_2(h)}, \tag{14}$$

where the first inequality holds taking limits when h tends to T, and the second inequality is true because by hypothesis the derivative of $h \mapsto \int_h^T \overline{F}_1(u)\,\mathrm{d}u / \int_h^T \overline{F}_2(u)\,\mathrm{d}u$ with respect to h is nonnegative.

Taking $h = 0$ in (14), we have $\overline{F}_1(T)/\overline{F}_2(T) \geq \overline{F}_1(0)/\overline{F}_2(0) = 1$, and thus, $[\overline{F}_1(T)/\overline{F}_2(T)]^n$ is increasing in n. Consequently, it only remains to prove that $l(h)$ is increasing in $h \in (0,T)$.

Deriving $l(h)$, we get it is increasing if and only if

$$\frac{\overline{F}_2(h)\displaystyle\int_h^T \overline{F}_1(u)\,\mathrm{d}u}{\overline{F}_1(T)\overline{F}_2(T)} + \frac{\overline{F}_2(h)}{\overline{F}_2(T)}\mathbb{E}[\tau_{X_1,T}] \geq \frac{\overline{F}_1(h)\displaystyle\int_h^T \overline{F}_2(u)\,\mathrm{d}u}{\overline{F}_1(T)\overline{F}_2(T)} + \frac{\overline{F}_1(h)}{\overline{F}_1(T)}\mathbb{E}[\tau_{X_2,T}]. \tag{15}$$

Note that the inequalities $\overline{F}_2(h)\int_h^T \overline{F}_1(u)\,\mathrm{d}u \geq \overline{F}_1(h)\int_h^T \overline{F}_2(u)\,\mathrm{d}u$ and $\overline{F}_2(h)/\overline{F}_2(T) \geq \overline{F}_1(h)/\overline{F}_1(T)$ hold due to (14). Taking $h = 0$ in (14), we have $\int_0^T \overline{F}_1(u)\,\mathrm{d}u \geq \int_0^T \overline{F}_2(u)\,\mathrm{d}u$, and using $F_1(T) \leq F_2(T)$, we get $\mathbb{E}[\tau_{X_1,T}] \geq \mathbb{E}[\tau_{X_2,T}]$. Thus, the inequality (15) holds.

(c) $\tau_{X_1,T} \geq_{\text{icv}} \tau_{X_2,T}$ holds when

$$\int_0^t \overline{F}_{X_1,T}(u)\,\mathrm{d}u \geq \int_0^t \overline{F}_{X_2,T}(u)\,\mathrm{d}u,$$

for all $t \geq 0$. This inequality is equivalent to

$$\mathbb{E}[\tau_{X_1,T}] - \int_t^\infty \overline{F}_{X_1,T}(u)\,\mathrm{d}u \geq \mathbb{E}[\tau_{X_2,T}] - \int_t^\infty \overline{F}_{X_2,T}(u)\,\mathrm{d}u. \tag{16}$$

Using (2) and (7), the inequality (16) becomes

$$\begin{aligned}
&\frac{\displaystyle\int_0^T \overline{F}_1(u)\,\mathrm{d}u}{F_1(T)} - [\overline{F}_1(T)]^{n-1}\int_h^T \overline{F}_1(u)\,\mathrm{d}u - \frac{[\overline{F}_1(T)]^n}{F_1(T)}\int_0^T \overline{F}_1(u)\,\mathrm{d}u \\
&\quad \geq \frac{\displaystyle\int_0^T \overline{F}_2(u)\,\mathrm{d}u}{F_2(T)} - [\overline{F}_2(T)]^{n-1}\int_h^T \overline{F}_2(u)\,\mathrm{d}u - \frac{[\overline{F}_2(T)]^n}{F_2(T)}\int_0^T \overline{F}_2(u)\,\mathrm{d}u,
\end{aligned}$$

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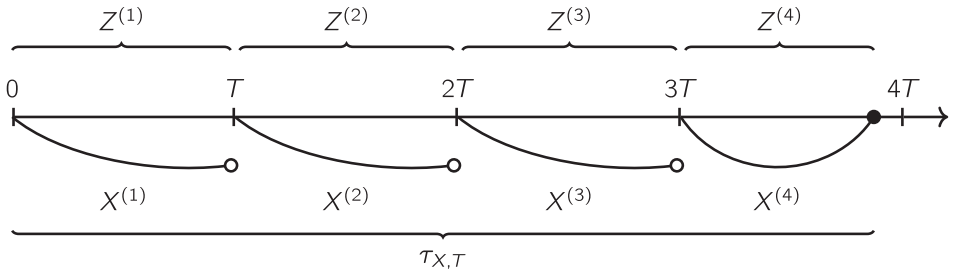


FIGURE **5.** Scheme of a trajectory of the stochastic process modeling the lifetime $\tau_{X,T}$.

which is equivalent to

$$\frac{\int_0^T \overline{F}_1(u)\,\mathrm{d}u}{F_1(T)}[1-[\overline{F}_1(T)]^{n-1}]+[\overline{F}_1(T)]^{n-1}\int_0^h \overline{F}_1(u)\,\mathrm{d}u$$
$$\geq \frac{\int_0^T \overline{F}_2(u)\,\mathrm{d}u}{F_2(T)}\left[1-[\overline{F}_2(T)]^{n-1}\right]+[\overline{F}_2(T)]^{n-1}\int_0^h \overline{F}_2(u)\,\mathrm{d}u.$$

Note that

$$\frac{1-[\overline{F}_i(T)]^{n-1}}{F_i(T)}=1+\overline{F}_i(T)+[\overline{F}_i(T)]^2+\cdots+[\overline{F}_i(T)]^{n-2},\quad \text{for } i=1,2.$$

Thus, the result comes from (11) and the inequality $[\overline{F}_1(T)]^k \geq [\overline{F}_2(T)]^k$ for all $k \geq 1$. ■

We remark that the use of the increasing concave order has not been so common in reliability. Recently, Mercier and Castro [26] use this stochastic order notion for comparing the lifetimes of systems with imperfect maintenance actions modeled as a Gamma process, which is, in some sense, a generalization of the classical system with age replacement.

Let $X^{(1)}, X^{(2)}, \ldots$ be independent random variables with the same distribution of X such that $X^{(n)}$ is the lifetime of the unit which starts to work after the $(n-1)$th replacement. Thus, the interval from the $(n-1)$th replacement to the nth replacement or the failure of the system, whichever occurs first, is given by $Z^{(n)} = \min\{X^{(n)}, T\}$, for all $n \geq 1$. The random variables $Z^{(1)}, Z^{(2)}, \ldots$ are independent and equally distributed. The number of replacements until the failure of the system is the random variable $\nu_X - 1$, where $\nu_X = \inf\{n \geq 1 : X^{(n)} < T\}$, which has geometric distribution with mean $1/F(T)$, that is,

$$\mathbb{P}[\nu_X = n] = [\overline{F}(T)]^{n-1}F(T),$$

for all $n \geq 1$. A scheme of a trajectory of the stochastic process modeling the lifetime of the system with age replacement is shown in Figure 5.

Note that the lifetime $\tau_{X,T}$ can be expressed as a random sum of independent random variables as follows:

$$\tau_{X,T} = \sum_{n=1}^{\nu_X} Z^{(n)}.$$

However, the summands in the previous random sum are not independent of the number of summands. This problem can be solved noting that

$$\tau_{X,T} =_{\mathrm{st}} (\nu_X - 1)T + (X \mid X < T),$$

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where $=_{\text{st}}$ stands for the equality in distribution. Moreover, the two random variables in the right side of (17) are independent.

