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Self-maps of BU

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1. *Introduction.* One of the results proved by Lance in (7) is the following.

THEOREM 1. *Let* $\phi: BU_{(p)} \to BU_{(p)}$ *be an H-map, where* $BU_{(p)}$ *denotes the localization of BU with respect to an odd prime p. Let* $\phi_*: \pi_{2k}(BU_{(p)}) \to \pi_{2k}(BU_{(p)}) \cong \mathbb{Z}_{(p)}$ *be multiplication by* λ_k, *then*

$$\lambda_{n_1 p^j} \equiv \lambda_{n_2 p^j} \bmod p^{j+1},$$

if $n_1 \equiv n_2 \bmod (p-1)$.

In this paper we generalize this result in several directions.

In the first place our results will apply without localization. Results for $BU_{(p)}$ follow from our integral results and methods; in particular we have information for $p = 2$.

Secondly we prove a larger class of congruences among the λ_j. One of our results (Theorem 2) specializes to show that $\lambda_{q_1} \equiv \lambda_{q_2} \bmod p^{j+1}$ if $q_1 \equiv q_2 \bmod (p-1)p^j$ and $q_1, q_2 \geqslant j+1$, which clearly includes theorem 1. But our main result concerns congruences involving an arbitrary number of the λ_j, rather than just two.

Thirdly we show that almost all of our congruences apply to any self-map of BU whether it is an H-map or not. This settles a conjecture which Lance made in (7) (Remark 1·8).

Our main results are summarized in

THEOREM 2. *Let* $\phi: BU \to BU$ *and suppose* $\phi_*: \pi_{2k}(BU) \to \pi_{2k}(BU) = \mathbb{Z}$ *is multiplication by* $\lambda_k \in \mathbb{Z}$.

If p is an odd prime, l and m are positive and $p^j(p-1)$ *divides l, then*

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{q+rl} \equiv 0 \bmod p^{(j+1)m},$$

if $q > (j+1)m$.

(*Note that theorem* 1 *is the case where* $m = 1$, $q = n_1 p^j$, $l = (n_2 - n_1)p^j$.)

If l and m are positive and 2^j *divides l, then*

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{q+rl} \equiv 0 \bmod \begin{cases} 2^{(j+2)m} & \text{if } j \geqslant 1, \\ 2^m & \text{if } j = 0, \end{cases}$$

if

$$q > \begin{cases} (j+2)m & \text{if } j \geqslant 1, \\ m & \text{if } j = 0. \end{cases}$$

If ϕ *is an H-map with respect to the Whitney sum structure on BU these congruences hold also for* $q = (j+1)m$, $(j+2)m$ *or* m *respectively.*

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The congruences in theorem 2 may all be expressed as

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{q+rl} \equiv 0 \bmod m(l)^m,$$

for sufficiently large q, where $m(l)$ is the numerical function of (2), § 2.

It is easy to see that an H-map $\phi = BU \to BU$ is characterized by its λ-sequence $(\lambda_k(\phi))_{k \geqslant 1}$. We may ask whether the congruences are complete, in the sense that any λ-sequence satisfying the congruences arises from an H-map $BU \to BU$, or irredundant, in the sense that no proper subset of congruences is equivalent. We find that the congruences of theorem 2 are neither complete nor irredundant.

The question of redundancy is answered by the following result.

THEOREM 3. *All the H-map congruences in theorem 2 follow from the cases*:

for p odd:

$$l = p-1, \quad m \leqslant q < m+p,$$

that is

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{q+r(p-1)} \equiv 0 \bmod p^m$$

for $p = 2$:

$$l = 1,\ m \leqslant q < m+2 \text{ and } l = 2,\ 3m \leqslant q < 3m+5$$

that is

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{q+r} \equiv 0 \bmod 2^m,$$

and

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{q+2r} \equiv 0 \bmod 8^m.$$

The incompleteness of these congruences is demonstrated in Proposition 10 in section 5, in which specific λ-sequences are given which, though they satisfy all the congruences of theorem 2, do not arise from a self-H-map of BU.

