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# *Demand analysis with partial predicates*

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**Abstract**

To alleviate the inefficiencies caused by the interaction of the logic and functional sides, integrated languages may take advantage of *demand* information, i.e. knowing in advance which computations are needed and, to which extent, in a particular context. This work studies *demand analysis* – which is closely related to *backwards strictness analysis* – in a semantic framework of *partial predicates*, which in turn are constructive realizations of ideals in a domain. This will allow us to give a concise, unified presentation of demand analysis, to relate it to other analyses based on abstract interpretation or strictness logics, some hints for the implementation, and, more important, to prove the soundness of our analysis based on *demand equations*. There are also some innovative results. One of them is that a set constraint-based analysis has been derived in a stepwise manner using ideas taken from the area of program transformation. The other one is the possibility of using program transformation itself to perform the analysis, specially in those domains of properties where algorithms based on constraint solving are too weak.

*KEYWORDS*: functional-logic programming, demand analysis, strictness analysis, program transformation, abstract interpretation, set-constraint analysis

## 1 Introduction

Although the main idea of declarative programming is to use mathematical elements for programming, the area is split in two main paradigms based on the subset of mathematics they are focused on: functional programming (functions: lambda calculus) and logic programming (predicate logic). However, it is obvious that both paradigms have a common core and can be seen as different faces of a single idea.

*Functional-logic languages* aim at bringing together the advantages of functional programming and logic programming, see (Hanus 1994; Moreno-Navarro 1994), i.e., from functional programming they take higher-order features, polymorphic types, lazy evaluation, etc., while logic programming provides partial information, constraints, logical variables, search, etc. The language Curry (Hanus *et al*. 2003) is the de facto standard of functional-logic languages.

Probably, the combination of the last mentioned features of each paradigm (laziness, search) seems the more problematic to achieve in an efficient implementation. In other

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words, for executing a program it may be necessary to evaluate a functional-like expression containing uninstantiated (i.e. existentially quantified) logic variables.

The operational principle proposed for this situation is narrowing. Roughly speaking, narrowing guesses an instantiation for these variables. Functional nesting, nondeterminism, semantic unification, functional inversion, and lazy evaluation, which are key part for the expressiveness of functional-logic programs, are supported by this operational mechanism. In order to apply the general idea of narrowing to functional-logic languages, a functional-logic program is considered as a set of rewrite rules (plus some additional restrictions described later).

However, narrowing by itself is not enough: a brute-force approach to finding all the solutions would attempt to unify each rule with each nonvariable subterm of the given equation in every narrowing step. Even for small programs a huge search space would result. Therefore, an additional aspect to take into account in the implementation of functional-logic languages is the definition of an appropriate narrowing strategy. This strategy should be sound (i.e., only correct solutions are computed) and complete (i.e., all solutions or more general representatives of all solutions are computed).

Many narrowing strategies for limiting the size of the search space have been proposed, but we are interested on those with a lazy behaviour. To preserve completeness, see Moreno-Navarro and Rodríguez-Artalejo (1992), a lazy narrowing step is applied at outermost positions with the exception that inner arguments of a function are evaluated, by narrowing them to their head normal forms, if their values are required for an outermost narrowing step. This property can only be ensured by looking-ahead on the rules tried in following steps. Unfortunately, a potentially infinite number of substitutions could arise but, in the case of inductively sequential programs it is possible to compute the property in an efficient way. This is the idea of needed narrowing introduced in Antoy *et al*. (2000). The paper also proves completeness and optimality of the strategy.

But beyond this remarkable contribution to the implementation of functional-logic languages two problems remain: (i) the strategy is optimal with respect to the length of derivations but not in the size of the search space and additional improvements could be achieved, and (ii) it is defined only for a restricted class of programs (inductively sequential).

### *1.1 Demand analysis*

Our proposal is to use a static analysis to improve the look-ahead approach. The analysis is able to extract demand information from a functional-logic program. This information can be used to guide and improve needed narrowing (thus cutting the search space and avoiding reevaluations). Additionally, we have some other advantages: transform non-sequential programs into sequential ones, following the ideas of Mariño and Moreno-Navarro (2000) where strictness analysis is used; safe replacement of strict equality by unification; implementation of default rules (Moreno-Navarro 1996); improvement of the accuracy of groundness analysis; translation into Prolog (Mariño and Moreno-Navarro 1992; Mariño and Rey 1998), etc.

*Demand analysis* was introduced in Mariño and Moreno-Navarro (1992) and then used in Moreno-Navarro *et al*. (1993), Mariño and Herranz (1993) and Mariño *et al*. (1993) as a way to improve the compilation of functional-logic programs. The essential idea was to perform backwards strictness analysis. The proposed solution consisted in generating, for a

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```
not False = True
not True  = False