The representation of $\tau_{X,T}$ as a sum of independent random variables could be used to establish stochastic orderings similar to those in Theorem 2.8, using the existing theory for comparing the sum of independent random variables (cf. [7,37]). Moreover, this different approach has a practical interpretation: when we collect data of the lifetime of a system with replacement, it gives us information about the lifetime of its units conditioned to be smaller than T and about the number of replacements until failure, not about the unconditional lifetime of the units. Suppose we have two different kinds of units with lifetimes X_1 and X_2, which have survival functions $\overline{F}_1$ and $\overline{F}_2$ and density functions f_1 and f_2, respectively. From a practical point of view, it would be more useful to take advantage of the stochastic relations between $X_1' = (X_1 \mid X_1 < T)$ and $X_2' = (X_2 \mid X_2 < T)$ to decide which system has a greater lifetime, instead of the relations between the unconditional lifetimes of the units. The next result is similar to Theorem 2.8 but under this different approach.

THEOREM 2.8: *The following statements hold*

(a) *If* $\overline{F}_1(T) \geq \overline{F}_2(T)$ *and* $(X_1 \mid X_1 < T) \geq_{\text{icx}} (X_2 \mid X_2 < T)$, *then* $\tau_{X_1,T} \geq_{\text{icx}} \tau_{X_2,T}$,

(b) *If* $\overline{F}_1(T) \geq \overline{F}_2(T)$ *and* $(X_1 \mid X_1 < T) \geq_{\text{icv}} (X_2 \mid X_2 < T)$, *then* $\tau_{X_1,T} \geq_{\text{icv}} \tau_{X_2,T}$,

(c) *If* $\overline{F}_1(T) \geq \overline{F}_2(T)$ *and* $(X_1 \mid X_1 < T) \geq_{\text{st}} (X_2 \mid X_2 < T)$, *then* $\tau_{X_1,T} \geq_{\text{st}} \tau_{X_2,T}$,

(d) *If* $f_1(t) \leq f_2(t)$ *for all* $t \in [0,T]$ *and* $(X_1 \mid X_1 < T) \geq_{\text{hr}} (X_2 \mid X_2 < T)$, *then* $\tau_{X_1,T} \geq_{\text{hr}} \tau_{X_2,T}$,

(e) *If* $f_1(t) \leq f_2(t)$ *for all* $t \in [0,T]$ *and* $(X_1 \mid X_1 < T) \geq_{\text{rh}} (X_2 \mid X_2 < T)$, *then* $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$,

(f) $(X_1 \mid X_1 < T) \geq_{\text{lr}} (X_2 \mid X_2 < T)$ *if and only if* $\tau_{X_1,T} \geq_{\text{lr}} \tau_{X_2,T}$.

PROOF: Note that as ν_{X_1} and ν_{X_2} are geometric random variables with means $1/F_1(T)$ and $1/F_2(T)$, respectively. The ordering $\nu_{X_1} \geq_{\text{st}} \nu_{X_2}$ is thus equivalent to $\overline{F}_1(T) \geq \overline{F}_2(T)$. In fact, the ordering $\nu_{X_1} \geq_{\text{st}} \nu_{X_2}$ is also equivalent to the same ordering in the sense of all the stochastic orders defined in Definition 1.1, and to the inequality $\mathbb{E}[\nu_{X_1}] \geq \mathbb{E}[\nu_{X_2}]$.

The increasing concave, the increasing convex and the usual stochastic orders remain valid under sums of independent random variables, see for example [37]. Thus, the results in (a), (b) and (c) come from (17).

Let us denote by $\overline{F}_{X_i'}$ and $f_{X_i'}$ the survival and the density functions of $X_i' = (X_i \mid X_i < T)$, respectively, for $i = 1, 2$. Using (17), we have

$$\begin{aligned}\overline{F}_{X_i,T}(t) &= \mathbb{P}\left[\nu_{X_i} = \left\lfloor \frac{t}{T} \right\rfloor + 1\right] \overline{F}_{X_i'}\left(t - \left\lfloor \frac{t}{T} \right\rfloor\right) + \mathbb{P}\left[\nu_{X_i} > \left\lfloor \frac{t}{T} \right\rfloor + 1\right] \\ &= p_i(1-p_i)^{\lfloor t/T \rfloor} \overline{F}_{X_i'}\left(t - \left\lfloor \frac{t}{T} \right\rfloor\right) + (1-p_i)^{\lfloor t/T \rfloor + 1}, \end{aligned} \tag{18}$$

where $p_i = F_i(T)$, for $i = 1, 2$. From (18), we get

$$f_{X_i,T}(t) = p_i(1-p_i)^{\lfloor t/T \rfloor} f_{X_i'}\left(t - \left\lfloor \frac{t}{T} \right\rfloor\right). \tag{19}$$

To prove (d) and (e), note that the hazard rate function of $\tau_{X_i,T}$ can be written as

$$\lambda_{X_i,T}(t) = \frac{p_i(1-p_i)^{\lfloor \frac{t}{T} \rfloor} f_{X_i'}(t - \lfloor \frac{t}{T} \rfloor)}{p_i(1-p_i)^{\lfloor t/T \rfloor} \overline{F}_{X_i'}(t - \lfloor \frac{t}{T} \rfloor) + (1-p_i)^{\lfloor t/T \rfloor + 1}} = \frac{p_i f_{X_i'}(t - \lfloor \frac{t}{T} \rfloor)}{p_i \overline{F}_{X_i'}(t - \lfloor \frac{t}{T} \rfloor) + (1-p_i)},$$

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for all $t \geq 0$ and $i = 1, 2$. The inequality $\lambda_{X_1,T}(t) \leq \lambda_{X_2,T}(t)$ becomes equivalent to

$$p_1 p_2 f_{X_1'}(x)\overline{F}_{X_2'}(x) + p_1(1-p_2) f_{X_1'}(x) \leq p_1 p_2 f_{X_2'}(x)\overline{F}_{X_1'}(x) + p_2(1-p_1) f_{X_2'}(x),$$

for all $x \in [0, T]$. Note that $p_1 p_2 f_{X_1'}(x)\overline{F}_{X_2'}(x) \leq p_1 p_2 f_{X_2'}(x)\overline{F}_{X_1'}(x)$ holds due to $X_1' \geq_{\text{hr}} X_2'$. Also, as $p_1 \leq p_2$, a sufficient condition for $p_1(1-p_2) f_{X_1'}(x) \leq p_2(1-p_1) f_{X_2'}(x)$ to hold is $f_1(x) = p_1 f_{X_1'}(x) \leq p_2 f_{X_2'}(x) = f_2(x)$, for all $x \in [0, T]$, which is true by hypothesis. Consequently, $\tau_{X_1,T} \geq_{\text{hr}} \tau_{X_2,T}$. Part (e) is analogously proved.

Moreover, (f) is trivially proved using (19). ■

REMARK 2.2: *The ordering* $(X_1 \mid X_1 < T) \geq_{\text{icv}} (X_2 \mid X_2 < T)$ *is equivalent to the inequality*

$$\frac{\int_0^t F_1(x)\,\mathrm{d}x}{F_1(T)} \leq \frac{\int_0^t F_2(x)\,\mathrm{d}x}{F_2(T)} \tag{20}$$

for all $t \in [0, T]$. *If* $\overline{F}_1(T) \geq \overline{F}_2(T)$ *and (11) hold, so does (20). Consequently, Theorem 2.9*(b) *is a particular case of Theorem 2.8*(c). *Also, note that* $X_1 \geq_{\text{icv}} X_2$ *is sufficient for (11) to hold.*

Weibull distribution with the increasing hazard rate is one of the most commonly used distributions to model lifetimes of components in a system. This fact makes interesting the comparison of the lifetimes of system with age replacement whose working units have Weibull lifetimes, as we consider in the following example.