However an alternative set of congruences for λ-sequences of H-maps turns out to be both complete and irredundant. These are given in the following result.

THEOREM 4. *If $\phi: BU \to BU$ is an H-map and $\phi_*: \pi_{2k}(BU) \to \pi_{2k}(BU)$ is multiplication by λ_k, then*

$$\sum_{k=1}^{i} s_i^k \lambda_k \equiv 0 \bmod i! \quad (i \geqslant 1),$$

where s_i^k is the Stirling number of the first kind (5), *determined by*

$$w(w-1)\dots(w-i+1) = \sum_{k=1}^{i} s_i^k w^k.$$

Moreover any sequence $(\lambda_k)_{k \geqslant 1}$ satisfying these congruences for all $i \geqslant 1$ arises from a unique H-map $BU \to BU$.

In the remainder of this section we outline the structure of the proofs of theorems 2, 3 and 4.

The methods are quite simple. In essence they depend on the fact that any H-map $BU \to BU$, agrees on any finite skeleton with a linear combination of the Adams

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operations ψ^j, and that the congruences of theorem 2 are easily verified for the Adams operations. However our proof will not be phrased in this language.

We recall from (3) and (4) that an H-map $BU \to BU$, thought of as an additive K-theory operation, is uniquely determined by its effect on the Hopf bundle over $BU(1)$. The space of self-H-maps of BU is the maximal ideal $x\mathbb{Z}[[x]] \Delta \mathbb{Z}[[x]] = K^*(BU(1))$.

Given a power series $f(x)$, we show in § 2 how to compute the corresponding sequence of λ_k's. The formula (proposition 5) involves Stirling numbers of the second kind S_k^i, which are determined by

$$w^k = \sum_{i=1}^{k} S_k^i w(w-1)\dots(w-i+1).$$

We prove in § 3, that certain congruences hold among the Stirling numbers. These congruences are analogous to those of theorem 2, which then follow directly for the case of H-maps. The Stirling number congruences go back essentially to Kummer (6), and imply a weak version of the Kummer congruences.

General self-maps of BU are classified by $\tilde{K}^0(BU) = [BU, BU]$. In § 4 we use the Hopf algebra structure of $K^0(BU)$ to extend the results. The H-maps are identified with $PK^0(BU) \subset \tilde{K}^0(BU)$. We prove that a map $BU \to BU$ which is decomposable as an element of $\tilde{K}^0(BU)$ has $\lambda_k = 0$ for all k. Since $K^0(BU)$ is rationally (and topologically) primitively generated, this enables us to give another formula (proposition 8) for λ_k, from which the part of theorem 2 which applies to general self-maps of BU follows.

At the end of section 4 we give the examples which show that, in theorem 2, q is as small as possible, and settle Lance's conjecture.

Theorems 3 and 4 are proved in section 5. Theorem 4 is obtained simply by inverting the formula of proposition 6. A λ-sequence can then be reinterpreted as a homomorphism $A \to \mathbb{Z}$, where A denotes the ring consisting of the rational polynomials in a single variable w which are integer valued on the integers. λ_k is the value of the homomorphism on the element $w^k \in A$. The congruences of theorems 2 and 4 appear by evaluating the homomorphism on certain simple elements of A. The redundancy question to which theorem 3 is addressed is solved by analysing the structure of a certain subring of A.

Finally we reveal that the ring A is simply $K_0(BU(1))$ with the product arising from the tensor product H-space structure on $BU(1)$.

2. *The formula for* λ_k. An H-map $BU \to BU$ is uniquely determined by its restriction to $BU(1) \subset BU$, and any map $BU(1) \to BU$ extends to an H-map $BU \to BU$, (4), (3).