f x False = not x
f x True  = x

?- f (not True) (not y)
```

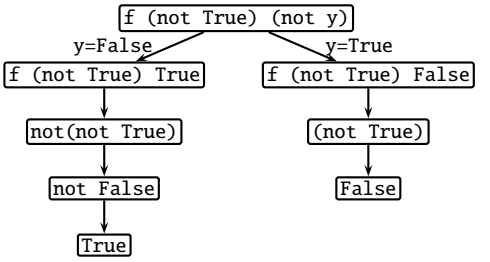



Fig. 1. Reevaluation example.

given program, a set of so called *demand equations* that were solved in a domain of regular trees (*demand patterns*). A strong point of demand patterns compared to existing strictness analyzers based on abstract interpretation was that they allowed, at least theoretically, inference in an infinite domain of properties.

The aim of this paper is to provide a semantic framework (*partial predicates*) for demand analysis, to prove the correctness of demand equations and to introduce a novel approach to the implementation of analyzers. In fact, the whole process is simplified and optimized: while it is easier to reason about demand analysis with partial predicates, the implementation is also simpler because we can reuse a lot of work already done in partial evaluation tools. Moreover, we claim that the method can be applied to different contexts with a similar success.

Partial predicates can be used to specify demand analysis as well as other classical analyses. One of the important point of this formalism is that it can be expressed by functional-logic programs. This has important consequences, allowing for reasoning about demand properties (for instance checking and inference of them) by using program transformation techniques.

Applications of demand analysis go beyond the usual applications of strictness analysis in functional programming, where strictness is used for trying to provide bigger computations that can be computed eagerly. Advances on this subject can have effects in getting a more efficient low level implementation, and in making easier and profitable the parallelization of programs. In functional-logic programming the gain in effiency means that some computations are not reevaluated or even that a wider class of programs can be used.

*The Reevaluation Problem.* There is an efficiency problem caused by the interaction of laziness and backtracking which are, in some sense, antagonistic in nature. The former tries to delay some computations while the second drives different computations through a tree of branching paths. The problem appears clear – if we place some computation beyond the branching point there exists the risk that the evaluation may be performed several times. This can be rather annoying, because laziness is intended to save work, not to waste it.

The toy example in Figure 1 shows how combining lazy evaluation and backtracking can lead to the repeated evaluation of delayed redexes. Observe that the redex `(not True)` is needed for the final result but, due to the outermost reduction strategy, its evaluation is delayed and evaluated twice, in different branches of the search tree. In this simple example, this is not serious, but in general, the redex reevaluated could have an expensive operation and the reevaluation can take place not only two, but an unbounded (even infinite) number of times.

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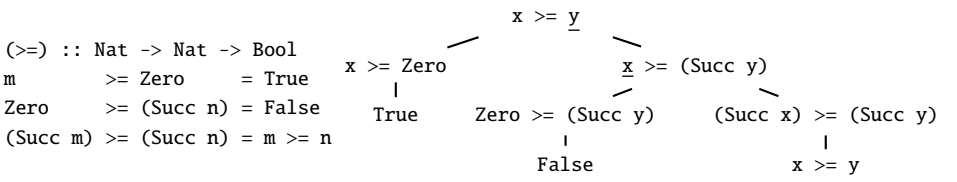



Fig. 2. Sample definition and its associated definitional tree.

*Sequentiality Analysis*. One of the stages in compiling the code for a function definition in a lazy language is to decide (i) which arguments need reduction to perform the matching against the patterns in the left hand sides of the rules, and (ii) in which order are these arguments to be reduced. For instance, the code for the *greater or equal* predicate (Figure 2) needs to obtain a topmost data constructor for the second argument to choose a rule. If this constructor is `Zero`, the only match is with the first rule and no evaluation is needed on the first argument, but if it is `Succ`, the first argument needs to be examined in order to choose between the second and third rules.

Sloth[1] (Mariño and Rey 1998), our implementation of Curry, uses *definitional trees* (Antoy *et al*. 2000) as the intermediate structure to store these decisions. Figure 2 shows the definition for `(>=)` and the definitional tree obtained from it. Underlined positions are those where the branching of the decisions are done. Next section will provide a definition of this concept but the graphical notation could be enough at this point. Observe that arguments are not necessarily examined in a left to right order.

Sometimes, it is difficult – or impossible – to discover a definitional tree from the left hand sides alone. A typical example is the merging of two sorted lists:

```
merge :: [Nat] -> [Nat] -> [Nat]
merge [] ys = ys          merge (x:xs) (y:ys) | x <= y = x:(merge xs (y:ys))
merge xs [] = xs                              | x >= y = y:(merge (x:xs) ys)
```

If we just look at the left hand sides there is no way of building a decision tree like the one for `(>=)`, but looking at the right hand sides we immediately see that both arguments to *merge* are demanded. A programmer would write, in fact, a modified version where the second rule is rewritten as

