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Limit points of uniform arithmetic bass notes

William Hide and Bram Petri

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Limit points of uniform arithmetic bass notes

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Abstract

We prove that the set of limit points of the set of all spectral gaps of closed arithmetic hyperbolic surfaces equals $[0, \frac{1}{4}]$.

1. Introduction

The *spectral gap* $\lambda_1(X)$ of a closed orientable hyperbolic surface X is the minimal non-zero eigenvalue of its Laplacian. A natural question, which fits into a larger context of the study of the bass note spectra of locally uniform geometries [Sar23], is which numbers appear as spectral gaps of hyperbolic surfaces. It can be derived in various ways from various combinations of [Hub74, Che75, Bus77, Gag80, Jen81, Bav96, SU13, Bon22, FP23, KMP24] that the *bass note spectrum* of the set of closed orientable hyperbolic surfaces satisfies

$$\mathrm{Bass}(\mathbf{hyp}_2^c) := \{\lambda_1(X) : X \text{ a closed orientable hyperbolic surface}\} = (0, \Lambda_2],$$

where $\Lambda_2 = \max\{\lambda_1(X) : X \text{ a closed hyperbolic surface of genus } 2\}$. The latter conjecturally equals the spectral gap of the Bolza surface (see § 2.1 for a description).

If we restrict to arithmetic surfaces, the set of spectral gaps we obtain becomes countable. Recently, Magee [Mag24a] proved that the spectral gaps, defined as the infimum of the non-zero spectrum in this setting, of non-compact arithmetic surfaces form a dense set in $[0, \frac{1}{4}]$. He also asked whether the same is true for closed arithmetic surfaces. The goal of this paper is to answer this question in the affirmative.

Theorem 1. *The set* $\{\lambda_1(X) : X \text{ a closed arithmetic surface}\} \cap [0, \frac{1}{4}]$ *is dense in* $[0, \frac{1}{4}]$.

In fact, we show that, as in the non-compact case, it suffices to consider only finite-index torsion-free subgroups of a single uniform arithmetic group. In our case this is the $(2, 3, 8)$-triangle group.

Similarly, the same method can be used to prove that if $X = \Gamma\backslash\mathbb{H}^2$ is a closed orientable surface then

$$\{\lambda_1(\Gamma'\backslash\mathbb{H}^2) : \Gamma' < \Gamma,\ [\Gamma : \Gamma'] < \infty\} \cap [0, \Lambda]$$

is dense in $[0, \Lambda]$, where $\Lambda = \min\{\lambda_1(\Gamma'\backslash\mathbb{H}^2) : \Gamma' < \Gamma,\ [\Gamma : \Gamma'] = 2\}$.

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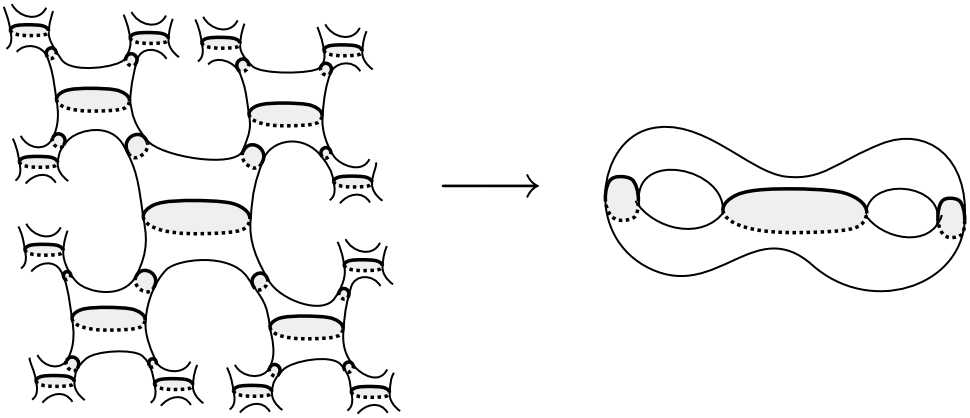


Figure 1. A tree cover of a surface of genus 2.

Finally, we also obtain a different proof of [LMH25, Theorem 1.8], the fact that there exists a sequence of finite-sheeted covers $Y_n \to X$ of a fixed closed orientable arithmetic hyperbolic surface X, whose genera tend to infinity and whose spectral gap tends to $\frac{1}{4}$.

1.1 Strategy of proof

Like Magee's proof in the non-uniform case, we build our arithmetic surfaces as covers of degree 2 of a certain type of random covers of a fixed base surface. In our case, this base surface is the Bolza surface X_B. As such, the overarching strategy of our proof is similar to that of Magee. However, several of the key ingredients of his proof are not present in our setting, so in what follows we will describe the similarities and differences.

The first thing that makes our case different is that the fundamental group $\Gamma_B = \pi_1(X_B)$ of the X_B is not free, which makes random finite-degree covers harder to control (see [MP23, MNP22] for the geometry of uniform random covers of a closed hyperbolic surface). We remedy this by considering random covers $Y^{(n)} \to X_B$ of degree n whose monodromy factors through an epimorphism from our surface group $\Gamma_B \to F_2$ to a free group F_2 on two generators. This allows us to use the results of Bordenave and Collins [BC19] on strong convergence of permutation representations combined with results by Hide and Magee [LMH25, Appendix] on the convergence of the spectral gaps of the associated covers. In [LMH25], Louder and Magee also did this. However, we need more control over the geometry of the associated cover, so we use a different argument.

We need to control the following things.

(1) In order to obtain near-optimal spectral gaps, we need to understand the spectral gap of the cover of the Bolza surface associated to the map $\Gamma_B \to F_2$ (see Figure 1). We tackle this using Cheeger's inequality. Using Mirzakhani's result on random pants decompositions of hyperbolic surfaces [Mir16] combined with some further geometric arguments, we show that by varying the map to the free group (see § 4 and Proposition 6.1) we can obtain a spectral gap as close to $\frac{1}{4}$ as we like.

(2) To reach intermediate spectral gaps, our aim is to use the covers of degree 2 of our random cover $Y^{(n)} \to X_B$. To make this work, we need there to be at least one cover among these degree 2 covers that has a large spectral gap. We obtain such a cover by restricting a non-constant homomorphism $\Gamma_B \to F_2 \to \mathbb{Z}/2\mathbb{Z}$ to the fundamental group of $Y^{(n)}$, seen as a subgroup of Γ_B. The fact that this indeed yields two-sheeted covers of $Y^{(n)}$ with large spectral gaps relies on a strong convergence argument combined with the fact that all

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connected two-sheeted covers $\widehat{Y} \to X_B$ satisfy $\lambda_1(\widehat{Y}) > \frac{1}{4}$. The latter fact we prove (in § 3) using the linear programming methods developed in [FP23, FGP+23].

(3) Like Magee, we then need to show that, when n is large enough, 'small' modifications of the degree 2 covers of $Y^{(n)}$ do not influence the spectral gap too much. These modifications consist of changing the cover along the lifts of a simple closed curve in $Y^{(n)}$ and will be called switches (see § 6.5). In Magee's paper, logarithmic tangle-freeness is used to prove that similar moves do not influence the spectral gap too much. Unfortunately, our covers are, by construction, much less tangle-free: the infinite cover associated to the map $\Gamma_B \to F_2$ contains many short curves. As a result, the injectivity radius of $Y^{(n)}$ is uniformly bounded.

We will show, however, again relying on Mirzakhani's results on random pants decompositions and also using Nica's work on cycles of random permutations coming from word maps [Nic94], that we can find maps $\Gamma_B \to F_2$ such that, with positive probability, the curves in $Y^{(n)}$ along which we switch admit uniformly wide collars (see § 6.6). This suffices to make the argument, based on the L^∞-bounds on eigenfunctions by Gilmore *et al.* [GLS+21], work. This implies the density result because any two covers of degree 2 of $Y^{(n)}$ can be connected by a finite sequence of switches.

1.2 Further notes and references

The study of spectral gaps of hyperbolic surfaces has a long history, which we will not attempt to summarize here. We refer to [Bus10, Ber16] for introductory texts. The shape of the bass note spectrum has been studied in the case of compact hyperbolic 2-orbifolds [Bon22, KMP24], non-compact arithmetic surfaces [Mag24a] and regular graphs [AW23, DM26]. In the case of orbifolds, Kravchuk, Mazac and Pal give a complete description of the set

$$\{\lambda_1(O) : O \text{ a compact orientable hyperbolic 2-orbifold}\}.$$

This description is conjectural, but a large part of it is proven (see [KMP24, Conjecture 4.2]).