Thus the set of H-maps $BU \to BU$ may be identified with

$$[BU(1), BU] = \tilde{K}^0(BU(1)).$$

If $\xi \in K^0(BU(1))$ is the Hopf bundle, $K^0(BU(1)) = \mathbb{Z}[[x]]$, where $x = \xi - 1$, and $\tilde{K}^0(BU(1))$ is the ideal generated by x.

If we think of an H-map $\phi\colon BU \to BU$ as an additive operation $\tilde{K}^0(X) \to \tilde{K}^0(X)$, the corresponding power series is obtained by applying ϕ to $x \in \tilde{K}^0(BU(1))$ (3) § 6.

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PROPOSITION 5. *If the H-map* $\phi: BU \to BU$ *corresponds to the power series*

$$\sum_{i \geqslant 1} a_i x^i \in \mathbb{Z}[[x]],$$

then $\phi_*: \pi_{2k}(BU) \to \pi_{2k}(BU)$ *is multiplication by*

$$\lambda_k(\phi) = \sum_{i=1}^{k} A_k^i a_i,$$

where $A_k^i = i!\, S_k^i$, *and* S_k^i *is the Stirling number of the second kind* (*see for example* (5)).

Proof. Let $\psi^j: BU \to BU$ be the Adams operation, then $\psi^j(\xi) = \xi^j$, so that $(1+x)^j - 1$ is the corresponding power series; while $\lambda_k(\psi^j) = j^k$ (1).

Define the map $\Lambda: x\mathbb{Z}[[x]] \to \mathbb{Z}[[y[[$ by sending $f(x)$ to $\Sigma_{k \geqslant 1} \lambda_k(\phi)\, y^k$, if $\phi x = f(x)$. It is clear that Λ is an additive homomorphism.

By considering the Adams operation ψ^j,

$$\Lambda((1+x)^j - 1) = \sum_{k \geqslant 0} j^k y^k = (1-jy)^{-1} - 1.$$

Thus since

$$x^i = \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} ((1+x)^j - 1),$$

$$\Lambda(x^i) = \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} [(1-jy)^{-1} - 1],$$

$$= \sum_{k \geqslant 1} \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^k y^k.$$

Now our proposition is proved since

$$S_k^i = \frac{1}{i!} \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^k,$$

is Stirling's formula (5).

3. *Kummer congruences.* Those parts of theorem 2 which concern H-maps follow immediately from proposition 5 and the following result (due essentially to Kummer (6)).

THEOREM 6. *Let* $A_k^i = i!\, S_k^i$, *where* S_k^i *is the Stirling number of the second kind. If* p *is an odd prime,* l *and* m *are positive,* $p^j(p-1)$ *divides* l *and* $q \geqslant (j+1)m$, *then*

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} A_{q+rl}^i \equiv 0 \bmod p^{(j+1)m}.$$

If l *and* m *are positive,* $j \geqslant 1$, 2^j *divides* l *and* $q \geqslant (j+2)m$, *then*

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} A_{q+rl}^i \equiv 0 \bmod 2^{(j+2)m}.$$

If l *is odd and positive and* $q \geqslant m > 0$, *then*

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} A_{q+rl}^j \equiv 0 \bmod 2^m.$$

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Proof. Stirling's formula is that

$$A^i_k = \sum_{s=1}^{i} (-1)^{i-s} \binom{i}{s} s^k.$$

Hence

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} A^i_{q+rl} = \sum_{s=1}^{i} (-1)^{i-s} \binom{i}{s} s^q \sum_{r=0}^{m} (-1)^r \binom{m}{r} s^{rl},$$

$$= \sum_{s=1}^{i} (-1)^{i-s} \binom{i}{s} s^q (1-s^l)^m.$$

The proof proceeds by showing that each term in this last sum is congruent to zero modulo the appropriate prime power. In each case, for p odd or $p = 2$, q is sufficiently large that s^q is congruent to zero if p divides s. That the other terms are zero depends on the structure of the group of units modulo a power of p. We give the proof for the case where 2^j divides l and $j \geqslant 1$. The two other cases are proved in a similar way.