EXAMPLE 2.3 (Weibull lifetimes): *Suppose the lifetimes of the units of two systems with age replacements have Weibull distribution. Namely,* $X_i \sim \text{Weibull}(\alpha_i, \beta_i)$, *where* $\alpha_i > 1$ *and* $\beta_i > 0$, *for* $i = 1, 2$. *Using Table 1, we obtain that if* $\alpha_1 = \alpha_2$ *and* $\beta_1 \leq \beta_2$, *then* $X_1 \geq_{\text{lr}} X_2$, *and consequently* $\tau_{X_1,T} \geq_{\text{lr}} \tau_{X_2,T}$, *for all* $T > 0$. *If* $\alpha_1 \neq \alpha_2$, *it can be proved that there is no usual stochastic ordering between* X_1 *and* X_2 *(cf. [41]). However, the condition* $(X_1 \mid X_1 < T) \geq_{\text{lr}} (X_2 \mid X_2 < T)$ *becomes equivalent to the function*

$$\psi(t) = \frac{\overline{F}_1'(t)}{\overline{F}_2'(t)} = \frac{\beta_1 \alpha_1 t^{\alpha_1}\, \mathrm{e}^{-\beta_1 t^{\alpha_1}}}{\beta_2 \alpha_2 t^{\alpha_2}\, \mathrm{e}^{-\beta_2 t^{\alpha_2}}}$$

being increasing for $t \in [0, T]$. *Deriving* ψ, *we get*

$$\psi'(t) =_{\text{sg}} g(t) = \alpha_1 - \alpha_2 + \alpha_2 \beta_2 t^{\alpha_2} - \alpha_1 \beta_1 t^{\alpha_1},$$

where $h_1 =_{\text{sg}} h_2$ *means that there exists a real positive function* h *such that* $h_1(t) = h(t) h_2(t)$, *for all* $t \geq 0$. *Assume* $\alpha_1 \geq \alpha_2$, *then a sufficient condition for* ψ *being increasing in* $[0, T]$ *is that* $T^{\alpha_1 - \alpha_2} \leq \alpha_2 \beta_2 / \alpha_1 \beta_1$.

Now, suppose that $T^{\alpha_1 - \alpha_2} \leq \beta_2 / \beta_1$. *Note that for all* $t \geq 0$, *we have*

$$\begin{aligned} g(t) &= (\alpha_1 - \alpha_2) + \alpha_2 \beta_2 t^{\alpha_1} \left(t^{\alpha_2 - \alpha_1} - \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right) \\ &\geq (\alpha_1 - \alpha_2) + \beta_1 \alpha_2 t^{\alpha_1} \left(1 - \frac{\alpha_1}{\alpha_2} \right) \\ &\geq (\alpha_1 - \alpha_2)\left(1 - \beta_1 T^{\alpha_1}\right), \end{aligned}$$

which is nonnegative if $T^{\alpha_1} \leq 1/\beta_1$. *Thus, if* $\alpha_1 \geq \alpha_2$ *and one of the following two conditions holds:*

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(a) $T^{\alpha_1-\alpha_2} \leq \alpha_2\beta_2/\alpha_1\beta_1$,
(b) $T^{\alpha_1-\alpha_2} \leq \beta_2/\beta_1$ *and* $T^{\alpha_1} \leq 1/\beta_1$,

then, Theorem 2.9(f) *ensures that* $\tau_{X_1,T} \geq_{\rm lr} \tau_{X_2,T}$, *even though* $X_1 \not\geq_{\rm st} X_2$.

2.2.1. Applications to parallel systems Next, we consider the age replacement models where the working unit is a parallel system. For such systems, optimization analysis has been carried out to determine the optimum replacement times assuming costs [29,44]. The results previously obtained related to stochastic orderings can be easily adapted for these models as we next show.

Consider two parallel systems whose units have lifetimes

$$X_1^{\{1\}}, X_1^{\{2\}}, \ldots, X_1^{\{n\}} \quad \text{and} \quad X_2^{\{1\}}, X_2^{\{2\}}, \ldots, X_2^{\{n\}},$$

respectively. For each system, the lifetimes of its units are assumed independent, but not necessarily equally distributed. Thus, the lifetimes of the parallel systems are given by

$$M_1^n = \max\{X_1^{\{1\}}, X_1^{\{2\}}, \ldots, X_1^{\{n\}}\} \quad \text{and} \quad M_2^n = \max\{X_2^{\{1\}}, X_2^{\{2\}}, \ldots, X_2^{\{n\}}\}.$$

Our goal is to find conditions among the random variables $X_1^{\{i\}}$ and $X_2^{\{j\}}$, for $i, j \in \{1, 2, \ldots, n\}$, that ensure an ordering between $\tau_{M_1^n,T}$ and $\tau_{M_2^n,T}$ in the sense of one of the stochastic orders in Definition 1.2. The following proposition uses well-known results on the stochastic ordering of parallel systems to achieve this purpose.

PROPOSITION 2.2: *The following statements hold for every replacement time* $T > 0$*:*

(a) If $X_1^{\{i\}} \geq_{\rm st} X_2^{\{i\}}$, *for all* $i \in \{1, 2, \ldots, n\}$, *then* $\tau_{M_1^n,T} \geq_{\rm st} \tau_{M_2^n,T}$,
(b) If $X_1^{\{i\}} \geq_{\rm rh} X_2^{\{i\}}$, *for all* $i \in \{1, 2, \ldots, n\}$, *then* $\tau_{M_1^n,T} \geq_{\rm rh} \tau_{M_2^n,T}$,
(c) If $X_1^{\{i\}} \geq_{\rm hr} X_2^{\{j\}}$, *for all* $i, j \in \{1, 2, \ldots, n\}$, *and the probability densities of* $X_1^{\{i\}}, X_2^{\{i\}}$, *for* $i \in \{1, 2, \ldots, n\}$, *all have support on the same interval, then* $\tau_{M_1^n,T} \geq_{\rm hr} \tau_{M_2^n,T}$,
(d) If $X_1^{\{i\}} \geq_{\rm lr} X_2^{\{j\}}$, *for all* $i, j \in \{1, 2, \ldots, n\}$, *then* $\tau_{M_1^n,T} \geq_{\rm lr} \tau_{M_2^n,T}$.

PROOF: It is immediate that $X_1^{\{i\}} \geq_{\rm st} X_2^{\{i\}}$, for all $i \in \{1, 2, \ldots, n\}$, implies that $M_1^n \geq_{\rm st} M_2^n$. Moreover, using Proposition 1.B.35 in [37], we get that $X_1^{\{i\}} \geq_{\rm rh} X_2^{\{i\}}$, for all $i \in \{1, 2, \ldots, n\}$, implies that $M_1^n \geq_{\rm rh} M_2^n$. Thus, using Theorem 3(c) in [13] and Theorem 2.8(a) above, we get (a) and (b).

Furthermore, under the hypothesis in (c) and (d), we have that $M_1^n \geq_{\rm hr} M_2^n$ and $M_1^n \geq_{\rm lr} M_2^n$, respectively (see Proposition 1.B.35 and Corollary 1.C.34 in [37]). Finally, using Theorem 32(a) and (b) in [13], we prove the statements (c) and (d). ■

Another model that has attracted the attention in studies about replacement in recent years is the random-size parallel system with age replacement [10,30,31,44]. For this model, it is considered a parallel system of independent and identically distributed units, but the number of units is assumed random. Let us denote by $M_i^{N_i}$ the lifetime of a parallel system of independent units with lifetimes with the same distribution as the random variable X_i,

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whereas the number of units is given by the positive integer-valued random variable N_i, for $i = 1, 2$. Then,

$$M_i^{N_i} = \max\{X_i^{\{1\}}, X_i^{\{2\}}, \ldots, X_i^{\{N_i\}}\},$$

for $i = 1, 2$. The distribution function of $M_i^{N_i}$ is given by

$$F_{M_i^{N_i}}(t) = \sum_{n=1}^{\infty} [F_i(t)]^n \mathbb{P}[N_i = n] = \varphi_{N_i}(F_i(t)), \tag{21}$$

where φ_{N_i} is the probability generating function of N_i, for $i = 1, 2$. The next result compares $\tau_{M_1{}^{N_1},T}$ and $\tau_{M_2{}^{N_2},T}$ in the sense of the usual stochastic order.