The case of arithmetic surfaces is very different. There are only finitely many arithmetic hyperbolic surfaces of any given area (up to isometry). As such, for any $\varepsilon > 0$, there can be at most finitely many closed arithmetic surfaces whose spectral gap exceeds $\frac{1}{4} + \varepsilon$, using, for instance, the results in [Hub74, Che75]. Hence, the theorem above can be stated as saying that the set of limit points of the set of spectral gaps of closed hyperbolic surfaces equals $[0, \frac{1}{4}]$. There are, however, isolated points outside of this interval, corresponding, for instance, to the Bolza surface and the Klein quartic. Sarnak [Sar23] conjectured that there should be infinitely many such isolated points.

2. The model

In this section we formally describe our model of random surfaces, which is a variation of the models appearing in [BCP21, LMH25, Mat24].

2.1 The Bolza surface

The model we will describe samples a certain type of covers of a closed hyperbolic surface. The model makes sense for any closed base surface, but because of the application we have in mind, we will assume the base surface is the Bolza surface. We will use this subsection to describe some of its properties.

The quickest way to define the Bolza surface, which we denote by X_B, is to say that it is a Riemann surface of genus 2 with 48 automorphisms. Indeed, it turns out that this is the maximal

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possible number of automorphisms of a Riemann surface of genus 2 and there is a unique such surface. More concretely, X_B can be obtained by taking a regular octagon O with interior angles $\pi/4$ in the hyperbolic plane $\mathbb{H}^2$ and identifying opposite sides. We will also fix a Fuchsian group Γ_B such that $X_B = \Gamma_B \backslash \mathbb{H}^2$ and we will, moreover, fix an identification of Γ_B with $\pi_1(X_B)$.

The surface X_B is also a $(2,3,8)$-triangle surface. That is, its fundamental group is a normal subgroup (of index 48) in the orientable triangle group

$$\Theta(2,3,8) = \langle x, y, z \,|\, x^2,\ y^3,\ z^8,\ xyz \rangle$$

whose generators act on $\mathbb{H}^2$ by the order 2, 3 and 8 rotations in the vertices of a hyperbolic triangle whose interior angles are $\pi/2$, $\pi/3$ an $\pi/8$, respectively. The quotient group is described explicitly as item `T2.1` in [Con15]. Since the $(2,3,8)$-triangle group is arithmetic [Tak77] (see also [MR03, § 13.3]), X_B is arithmetic.

Very good numerical estimates of the spectral gap of X_B have been obtained in [SU13]. That is,

$$\lambda_1(X_B) = 3.838887 \pm 10^{-6}.$$

The best known upper bound on the maximal spectral gap of a hyperbolic surface of genus 2 equals 3.8388976481 [Bon22, KMP24].

2.2 Handlebody attachments, maps to free groups and pants decompositions

A handlebody H is a 3-manifold that can be obtained by attaching 1-handles to a closed 3-ball. Alternatively, it can be seen as the closed regular neighbourhood of a finite graph in $\mathbb{R}^3$. In particular, its fundamental group is a free group and its boundary is a closed orientable surface. The genus of this surface will be called the *genus* of H. This genus coincides with the rank of the fundamental group of H. The inclusion map $\iota : \partial H \hookrightarrow H$ induces an epimorphism

$$\iota_* : \pi_1(\partial H) \longrightarrow F_g,$$

where g is the genus of H and F_g denotes the free group of rank g. In fact, we obtain a large family of such maps, because we can precompose the inclusion by a homeomorphism $\phi : \partial H \to \partial H$.

Any such homeomorphism yields an F_g-cover

$$T_\phi \to \partial H$$

corresponding to the subgroup $\ker((\iota \circ \phi)_*)$. The surface T_ϕ is what is sometimes called a Cantor tree: a surface homeomorphic to $\mathbb{S}^2 - C$, where C is a Cantor set. As such, we will call the F_g-covers constructed by the procedure above *tree covers*.

Finally, we observe that ∂H admits a pants decomposition $\mathcal{P}$ consisting entirely of *meridians*: simple closed curves that bound disks in H (see Figure 1 for the case of genus 2). This means in particular that all the curves in $\mathcal{P}$ are homotopically trivial in H and thus that all the curves in the pants decomposition $\phi^{-1}(\mathcal{P})$ lie in $\ker((\iota \circ \phi)_*)$. This in turn implies that $\phi^{-1}(\mathcal{P})$ lifts to a pants decomposition of T_ϕ (see Figure 1 again for a picture).

2.3 Random covers

We will now once and for all fix a handlebody attachment $\iota : X_B \hookrightarrow H$ of the Bolza surface X_B and a pants decomposition $\mathcal{P}_0 = (\alpha_1, \alpha_2, \alpha_3)$ of the X_B that consists entirely of meridians.

Given an orientation-preserving homeomorphism $\phi : X_B \to X_B$, we obtain, as in the previous section, a homomorphism $(\iota \circ \phi)_* : \Gamma_B \to F_2$ and a tree cover $T_\phi \to X_B$. The pants decomposition of T_ϕ that we obtain by lifting $\phi^{-1}(\mathcal{P}_0)$ will be called the *canonical pants decomposition* of T_ϕ in what follows.

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Now, given a homomorphism $\rho \in \mathrm{Hom}(F_2, \mathfrak{S}_n)$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters, we obtain a subgroup

$$H^{(n)} = \mathrm{Stab}_\rho(\{1\}) < F_2,$$

but also a homomorphism

$$\rho \circ (\iota \circ \phi)_* : \Gamma_B \longrightarrow \mathfrak{S}_n,$$

and hence a subgroup

$$\Lambda_{\phi,\rho} = \mathrm{Stab}_{\rho\circ(\iota\circ\phi)_*}(\{1\}) < \Gamma_B$$

of index at most n, and thus a cover

$$Y_{\phi,\rho} \longrightarrow X_B$$

of degree at most n.

Our random surface $Y_\phi^{(n)}$ will be obtained by taking the permutation representation $\rho \in \mathrm{Hom}(F_2, \mathfrak{S}_n)$ uniformly at random. It follows from results by Dixon [Dix69] that, as $n \to \infty$, the probability that the cover $Y_\phi^{(n)}$ is of degree exactly n tends to 1.

3. The bass notes of degree 2 covers of the Bolza surface

In this section we will explain how we prove, with the help of a computer, that all the degree 2 covers $Y \to X_B$ of the Bolza surface satisfy $\lambda_1(Y) > \frac{1}{4}$. To do so, we will employ the linear programming methods from [FP23, FGP+23].

In fact, just as in [Mag24a], this can be proved by verifying the spectral gap of a single surface Y_{17} obtained as the $\mathrm{H}_1(X_B, \mathbb{Z}/2\mathbb{Z})$-cover of X_B. Indeed, any homomorphism $\Gamma_B \to \mathbb{Z}/2\mathbb{Z}$ factors through $\mathrm{H}_1(X_B, \mathbb{Z}/2\mathbb{Z})$ and as such any degree 2 cover of X_B is intermediate to the cover $Y_{17} \to X_B$.

3.1 Identifying the cover

First of all, we need a description of Y_{17}. Since

$$|\mathrm{H}_1(X_B, \mathbb{Z}/2\mathbb{Z})| = 16,$$

the genus of Y_{17} equals 17 (whence the notation). Moreover, by construction, the fundamental group of Y_{17} is a characteristic[1] subgroup of Γ_B. This implies that it is a normal subgroup of the $(2,3,8)$-triangle group $\Theta(2,3,8)$. In other words, Y_{17} is a $(2,3,8)$-triangle surface of genus 17. It turns out there is only one such surface, which appears as item `T17.2` in Conder's list [Con15]. In particular, Y_{17} corresponds to the quotient

$$G = \langle x, y, z \,|\, x^2,\ y^3,\ z^8,\ xyz,\ z^2yxz^2yxzy^{-1}z^{-1}xzy^{-1}z^{-1}x \rangle$$

of $\Theta(2,3,8)$.

3.2 Excluding small eigenvalues

In order to prove that Y_{17} has no small eigenvalues, we use the method from [FGP+23] which we will very briefly describe here. For more details, we refer to *loc. cit.* and [CP23].

[1] Recall that a characteristic subgroup of Γ_B is a subgroup that is preserved by all elements of $\mathrm{Aut}(\Gamma_B)$. This is a stronger assumption than being a normal subgroup.

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First of all, the Deck group G of the orbifold cover $Y_{17} \to S_{2,3,8}$, where $S_{2,3,8} = \Theta(2,3,8)\backslash\mathbb{H}^2$ denotes the orientable $(2,3,8)$-triangle orbifold, acts on functions on Y_{17}. This action commutes with the Laplacian, so we can split the eigenspaces of the Laplacian into G-representations.