The group of units modulo 2^{j+2} is the product of a cyclic group of order 2^j with a cyclic group of order 2. Thus, if s is odd, $s^l \equiv 1 \bmod 2^{j+2}$. Then $2^{(j+2)m}$ will divide $(1-s^l)^m$.

Notes. (1) The coefficients A^i_k can be computed recursively from the relation $A^i_k = i(A^i_{k-1} + A^{i-1}_{k-1})$; the values for small i, k are given in the following table:

k \ i	1	2	3	4	5
1	1	0	0	0	0
2	1	2	0	0	0
3	1	6	6	0	0
4	1	14	36	24	0
5	1	30	150	240	120

(2) Since A^i_k is divisible by $i!$ the theorem only carries weight for suitably small values of i.

(3) The Bernoulli number B_k is equal to

$$\sum_{i=1}^{k} \frac{(-1)^i}{i+1} A^i_k,$$

where the Bernoulli numbers are indexed so that

$$\frac{t}{e^t - 1} = \sum_{k \geqslant 0} B_k \frac{t^k}{k!}.$$

Kummer proved his congruences among Bernoulli numbers this way (6). Generalizations of these congruences are known but they do not seem to apply to the Stirling numbers.

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(4) There are applications of these congruences to the construction of elements in the K-theory co-operation algebra $K_*(K)$. Details will appear elsewhere.

4. *General maps $BU \to BU$.* The previous section gave a formula for computing the λ-sequence for an H-map. Here we show how to compute the λ-sequence for an arbitrary self-map of BU.

$[BU, BU] = \tilde{K}^0(BU).K^0(BU)$ is a (completed) Hopf algebra. As an algebra it is a power-series ring. We will use the K-theory Chern classes γ_i as power-series generators. These are constructed as follows. Let $1, \beta_1, \beta_2, \ldots$ be basis elements for $K_0(BU(1))$ dual to $1, x, x^2, \ldots$ in $K^0(BU(1)) = \mathbb{Z}[[x]]$. Then $K_0(BU)$ is polynomially generated by β_1, $\beta_2, \ldots$ and γ_i is dual to $(\beta_1)^i$ in the β-monomial basis for $K_0(BU)$.

$K^0(BU)$ is filtered by the skeleton filtration, and the elements, $\gamma_1, \gamma_2, \ldots$ are in increasing filtration. $K^0(BU)$ is complete with respect to the filtration, so that elements such as $\gamma_1 + \gamma_2 + \ldots$ are defined.

We do not need to refer to the topology on $K^0(BU)$ very much, however implicit in our working is that the maps $\lambda_k : \tilde{K}^0(BU) \to \mathbb{Z}$ are continuous. In fact clearly λ_k is zero above filtration $2k$, so that it is sufficient to evaluate λ_k on the polynomial subalgebra generated by $\gamma_1, \gamma_2, \ldots$.

The H-maps in $\tilde{K}^0(BU) = [BU, BU]$ correspond to the primitive elements. The result which we used in § 3 was that the restriction map $\tilde{K}^0(BU) \to \tilde{K}^0(BU(1))$, when restricted to the primitive submodule $PK^0(BU) \subset \tilde{K}^0(BU)$ is an isomorphism.

We know from proposition 5 how to compute the map λ_k: $\tilde{K}^0(BU) \to \mathbb{Z}$ on $PK^0(BU)$. We are able to extend our calculation to $\tilde{K}^0(BU)$ because of the following result.

PROPOSITION 7. *λ_k maps decomposable elements to zero.*

Proof. If $\phi, \psi \in [BU, BU] = \tilde{K}^0(BU)$, and $i_{2k} \in \pi_{2k}(BU)$ is a generator, we have $\phi_* i_{2k} = \lambda_k(\phi) i_{2k}$, and $\psi_* i_{2k} = \lambda_k(\psi) i_{2k}$.