PROPOSITION 2.3: *If $X_1 \geq_{\text{st}} X_2$ and $N_1 \geq_{\text{pgf}} N_2$, then $\tau_{M_1{}^{N_1},T} \geq_{\text{st}} \tau_{M_2{}^{N_2},T}$.*

PROOF: Using (21), we have that $M_1{}^{N_1} \geq_{\text{st}} M_2{}^{N_2}$, if $X_1 \geq_{\text{st}} X_2$ and $N_1 \geq_{\text{pgf}} N_2$ (cf. [32, Thms. 3.1 and 3.5]). Finally, the ordering $\tau_{M_1{}^{N_1},T} \geq_{\text{st}} \tau_{M_2{}^{N_2},T}$ holds due to the results in [8]. ■

We next consider the case where the number of components in the parallel systems, N_i, is distributed according to a truncated Poisson distribution with parameter $\rho_i > 0$, that is,

$$\mathbb{P}[N_i = n] = \frac{\mathrm{e}^{-\rho_i}}{1 - \mathrm{e}^{-\rho_i}} \frac{\rho_i^n}{n!},$$

for all $n \geq 1$, and $i = 1, 2$. This particular choice of distribution for the random number of units has been recurrent in the literature related to random-size parallel systems with age replacement, see for example [10,30,31,44].

It can be checked that

$$\phi_{N_i}(t) = \frac{\mathrm{e}^{\rho_i(t-1)} - \mathrm{e}^{-\rho_i}}{1 - \mathrm{e}^{-\rho_i}},$$

for $i = 1, 2$. Thus, the distribution function of $M_i{}^{N_i}$, denoted $F_{M_i{}^{N_i}}$, satisfies

$$F_{M_i{}^{N_i}}(t) = \frac{\mathrm{e}^{-\rho_i \overline{F}_i(t)} - \mathrm{e}^{-\rho_i}}{1 - \mathrm{e}^{-\rho_i}},$$

for $i = 1, 2$. Let us denote by $\overline{F}_{M_i{}^{N_i}}$ and $f_{M_i{}^{N_i}}$ the survival and density functions of $M_i{}^{N_i}$, for $i = 1, 2$. These functions satisfy,

$$\overline{F}_{M_i{}^{N_i}}(t) = \frac{1 - \mathrm{e}^{-\rho_i \overline{F}_i(t)}}{1 - \mathrm{e}^{-\rho_i}}, \quad f_{M_i{}^{N_i}}(t) = \rho_i f_i(t) \frac{\mathrm{e}^{-\rho_i \overline{F}_i(t)}}{1 - \mathrm{e}^{-\rho_i}},$$

for $i = 1, 2$.

The next result compares two age replacement models where the lifetimes of the working units are given by parallel systems with truncated Poisson number of components.

PROPOSITION 2.4 (Parallel system with truncated Poisson number of components): *Consider N_1 and N_2 two random variables with truncated Poisson distribution with positive parameters ρ_1 and ρ_2, respectively, such that $\rho_1 \geq \rho_2$. The following statements hold:*

(a) If $X_1 \geq_{\text{st}} X_2$, then $\tau_{M_1^{N_1},T} \geq_{\text{st}} \tau_{M_2^{N_2},T}$,

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(b) If $X_1 \geq_{\mathrm{hr}} X_2$, *then* $\tau_{M_1^{N_1},T} \geq_{\mathrm{hr}} \tau_{M_2^{N_2},T}$.

PROOF: The statement (a) is proved using Proposition 2.11. Indeed, we only need to check that $N_1 \geq_{\mathrm{pgf}} N_2$, when $\rho_1 \geq \rho_2$. In fact, it can be proved that the stronger ordering $N_1 \geq_{\mathrm{lr}} N_2$ holds, which is equivalent to $\mathbb{P}[N_1 = n]/\mathbb{P}[N_2 = n]$ being increasing for $n \in \mathbb{N}$.

To prove statement (b) note that the hazard rate function of $M_i^{N_i}$, denoted $\lambda_{M_i^{N_i}}$, satisfies

$$\lambda_{M_i^{N_i}}(t) = \rho_i f_i(t) \frac{\mathrm{e}^{-\rho_i \overline{F}_i(t)}}{1 - \mathrm{e}^{-\rho_i \overline{F}_i(t)}} = \lambda_i(t) \frac{\rho_i \overline{F}_i(t)\, \mathrm{e}^{-\rho_i \overline{F}_i(t)}}{1 - \mathrm{e}^{-\rho_i \overline{F}_i(t)}}.$$

The prove $\tau_{M_1^{N_1},T} \geq_{\mathrm{hr}} \tau_{M_2^{N_2},T}$ we need to check that the inequality $\lambda_{M_1^{N_1}}(t) \leq \lambda_{M_2^{N_2}}(t)$ holds for all $t \geq 0$. By hypothesis $\lambda_1(t) \leq \lambda_2(t)$ holds, for all $t \geq 0$. Hence, it is sufficient to prove that

$$\frac{\rho_1 \overline{F}_1(t)\, \mathrm{e}^{-\rho_1 \overline{F}_1(t)}}{1 - \mathrm{e}^{-\rho_1 \overline{F}_1(t)}} \leq \frac{\rho_2 \overline{F}_2(t)\, \mathrm{e}^{-\rho_2 \overline{F}_2(t)}}{1 - \mathrm{e}^{-\rho_2 \overline{F}_2(t)}},$$

for all $t \geq 0$. The last inequality follows from $\rho_1 \overline{F}_1(t) \geq \rho_2 \overline{F}_2(t)$, for all $t \geq 0$, and the fact that the function $x \mapsto x\, \mathrm{e}^{-x}/(1 - \mathrm{e}^{-x})$ is decreasing for all $x \geq 0$. ■

The following example shows that a similar result to those in the previous proposition does not hold for the reversed hazard rate order, and thus neither for the likelihood ratio order.

EXAMPLE 2.4: *Consider* X_1 *and* X_2 *with exponential distribution with hazard rates* μ_1 *and* μ_2, *respectively. Assume* $\rho_1 = \rho_2 = \rho$ *and* $\mu_1 < \mu_2$, *which imply* $N_1 =_{\mathrm{st}} N_2$ *and* $X_1 \geq_{\mathrm{lr}} X_2$, *respectively. Let us denote by* $r_{M_i^{N_i}}$ *the reversed hazard rate function of* $\tau_{M_i^{N_i}}$, *for* $i = 1, 2$. *Then, the ordering* $\tau_{M_1^{N_1},T} \geq_{\mathrm{rh}} \tau_{M_2^{N_2},T}$ *is equivalent to the inequality* $r_{M_1^{N_1}}(t) \geq r_{M_2^{N_2}}(t)$, *for all* $t \geq 0$, *which can be written as*

$$\Delta(t) = \frac{\mu_1}{\mu_2} \frac{\mathrm{e}^{-\mu_1 t}}{\mathrm{e}^{-\mu_2 t}} \frac{\exp\{-\rho\, \mathrm{e}^{-\mu_1 t}\}}{\exp\{-\rho\, \mathrm{e}^{-\mu_2 t}\}} \frac{\exp\{-\rho\, \mathrm{e}^{-\mu_2 t}\} - \mathrm{e}^{-\rho}}{\exp\{-\rho\, \mathrm{e}^{-\mu_1 t}\} - \mathrm{e}^{-\rho}} \geq 1,$$

for all $t \geq 0$. *However, it is possible to check that the function* Δ *satisfies*

$$\Delta(t) = 1 + (\mu_1 - \mu_2) \left(\rho - \frac{1}{2} \right) t + t\epsilon(t)$$

where $\epsilon(t)$ *denotes a function such that* $\lim_{t \to 0^+} \epsilon(t) = 0$. *Then, since* $\mu_1 < \mu_2$, *the function* Δ *is initially decreasing when* $\rho > 1/2$ *and thus, in this case, there is no reversed hazard rate ordering between* $M_1^{N_1}$ *and* $M_2^{N_2}$, *and neither between* $\tau_{M_1^{N_1},T}$ *and* $\tau_{M_2^{N_2},T}$, *because of Theorem 2.8(a).*