```
merge (x:xs) [] = x:xs
```

instead. In our paper (Mariño and Moreno-Navarro 2000) we show that for the vast majority of programs, the information obtained from type inference and demand analysis can help a compiler to generate sequential definitional trees for programs even when they are not syntactically sequential.

*Avoiding Redundant Tests in Target Code*. A substantial part of the overhead in implementing a lazy functional-logic language is precisely due to the code that implements the lazy evaluation of arguments (this was, in fact, what motivated research on strictness analysis of functional programs in the first place). This is particularly evident when looking at the

[1] `http://babel.ls.fi.upm.es/software`.

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translation scheme into Prolog of Sloth (Mariño and Rey 1998), but the problem appears also in abstract machine based implementations.

Demand information can help to generate, for most functions, eager code, with a drastic effect on efficiency. In the case of the translator to Prolog, this allows an almost verbatim translation with very little overhead.

*Relation with Freeness and Sharing Analysis.* *Freeness* and *sharing* (Jacobs and Langen 1992; Muthukumar and Hermenegildo 1991) are two operational properties of logic programs which have been extensively studied. Freeness analysis can tell whether a free variable present in a goal does not get bound during its evaluation. Sharing analysis can tell whether two variables present in a goal will not eventually share some structure. Freeness and sharing are important, for instance, for the parallel execution of logic programs. Moreover, freeness can be useful to detect deterministic computations in functional-logic programs.

Freeness and sharing are much more difficult to study in a lazy functional-logic language than in Prolog. From an operational point of view, the techniques used for Prolog can predict whether a given expression may bind some variables *provided that* this expression gets *actually evaluated*.

From a denotational point of view, this connection can be explained because the lazy semantics can be seen as a restricted form of free-variable semantics where all variables are collapsed into one symbol ($\perp$). In this setting, variable propagation (freeness) and $\perp$-propagation (strictness) share a common mechanism (Mariño 2002).

*Transformation of Strict Equality into Unification.* The behaviour of the equality operator is slightly different in logic programming and in functional-logic programming. While in the first case the expression $x = t$, where $x$ is a free variable, can be dealt with by assigning $t$ to $x$ (provided that $x$ does not occur in $t$), in functional-logic programming it is necessary to ensure that $t$ can be evaluated to a *total* value, i.e. a term with no undefined subterms, otherwise the whole expression will be undefined. When logic programs are translated into functional-logic programs, this has two undesired effects: the expression cannot be resolved in constant time – which precludes the use of Prolog techniques such as difference lists – and the computed answers can be unnecessarily detailed.

The connection with strictness is twofold. On one hand, totality is a much stronger condition than not being undefined, which makes rules with equality expressions a source of useful information for a strictness analyzer. On the other hand, the same techniques employed to avoid redundant tests when applying strictness analysis can be used to avoid the test that $t$ does not contain an occurrence of a defined function symbol, which is a sufficient condition to perform the assignment.

Some of these problems can be tackled by nonstandard implementation architectures like *memoization* or *bottom-up execution* (Mariño and Moreno-Navarro 1995) for the reevaluation problem, or *parallel definitional trees* (Antoy *et al*. 1997; Genius 1996) to cope with the lack of sequentiality but, in practice, these methods introduce their own overheads and actual implementations are based either on extensions of abstract machines for the execution of functional languages or logic languages, or on the translation to another declarative language, like Prolog, see Mariño and Moreno-Navarro (1992) and Mariño

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and Rey (1998). This is why we chose to attack the problem at compile time or, in other words, how our research on demand analysis began.

### 1.2 Paper organization

Section 2 introduces the subset of the Curry language we are going to use along the paper as well as the operational semantics used, namely narrowing. Section 3 introduces the formalism of partial predicates, their structure and how demand properties can be represented by using them. Section 4 discusses the problem of checking and inferring demand properties of a program. Different sublattices of demand properties, with increasing complexity, are introduced. Checking is demonstrated in an abstract way, by means of syntactic transformations (fold/unfold). Later, the harder problem of inferring demand properties is considered, firstly (Section 4.2), in an abstract way and then (Section 5) in connection with a particular analysis tool. Correctness of the aforementioned demand equations is shown there as well as the algorithms to compute approximate solutions to them. Code generation based on the information from the analyzer is discussed in Section 6 and a few experimental results showing the feasibility of the method are shown in Section 6. Some related work is discussed in Section 7 and open issues in Section 8. Finally, Section 9 concludes.[2]

## 2 Preliminaries

This section is devoted to fix the subset of Curry that will be used along the paper. The operational semantics assumed is discussed too.

### 2.1 Kernel language

The language of choice is largely immaterial but a little syntax is needed in order to keep some coherency throughout the paper. We will use a simplified version of the functional-logic language Curry (Hanus *et al*. 2003), basically a language of recursion equations, with a Haskell-like syntax. In the sequel we assume some knowledge of functional-logic languages and Curry operational semantics.

We assume a ranked set $TC = \bigcup_n TC^n$ of *type constructors* $K$ and a countably infinite set $TV$ of *type variables* $\alpha$. Any data type is uniquely denoted by an algebraic term $\tau \in \mathcal{T}(TC \cup TV)$ or a function type $(\tau_1 \to \tau_2)$. Next, we assume a set $DC = \bigcup_n DC^n$ of typed *data constructors* $C$, a countably infinite set $VS$ of *variable symbols* $x$, and a set $FS$ of *function symbols* $f$ with declared principal type $f : \tau_1 \to \cdots \to \tau_n \to \tau$ where $\tau$ is not a function type. $TC$, $DC$, $VS$ and $FS$ are disjoint. The *arity* of a data constructor $C \in DC^n$ is $n$ and is denoted $ar(C)$. In practice, type and data constructors are both defined via *data* declarations of the form

$$\texttt{data}\ K\ \alpha_1 \ldots \alpha_l = C_1\ \tau_{11}\ \ldots\ \tau_{1m_1} \mid \ldots \mid C_n\ \tau_{n1}\ \ldots\ \tau_{nm_n}.$$

[2] The electronic version of this paper, which can be found at the CoRR (Computing Research Repository, http://arxiv.org/corr) with ID cs.PL/0602008, contains some proofs that have been removed from the printed version due to lack of space.

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A type constructor *Bool* with data constructors *True* and *False* is always assumed. Expressions are given by the grammar

$$e ::= C \mid x \mid f \mid e_1\ e_2.$$

Expressions must be well typed. A program is a set of defining rules of the form