Now if $\phi.\psi$ denotes the product in $\tilde{K}^0(BU)$, $\lambda_k(\phi.\psi) i_{2k}$ is represented by the composition

$$S^{2k} \xrightarrow{i_{2k}} BU \xrightarrow{\Delta} BU \times BU \xrightarrow{\phi \times \psi} BU \times BU \xrightarrow{t} BU,$$

where t is the tensor product map. But looked at another way this composition represents the product of the elements $\phi_* i_{2k}$ and $\psi_* i_{2k}$ in $\tilde{K}^0(S^{2k}) = \pi_{2k}(BU)$. The proposition is proved because all products are zero in $\tilde{K}^0(S^{2k})$ (4).

Since the coproduct

$$\Delta\gamma_k = \sum_{i+j=k} \gamma_i \otimes \gamma_j \quad (\gamma_0 = 1),$$

$PK^0(BU)$ is generated (topologically) by the elements p_j which are the Newton polynomials in $\gamma_1, \gamma_2, \ldots, \gamma_j$;

$$p_j = (\gamma_1)^j + \ldots + (-1)^{j-1} j\gamma_j,$$

where the omitted terms involve elements γ_i with $1 < i < j$.

$$K^0(BU) \to K^0(BU(1)) = \mathbb{Z}[[x]],$$

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maps γ_1 to x, and γ_i to zero if $i > 1$, so that p_j is mapped to x^j. But, by proposition 7,

$$\lambda_k(p_j) = (-1)^{j-1} j \lambda_k(\gamma_j).$$

Together with proposition 5 this yields the following result.

PROPOSITION 8. *λ_k: $\tilde{K}^0(BU) = [BU, BU] \to \mathbb{Z}$ is the continuous homomorphism of abelian groups which is zero on decomposable elements and which maps the K-theory Chern class γ_j to*

$$\frac{(-1)^{j-1}}{j} A_k^j = (-1)^{j-1}(j-1)!\, S_k^j = (-1)^{j-1}(A_{k-1}^j + A_{k-1}^{j-1}).$$

It is now clear from the last formula and from Theorem 6 that the congruences of Theorem 2 hold for a general self-map of BU with the strict inequalities on q.

This settles Lance's conjecture in Remark 1·8 of (7). He conjectured that his class of congruences held for general self-maps of BU. Theorem 2 shows exactly which of these do or do not hold. For the results there are best possible, in the sense that the restrictions on q cannot be relaxed.

Lance's conjecture is that, for an odd prime p and a general self-map of BU, $\lambda_{np^k} \equiv \lambda_{p^k} \bmod p^{k+1}$ if $n \equiv 1 \bmod (p-1)$. This holds if $k > 0$, for this is the case $m = 1, q = p^k, l = (n-1)p^k, j = k$ of theorem 2, in which $q > (j+1)m$ is satisfied. But it need not hold for $k = 0$, unless the map is an H-map. An example which shows this is provided by the map of BU which represents the K-theory Chern class γ_p (or any element congruent to it modulo decomposables). This map has

$$\lambda_k = \frac{A_k^p}{p} = (p-1)!\, S_k^p.$$

But $S_p^p = 1$, $S_1^p = 0$, so that $\lambda_1 = 0$, $\lambda_p = (p-1)!$ and $\lambda_1 \not\equiv \lambda_p \bmod p$.

5. *The proofs of theorems* 3 *and* 4, *and a reinterpretation of our results.* If we invert the formula of proposition 5 we find that if ϕ is an H-map $BU \to BU$ such that

$$\phi x = \Sigma a_i x^i \in \tilde{K}^0(BU(1)) = \mathbb{Z}[[x]],$$

then

$$a_i = \frac{1}{i!} \sum_{k=1}^{i} s_i^k \lambda_k(\phi),$$

where s_i^k is the Stirling number of the first kind (5). Theorem 4 follows immediately.