3. RESULTS ON SYSTEMS WITH REPLACEMENT AT RANDOM TIME

Consider a nonnegative absolutely continuous random variable Y, with finite mean, distribution function G, survival function $\overline{G}$ and density function g. Let us suppose that $X^{(n)}$ is the lifetime of the unit which starts to work after the $(n-1)$th replacement and $Y^{(n)}$ the random time between the $(n-1)$th and the nth replacements, which has the same distribution of Y, for $n \geq 1$. Assume that $X^{(n)}$ and $Y^{(n)}$ are independent for all $n \geq 1$. Thus, the interval from

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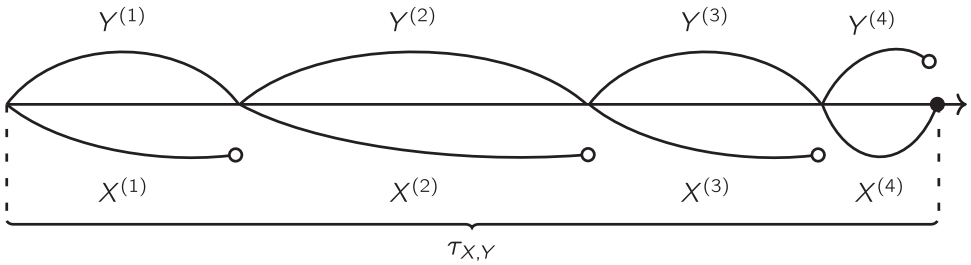


FIGURE **6.** Sketch of a trajectory of the stochastic process modeling the lifetime $\tau_{X,Y}$.

the nth replacement to the $(n+1)$th replacement or the failure of the system, whichever occurs first, is $\min\{X^{(n)}, Y^{(n)}\}$, for all $n \geq 1$. The number of replacements before the failure of the system is the random variable $\nu_{X,Y} - 1$, where $\nu_{X,Y} = \min\{n \geq 1 : X^{(n)} < Y^{(n)}\}$ follows a geometric distribution starting at 1, with mean $1/\mathbb{P}[X < Y]$. Then, the lifetime of the system with random time replacement, denoted by $\tau_{X,Y}$, satisfies

$$\tau_{X,Y} = \sum_{n=0}^{\nu_{X,Y}} \min\{X^{(n)}, Y^{(n)}\}. \tag{22}$$

A scheme of a trajectory of the stochastic process modeling the lifetime of the system with replacement at random time is shown in Figure 6.

The summands and the number of summands in the representation of $\tau_{X,Y}$ as a random sum in (22) are not independent. In order to solve the problem, we give the following alternative representation of $\tau_{X,Y}$ as a random sum of independent random variables:

$$\tau_{X,Y} =_{\text{st}} \sum_{n=0}^{\nu_{X,Y}-1} (Y^{(n)} \mid X^{(n)} \geq Y^{(n)}) + (X^{(n)} \mid X^{(n)} < Y^{(n)}), \tag{23}$$

where $Y_0 = 0$, $X^{(n)} =_{\text{st}} X$ and $Y^{(n)} =_{\text{st}} Y$, for all $n \geq 1$. Observe that the summands in (23) and $\nu_{X,Y}$ are independent.

From Theorem 1.A.4 in [37], we know that the usual stochastic ordering between random variables is preserved by random sums if the random number of summands also satisfy the usual stochastic ordering in the same sense, the summands are independent among them and independent of the number of summands. Thus, it would be interesting to compare random variables with the same distribution as $(Y \mid X \geq Y)$ and $(X \mid X < Y)$. In order to do that we have the following result which has an independent interest.

LEMMA 3.1: *Consider X_1, X_2, Y_1 and Y_2 as absolutely continuous random variables such that X_i and Y_i are independent for $i = 1, 2$. Let us suppose $Y_1 =_{\text{st}} Y_2 =_{\text{st}} Y$. The following two statements hold:*

(a) If $X_1 \geq_{\text{hr}} X_2$, then $(Y_1 \mid X_1 \geq Y_1) \geq_{\text{st}} (Y_2 \mid X_2 \geq Y_2)$.
(b) If $X_1 \geq_{\text{rh}} X_2$, then $(X_1 \mid X_1 < Y_1) \geq_{\text{st}} (X_2 \mid X_2 < Y_2)$.

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PROOF: (a) It must be proved that for all $t \geq 0$

$$\mathbb{P}[Y_1 > t \mid X_1 \geq Y_1] \geq \mathbb{P}[Y_2 > t \mid X_2 \geq Y_2],$$

or equivalently

$$\frac{\mathbb{P}[Y_1 > t;\, X_1 \geq Y_1]}{\mathbb{P}[X_1 \geq Y_1]} \geq \frac{\mathbb{P}[Y_2 > t;\, X_2 \geq Y_2]}{\mathbb{P}[X_2 \geq Y_2]}.$$

This inequality can be written as

$$\frac{\displaystyle\int_t^\infty \overline{F}_1(x)\,\mathrm{d}G(x)}{\displaystyle\int_0^t \overline{F}_1(x)\,\mathrm{d}G(x) + \int_t^\infty \overline{F}_1(x)\,\mathrm{d}G(x)} \geq \frac{\displaystyle\int_t^\infty \overline{F}_2(x)\,\mathrm{d}G(x)}{\displaystyle\int_0^t \overline{F}_2(x)\,\mathrm{d}G(x) + \int_t^\infty \overline{F}_2(x)\,\mathrm{d}G(x)}.$$

Using Fubini's Theorem, it is obtained the equivalent inequality

$$\int_t^\infty \int_0^t \overline{F}_1(x)\overline{F}_2(y)\,\mathrm{d}G(y)\,\mathrm{d}G(x) \geq \int_t^\infty \int_0^t \overline{F}_1(y)\overline{F}_2(x)\,\mathrm{d}G(y)\,\mathrm{d}G(x). \tag{24}$$

Thus, (24) holds if $\overline{F}_1(x)\overline{F}_2(y) \geq \overline{F}_1(y)\overline{F}_2(x)$, for $x \geq t$ y $0 \leq y \leq t$, which is true due to $X_1 \geq_{\mathrm{hr}} X_2$.

(b) In a similar way to the previous case, it must be proved

$$\frac{\displaystyle\int_t^\infty \overline{G}(x)\,\mathrm{d}F_1(x)}{\displaystyle\int_0^\infty \overline{G}(x)\,\mathrm{d}F_1(x)} \geq \frac{\displaystyle\int_t^\infty \overline{G}(x)\,\mathrm{d}F_2(x)}{\displaystyle\int_0^\infty \overline{G}(x)\,\mathrm{d}F_2(x)}.$$

That is equivalent to

$$\int_t^\infty \int_0^t \overline{G}(x)\overline{G}(y)\,\mathrm{d}F_1(x)\,\mathrm{d}F_2(y) \geq \int_t^\infty \int_0^t \overline{G}(x)\overline{G}(y)\,\mathrm{d}F_2(x)\,\mathrm{d}F_1(y). \tag{25}$$

Denote $\phi(x,y) = 1_{[t,\infty)}(x)1_{(-\infty,t]}(y)\overline{G}(x)\overline{G}(y)$. The inequality (25) is equivalent to $\mathbb{E}[\phi(X_1,X_2)] \geq \mathbb{E}[\phi(X_2,X_1)]$. When $x \leq y$, we have

$$\phi(x,y) - \phi(y,x) = -\overline{G}(x)\overline{G}(y)1_{(-\infty,t]}(x)1_{[t,\infty)}(y)$$

which is increasing in x. Thus, $\mathbb{E}[\phi(X_1,X_2)] \geq \mathbb{E}[\phi(X_2,X_1)]$ holds due to $\phi \in \mathcal{G}_{\mathrm{rh}}$ and Proposition 1.1(a). ■

We next establish sufficient conditions to compare two systems with replacement at random time in the sense of the usual stochastic order, when the replacement times have the same distribution for both systems.