Our goal is now to exclude small eigenvalues from these representations. To do so, we use a criterion based on the twisted Selberg trace formula corresponding to every such representation.

We start with some setting up. We will write $\operatorname{spec}(Y_{17})$ for the spectrum of the Laplacian on Y_{17}, as a multi-set. Moreover, $\operatorname{Irr}(G)$ will denote the finite set of real G-representations $\rho : G \to \operatorname{GL}(V^\rho)$. The splitting of the eigenspaces into G-representations described above yields a splitting of the spectrum

$$\operatorname{spec}(Y_{17}) = \bigcup_{\rho \in \operatorname{Irr}(G)} m(\rho) \cdot \operatorname{spec}(Y_{17}, \rho),$$

where multiplication of a multi-set by a positive integer means multiplying the multiplicities by that integer, $m(\rho)$ is the multiplicity of ρ in $\mathbb{R}[G]$ and $\operatorname{spec}(Y_{17}, \rho)$ is the spectrum of the Laplacian acting on functions $\mathbb{H}^2 \to V^\rho$ that are equivariant with respect to the action of $\Theta(2,3,8)$ on the left and $\rho(G)$ on the right. This spectrum can also be interpreted in terms of the associated bundle.

For $d > 0$, we define $f_d : \mathbb{R} \to \mathbb{R}$ by

$$f_d(x) = \left(\frac{1}{2d}\chi_{[-d,d]}\right)^{*4}(x) = \begin{cases} \dfrac{1}{12d}\left(4 - \dfrac{3}{2d^2}x^2 + \dfrac{3}{8d^3}|x|^3\right) & \text{if } 0 \le |x| \le 2d, \\ \dfrac{1}{12d}\left(2 - \dfrac{|x|}{2d}\right)^3 & \text{if } 2d \le |x| \le 4d, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\chi_{[-d,d]}$ denotes the characteristic function of the interval $[-d,d]$ and '$*$' denotes the convolution product.

The Fourier transform $\widehat{f_d} : \mathbb{C} \to \mathbb{C}$ of f is given by

$$\widehat{f_d}(y) = \int_{-\infty}^{\infty} f_d(x) e^{-iy\cdot x}\, dx = \frac{\sin(d\cdot y)^4}{(d\cdot y)^4}, \quad y \in \mathbb{C}.$$

Given $\rho \in \operatorname{Irr}(G)$, we define

$$\begin{aligned} \mathcal{G}_d(\rho) = &-\frac{\dim(V^\rho)}{96}\int_{-\infty}^{\infty} \frac{f_d'(x)}{\sinh(x/2)}\, dx \\ &+ \sum_{[\gamma]\in\mathcal{E}(\Gamma)} \frac{\operatorname{tr}(\widetilde{\rho}(\gamma))}{m(\gamma)} \int_0^{\infty} \frac{\cosh(x/2)}{\cosh(x) - 1 + 2\sin(\theta(\gamma))^2}\, f_d(x)\, dx \\ &+ \sum_{[\gamma]\in\mathcal{P}(\Gamma)} \ell(\gamma) \sum_{n\ge 1} \frac{\operatorname{tr}(\widetilde{\rho}(\gamma^n))}{2\sinh(n\ell(\gamma)/2)} f_d(n\ell(\gamma)), \end{aligned} \tag{3.1}$$

where:

- $\widetilde{\rho}$ is the composition of $\rho : G \to \operatorname{GL}(V^\rho)$ with the quotient map $\Theta(2,3,8) \to G$;
- $\mathcal{E}(\Gamma)$ denotes the set of conjugacy classes of elliptic elements in $\Theta(2,3,8)$;
- for an elliptic element $\gamma \in \Theta(2,3,8)$, $m(\gamma)$ denotes its order and $\theta(\gamma)$ denotes half the angle of rotation, that is, is such that γ is conjugate to

$$\begin{bmatrix} \cos(\theta(\gamma)) & \sin(\theta(\gamma)) \\ -\sin(\theta(\gamma)) & \cos(\theta(\gamma)) \end{bmatrix} \in \operatorname{PSL}(2,\mathbb{R});$$

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– $\mathcal{P}(\Gamma)$ denotes the set of conjugacy classes of primitive hyperbolic elements in $\Theta(2,3,8)$;
– for a hyperbolic element $\gamma \in \Theta(2,3,8)$, $\ell(\gamma)$ denotes its translation length on $\mathbb{H}^2$.

We observe that, because f_d has finite support, the sum and the integrals appearing on the right-hand side of (3.1) are all finite.

Given these definitions, the criterion we use is as follows.

PROPOSITION 3.1 [FGP+23, Proposition 3.1]. *Let $\rho \in \mathrm{Irr}(G)$ and let $\lambda > 0$. Suppose that*

$$\widehat{f}_d\Big(\sqrt{\lambda - \tfrac{1}{4}}\Big) > \begin{cases} \mathcal{G}_d(\rho) & \text{if } \rho \text{ is non-trivial,} \\ \mathcal{G}_d(\rho) - \widehat{f}(i/2) & \text{if } \rho \text{ is trivial,} \end{cases}$$

for some $d > 0$. Then $\lambda \notin \mathrm{spec}(Y_{17}, \rho)$.

We prove the following lower bound on the spectral gap of Y_{17}.

PROPOSITION 3.2. *The criterion from Proposition 3.1 holds for the parameter $d = \frac{3}{4}$ for all $\rho \in \mathrm{Irr}(G)$ and for all $\lambda \le 0.2501$. In particular,*

$$\lambda_1(Y_{17}) \ge 0.2501.$$

Proof. The final proof of this proposition is performed using `SageMath` and its interface to `GAP`. Here we will discuss the main ingredients.

(1) We compute the character table of G, using `GAP`.
(2) Now we observe that when $d = \frac{3}{4}$, the support $\mathrm{supp}(f_d) = [-3, 3]$. As such, we need to know all the hyperbolic conjugacy classes of translation length at most 3 and all the elliptic conjugacy classes of $\Theta(2,3,8)$ in order to compute $\mathcal{G}_d(\rho)$. The latter are the just the conjugacy classes of the generators x, y and z and their powers. The hyperbolic conjugacy classes we need were determined in [FGP+23].
(3) The integrals appearing in $\mathcal{G}_d(\rho)$ are treated using interval arithmetic as implemented in the `Arb` package.

The claim in our proposition is verified in the `Jupyter` notebook `computation_spectral_gap.ipynb` that is supplied as supplementary material with the online version of this article. □

Remark 3.3. Because 0.2501 suffices for what follows, we have made no effort to optimize the bound in the proposition above.

4. Bass notes of planar surfaces of infinite area

Given a hyperbolic surface X, we define its *Cheeger constant* by

$$h(X) = \inf\left\{ \frac{\ell(\partial A)}{\mathrm{area}(A)} : \begin{matrix} A \subset X \text{ compact,} \\ \partial A \text{ rectifiable} \end{matrix} \right\}.$$

The *systole*, the length of the shortest closed geodesic on X, of a hyperbolic surface X will be denoted $\mathrm{sys}(X)$. We call a surface *planar* if it has genus 0. Note that such a surface either has cusps or infinite area.

We make the following claim.

LEMMA 4.1. *Let X be a planar orientable hyperbolic surface without cusps. Then*

$$h(X) \ge 1 - \frac{2\pi}{\mathrm{sys}(X) + 2\pi}.$$

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Proof. The case in which the fundamental group of X is elementary, which in our setting implies that X is either the hyperbolic plane or an annulus $\langle g\rangle\backslash\mathbb{H}^2$ with $g\in \mathrm{PSL}(2,\mathbb{R})$ hyperbolic, is covered by the results of Adams and Morgan [AM99, Theorem 2.2, Lemma 2.3].

As such, we may assume the fundamental group of X is non-elementary. We define

$$H(X)=\inf\left\{\frac{\ell(\partial A)}{\mathrm{area}(A)}:\begin{array}{c}A\subset X \text{ compact,}\\ \partial A \text{ consists of simple closed geodesics}\end{array}\right\}.$$

Clearly $h(X)\le H(X)$. It turns out that

$$h(X)\ge \frac{H(X)}{H(X)+1};$$

see [Mat05, Theorem 7][2] and [Mir13, Proposition 4.7].