Let $A \subset \mathbf{Q}[w]$ be the subring consisting of those polynomials which are integer valued on the integers. As an abelian group A is free on the basis

$$\left\{1, w, \binom{w}{2}, \binom{w}{3}, \ldots\right\} \quad \text{where } \binom{w}{i}$$

is the binomial coefficient polynomial

$$\binom{w}{i} = \frac{w(w-1)\ldots(w-i+1)}{i!} = \frac{1}{i!} \sum_{k=1}^{i} s_i^k w^k, \text{ (3)}.$$

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Let $\tilde{A}$ be the ideal of A consisting of those polynomials in A with zero constant term.

Theorem 4 shows that a self-H-map ϕ of BU determines an additive homomorphism $\tilde{\phi}\colon \tilde{A} \to \mathbb{Z}$ by mapping w^k to $\lambda_k(\phi)$, and that every such homomorphism arises in this way. For example corresponding to the Adams operation ψ^j is the *ring* homomorphism $f(w) \mapsto f(j)$.

The congruences of theorem 2 arise from the evaluation of $\tilde{\phi}$ on certain simple elements of $\tilde{A}$. Specifically the odd primary congruences there hold because the elements $w^q(1-w^l)^m/p^{(j+1)m}$ belong to A, under the given hypotheses on q, l, m and j. This is so simply because A is a ring and $w^{j+1}(1-w^l)/p^{j+1}$ belongs to A if $p^j(p-1)$ divides l.

Having established this notation we turn to the proof of Theorem 3. We prove the part relating to an odd prime p; the case of $p = 2$ has a similar proof.

Let B be the subring of A generated by the elements w and $w^{j+1}(1-w^l)/p^{j+1}$ where $p^j(p-1)$ divides l, so that the congruences of theorem 2 follow by evaluating $\tilde{\phi}$ on $\tilde{B} = \tilde{A} \cap B$.

LEMMA 9. *B is generated as a ring by the elements w and $(w^p - w)/p$.*

Proof. We need only show that if $p^j(p-1)$ divides l then $w^{j+1}(1-w^l)/p^{j+1}$ can be expressed as a polynomial in w and $(w^p - w)/p$, where we may assume inductively that this is possible for all elements of B of strictly smaller degree. Writing $l = ap^j(p-1)$,

$$1 - w^l = (1 + w^{p^j(p-1)} + \ldots + w^{(a-1)p^j(p-1)})(1 - w^{p^j(p-1)}),$$

so that it suffices to deal with the case $l = p^j(p-1)$.

Now

$$\frac{w^{j+1}(1-w^{p^j(p-1)})}{p^{j+1}} + w^{(p^j-j-1)(p-1)}\left(\frac{w^p - w}{p}\right)^{j+1},$$

belongs to B, since $p^j \geqslant j+1$, and is a polynomial in w of degree $p^j(p-1)+j+1-(p-1)$, which can, by our induction hypothesis, be expressed in the required form.

Lemma 9 shows that B has a basis consisting of the elements

$$w^s\left(\frac{w^p - w}{p}\right)^m, \quad 0 \leqslant s < p,\ m \geqslant 0.$$

Theorem 3 follows, for the case of an odd prime.

The redundancy of the congruences of Theorem 2 relates to linear dependence relations between the corresponding elements of A. Their incompleteness follows from the fact that these elements do not span A. The point is that there is more p-primary divisibility in A than is accounted for by the subring B. In particular the element

$$\binom{w}{p^2} \in A$$

is a polynomial of degree p^2 with denominator divisible by p^{p+1}. But it follows from Lemma 9 that every element in B of degree p^2 can be written as an integral polynomial in w divided by p^p. This is the algebra behind the following result.