THEOREM 3.2: *If* $X_1 \geq_{\mathrm{hr}} X_2$, $X_1 \geq_{\mathrm{rh}} X_2$ *and* $Y_1 =_{\mathrm{st}} Y_2$, *then* $\tau_{X_1,Y_1} \geq_{\mathrm{st}} \tau_{X_2,Y_2}$.

PROOF: Note that $\nu_{X_1,Y_1} \geq_{\mathrm{st}} \nu_{X_2,Y_2}$ due to $\mathbb{P}[X_1 < Y_1] \leq \mathbb{P}[X_2 < Y_2]$. Now, using (23), Lemma 3.1 and Theorem 1.A.4 in [37], we get the desired result. ■

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EXAMPLE 3.1 (Gamma lifetimes): *The conditions in Theorem 3.2 hold, for instance, when* $X_1 \geq_{\text{lr}} X_2$ *and* $Y_1 =_{\text{st}} Y_2$. *As an example, consider* X_i *with distribution* $\text{Gamma}(\alpha_i, \beta_i)$ *for* $i = 1, 2$. *If* $\alpha_1 \geq \alpha_2$ *and* $\beta_1 \leq \beta_2$, *we get that the ordering* $X_1 \geq_{\text{lr}} X_2$ *holds, see Table 1. Thus,* $\tau_{X_1,Y_1} \geq_{\text{st}} \tau_{X_2,Y_2}$ *for any pair of random variables* Y_1 *and* Y_2, *such that* $Y_1 =_{\text{st}} Y_2$.

Let $\Phi_{X,Y}$ be the survival function of $\tau_{X,Y}$ and consider the Laplace transform of $\Phi_{X,Y}$, denoted by $\hat{\Phi}_{X,Y}$, defined by

$$\hat{\Phi}_{X,Y}(s) = \int_0^\infty \Phi_{X,Y}(t)\, \mathrm{e}^{-st} \mathrm{d}t,$$

for all $s \geq 0$. It is not difficult to see that

$$\hat{\Phi}_{X,Y}(s) = \frac{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x}{\displaystyle\int_0^\infty f(x)\overline{G}(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x + s\int_0^\infty \overline{F}(x)\overline{G}(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x}. \tag{26}$$

The next theorem compares the lifetimes of two systems with different replacement time distributions in the sense of the Laplace transform order.

THEOREM 3.3: *If* $X_1 =_{\text{st}} X_2$, $Y_1 \leq_{\text{hr}} Y_2$ *(*$Y_1 \geq_{\text{hr}} Y_2$*) and* $X_1, X_2 \in \text{IFR}$ *(*$X_1, X_2 \in \text{DFR}$*), then* $\tau_{X_1,Y_1} \geq_{\text{Lt}} \tau_{X_2,Y_2}$.

PROOF: Let $\hat{\Phi}_{X_1,Y_1}$ and $\hat{\Phi}_{X_2,Y_2}$ be the Laplace transforms of Φ_{X_1,Y_1} and Φ_{X_2,Y_2}, respectively. Let $\overline{G}_1$ and $\overline{G}_2$ be the survival functions of Y_1 and Y_2, respectively. We need to prove that, $\hat{\Phi}_{X_1,Y_1}(s) \geq \hat{\Phi}_{X_2,Y_2}(s)$, for all $s \geq 0$ which, due to (26), is equivalent to

$$\frac{\displaystyle\int_0^\infty f(x)\overline{G}_1(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x}{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}_1(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x} \leq \frac{\displaystyle\int_0^\infty f(x)\overline{G}_2(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x}{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}_2(x)\, \mathrm{e}^{-sx}\, \mathrm{d}x}.$$

Rearranging the terms and using Fubini's Theorem, the previous expression becomes equivalent to

$$\iint_{\mathbb{R}_+^2} \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(x)f(y)\, \mathrm{d}x\, \mathrm{d}y \leq \iint_{\mathbb{R}_+^2} \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(y)f(x)\, \mathrm{d}x\, \mathrm{d}y,$$

which is equivalent to

$$I = \iint_{\mathbb{R}_+^2} \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(x)\overline{F}(y)[\lambda(x) - \lambda(y)]\, \mathrm{d}x\, \mathrm{d}y \geq 0,$$

where λ is the hazard rate of X_1 and X_2. Let us define

$$h(x, y) = \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(x)\overline{F}(y)[\lambda(x) - \lambda(y)].$$

Then,

$$\begin{aligned} I &= \iint_{x\geq y} h(x, y)\, \mathrm{d}x\, \mathrm{d}y + \iint_{x\leq y} h(x, y)\, \mathrm{d}x\, \mathrm{d}y = \iint_{x\geq y} [h(x, y) + h(y, x)]\, \mathrm{d}x\, \mathrm{d}y \\ &= \iint_{x\geq y} \mathrm{e}^{-s(x+y)}\overline{F}(x)\overline{F}(y)[\lambda(x) - \lambda(y)] \times [\overline{G}_1(x)\overline{G}_2(y) - \overline{G}_1(y)\overline{G}_2(x)]\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

Finally, the last expression is positive when $Y_1 \leq_{\text{hr}} Y_2$ and $X_1 \in \text{IFR}$ or $Y_1 \geq_{\text{hr}} Y_2$ and $X_1 \in \text{DFR}$. ■

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As a consequence of Theorems 3.2 and 3.3, we have the following result.

COROLLARY 3.4: *If* $X_1 \geq_{\mathrm{hr}} X_2$, $X_1 \geq_{\mathrm{rh}} X_2$ *and one of the following conditions holds*

(a) $Y_1 \leq_{\mathrm{hr}} Y_2$ *and* $X_1 \in \mathrm{IFR}$ *or* $X_2 \in \mathrm{IFR}$,

(b) $Y_1 \geq_{\mathrm{hr}} Y_2$ *and* $X_1 \in \mathrm{DFR}$ *or* $X_2 \in \mathrm{DFR}$,

then $\tau_{X_1,Y_1} \geq_{\mathrm{Lt}} \tau_{X_2,Y_2}$.

EXAMPLE 3.2 (Replacement first and replacement last): *Recently, some authors have studied a model where the time to replacement combines the constant and the random policies, that is, the model with replacement first and replacement last [30,43,44]. For the model with replacement first, it is assumed that the unit with lifetime* X_i *is replaced at time* T *or at times* $Y_i^{\{j\}}$, $j = 1, 2, \ldots, n$, *whichever occurs first, for* $i = 1, 2$. *The random variables* X_i *and* $Y_i^{\{j\}}$, *for* $i = 1, 2$ *and* $j = 1, 2, \ldots, n$, *are considered independent. Then, the replacement time under this policy, denoted* Y_i^{First}, *satisfies*

$$Y_i^{\mathrm{First}} = \min\{Y_i^{\{1\}}, Y_i^{\{2\}}, \ldots, Y_i^{\{n\}}, T\},$$

for $i = 1, 2$. *Analogously, it is defined the model with replacement last, that is, the replacement occurs at time* T *or at* $Y_i^{\{j\}}$, $j = 1, 2, \ldots, n$ *and* $i = 1, 2$, *whichever occurs last. In this case, the replacement time is given by*

$$Y_i^{\mathrm{Last}} = \max\{Y_i^{\{1\}}, Y_i^{\{2\}}, \ldots, Y_i^{\{n\}}, T\},$$

for $i = 1, 2$. *Then, we can use the existing results related to comparisons of the maximum or the minimum of random variables to establish stochastic orderings between two systems with first replacement or last replacement. Let us denote by* $\tau_{X_i,Y_i^{\mathrm{First}}}$ *(resp.* $\tau_{X_i,Y_i^{\mathrm{Last}}}$*) the lifetime of the system whose units have lifetimes* X_i *and first replacement at* Y_i^{First} *(resp. last replacement at* Y_i^{Last}*), for* $i = 1, 2$. *Moreover, let us denote by* $\lambda_{Y_i^{\{j\}}}(t)$ *the hazard rate of* $Y_i^{\{j\}}$, *for* $j = 1, 2, \ldots, n$ *and* $i = 1, 2$. *Then, using Theorems 1.B.33 and 1.B.35 in [37] and Theorem 3.3 above, we get the following results:*