Because X is planar, any candidate subsurface A for $H(X)$ is necessarily planar as well. Suppose A is an n-holed sphere. This means that

$$\mathrm{area}(A)=2\pi\cdot(n-2)\quad\text{and}\quad \ell(\partial A)\ge n\cdot\mathrm{sys}(X).$$

So, $H(X)\ge (n/(n-2))(\mathrm{sys}(X)/2\pi)\ge \mathrm{sys}(X)/2\pi$ and hence

$$h(X)\ge 1-\frac{2\pi}{\mathrm{sys}(X)+2\pi}.$$ □

Using Cheeger's inequality [Che70] (see also [Cha84, § IV.3]), we obtain the following proposition.

Proposition 4.2. *Let X be a planar hyperbolic surface without cusps. Then*

$$\lambda_1(X)\ge\frac{1}{4}\cdot\left(1-\frac{2\pi}{\mathrm{sys}(X)+2\pi}\right)^2.$$

5. Eigenfunctions near pants curves

In this section, we describe the effect of flattening an eigenfunction near a pants curve on the Rayleigh quotient of that function.

5.1 Delocalization

One of the inputs we need is a delocalization result by Gilmore *et al.* [GLS+21, Theorem 1.5].

Theorem 5.1 (Gilmore *et al.*). *Suppose that $X=\Gamma\backslash\mathbb{H}^2$ is a closed hyperbolic surface of genus $g\ge 2$. Then there exists a universal constant $A>0$ such that*

$$|f(z)|^2\le A\cdot e^{-t\cdot\sqrt{\frac{1}{4}-\lambda}}\cdot|\{\gamma\in\Gamma:\mathrm{d}_{\mathbb{H}^2}(z,\gamma\cdot z)\le t\}|$$

for all $t>0$, $\lambda\in(0,\frac{1}{4})$ and all Laplacian eigenfunctions f of eigenvalue λ with $\|f\|_{L^2}=1$.

5.2 Flattening an eigenfunction near a pants curve

Given the result above, we need a bound on the Rayleigh quotient of a flattened eigenfunction in terms of the L^∞-norm of the eigenfunction. Our main technical input is the Ismagilov–Morgan–Simo–Sigal localization formula [CFK+87, Theorem 3.2].

Theorem 5.2 (Ismagilov–Morgan–Simo–Sigal). *Let M be a Riemannian manifold and suppose $\{J_i\}_{i\in\mathcal{I}}$ is a family of smooth functions $J_i:M\to[0,1]$ such that:*

[2]Note that Matsuzaki uses the inverse convention for the Cheeger constant.

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(1) $\sum_{i\in\mathcal{I}} J_i^2 \equiv 1$;
(2) *on any compact subset* $K \subset M$, *only finitely many* J_i *are non-zero;*
(3) $\sup_{x\in M} \sum_{i\in\mathcal{I}} |\nabla J_i(x)|^2 < \infty$.

Then

$$\Delta = \sum_{i\in\mathcal{I}} J_i \Delta J_i - \sum_{i\in\mathcal{I}} |\nabla J_i|^2.$$

Given a simple closed geodesic γ in a hyperbolic surface X and $w > 0$, we write

$$C_w(\gamma) = \{x \in X : \mathrm{d}(x,\gamma) \le w\}$$

for the *collar* of width w around γ. If $C_w(\gamma)$ is isometric to a cylinder $[-w,w]\times\mathbb{S}^1$ with the Riemannian metric

$$ds^2 = d\rho^2 + \ell^2 \cosh^2(\rho)\, dt^2,$$

where ℓ is the length of γ, we call this collar *standard*. The collar lemma (see [Kee74] and [Bus10, Chapter 4]) states that the collar of width $w(\ell) = \operatorname{arcsinh}(1/\sinh(\ell/2))$ is always standard. Note that $w(\ell) \sim e^{-\ell/2}$ as $\ell \to \infty$. For our arguments, we need to work with (long) geodesics that admit standard collars of uniform width.

We use the following bound, which is the analogue of [Mag24a, Proposition 8.1] in our setting.

PROPOSITION 5.3. *For all* $w > 0$ *there exists a constant* $\varepsilon > 0$ *such that the following statement holds. Let* X *be a closed hyperbolic surface and let* $\gamma_1, \dots, \gamma_k \subset X$ *be simple closed geodesics of length* ℓ *such that the collars of width* $w > 0$ *around* $\gamma_1, \dots, \gamma_k$ *are standard and disjoint from each other. Moreover, let* $f : X \to \mathbb{R}$ *be an* L^2*-normalized Laplacian eigenfunction of eigenvalue* $\lambda > 0$. *Suppose, furthermore, that*[3]

$$\|f\|_\infty^2 \cdot \ell < \varepsilon.$$

Then there exists a smooth function $f_1 : X \to \mathbb{R}$ *such that:*

– f_1 *restricts to the same constant on every* $C_{w/2}(\gamma_j)$;
– f_1 *has zero average on* X;
– *we have that*

$$\frac{\langle \Delta f_1, f_1\rangle}{\langle f_1, f_1\rangle} - \lambda \le B \cdot \|f\|_\infty^2 \cdot \ell \cdot k,$$

where $B = B(w) > 0$ *is a constant depending only on* w.

Proof. We work with one curve (i.e. $k = 1$). Because we assume disjointness, the argument in the general case is the same, except that the final errors multiply by k.

Once and for all, fix a symmetric smooth function $J_0 : \mathbb{R} \to [0,1]$ such that

$$J_0(x) = \begin{cases} 1 & \text{if } |x| \ge 2, \\ 0 & \text{if } |x| \le 1. \end{cases}$$

Using this function, we define $J : C_w(\gamma) \to \mathbb{R}$ by

$$J(x) = J_0\left(\frac{2\cdot \mathrm{d}(x,\gamma)}{w}\right)$$

[3]This is a technical condition we set to make the bound come out nicer. The regime we will be interested in is when this quantity tends to 0 in any event.

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and define $\tilde{f}: X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} J(x) \cdot f(x) & \text{if } x \in C_w(\gamma), \\ f(x) & \text{otherwise.} \end{cases}$$

Finally, we define $f_1: X \to \mathbb{R}$ by

$$f_1 = \tilde{f} - \frac{1}{\mathrm{Vol(X)}} \int_X \tilde{f} = \tilde{f} - \mathrm{av}(\tilde{f}).$$

Here f_1 satisfies the first two parts of the claim, so we need to verify the third part.

We now first estimate the numerator of the Rayleigh quotient. Since $\langle \Delta f_1, f_1 \rangle = \langle \Delta \tilde{f}, \tilde{f} \rangle$, we obtain

$$\begin{aligned} \langle \Delta f_1, f_1 \rangle - \langle \Delta f, f \rangle &= \int_{C_w(\gamma)} (\Delta Jf) \cdot Jf - (\Delta f) \cdot f \\ &= \int_{C_w(\gamma)} (|\nabla J|^2 + |\nabla(\sqrt{1-J^2})|^2) \cdot f^2 \\ &\quad - \int_{C_w(\gamma)} \Delta(\sqrt{1-J^2} f) \cdot \sqrt{1-J^2} f \\ &\leq \int_{C_w(\gamma)} (|\nabla J|^2 + |\nabla(\sqrt{1-J^2})|^2) \cdot f^2 \\ &\leq \|f\|_\infty^2 \cdot \int_{C_w(\gamma)} (|\nabla J|^2 + |\nabla(\sqrt{1-J^2})|^2). \end{aligned}$$

Here we applied Theorem 5.2 to pass from the first to the second line and used the fact that Δ is a positive operator in order to pass from the second to the third line.

Now, using the fact that $C_w(\gamma)$ is standard and that the function J depends on the distance to the core curve only, we get that

$$\begin{aligned} &\int_{C_w(\gamma)} (|\nabla J|^2 + |\nabla(\sqrt{1-J^2})|^2) \\ &= \ell \cdot \int_{-w}^{w} \left[\left(\frac{d}{d\rho} J_0\left(\frac{2\rho}{w}\right) \right)^2 + \left(\frac{d}{d\rho} \sqrt{1 - J_0\left(\frac{2\rho}{w}\right)^2} \right)^2 \right] \cdot \cosh(\rho)\, d\rho \\ &= \frac{\ell}{w} \int_{-1}^{1} \left[\left(\frac{dJ_0(2y)}{dy} \right)^2 + \left(\frac{d}{dy} \sqrt{1 - J_0(2y)^2} \right)^2 \right] \cdot \cosh(w \cdot y)\, dy \\ &\leq A_1 \cdot \frac{\ell \cdot \cosh(w)}{w}, \end{aligned}$$

where $A_1 > 0$ is a constant depending on our choice of J_0 alone. The division by w in the penultimate line comes from the combination of the chain rule and the substitution rule. All in all, we obtain

$$\langle \Delta f_1, f_1 \rangle \leq \langle \Delta f, f \rangle + A_1 \cdot \|f\|_\infty^2 \cdot \frac{\ell \cdot \cosh(w)}{w}.$$

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Likewise, for the denominator, we have

$$\begin{aligned}\langle f_1, f_1\rangle &= \langle \tilde{f}, \tilde{f}\rangle - \mathrm{Vol}(X)\mathrm{av}(\tilde{f})^2 \\ &= \langle f, f\rangle - \int_{C_w(\gamma)} (1-J^2)f^2 - \frac{1}{\mathrm{Vol}(X)}\left(\int_{C_w(\gamma)} (J-1)f\right)^2 \\ &\geq 1 - \left(1 + \frac{1}{\mathrm{Vol}(X)}\right)\mathrm{Vol}(C_w(\gamma))\cdot\|f\|_\infty^2 \geq 1 - 4\cdot\|f\|_\infty^2\cdot\ell\cdot\sinh(w),\end{aligned}$$

where we have used that $\mathrm{Vol}(C_w(\gamma)) = 2\ell(\gamma)\sinh(w)$.