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PROPOSITION 10. *If p is prime, $a > 1$, the sequence*

$$\lambda_k = \frac{1}{p^{a-1}} A_k^{p^a} = (p^a - 1)!\, p S_k^{p^a} \in \mathbb{Z},$$

satisfies all the congruences of theorem 2 satisfied by the λ-sequence of an H-map, but does not arise from an H-map.

Proof. That the sequence does not arise from an H-map follows from the inversion formula at the beginning of this section, which tells us that the corresponding power series is x^{p^a}/p^{a-1}, which is not integral if $a > 1$.

Noting that if p^k is the highest power of p to divide s then p^{a-k} divides

$$\binom{p^a}{s},$$

we consider again the sum

$$\sum_{s=1}^{p^a} (-1)^{p^a - s} \binom{p^a}{s} s^q (1 - s^l)^m$$

where we shall assume for simplicity that p is odd, with $p^j(p-1)$ dividing l, and $q \geqslant (j+1)m$.

If $(p, s) = 1$, then p^a divides $\binom{p^a}{s}$, p^{j+1} divides $(1 - s^l)$ so that $p^{(j+1)m+a}$ divides the s-th term in the sum. If $s = p^k r$ with $(p, r) = 1$ we find that p^{kq+a-k} divides the term. It follows then that $p^{(j+1)m+a-1}$ divides the whole sum, and hence that the congruences of Theorem 6 hold modulo an extra factor p^{a-1}, which proves the proposition.

We now show that the ring A has a very natural interpretation in the context in which we have been working.

THEOREM 11. *The ring $A = \{f(w) \in Q[w] : f(\mathbb{Z}) \subset \mathbb{Z}\}$ is isomorphic to $K_0(BU(1))$. where $K_0(BU(1))$ has the Pontryagin ring structure induced by the tensor product map $BU(1) \times BU(1) \to BU(1)$.*

Proof. As an abelian group $K_0(BU(1))$ has basis $1, \beta_1, \beta_2, \ldots$. Since A is generated as free abelian group by the elements 1, w, $\binom{w}{2}$, $\binom{w}{3}$, ..., the map $K_0(BU(1)) \to A$ which maps β_i to $\binom{w}{i}$ is an additive isomorphism. We need to show that it preserves the multiplication.

Dual to the multiplication in $K_0(BU(1))$ is the coproduct in the (completed) Hopf algebra $K^0(BU(1)) = \mathbb{Z}[[x]]$. This coproduct is the K-theory formal group law:

$$\Delta x = x \otimes 1 + 1 \otimes x + x \otimes x,$$

$$= (x+1) \otimes (x+1) - 1 \otimes 1.$$

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Thus
$$\Delta x^i = [(x+1)\otimes(x+1) - 1\otimes 1]^i,$$
$$= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} (x+1)^j \otimes (x+1)^j,$$
$$= \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{j} (-1)^{i-j} \binom{i}{j} \binom{j}{k} \binom{j}{l} x^k \otimes x^l.$$

Since $1, \beta_1, \beta_2, \ldots$ in $K_0(BU(1))$ are dual to $1, x, x^2, \ldots$ in $K^0(BU(1))$, we have for the product in $K_0(BU(1))$
$$\beta_k \beta_l = \sum_{i\geqslant 0} \left[\sum_{j\geqslant k,l} (-1)^{i-j} \binom{i}{j} \binom{j}{k} \binom{j}{l} \right] \beta_i,$$
$$= \sum_{i\geqslant 0} \binom{i}{l} \binom{l}{i-k} \beta_i,$$

where the last step follows from an identity which is to be found in (8), page 15.

But now another formula on the same page of (8) tells us that
$$\binom{w}{k} \binom{w}{l} = \sum_{i\geqslant 0} \binom{i}{l} \binom{l}{i-k} \binom{w}{i},$$

so that the isomorphism is multiplicative.

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