$$f\ e_1 \ldots e_n = [b \rightarrow] e.$$

The optional condition $b$ of type *Bool* is called the *guard* of the rule. Several restrictions are imposed on the rules in a program in order to ensure confluence of reduction, and the following are used somewhere in the paper: (i) for every rule $(l = r)$, $l$ is a *pattern*, i.e. it has a single function symbol at its top and no variable occurs twice in $l$; (ii) rules must be well typed; (iii) for every pair of program rules $(l_1 = r_1)$, $(l_2 = r_2)$, if $l_1$ and $l_2$ have a unifier $\sigma$ then $\sigma(r_1) = \sigma(r_2)$; (iv) free variables – i.e. those occurring in the right hand side but not in the left hand side – are allowed only if their rightmost occurrence is in the guard. Moreover, they must be of first order type. Observe that we are not forbidding overlaps. The set of rules defining function symbol $f$ in program $P$ is denoted $Rules_P(f)$.

When looking at the syntactic shape of the left hand sides of defining rules, a total application $(f\ e_1\ \ldots\ e_n)$ is treated as the algebraic term $f(e_1, \ldots, e_n)$ and then the standard notation for positions and substitutions is used. A position is a string of natural numbers that identifies a path to a subterm in a term. The expression $t|_p$ denotes the subterm of $t$ at position $p$, i.e. $f(e_1, \ldots, e_n)|_{i.p} = e_i|_p$ and $t|_\epsilon = t$, with $\epsilon$ the empty string. Replacement of $t|_p$ by $t'$ is abbreviated $t[t']_p$. The topmost symbol of term $t$ is denoted $root(t)$.

A denotational semantics for the kernel language can be found in the electronic version. Although not strictly necessary to understand the techniques proposed here, this is the ultimate foundation for the validity of the equations and inequalities used and supports the validity of the fold/unfold transformations.

### *2.2 Narrowing*

The fundamental computation mechanism of functional-logic languages is narrowing. Informally, to *narrow* an expression $e$ means to apply a substitution that makes it reducible, and then reduce it. An expression $e$ narrows to $e'$ with substitution $\sigma$, if $p$ is a nonvariable position of $e$, $l = r$ is a variant of a program rule sharing no variables with $e$, and $\sigma$ is a substitution such that $\sigma(l) = \sigma(e|_p)$, and $e' = \sigma(e[r]_p)$.

As we have mentioned, unrestricted application of the narrowing rule is too nondeterministic and many strategies have been proposed to improve this. From an expressiveness point of view, we prefer those with a lazy behaviour because functions can be defined more independently, without interaction among them, thus increasing modularity and reusability and allowing programming techniques like infinite objects. A general description of lazy narrowing for functional-logic languages can be found in Moreno-Navarro and Rodríguez-Artalejo (1992).

The task of a narrowing strategy is the computation of the step, or steps, that must be applied to a term. A narrowing strategy suitable for functional-logic languages must be sound, complete, and efficient. The intuition behind the soundness and the completeness of

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a strategy when the initial term of a derivation is an equation containing unknown variables is easy to describe: Soundness guarantees that any instantiation of the variables computed by the strategy is a solution of the equation. Completeness ensures that for any solution of the equation, the strategy computes another solution which is at least as general.

However, efficiency is more difficult to state. As usual, the goal is to minimize the overall time and memory consumed when finding one or all the values of an expression. In the narrowing context, it is related with the length of the derivations, and, specially, with the size of the search space. Basically, the two factors affecting the efficiency of a strategy are: (i) unnecessary steps should be avoided, and (ii) steps should be computed without consuming unnecessary resources. Lazy narrowing steps try to be applied at outermost positions but to preserve completeness (see Moreno-Navarro and Rodríguez-Artalejo (1992)) inner arguments of a function are evaluated, by narrowing them to their head normal forms, if their values are required for an outermost narrowing step.

In general, a strategy cannot easily determine if a computation is unnecessary without look-ahead.

The strategy used in Curry is *needed narrowing* (Antoy *et al*. 2000), a lazy strategy where the program is translated into a set of definitional trees, one for every function symbol being defined. Definitional trees are given by the grammar

$$DT \quad ::= \quad \text{branch}\,(\mathit{Pattern}, \text{Pos}\,[,\,\text{DT}\,]^{+}) \mid \text{rule}\,\mathit{Rule} \mid \text{or}\,(DT[,DT]^{+}),$$

where *Pattern* stands for patterns made up of data constructors and different variables as in the left hand sides of program rules, *Pos* are positions defined in the standard way and *Rules* are program rules. Trees without *or* nodes are called (inductively) sequential, otherwise they are parallel definitional trees.

Given an expression $e$ and a set of definitional trees for the defined symbols of the program, a position in $e$ can be chosen to apply narrowing. This is done by first looking for an outermost application $fe_1 \ldots e_n$ where $f$ is a defined function symbol, and then descending some $e_i$ according to $f$'s definitional tree. Therefore, needed narrowing with sequential definitional trees establishes an efficient algorithm for implementing the look-ahead for required evaluations. In Antoy *et al*. (2000) the formal definition is given but also an interesting property is shown: the strategy is optimal with respect to the length of derivations.

To overcome the restriction to inductively sequential programs, other strategies have been proposed. For instance, the strategy of Loogen *et al*. (1987) is based on some form of generalized definitional trees. The completeness of this strategy is unknown. These strategies are *demand driven*, which informally means the following: a subterm $v$ of a term $t$ is evaluated if there is a rule $R$ potentially applicable to $t$ that demands the evaluation of $v$.

The lack of well-defined strategies with provable properties motivated alternative efforts for computations in this class. In our case, we use the information provided by demand analysis to guide the computation. The analysis, its formal properties and the use of demand information for implementing efficiently functional-logic languages are the goal of the following sections.

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## 3 Partial predicates

This section is devoted to introduce the formalism of *partial predicates*. Informally speaking, they are logic predicates that represent the degree of evaluation of an expression (demand properties). The main feature is that they can be described using the language under analysis, so the language itself is used to abstract some properties of a given program. This fact is essential for their use as an analysis tool, as will be shown later.[3]

**Definition 1 (partial predicate)** Let *Two* be a two point domain; a *partial predicate* $\pi$ defined on type $\tau$ is any continuous map $\pi \in \tau \to Two$ which can be defined in the kernel language. In the following, the type scheme $PP\ \alpha$ will be used as synonymous with $\alpha \to Two$.

A two point domain *Two* is isomorphic to the subset of the domain $Bool = \{True, False, \bot\}$ after removing *False*[4] and can be defined in the kernel language:

```
data Two = True
```

The definition of conjunction and disjunction functions in *Bool* can be restricted to this domain:

```
(&&), (||) :: Two -> Two -> Two
True && True = True
True || y = True;                    x || True = True
```

Observe that disjunction in *Two* is given by a parallel definition. While this could be problematic in case of trying to execute the program, it is not the case in our context as long as the definitions of the predicate transformers will mainly be used for program transformation.

Every partial predicate $\pi$ of type $PP\ \tau$ represents subsets of the domain $\tau$:

$$\pi^{-1}(True) = \{x \in \tau \mid \pi(x) = True\}.$$

**Example 2 (Peano naturals)** Some of the examples throughout the paper will make use of a data type for Peano naturals:

```
data Nat = Zero | Succ Nat
```

A pair of predicates *hnfNat* and *nfNat* can be introduced:

```
hnfNat, nfNat :: PP Nat
hnfNat Zero = True;                  hnfNat (Succ _) = True
nfNat Zero = True;                   nfNat (Succ n) = nfNat n
```

The former yields *True* when its argument is evaluated enough to identify its topmost constructor. The latter yields *True* when its argument is evaluated to normal form. Observe that $hnfNat\ n = \bot \Leftrightarrow n = \bot$.

[3] From this point on, notation based on *domain theory* is extensively used, and a denotational semantics for the kernel language is assumed. Readers less familiar with these topics are referred to van Leeuwen (1990), Chapters 11 and 12.

[4] Hence the name of partial predicates.

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**Example 3 (partial predicates *hnfList* and *spine*)** Less trivial are those predicates that can be used to express properties of polymorphic types. Consider, for instance, *hnfList* or *spine* in the domain of polymorphic lists:

```
hnfList, spine :: PP [a]
hnfList [] = True;                  hnfList (h:ts) = True
spine [] = True;                    spine (h:ts) = spine ts
```

Analogously to the example above, the former yields *True* when its list argument is evaluated enough to identify its topmost constructor. The latter yields *True* when the argument is evaluated enough to reach the end of the list. In both cases, the degree of definition of the elements in the list is immaterial. Observe again that $hnfList\ xs = \bot \Leftrightarrow xs = \bot$.

**Example 4 (partial predicates *any* and *nothing*)** A pair of polymorphic partial predicates *any* and *nothing* can be defined for all types:

```
any, nothing :: PP a
any x = True;                       nothing x = nothing x
```

### 3.1 Demand typings

Partial predicates can be used to express a great number of program properties, including classic strictness. For instance, suppose we are interested in proving $f \in [\tau_1] \to Nat$ strict:[5]

$$\begin{aligned} f \text{ is strict} &\Leftrightarrow f\ \bot = \bot \Leftrightarrow \forall x.\ x = \bot \Rightarrow f\ x = \bot \\ &\Leftrightarrow \forall x.\ f\ x \sqsupset \bot \Rightarrow x \sqsupset \bot \\ &\Leftrightarrow \forall x.\ hnfNat(f\ x) = True \Rightarrow hnfList\ x = True \\ &\Leftrightarrow \forall x.\ hnfNat(f\ x) \sqsubseteq hnfList\ x \qquad \Leftrightarrow\ hnfNat \circ f \sqsubseteq hnfList \end{aligned}$$

So the property '$f$ is strict' is equivalent to $hnfNat \circ f \sqsubseteq hnfList$.

If we are interested in studying how much information is needed in a function's argument in order to obtain a certain amount in the result, this can be generalized to properties of the form $\pi_2 \circ f \sqsubseteq \pi_1$.

**Definition 5 (demand properties, demand types and demand typings)** A *demand property* is an inequality of the form $\pi_2 \circ f \sqsubseteq \pi_1$ where $\pi_1$ and $\pi_2$ are partial predicates defined on the domain and codomain types of $f$, respectively. It is denoted in the following way:

$$f : \pi_1 \Leftarrow \pi_2 \quad \stackrel{\text{def}}{=} \quad \pi_2 \circ f \sqsubseteq \pi_1.$$

$\pi_1 \Leftarrow \pi_2$ is a *demand type* and $f : \pi_1 \Leftarrow \pi_2$ is a *demand typing*.

Demand typings $f : \pi_1 \Leftarrow \pi_2$ are usually read '$f$ demands $\pi_1$ to its argument in order to give a result as evaluated as $\pi_2$.' Our previous strictness example would be rewritten $f : hnfList \Leftarrow hnfNat$ . It is rather simple to show that for any partial predicate $\pi^{-1}(True)$ is an ideal – it is the inverse image of a closed set. In fact, 'partial predicate' can be taken as synonymous with 'computable ideal.' The lattice of partial predicates induces another on demand types $\pi_1 \Leftarrow \pi_2$ : covariantly on $\pi_1$ and contravariantly on $\pi_2$.

[5] A formal definition of *strict* can be found in Definition 16.

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### *3.2 Polymorphism*

One advantage of partial predicates over other approaches to program analysis, such as abstract interpretation on finite domains, is a natural treatment of polymorphism. Being written in source code they have the same type constraints and expressiveness.

Some partial predicates presented so far are polymorphic and the same can be said of the predicate transformers to be introduced below.

**Definition 6 (predicate transformers)** *Predicate transformers* are higher order functions that take partial predicates as (part of) its argument and yield partial predicates as result.

Data constructors can be seen as predicate transformers, in particular, polymorphic constructors can be represented by polymorphic transformers.

**Definition 7 (constructor predicate transformer)** Consider a type definition of the form

$$\texttt{data}\ K\ \alpha_1\ \ldots\ \alpha_l\ =\ \ldots\ |\ C_i\ \tau_1\ \ldots\ \tau_{m_i}\ |\ \ldots$$

The *constructor predicate transformer* $c_i$[6] associated with each data constructor is defined as follows:

$$c_i\ ::\ PP\ \tau_1\ \rightarrow \ldots \rightarrow\ PP\ \tau_{m_i}\ \rightarrow\ PP\ (K\ \alpha_1\ \ldots\ \alpha_l)$$
$$c_i\ \mathtt{p}_1\ \ldots\ \mathtt{p}_{m_i}\ (C_i\ \mathtt{x}_1\ \ldots\ \mathtt{x}_{m_i}) = (\mathtt{p}_1\ \mathtt{x}_1)\ \&\&\ \ldots\ \&\&\ (\mathtt{p}_{m_i}\ \mathtt{x}_{m_i}).$$

Interesting partial predicates associated to data constructors and data types can be defined by using constructor predicate transformers (e.g. Definitions 8 and 10).

**Definition 8 (matching predicates)** For every data constructor $C$ the partial predicate *isC* defined

$$isC\ ::\ PP\ (K\ \alpha_1\ \ldots\ \alpha_l)$$
$$isC\ =\ (c\ \mathtt{any}\ \ldots\ \mathtt{any})$$

is called the *matching predicate for constructor C*.

**Definition 9 (meets and joins)** The greatest lower bound operator $\sqcap$ (/\) and the least upper bound operator $\sqcup$ (\/) can be defined:

```
(/\), (\/) :: PP a -> PP a -> PP a
(p /\ q) x = (p x) && (q x);        (p \/ q) x = (p x) || (q x)
```

**Definition 10 (*hnf* predicates)** For every type constructor $K$ with data constructors $C_1, \ldots, C_n$, the partial predicate *hnfK* defined

$$hnfK\ ::\ PP\ (K\ \alpha_1\ \ldots\ \alpha_l)$$
$$hnfK\ =\ isC_1\ \sqcup\ \ldots\ \sqcup\ isC_n$$

is the *hnf predicate of K*.

[6] We are not actualy overloading data constructor names but just changing first letter of the name to lowercase.

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**Example 11** Constructor predicate transformers and matching predicates associated with the list constructors `[] :: [α]` and `(:) :: α → [α] → [α]`[7] are:

```
nil :: PP a                                    isNil :: PP [a]
nil [] = True                                  isNil = nil
cons :: PP a -> PP [a] -> PP [a]               isCons :: PP [a]
cons p q (x : xs) = (p x) && (q xs)            isCons = cons any any
```

The definition of *hnfList* in Example 3 can be rewriten

```
hnfList :: PP [a]                  hnfList = isNil \/ isCons
```

**Example 12** The constructor predicate transformer and the matching predicate associated with the tuple constructor `(,) :: α → β → (α,β)`[8] are:

```
tup2 :: PP a -> PP b -> PP (a,b)               isTup2 :: PP (a,b)
tup2 p q (x,y) = (p x) && (q y)                isTup2 = tup2 any any
```

And the definition of *hnfTup2* is:

```
hnfTup2 :: PP (a,b)                hnfTup2 = isTup2
```

**Definition 13 (cartesian products)** The *cartesian product* of two partial predicates is defined as the constructor predicate transformer *tup2*. In the following we will use the infix operator `(×) :: α → β → (α,β)` as synonymous with *tup2*. The definition is generalised to arbitrary length tuples:

$$(p_1 \times \ldots \times p_n)\ (x_1, \ldots, x_n) = (p_1\ x_1)\ \&\&\ \ldots\ \&\&\ (p_n\ x_n)$$

**Example 14 (projections on tuples)** A cartesian product implies the existence of projections. We will show that there are actually two predicate transformers *prj*1 and *prj*2 with types

```
prj1 :: PP (a,b) -> PP a           prj2 :: PP (a,b) -> PP b
```

although their definition is somewhat special. Mathematically, the following must hold:

$$\begin{aligned}(prj1\ p)^{-1} &= \{x \mid \exists y.\ p(x,y) = True\}\\ (prj2\ p)^{-1} &= \{y \mid \exists x.\ p(x,y) = True\}.\end{aligned}$$

As has been said in the introduction, the kernel language does not forbid free variables in the equations. In fact, the denotational semantics of rules treats them via a least upper bound quantified over all the possible values in their type. This means that an implementation of projections will be:

```
prj1 p x = p (x, y) -> True;       prj2 p y = p (x, y) -> True
```

[7] We are using *List*, *Nil* and *Cons* as names for `([])`, `[]` and `(:)`, respectively.
[8] We are using *Tup2* as the name for `(,)`.

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This can be surprising to the reader more biased towards functional programming but is by no means strange if we look at Prolog or functional-logic languages as Curry itself. In the special case that the variable being quantified is of a first order type, implementing such an equation is not a problem.

The extension of projections to arbitrary data types and data constructors is trivial. In particular projections on arbitrary length tuples will be used in the following section.

**Definition 15 (projections)** Given the data declaration scheme

$$\texttt{data}\ K\ \alpha_1\ \ldots\ \alpha_l\ =\ \ldots\ |\ C_i\ \tau_1\ \ldots\ \tau_{m_i}\ |\ \ldots$$

for each data constructor $C_i$ and for each $k \in \{1, \ldots, m_i\}$, the *projection partial predicate* $prjkC_i$ is defined as[9]

$prjkC_i$ `::` $PP\ \tau_k \rightarrow PP\ (K\ \alpha_1\ \ldots\ \alpha_l)$
$prjkC_i$ `p x = p (`$C_i$ `x`$_1$ `...  x  ...  x`$_{m_i}$`)` $\rightarrow$ `True.`

## 4 Checking and inference of demand properties

This section studies some analyses and their domains of properties under the prism of partial predicates. The novelty of our approach is that, by expressing partial predicates in a subset of the programming language under analysis, a *program transformation approach* is feasible. All the examples below use the well known fold/unfold transformations and are, thus, trivially correct for a language with a lazy declarative semantics.

### *4.1 Checking*

The problem of deciding if a given partial predicate fulfills the demand information of a given function is the *checking problem*: to prove that given $f, \pi_1$ and $\pi_2$, $f : \pi_1 \Leftarrow \pi_2$ holds.

Concrete analyses fix a specific domain of checking properties, i.e. only a limited number of partial predicates are allowed. Let us show the translation of several domains of properties into the language of partial predicates and exemplify checking by means of equational reasoning.

*Classic Strictness Analysis.* The first attempt to mechanize strictness analysis is found in Mycroft (1980). The aim is to detect when an argument can be safely reduced in advance without affecting the termination properties of the program.

**Definition 16** A function $f$ is said to be *strict* iff $f\ \bot = \bot$.

Due to evident practical reasons, this definition is relaxed to cope with the usual case of the argument belonging to a product type:

**Definition 17** A function $f$ is said to be *strict in its i-th argument* iff

$$\forall\ x_1 \ldots x_{i-1}\ x_{i+1} \ldots x_n.\ f(x_1, \ldots, x_{i-1}, \bot, x_{i+1}, \ldots, x_n) = \bot.$$

[9] Observe that $k$ actualy expands: $prj1C_i, prj2C_i, \ldots$.

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As we have seen in Section 3.1, the first case can be expressed in our setting by saying that the function demands an argument strictly more evaluated than $\perp$ in order to produce a result strictly more evaluated than $\perp$. When both the argument and result types are constructed, values other than $\perp$ can be finitely presented by enumeration of the different constructors in the type:

**Lemma 18** If $f$ is a function with type $f :: K\tau \to K'\tau'$ then it is strict iff $f : hnfK \Leftarrow hnfK'$.

*Proof*
See proof in Section 3.1 and replace *Nat* and *List* with $K'$ and $K$. □

Let us see an example that will also show how to use equational reasoning in order to prove the demand typing correct.

**Example 19** Function *length*