This means that, setting $E = A_2\cdot\|f\|_\infty^2\cdot\ell$ (where $A_2 = A_2(w) > 0$ is large enough to absorb the previous constants and the factors depending on w),

$$\frac{\langle \Delta f_1, f_1\rangle}{\langle f_1, f_1\rangle} \leq \frac{\langle \Delta f, f\rangle + E}{\langle f, f\rangle - E} = \frac{\lambda + E}{1-E} \leq \lambda + A_3\cdot E,$$

for some constant $A_3 > 0$, where we have used the geometric series around $E = 0$ and our assumption that $\|f\|_\infty^2\cdot\ell < \varepsilon$. Here we can choose any $\varepsilon \in (0, 1/A_2)$. □

6. Building good surfaces

The goal of this section is to prove the main result of this paper: we can produce sequences of covers of the Bolza surface whose spectral gaps are dense in $[0, \frac{1}{4}]$. We use a variant of the switching technique from [Mag24a], based on the model of random surfaces described in § 2.

6.1 The order of constants

There are many constants involved in our construction and, even though they are uniform, the order in which they are fixed matters. So we will record this order here.

C1 We fix a constant $\eta > 0$. Our goal will be to build a set of covers whose spectral gaps are η-dense in $[0, \frac{1}{4}]$.

C2 We also once and for all fix a parameter $w > 0$ (independent of η). This number appears as a collar width in Proposition 5.3 and determines another parameter $\varepsilon > 0$ and a third number $B = B(w)$ that both appear in that same proposition.

C3 Given all the parameters above, we fix any $\ell > 4w$ such that

$$\frac{1}{4}\cdot\left(1 - \frac{2\pi}{\ell + 2\pi}\right)^2 > \frac{1}{4} - \frac{1}{2}\cdot\eta \quad \text{and} \quad \ell\cdot e^{-\ell\cdot\sqrt{\eta}} < \min\left\{\frac{\varepsilon}{A}, \frac{\eta}{2AB}\right\},$$

where $B = B(w)$ is the constant that appears in Proposition 5.3 and A the constant that appears in Theorem 5.1.

6.2 Building a good tree cover

From now on the parameters chosen in the previous subsection will be considered fixed.

We now first prove that we can find good tree covers.

PROPOSITION 6.1. *The Bolza surface admits a tree cover $T_\phi \to X_B$, coming from a homeomorphism $\phi : X_B \to X_B$ such that the following conditions hold:*

(1) $\mathrm{sys}(T_\phi) > \ell$;

(2) *the collars of width w around all geodesics in the canonical pants decomposition of T_ϕ are all standard;*

(3) $\lambda_1(T_\phi) > \frac{1}{4} - \frac{1}{2}\cdot\eta$.

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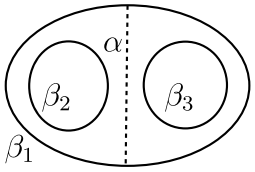


Figure 2. A pair of pants.

Proof. First of all, Mirzakhani [Mir16, Theorem 1.2] determined the asymptotic distribution of the normalized length vector of a random pants decomposition $\mathcal{P}_L$ sampled uniformly at random from

$$\left\{\mathcal{P} \in \mathrm{MCG}(X_B)\cdot\mathcal{P}_0 : \sum_{\alpha\in\mathcal{P}} \ell_\alpha(X_B) \le L\right\},$$

where $\mathrm{MCG}(X_B)$ denotes the mapping class group of X_B. The normalized length vector of a pants decomposition is the vector

$$\bar{\ell}(\mathcal{P}_L) = \left(\ell_\alpha(X_B)/\sum_{\beta\in\mathcal{P}_L}\ell_\beta(X_B)\right)_{\alpha\in\mathcal{P}_L} \in \sigma_3 = \left\{x\in\mathbb{R}^3 : \begin{array}{c} x_i\ge 0,\ i=1,2,3 \\ \text{and } x_1+x_2+x_3=1\end{array}\right\}.$$

Mirzakhani proved that, as $L\to\infty$, this random vector converges in distribution to a random vector in σ_3 distributed according to a measure that can be explicitly computed. This was generalized to other multi-curves by Arana and Herrera [Ara22] and Liu [Liu22].

For us, the only thing that matters is that the limit measure dominates the Lebesgue measure on the interior of σ_3; that is, it assigns positive mass to measurable sets that have positive Lebesgue mass. This implies that for all $L>0$ and all $\delta>0$, we can find a homeomorphism $\phi : X_B \to X_B$ such that the pants decomposition $\phi^{-1}(\mathcal{P}_0) = (\beta_1,\beta_2,\beta_3)$ satisfies

$$\ell(\beta_i)\ge L \quad\text{and}\quad 1-\delta < \frac{\ell(\beta_i)}{\ell(\beta_j)} < 1+\delta \quad \text{for all } i,j\in\{1,2,3\}.$$

We claim that the tree cover $T_\phi \longrightarrow X_B$ has the properties we require, as soon as $L > \ell/(1-\delta)$.

We start with item (1): the systole. First observe that, if the systole corresponds to a pants curve in the canonical pants decomposition of T_ϕ, then it automatically satisfies the inequality above. If it does not, the geodesic realizing it contains at least two arcs that each run between two points on one of the boundary components of one of the pairs of pants in the pants decomposition. Indeed, the systole is necessarily simple, so it crosses at least two pairs of pants. As such the path it traces in the tree dual to the canonical pants decomposition has at least two leaves. So the length of the systole is at least twice the length of such an arc. All that remains is to find a lower bound on this length.

To do this, consider the arc α in Figure 2. The arc α cuts the boundary component β_1 into two arcs, the shortest one of which has length at most $(\ell(\beta_1))/2 \le ((1+\delta)/2)\ell(\beta_2)$. If we suppose this shortest arc lies on the side of β_2, we obtain

$$\ell(\alpha) + \frac{1+\delta}{2}\ell(\beta_2) \ge \ell(\beta_2),$$

which gives us $\ell(\alpha)\ge((1-\delta)/2)\ell(\beta_2)\ge((1-\delta)/2)L$ and hence

$$\mathrm{sys}(T_\phi)\ge(1-\delta)\cdot L > \ell.$$

We prove item (2) using essentially the same argument. Indeed, if a collar of width u around one of the pants curves is not standard, then this yields a simple arc of length at most $2u$ between

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two points on the corresponding pants curve. This arc must be essential relative to its endpoints and, by the same argument as above, such an arc has length at least $\ell/2$. So, the collars of width $\ell/4 > w$ (by assumption **C3**) are standard.

Finally, item (3) follows from the first part of the same assumption **C3**, combined with Proposition 4.2. □

6.3 Strong convergence

To obtain covers of X_B whose spectral gap is nearly optimal we will rely on the techniques developed by Hide and Magee [HM23, LMH25] based on strong convergence of permutation representations.

Let Γ be a finitely generated group and suppose $(\mathcal{H}_n)_n$ is a sequence of Hilbert spaces. Given a sequence of unitary representations $\rho_n : \Gamma \to U(\mathcal{H}_n)$, where $U(\mathcal{H}_n)$ denotes the unitary group of $\mathcal{H}_n$, we say that this sequence *strongly converges* to $\rho_\infty : \Gamma \to U(\mathcal{H}_\infty)$ if

$$\lim_{n\to\infty} \|\rho_n(z)\| = \|\rho_\infty(z)\|$$

for all $z \in \mathbb{C}[\Gamma]$.

We will need the following result which is a version of the results in [Mag24b], specialized to our set-up.