- *if* $X_1 =_{\mathrm{st}} X_2 \in \mathrm{IFR}$ *and* $\lambda_{Y_1^{\{j\}}}(t) \geq \lambda_{Y_2^{\{j\}}}(t)$ *for all* $j \in \{1, 2, \ldots, n\}$ *and* $t \in [0, T]$, *then* $\tau_{X_1,Y_1^{\mathrm{First}}} \geq_{\mathrm{Lt}} \tau_{X_2,Y_2^{\mathrm{First}}}$,
- *if* $X_1 =_{\mathrm{st}} X_2 \in \mathrm{IFR}$ *and* $\lambda_{Y_1^{\{i\}}}(t) \geq \lambda_{Y_2^{\{j\}}}(t)$ *for all* $i, j \in \{1, 2, \ldots, n\}$ *and* $t \in [0, T]$, *then* $\tau_{X_1,Y_1^{\mathrm{Last}}} \geq_{\mathrm{Lt}} \tau_{X_2,Y_2^{\mathrm{Last}}}$.

EXAMPLE 3.3: *Let us suppose that*

$$X_1 \geq_{\mathrm{hr}} X_2, \quad X_1 \geq_{\mathrm{rh}} X_2 \quad \textit{and} \quad X_1 \in \mathrm{IFR}. \tag{27}$$

Then, using Corollary 3.4, we get $\tau_{X_1,Y} \geq_{\mathrm{Lt}} X_2$, *for every random variable* Y. *From a practical point of view, this means that any replacement policy implemented will increase the lifetime of the system in the sense of Laplace order. Conditions in (27) holds for example in the following cases (see Table 1):*

- $X_i \sim \mathrm{Gamma}(\alpha_i, \beta_i)$, *for* $i = 1, 2$, *satisfying* $\alpha_1 \geq \alpha_2$, $\beta_1 \leq \beta_2$. *Moreover, we must demand* $\alpha_1 \geq 1$ *for* $X \in \mathrm{IFR}$ *to hold.*
- $X_i \sim \mathrm{Weibull}(\alpha, \beta_i)$, *for* $i = 1, 2$, *satisfying* $\beta_1 \leq \beta_2$ *and* $\alpha \geq 1$.

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3.1. Comparing Mean Lifetimes and Stationary Pointwise Availabilities

We next proceed to compare the mean lifetimes of systems with replacement at random time. Taking $s = 0$ in (26), we obtain

$$\mathbb{E}[\tau_{X,Y}] = \frac{\mathbb{E}[X \wedge Y]}{\mathbb{P}[Y > X]}. \tag{28}$$

LEMMA 3.5: *Let X and Y be independent random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. Then,*

$$\mathbb{E}[X \wedge Y] = \mathbb{E}[(\mathcal{H}\overline{G})(X)]. \tag{29}$$

$$= \mathbb{E}[(\mathcal{H}\overline{F})(Y)], \tag{30}$$

where $\mathcal{H}$ is the integral operator satisfying

$$(\mathcal{H}h)(x) = \int_0^x h(u)\,\mathrm{d}u, \quad \textit{for all } x \geq 0.$$

PROOF: Note that

$$\begin{aligned}\mathbb{E}[X \wedge Y] &= \int_0^\infty \overline{G}(x)\overline{F}(x)\,\mathrm{d}x = \int_0^\infty \overline{F}(x)\,\mathrm{d}\left(\int_0^x \overline{G}(u)\,\mathrm{d}u\right)\\ &= \int_0^\infty \left(\int_0^x \overline{G}(u)\,\mathrm{d}u\right)\mathrm{d}F(x) = \mathbb{E}[(\mathcal{H}\overline{G})(X)].\end{aligned}$$

Equality (30) is analogously proved. ■

In order to compare the mean values of τ_{X_1,Y_1} and τ_{X_2,Y_2}, we have the following proposition.

PROPOSITION 3.1: *If $X_1 \geq_{\mathrm{st}} X_2$, $Y_1 \leq_{\mathrm{lr}} Y_2$ ($Y_1 \geq_{\mathrm{lr}} Y_2$) and $X_1 \in$ DMTTF or $X_2 \in$ DMTTF ($X_1 \in$ IMTTF or $X_2 \in$ IMTTF), then $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$.*

PROOF: Suppose first that $X_1 =_{\mathrm{st}} X_2 =_{\mathrm{st}} X \in$ DMTTF and $Y_1 \leq_{\mathrm{lr}} Y_2$. Using (28) and (30), we get

$$\mathbb{E}[\tau_{X_i,Y_i}] = \frac{\mathbb{E}[(\mathcal{H}\overline{F})(Y_i)]}{\mathbb{E}[F(Y_i)]}.$$

Thus, the inequality $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ is equivalent to

$$\mathbb{E}\left[F(Y_1)F(Y_2)\frac{(\mathcal{H}\overline{F})(Y_1)}{F(Y_1)}\right] \geq \mathbb{E}\left[F(Y_1)F(Y_2)\frac{(\mathcal{H}\overline{F})(Y_2)}{F(Y_2)}\right]. \tag{31}$$

Consider $\phi(x,y) = F(x)F(y)(\int_0^y \overline{F}(u)\,\mathrm{d}u/F(y))$. As $X_1 \in$ DMTTF we have that

$$\phi(x,y) - \phi(y,x) = F(x)F(y)\left(\frac{\displaystyle\int_0^y \overline{F}(u)\,\mathrm{d}u}{F(y)} - \frac{\displaystyle\int_0^x \overline{F}(u)\,\mathrm{d}u}{F(x)}\right) \geq 0,$$

for all $x \geq y$. Thus, the inequality (31) is equivalent to $\mathbb{E}[\phi(Y_1,Y_2)] \leq \mathbb{E}[\phi(Y_2,Y_1)]$ for $\phi \in \mathcal{G}_{\mathrm{lr}}$, which holds from Proposition 1.1(b).

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Suppose now that $Y_1 =_{\mathrm{st}} Y_2$. A sufficient condition for the inequality $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ to hold is $\mathbb{E}[X_1 \wedge Y_1] \geq \mathbb{E}[X_2 \wedge Y_2]$ and $\mathbb{P}[Y_1 > X_1] \leq \mathbb{P}[Y_2 > X_2]$. It is not difficult to check that these inequalities hold when $X_1 \geq_{\mathrm{st}} X_2$.

In the general case let us assume, without lost of generality, that $X_1 \in \mathrm{DMTTF}$. Then, $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_1,Y_2}]$ using (a), and $\mathbb{E}[\tau_{X_1,Y_2}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ using (b). Consequently, $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$. ■

Note that when $X_1 =_{\mathrm{st}} X_2 =_{\mathrm{st}} X \in \mathrm{DMTTF}$, then for any $Y_1 \leq_{\mathrm{lr}} Y_2$, we get $\mathbb{E}[\tau_{X,Y_1}] \geq \mathbb{E}[\tau_{X,Y_2}]$. Thus, Proposition 3.6 generalizes the aging notion of the class DMTTF, showing that the expected lifetime of the systems with replacement at random time increases when the times between replacement are ordered in the likelihood ratio sense.