THEOREM 6.2. *Let $\Gamma < PSL(2,\mathbb{R})$ be a uniform lattice and let Λ be a group. Moreover, suppose that:*

- *$p : \Gamma \to \Lambda$ is an epimorphism and write $K = \ker(p) \lhd \Gamma$;*
- *$(\rho_n \in \mathrm{Hom}(\Lambda, \mathfrak{S}_n))_{n\geq 1}$ is a sequence of permutation representations such that*

$$\mathrm{std}_n \circ \rho_n \overset{\text{strong}}{\longrightarrow} \rho_{\mathrm{reg}} \quad \text{as } n \to \infty,$$

  where $\mathrm{std}_n : \mathfrak{S}_n \to U(\mathbb{C}^{n-1})$ denotes the $(n-1)$-dimensional irreducible representation and $\rho_{\mathrm{reg}} : \Lambda \to \mathcal{B}(\ell^2(\Lambda))$ denotes the left regular representation; and
- *we can write $\Gamma_n = \mathrm{Stab}_{\rho_n \circ p}(\{1\}) < \Gamma$.*

Then

$$\lim_{n\to\infty} \lambda_1(\Gamma_n \backslash \mathbb{H}^2) = \min\{\lambda_1(K\backslash\mathbb{H}^2), \lambda_1(\Gamma\backslash\mathbb{H}^2)\}.$$

Proof sketch. This statement can be derived using small modifications of the arguments in [HM23, LMH25, MT26]. A representation-theoretic argument due to Magee [Mag24b] is perhaps the fastest way to obtain it. Indeed, our statement is a special case of [Mag24b, Theorem 3.1], which is phrased in a different language. We will very briefly describe how this works here.

The fact that $\mathrm{std} \circ \rho_n \overset{\text{strong}}{\longrightarrow} \rho_{\mathrm{reg}}$, implies that

$$\mathrm{std}_n \circ \rho_n \circ p \overset{\text{strong}}{\longrightarrow} \rho_{\mathrm{reg}} \circ p : \Gamma \to \mathcal{B}(\ell^2(\Lambda)).$$

Magee [Mag24b, Theorem 3.1] proves that this implies that

$$\mathrm{Ind}_\Gamma^{\mathrm{PSL}(2,\mathbb{R})}(\mathrm{std}_n \circ \rho_n \circ p) \overset{\text{strong}}{\longrightarrow} \mathrm{Ind}_\Gamma^{\mathrm{PSL}(2,\mathbb{R})}(\rho_{\mathrm{reg}} \circ p).$$

By [Mag24b, Proposition 2.9], this implies our statement. □

Strongly convergent sequences of representations are notoriously hard to find. However, the following theorem by Bordenave and Collins [BC19] gives us what we need.

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Theorem 6.3 (Bordenave–Collins). *Let F be a finitely generated non-abelian free group and let $\rho_n \in \mathrm{Hom}(F, \mathfrak{S}_n)$ be chosen uniformly at random. Then*

$$\mathrm{std}_n \circ \rho_n \xrightarrow{\text{strong}} \rho_{\mathrm{reg}} \quad \text{as } n \to \infty$$

in probability.

6.4 Covers of degree 2 and Hamming distance

Using good tree covers, combined with the existence of strongly convergent sequences of permutation representations of F_2 and Theorem 6.2 above, we can already use our construction to build sequences of finite covers of X_B whose spectral gaps are arbitrarily close to $\frac{1}{4}$. As such, we need to explain how to get intermediate spectral gaps.

As mentioned in § 1.1 of the proof strategy in the introduction, we will use a variation of the switching argument of Magee in order to do this. We will fix a homeomorphism $\phi : X_B \to X_B$ such that T_ϕ has satisfies the result from Proposition 6.1. This gives us a sequence $(Y_\phi^{(n)} \to X_B)_{n \geq 1}$ of random degree n covers (see § 2.3). The spectral gaps of $Y_\phi^{(n)}$ converge to that of T_ϕ in probability. We will further condition on $Y_\phi^{(n)}$ not having any short (to be quantified below) curves. We claim that this suffices for the set of spectral gaps of the degree 2 covers[4] $\widehat{Y} \to Y_\phi^{(n)}$ to become arbitrarily dense in $[0, \lambda_0(T_\phi)]$. The rest of this text is devoted to working this out.

First of all, we will restrict to covers that factor through the handlebody attachment. That is, the groups defined in § 2.3 fit in the commutative diagram

$$\begin{array}{ccc} \Lambda_\phi^{(n)} & \hookrightarrow & \Gamma_B \\ \downarrow{\scriptstyle (\iota\circ\phi)_*} & & \downarrow{\scriptstyle (\iota\circ\phi)_*} \\ H^{(n)} & \hookrightarrow & F_2 \end{array}$$

which allows us to define the set

$$\mathcal{H}_\phi^{(n)} = \{\rho \circ (\iota \circ \phi)_* : \rho \in \mathrm{Hom}(H^{(n)}, \mathbb{Z}/2\mathbb{Z})\}.$$

Given a free generating set $S = (s_1, \ldots, s_k)$ of $H^{(n)}$, we can identify the set of maps $\mathcal{H}_\phi^{(n)}$ with $(\mathbb{Z}/2\mathbb{Z})^k$ through the map that associates $(\rho(s_1), \ldots, \rho(s_k))$ to $\rho \circ (\iota \circ \phi)_* \in \mathcal{H}_\phi^{(n)}$. This in turn allows us to define a *Hamming distance* $\mathrm{d}_S : \mathcal{H}_\phi^{(n)} \times \mathcal{H}_\phi^{(n)} \to \mathbb{N}$ by

$$\mathrm{d}_S(\rho_1 \circ (\iota \circ \phi)_*,\ \rho_2 \circ (\iota \circ \phi)_*) = |\{j : \rho_1(s_j) \neq \rho_2(s_j)\}|.$$

We note that every element $\rho \in \mathcal{H}_\phi^{(n)}$ corresponds to a cover of $Y_\phi^{(n)}$. If the image of ρ is trivial, we will let this cover be the cover of $Y_\phi^{(n)}$ by two disjoint copies of itself. This way, all the corresponding covers are of degree 2.

6.5 Switching

In this section we describe how two covers of degree 2 of $Y_\phi^{(n)}$ whose Hamming distance is 1 differ from each other.

[4] Magee phrases the switching argument in terms of bundles over $Y_\phi^{(n)}$ coming from homomorphisms $\pi_1(Y_\phi^{(n)}) \to \mathbb{Z}/2\mathbb{Z}$, which is an equivalent point of view.

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To make this precise, we need to first describe a free generating set of $H^{(n)}$. One way to do this is to identify F_2 with the fundamental group of a graph $W^{(1)}$. We will let $W^{(1)}$ be a wedge of two circles. Because the pants curves in $\mathcal{P}_0$ are meridians, we can set everything up in such a way that the two edges of $W^{(1)}$ are dual to two of the pants curves in the pants decomposition $\mathcal{P}_0$ we have fixed. If we change the handlebody attachment by a homeomorphism $\phi : X_B \to X_B$, this changes the identification. That is, we can think of the edges of $W^{(1)}$ as corresponding to two of the three pants curves in $\phi^{-1}(\mathcal{P}_0)$. Here $H^{(n)}$ is the fundamental group of a graph $W^{(n)}$ that covers $W^{(1)}$. The edges in this graph are dual to the canonical pants decomposition of $Y_\phi^{(n)}$.

In order to obtain a preferred free generating set of $H^{(n)}$, we pick an arbitrary spanning tree in $W^{(n)}$ and contract it. The edges that do not appear in the tree yield a generating set $S^{(n)} = (s_1, \dots, s_k)$ of $H^{(n)}$. Because of the correspondence between edges and pants curves described above, each of these generators is dual to a pants curve in the canonical pants decomposition of $Y_\phi^{(n)}$. We will denote these preferred pants curves by $(\alpha_1^{(n)}, \dots, \alpha_k^{(n)})$.

Given a map $\rho : H^{(n)} \to \mathbb{Z}/2\mathbb{Z}$, the corresponding cover of degree 2 of $Y_\phi^{(n)}$ can be built as follows.

- Write $Z_\phi^{(n)} = \overline{Y_\phi^{(n)} - \bigcup_{i=1}^k \alpha_i}$, where $\overline{X}$ denotes the completion of a hyperbolic surface X, so $Z_\phi^{(n)}$ is a surface with boundary.
- Take two disjoint copies Z_1 and Z_2 of $Z_\phi^{(n)}$.
- Then, for each pants curve α_i:
  - if $\rho(s_i) = [0]$, glue the two boundary components of Z_1 corresponding to α_i back together and do the same for those in Z_2;
  - if $\rho(s_i) = [1]$, glue the 'left' boundary component of Z_1 corresponding to α_i to the 'right' boundary component of Z_2 and, likewise, glue the corresponding right boundary component of Z_1 to the left boundary component of Z_2.

  All these gluings above should be performed with the correct twist, so as to guarantee that the resulting surface covers $Y_\phi^{(n)}$.

It follows from this description that if

$$\mathrm{d}_{S^{(n)}}(\rho_1 \circ (i \circ \phi)_*,\ \rho_2 \circ (i \circ \phi)_*) = 1,$$

then the two corresponding covers can be obtained from each other by opening up the lifts corresponding to one of the preferred pants curves $\alpha_i^{(n)} \subset Y_\phi^{(n)}$ and gluing them back together with the opposite pattern. We will call this operation a *switch*; this same terminology is used in graph theory for what happens at the level of the 2-covers of the graph $W^{(n)}$. Figure 3 shows a local picture of this move.

6.6 Finishing the proof

We now have everything we need in place to finish the proof of our main theorem. Indeed, Theorem 1 is implied directly by the following result.

Theorem 6.4. *For all $n \in \mathbb{N}$ large enough, the probability that the set*

$$\{\lambda_1(\widehat{Y}) : \widehat{Y} \text{ is a connected cover of } Y_\phi^{(n)} \text{ of degree } 2\}$$

is η-dense in $[0, \frac{1}{4}]$, is strictly positive.

Proof. First of all, we use Proposition 6.1 to choose a homeomorphism $\phi : X_B \to X_B$ that gives us a tree cover with the properties in that same proposition.

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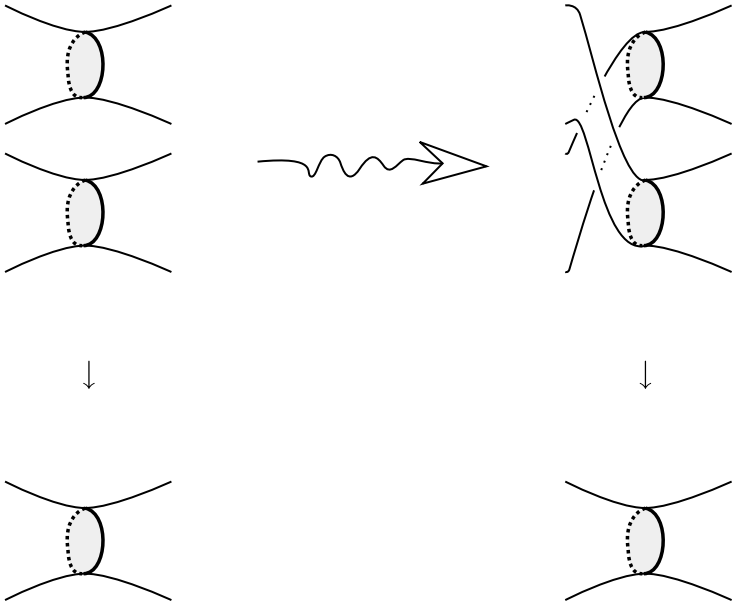


Figure 3. A local picture of a switch. The surface at the bottom represents a neighbourhood of one of the preferred pants curves $\alpha_i^{(n)}$ of $Y_\phi^{(n)}$. The surfaces at the top (one on the left and one on the right) are the neighbourhoods of the lifts of $\alpha_i^{(n)}$ in two distinct covers of degree 2. The shading is there to indicate that the curves that are involved are meridians.

It now follows from Theorems 6.2 and 6.3 that $\lambda_1(Y_\phi^{(n)}) > \frac{1}{4} - \eta$ with high probability. In fact, using the same argument as in [Mag24a, § 4],

$$\theta \otimes \rho_n \overset{\text{strong}}{\longrightarrow} \rho_{\text{reg}}$$

in probability as $n \to \infty$, for any $\theta \in \text{Hom}(F_2, \mathbb{Z}/2\mathbb{Z})$. For us this means that, again using Theorem 6.2, we obtain covers of degree 2, corresponding to some element in $\mathcal{H}_\phi^{(n)}$, of $Y_\phi^{(n)}$ whose spectral gap is at least

$$\lambda_1(T_\phi) - \eta/2 > \tfrac{1}{4} - \eta$$

for n large enough.

The set $\mathcal{H}_\phi^{(n)}$ also contains the trivial homomorphism, the corresponding degree 2 cover of which has $\lambda_1(\widehat{Y}) = 0$.

We can go between these covers using a finite number of switches, so as soon as we prove that a single switch does not change the spectral gap by more than η, we are done.

In order to do this, we need to make one further assumption on the surfaces $Y^{(n)}$: the preferred pants curves in $Y^{(n)}$ need to have collar width at least w and the systole of $Y^{(n)}$ needs to be at least ℓ.

Both of these statements are true in the tree cover T_ϕ. This means, that if they are not true in $Y^{(n)}$, there needs to be a closed geodesic $\gamma \subset X_B$ of uniformly bounded length $L = L(w, \ell)$ such that

$$\iota_*(\gamma) \neq e \quad \text{but} \quad \rho_n \circ (\iota \circ \phi)_*(\gamma) \cdot 1 = 1. \tag{6.1}$$

It follows from Nica's work [Nic94] that, with asymptotically strictly positive probability, $\rho_n(g)$ is fixed-point-free for all g in any fixed finite set of non-trivial conjugacy classes in F_2.

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The statement we use here is as follows. First of all, let $K_1, \dots K_m \subset F_2$ be a finite collection of distinct non-trivial conjugacy classes, none of which contain a power of a non-trivial element of F_2. Moreover, let $(c_1, \dots, c_k)$ be a finite set of cycle lengths. Then the vector of random variables $(X_{c_i}(K_j))_{1\leq i\leq m,\, 1\leq j\leq k}$, where $X_{c_i}(K_j)$ denotes the number of c_i-cycles in the image of an element of K_j under a uniformly random homomorphism $\rho \in \mathrm{Hom}(F_2, \mathfrak{S}_n)$, converges in distribution to a vector of independent random variables. Furthermore, the limit of $X_{c_i}(K_j)$ is Poisson$(1/c_i)$-distributed. This statement is not literally contained in Nica's paper, but it can be proven using similar methods. Versions of this statement are also available in [LP10, § 4], [BP23, Theorem 1.4] and [PZ24, Corollary 1.7 and Theorem 1.14]. The only consequence we need is that there exists a uniform $a > 0$ such that for all n large enough,

$$\mathbb{P}(X_{c_i}(K_j) = 0 \text{ for } i = 1, \dots, k \text{ and } j = 1, \dots, m) > a.$$

Also observe that if $g = u^k \in F_2$ and if, under a permutation representation of F_2, u has no d-cycles for any divisor d of k, then g has no fixed points.

Since Γ_B contains a finite number of conjugacy classes of elements of bounded translation length, we can avoid the existence of a closed geodesic $\gamma \subset X_B$ as in (6.1) with strictly positive probability. Moreover, the existence of a 2-cover with spectral gap above $\frac{1}{4} - \eta$ has probability tending to 1 as $n \to \infty$, so this property is preserved when we restrict to covers without short geodesics that do not lift to T_ϕ. In other words, the probability that $Y_\phi^{(n)}$ has systole at least ℓ, all the curves in the canonical pants decomposition have standard collars of width w and $Y_\phi^{(n)}$ has a 2-cover with spectral gap at least $\frac{1}{4} - \eta$ is asymptotically strictly positive.

So now suppose $\widehat{Y}_1$ and $\widehat{Y}_2$ are two covers of degree 2 of $Y_\phi^{(n)}$ that differ by a switch move. Moreover, let f be a $\lambda_1(\widehat{Y}_1)$-eigenfunction on $\widehat{Y}_1$ and let α_i be the curve in which we need to switch to obtain $\widehat{Y}_2$. Using Proposition 5.3, there exists a smooth function f_1 on $\widehat{Y}_1$ of zero average that is constant on a neighbourhood of both lifts of α_i and

$$\frac{\langle \Delta f_1, f_1\rangle}{\langle f_1, f_1\rangle} - \lambda_1(\widehat{Y}_1) \leq 2B\ell \|f\|_\infty^2 \leq 2AB\ell e^{-\ell\sqrt{\eta}} \leq \eta.$$

Here we have used Theorem 5.1 together with the assumption on the collar widths and the systole of $Y_\phi^{(n)}$ and the relations between our constants (§ 6.1). Because f_1 attains the same constant value on neighbourhoods of the lifts of α_i, it defines a function of zero average on the cover $\widehat{Y}_2$ and hence

$$\lambda_1(\widehat{Y}_2) \leq \lambda_1(\widehat{Y}_1) + \eta,$$

by the variational characterization of λ_1, which concludes the proof. □

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Conflicts of interest

None.