Let us assume we only can detect if the system is not working when a replacement is done. After a system failure, when the next replacement takes place, a renewal occurs and the system starts to work. Thus, at every instant, there is a probability that the system is working or not. We define the *pointwise availability* of the system at instant t, denoted by $A_{X,Y}(t)$ as the probability that it is working at the instant t. The *stationary pointwise availability*, denoted by $A_{X,Y}$, is defined as the limit of $A_{X,Y}(t)$ as t tends to infinity. Let us denote by $\overline{F}$ and $\overline{G}$ the survival functions of the random variables X and Y, respectively, and assume X and Y have finite means. Then, the following integral equation holds,

$$A_{X,Y}(t) = \overline{F}(t)\overline{G}(t) + \int_0^t A_{X,Y}(t-x)\,\mathrm{d}G(x).$$

Thus, using the well-known *Key Renewal Theorem*, see for example Theorems 3.4.2 and 3.4.4 in [35], we get

$$A_{X,Y} = \frac{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}(x)\,\mathrm{d}x}{\displaystyle\int_0^\infty \overline{G}(x)\,\mathrm{d}x} = \frac{\mathbb{E}[X \wedge Y]}{\mathbb{E}[Y]}. \tag{32}$$

Consider the random variable *asymptotic equilibrium age*, denoted by Y^{e}, associated with a random variable Y with finite mean, which has distribution function

$$G^{\mathrm{e}}(t) = \frac{\displaystyle\int_0^t \overline{G}(x)\,\mathrm{d}x}{\mathbb{E}[Y]}.$$

The ordering $Y \leq_{\mathrm{hmrl}} Y$ is equivalent to $Y_1^{\mathrm{e}} \leq_{\mathrm{st}} Y_2^{\mathrm{e}}$ (cf. [37] Thm. 2.B.2). We highlight that there is no implications between the harmonic mean residual lifetime and the usual stochastic orderings of two random variables. See for example [37] for more details about the asymptotic equilibrium age.

The following theorem deals with the comparison of the pointwise stationary availabilities of two systems with different distributions of lifetimes and replacement times.

THEOREM 3.6: *If* $X_1 \geq_{\mathrm{icv}} X_2$ *and* $Y_1 \leq_{\mathrm{hmrl}} Y_2$, *then* $A_{X_1,Y_1} \geq A_{X_2,Y_2}$.

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PROOF: Consider $X_1 =_{\text{st}} X_2 =_{\text{st}} X$ and $Y_1 \leq_{\text{hmrl}} Y_2$. By (32), the inequality $A_{X_1,Y_1} \geq A_{X_2,Y_2}$ is equivalent to

$$\int_0^\infty \overline{F}(x)\frac{\overline{G}_1(x)}{\mathbb{E}[Y_1]}\mathrm{d}x = \mathbb{E}[\overline{F}(Y_1^{\text{e}})] \geq \mathbb{E}[\overline{F}(Y_2^{\text{e}})] = \int_0^\infty \overline{F}(x)\frac{\overline{G}_2(x)}{\mathbb{E}[Y_2]}\,\mathrm{d}x. \tag{33}$$

The stochastic ordering $Y_1^{\text{e}} \leq_{\text{st}} Y_2^{\text{e}}$ is equivalent to $\mathbb{E}[\phi(Y_1^{\text{e}})] \leq \mathbb{E}[\phi(Y_2^{\text{e}})]$ for all increasing function ϕ such that the expectations exist (cf. [37]). As $\overline{F}$ is decreasing, the inequality (33) holds because $Y_1 \leq_{\text{hmrl}} Y_2$, which is equivalent to $Y_1^{\text{e}} \leq_{\text{st}} Y_2^{\text{e}}$.

Now assume $Y_1 =_{\text{st}} Y_2$. In this case, the inequality $A_{X_1,Y_1} \geq A_{X_2,Y_2}$, using (32), is equivalent to $\mathbb{E}[X_1 \wedge Y_1] \geq \mathbb{E}[X_2 \wedge Y_2]$. From the proof of Proposition 3.6, we know that this inequality holds when

$$\mathbb{E}[X_1 \wedge Y_1] = \mathbb{E}[(\mathcal{H}\overline{G})(X_1)] \geq \mathbb{E}[(\mathcal{H}\overline{G})(X_2)] = \mathbb{E}[X_2 \wedge Y_2].$$

Note that $\mathcal{H}\overline{G}$ is increasing and concave. Moreover, $X_1 \geq_{\text{icv}} X_2$ is equivalent to $\mathbb{E}[\phi(X_1)] \geq \mathbb{E}[\phi(X_2)]$ for every real function ϕ increasing and concave such that the expectation exists (cf. [37]). Thus, $A_{X_1,Y_1} \geq A_{X_2,Y_2}$ holds. The general result is obtained by transitivity. ■

EXAMPLE 3.4 (Lifetimes and times until replacements with Gamma distribution): *Consider* $X_i \sim \text{Gamma}(\alpha_i, \beta_i)$ *and* $Y_i \sim \text{Gamma}(\gamma_i, \mu_i)$, *for* $i = 1, 2$. *According to Table 1, we have that the ordering* $X_1 \geq_{\text{icv}} X_2$ *holds when it is satisfied one of the following two conditions:*

- $\alpha_1 \geq \alpha_2$ *and* $\beta_1 \leq \beta_2$ *(which actually imply* $X_1 \geq_{\text{lr}} X_2$*), or*
- $\beta_1 \geq \beta_2$ *and* $\alpha_1/\beta_1 \geq \alpha_2/\beta_2$.

Note that if $Y_1 \leq_{\text{mrl}} Y_2$, *we get* $Y_1 \leq_{\text{hmrl}} Y_2$. *Thus, according to Table 1, the ordering* $Y_1 \leq_{\text{mrl}} Y_2$ *holds when*

- $\gamma_1 \leq \gamma_2$ *and* $\mu_1 \geq \mu_2$ *(which actually imply* $X_1 \leq_{\text{lr}} X_2$*), or*
- $\gamma_1 \geq \gamma_2$ *and* $\gamma_1/\mu_1 \leq \gamma_2/\mu_2$.

Using Proposition 3.6 and Theorem 3.7, we can compare two systems with replacement at random time in the sense of the mean lifetime and the stationary availability under weaker conditions that those of Corollary 3.4, which ensures a stochastic ordering in the Laplace sense. This is one of the reasons why this comparison becomes interesting. Note also that when $X_1 \geq_{\text{st}} X_2$, $Y_1 \leq_{\text{lr}} Y_2$ and $X_1 \in \text{DMTTF}$ or $X_2 \in \text{DMTTF}$ we get both $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ and $A_{X_1,Y_1} \geq A_{X_2,Y_2}$, but we cannot ensure that $\tau_{X_1,Y_1} \geq_{\text{Lt}} \tau_{X_2,Y_2}$ holds.

4. CONCLUSIONS

Standing out as a novelty concerning previous research on this topic, and, consequently filling a gap on the matter, this paper continues the research on the aging properties of the lifetime of the classical system with age replacement using properties of the lifetimes of the working units. We mainly focused on the aging class to which the lifetime of the system belongs, according to the aging properties of the lifetime of the working unit. As a matter of fact, Theorem 2.8 extends the comparisons of the lifetimes of systems with age replacement to other notions of stochastic orders than the previous considered. Moreover, in Theorem 2.9, we also compare the lifetimes of the systems with age replacement using comparisons between the number of replacements and the lifetimes of the working units

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conditioned to be less or equal than the replacement time, which seems interesting from the practical point of view. Moreover, using these results, we provide several examples. In particular, our approach allow us to establish stochastic orderings between parallel systems with age replacement with a constant or random number of components. Our results on this direction are inspired by real-life applications. We refer the interested reader to Zhao *et al.* [44] Sect. 10 for some applications of age replacement models to real-life systems.

As for possible extensions and future work, we believe that the recent developments on stochastic ordering for the lifetimes of coherent systems could be used to establish stochastic ordering between more complicated systems with age replacement, as the K-out-of-n system considered by Ito *et al.* [11], Safaei *et al.* [36] and Zhao *et al.* [45].

We have additionally worked upon the lifetimes of systems with age replacement, studying the case where the replacement time is random. In this concern, our results include usual stochastic orderings, Laplace transform orderings and comparisons of the expected lifetimes and the stationary availabilities. The existence of stronger stochastic orderings for this model is a topic of further research.

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Conflict of interest

The authors declare no conflict of interest.

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