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Data availability

The code for this article, which is used in the proof of Proposition 3.2, is available as supplementary material.

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Supplementary material

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References

AM99 C. Adams and F. Morgan, *Isoperimetric curves on hyperbolic surfaces*, Proc. Amer. Math. Soc. **127** (1999), 1347–1356.

AW23 N. Alon and F. Wei, *The limit points of the top and bottom eigenvalues of regular graphs*, Preprint (2023), arXiv:2304.01281.

Ara22 F. Arana-Herrera, *Counting hyperbolic multigeodesics with respect to the lengths of individual components and asymptotics of Weil-Petersson volumes*, Geom. Topol. **26** (2022), 1291–1347.

BP23 E. Baker and B. Petri, *Statistics of finite degree covers of torus knot complements*, Ann. H. Lebesgue **6** (2023), 1213–1257.

Bav96 C. Bavard, *Disques extrémaux et surfaces modulaires*, Ann. Fac. Sci. Toulouse Math. (6) **5** (1996), 191–202.

Ber16 N. Bergeron, *The spectrum of hyperbolic surfaces*, Universitext (Springer, Cham; EDP Sciences, Les Ulis, 2016), Appendix C by Valentin Blomer and Farrell Brumley, translated from the 2011 French original by Brumley; MR 2857626.

Bon22 J. Bonifacio, *Bootstrapping closed hyperbolic surfaces*, J. High Energy Phys. **93** (2022), 093.

BC19 C. Bordenave and B. Collins, *Eigenvalues of random lifts and polynomials of random permutation matrices*, Ann. of Math. (2) **190** (2019), 811–875.

BCP21 T. Budzinski, N. Curien and B. Petri, *On the minimal diameter of closed hyperbolic surfaces*, Duke Math. J. **170** (2021), 365–377.

Bus77 P. Buser, *Riemannsche Flächen mit Eigenwerten in* (0, 1/4), Comment. Math. Helv. **52** (1977), 25–34.

Bus10 P. Buser, *Geometry and spectra of compact Riemann surfaces*, Modern Birkhäuser Classics (Birkhäuser, Boston, 2010), reprint of the 1992 edition.

https://doi.org/10.1017/S0010437X26103340 Published online by Cambridge University Press

Cha84 I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115 (Academic Press, Orlando, FL, 1984), including a chapter by Burton Randol, with an appendix by Jozef Dodziuk.

Che70 J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, in *Problems in analysis: a symposium in honor of Salomon Bochner* (Princeton University Press, Princeton, NJ, 1970), 195–199.

Che75 S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975), 289–297.

Con15 M. D. E. Conder, *Quotients of triangle groups acting on surfaces of genus 2 to 101* (2015), math.auckland.ac.nz/∼conder/TriangleGroupQuotients101.txt.

CP23 G. Cornelissen and N. Peyerimhoff, *Twisted isospectrality, homological wideness, and isometry—a sample of algebraic methods in isospectrality*, SpringerBriefs in Mathematics (Springer, Cham, 2023).

CFK+87 H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Texts and Monographs in Physics, study edition (Springer-Verlag, Berlin, 1987).

Dix69 J. D. Dixon, *The probability of generating the symmetric group*, Math. Z. **110** (1969), 199–205.

DM26 D. Dong and T. McKenzie, *Limit vectors of the top k eigenvalues of d-regular graphs*, Adv. Math. **499** (2026), 111047.

FGP+23 M. Fortier Bourque, É. Gruda-Mediavilla, B. Petri and M. Pineault, *Two counterexamples to a conjecture of Colin de Verdière on multiplicity*, Preprint (2023), arXiv:2312.03504.

FP23 M. Fortier Bourque and B. Petri, *Linear programming bounds for hyperbolic surfaces*, Preprint (2023), arXiv:2302.02540.

Gag80 M. E. Gage, *Upper bounds for the first eigenvalue of the Laplace-Beltrami operator*, Indiana Univ. Math. J. **29** (1980), 897–912.

GAP22 The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.12.2* (2022).

GLS+21 C. Gilmore, E. Le Masson, T. Sahlsten and J. Thomas, *Short geodesic loops and L^p norms of eigenfunctions on large genus random surfaces*, Geom. Funct. Anal. **31** (2021), 62–110.

HM23 W. Hide and M. Magee, *Near optimal spectral gaps for hyperbolic surfaces*, Ann. of Math. (2) **198** (2023), 791–824.

Hub74 H. Huber, *Über den ersten Eigenwert des Laplace-Operators auf kompakten Riemannschen Flächen*, Comment. Math. Helv. **49** (1974), 251–259.

Jen81 F. Jenni, *Über das Spektrum des Laplace-Operators auf einer Schar konmpakter Riemannscher Flächen*, PhD thesis, University of Basel (1981).

Joh17 F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), 1281–1292.

Kee74 L. Keen, *Collars on Riemann surfaces*, In *Discontinuous groups and Riemann surfaces*, Proc. Conf., University of Maryland, College Park, MD, 1973, Annals of Mathematics Studies, vol. 79 (Princeton University Press, Princeton, NJ, 1974), 263–268.

KMP24 P. Kravchuk, D. Mazáč and S. Pal, *Automorphic spectra and the conformal bootstrap*, Commun. Am. Math. Soc. **4** (2024), 1–63.

LP10 N. Linial and D. Puder, *Word maps and spectra of random graph lifts*, Random Structures Algorithms **37** (2010), 100–135.

https://doi.org/10.1017/S0010437X26103340 Published online by Cambridge University Press

Liu22 M. Liu, *Length statistics of random multicurves on closed hyperbolic surfaces*, Groups Geom. Dyn. **16** (2022), 437–459.

LMH25 L. Louder, M. Magee and W. Hide, *Strongly convergent unitary representations of limit groups*, J. Funct. Anal. **288** (2025), 110803.

MR03 C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol. 219 (Springer-Verlag, New York, 2003).

Mag24a M. Magee, *The limit points of the bass notes of arithmetic hyperbolic surfaces*, Preprint (2024), arXiv:2403.00928.

Mag24b M. Magee, *Strong convergence of unitary and permutation representations of discrete groups*, to appear (2024).

MNP22 M. Magee, F. Naud and D. Puder, *A random cover of a compact hyperbolic surface has relative spectral gap* $\frac{3}{16} - \varepsilon$, Geom. Funct. Anal. **32** (2022), 595–661.

MP23 M. Magee and D. Puder, *The asymptotic statistics of random covering surfaces*, Forum Math. Pi **11** (2023), e15.

MT26 M. Magee and J. Thomas, *Strongly convergent unitary representations of right-angled Artin groups*, Duke Math. J. **175** (2026), 1305–1343.

Mat24 J. Mathien, *Diameter of a new model of random hyperbolic surfaces*, Preprint (2024), arXiv:2403.01925.

Mat05 K. Matsuzaki, *Isoperimetric constants for conservative Fuchsian groups*, Kodai Math. J. **28** (2005), 292–300.

Mir13 M. Mirzakhani, *Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus*, J. Differential Geom. **94** (2013), 267–300.

Mir16 M. Mirzakhani, *Counting mapping class group orbits on hyperbolic surfaces*, Preprint (2016), arXiv:1601.03342.

Nic94 A. Nica, *On the number of cycles of given length of a free word in several random permutations*, Random Structures Algorithms **5** (1994), 703–730.

PZ24 D. Puder and T. Zimhoni, *Local statistics of random permutations from free products*, Int. Math. Res. Not. IMRN **2024** (2024), 4242–4300.

Sag21 The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 9.3)* (2021), https://www.sagemath.org.

Sar23 P. Sarnak, *Spectra of locally uniform geometries*, Chern Lectures (University of California, Berkeley, 2023), http://publications.ias.edu/sarnak/paper/2728.

SU13 A. Strohmaier and V. Uski, *An algorithm for the computation of eigenvalues, spectral zeta functions and zeta-determinants on hyperbolic surfaces*, Comm. Math. Phys. **317** (2013), 827–869.

Tak77 K. Takeuchi, *Arithmetic triangle groups*, J. Math. Soc. Japan **29** (1977), 91–106.

William Hide william.hide@maths.ox.ac.uk
Mathematical Institute, University of Oxford, Andrew Wiles Building, Oxford OX2 6GG, UK

Bram Petri bram.petri@imj-prg.fr
Institut de Mathématiques de Jussieu–Paris Rive Gauche; Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France

and

Institut Universitaire de France, France

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