```
length :: [a] -> Nat
length [] = Zero;                    length (h:ts)  = Succ (length ts)
```

is strict. This will be expressed as $length : hnfList \Leftarrow hnfNat$. Using the definitions of *hnfNat* and *hnfList* seen before, we have to prove that $hnfNat \circ length \sqsubseteq hnfList$. Then these equivalences follow from the semantics of the kernel language:[10]

```
(hnfNat . length ) [] = hnfNat (length []) = hnfNat Zero = True
(hnfNat . length ) (x:xs) = hnfNat (length (x:xs))
                          = hnfNat (Succ (length xs)) = True
```

Both *rules* coincide with the equations of *hnfList*.

**Lemma 20** If $f$ is a function with type $f :: (K_1\tau_1, \ldots, K_n\tau_n) \to K\tau'$ then it is strict in the $i$-th argument iff $f : any \times \ldots \times hnfK_i \times \ldots \times any \Leftarrow hnfK$.

*Proof*

$$
\begin{aligned}
& hnfK \circ f \sqsubseteq any \times \ldots \times hnfK_i \times \ldots \times any \\
\Leftrightarrow\ & \forall\, x_1 \ldots x_n.\ hnfK(f(x_1, \ldots, x_n)) \sqsubseteq (any \times \ldots \times hnfK_i \times \ldots \times any)(x_1, \ldots, x_n) \\
\Leftrightarrow\ & \forall\, x_1 \ldots x_n.\ hnfK(f(x_1, \ldots, x_n)) \sqsubseteq hnfK_i(x_i) \\
\Leftrightarrow\ & \forall\, x_1 \ldots x_{i-1}\ x_{i+1} \ldots x_n.\ hnfK(f(x_1, \ldots, x_{i-1}, \perp, x_{i+1}, \ldots, x_n)) \sqsubseteq hnfK_i(\perp) \\
\Leftrightarrow\ & \forall\, x_1 \ldots x_{i-1}\ x_{i+1} \ldots x_n.\ hnfK(f(x_1, \ldots, x_{i-1}, \perp, x_{i+1}, \ldots, x_n)) \sqsubseteq \perp \\
\Leftrightarrow\ & \forall\, x_1 \ldots x_{i-1}\ x_{i+1} \ldots x_n.\ hnfK(f(x_1, \ldots, x_{i-1}, \perp, x_{i+1}, \ldots, x_n)) = \perp \\
\Leftrightarrow\ & \forall\, x_1 \ldots x_{i-1}\ x_{i+1} \ldots x_n.\ f(x_1, \ldots, x_{i-1}, \perp, x_{i+1}, \ldots, x_n)) = \perp. \quad \square
\end{aligned}
$$

**Example 21** Function *plus*

```
plus :: (Nat, Nat) -> Nat
plus (Zero, m) = m;                  plus (Succ n, m) = Succ (plus (n, m))
```

[10] Composition in Curry is denoted by the symbol '.'.

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is strict in its first argument: $plus : hnfNat \times any \Leftarrow hnfNat$. Unfolding the definition of $hnfNat \times any$ we get

```
(hnfNat × any) (x,y) = (hnfNat x) && (any y) = hnfNat x
```

Unfolding $hnfNat \circ plus$:

```
(hnfNat . plus) (Zero,m) = hnfNat m ⊑ hnfNat Zero
(hnfNat . plus) (Succ n,m) = True = hnfNat (Succ n)
```

so $hnfNat \circ plus \sqsubseteq hnfNat \times any$ and, by Definition 5, $plus : hnfNat \times any \Leftarrow hnfNat$.

*Wadler's Four Point Domain.* Many interesting properties are related to the degree of evaluation required on recursive data structures, like lists. For instance, function *length* needs a nil-ending list to produce a definite result, but it is immaterial whether one or more of its elements is undefined.

In Wadler (1987) a four point abstract domain of degrees of definiteness of monomorphic lists is introduced. The domain, in increasing order, can be given as:

$$\mathbf{4} = \{\bot \sqsubset \infty \sqsubset \bot\in \sqsubset \top\in\}$$

representing, respectively, the undefined list, any list with an undefined suffix, finite lists with some undefined elements and total lists. The original paper is not very formal and does not make clear that this semantics for the four elements does not provide conjunctive nor disjunctive closeness. This, of course, can be achieved if their semantics is changed into:

| | | | |
|---|---|---|---|
| $\top\in$ | any list | $\bot\in$ | any nil ending list |
| $\infty$ | any list in head normal form | $\bot$ | an undefined list. |

These four levels of definiteness can be represented in our framework by the three partial predicates: *hnfList*, *spine*, and *nfListNat*:

```
nfListNat :: PP [Nat]
nfListNat (x:xs) = (nfNat x) && (nfListNat xs);   nfListNat [] = True;
```

The following can help demonstrate the transformational approach. Let us prove $length : spine \Leftarrow nfNat$. Let $R$ denote $nfNat \circ length$. Applying standard *fusion* techniques (see Section 4.2) we successively obtain:

```
R []     = nfNat (length [])   = nfNat Zero = True
R (x:xs) = nfNat (length x:xs) = nfNat (Succ (length xs))
         = nfNat (length xs)   = R xs
```

so we conclude that $R = spine$.

*Uniform Properties.* A number of proposals have been made to generalise Wadler's four point domain to any algebraic datatype. Intuitively, the *uniform properties* of a data structure are those invariant under any (type preserving) permutation of the elements of the structure. For instance, if $p$ is an uniform property of lists of naturals, then $p(Zero : (Succ\ Zero) : \bot) \Leftrightarrow p((Succ\ Zero) : Zero : \bot)$.

In Jensen (1994) a powerdomain construction for uniform properties over algebraic datatypes is given, using modalities, along with a strictness logic for reasoning about

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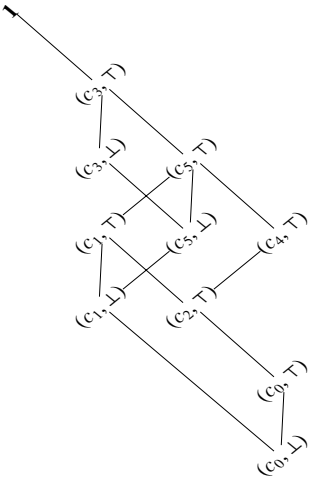



Fig. 3. The lattice of uniform properties on lists.

those properties. His domains are able to express certain properties that do not appear in Wadler's, like a list being empty or being finite with all their elements undefined, etc. The formalism is rather involved and the strictness logic does not lead very naturally to an implementation.

Trying to accommodate those ideas into our framework of partial predicates we immediately see that the domains that arise in Jensen's work correspond essentially to the *folds* on a given datatype. A fold on a data structure is a transformation that replaces every n-ary constructor by an n-ary function. Folds are generic programming constructs in the sense that folds can be defined for every algebraic datatype in an uniform way. For instance, folding lists is done using the following higher order operator:

```
foldl :: (a -> b -> b) -> b -> [a] -> b
foldl f b [] = b;                    foldl f b (x:xs) = f x (foldl f b xs)
```

Due to type restrictions, the number of partial predicates on natural lists that can be defined as folds is limited. If only two degrees of definiteness are considered for the naturals ($\bot$ and *True*) that leaves six possible combining functions of type $Two \to Two \to Two$:

$$(c_0)\ \lambda x.\ \lambda y.\ \bot \qquad (c_1)\ \lambda x.\ \lambda y.\ x \qquad (c_2)\ \lambda x.\ \lambda y.\ x \wedge y$$
$$(c_3)\ \lambda x.\ \lambda y.\ \texttt{True} \qquad (c_4)\ \lambda x.\ \lambda y.\ y \qquad (c_5)\ \lambda x.\ \lambda y.\ x \vee y$$

times two values for the base case gives a lattice (Figure 3) of at most 13 abstract values (adding *any*).[11] A similar domain appears in Benton (1992).

The possibility of using catamorphisms (fold-like functions) on algebraic types as a way of automatically constructing domains for the analysis of programs has been studied and implemented in Rey (2003).

[11] The notation $(c, v)$ stands for *foldl* $c\ v$. The exact cardinality of the abstract domain is 11, as some of the combinations coincide: *foldl* $c_2\ \bot$ = *foldl* $c_4\ \bot$ = *foldl* $c_0\ \bot$.

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### *4.2 Inference*

Here we study the problem dual to checking, i.e. how to infer a partial predicate that describes (with reasonable accuracy) demand information for a given function. The concrete inference problem considered in this paper is the following: given $f$ and $\pi_2$, to find the *best* $\pi_1$ such that $f : \pi_1 \Leftarrow \pi_2$ holds and, more important, to give a usable representation. The best $\pi_1$ such that $f : \pi_1 \Leftarrow \pi_2$ is $\pi_1 = \pi_2 \circ f$ and an explicit (recursive) definition can be obtained using the transformational approach used for checking. However, giving a compact representation of $\pi_1$ suitable for code generation can be difficult and here is where a purely symbolic approach is better suited than the program transformation one.

To get an informal understanding of the connection between the program transformation and the symbolic approaches to inference, let us revisit Example 19 recast as an inference problem:

**Example 22** The original program is

```
length []     = Zero;              length (h:ts) = Succ(length ts)
```

We want to infer the degree of definiteness demanded $\pi_1$ on its argument by a result in normal form, i.e. $\pi_1 = nfNat \circ length$, so the following must hold:

$$\begin{aligned} \pi_1\,([]) &= (nfNat \circ length)\,([]) \\ &= nfNat\,(Zero) = True \\ \pi_1\,(h : ts) &= (nfNat \circ length)\,(h : ts) \\ &= (nfNat \circ Succ)\,(length\,(ts)). \end{aligned}$$

It is easy to see that $nfNat \circ Succ$ simplifies to $nfNat$ so

$$\begin{aligned} \pi_1\,(h : ts) &= nfNat\,(length\,(ts)) \\ &= (nfNat \circ length)\,(ts) = \pi_1\,(ts), \end{aligned}$$

resulting in a rather generative set of equations for $\pi_1$ (that coincide with equations for *spine*).

The following section shows a more systematic method to manipulate partial predicates properties in a fully symbolic way. From the program under analysis and the inference question, a set of inequalities among symbolic representations of partial predicates is generated, and is this set which is manipulated – although the meaning of these rewritings must mimic the original program transformations.

## 5 From partial predicates to set expressions

The most natural interpretation of a partial predicate $\pi \in \alpha \to Two$ is a subset of the domain $D_\alpha$, a set of trees, more exactly an *ideal* set of *partial* trees. This section is devoted to show that set expressions and set constraint based analysis (Pacholski and Podelski 1997) can be used as a framework for the checking and inference of partial predicate typings.

### *5.1 Basic notions*

*'Set constraints are first-order logic formulae interpreted over the domain of sets of trees'* (Pacholski and Podelski 1997). Set expressions ($e$) are expressions built from variables

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interpreted over sets of trees, function symbols intepreted as functions over sets of trees and standard set operators (union, intersection, inclusion, complement, etc.). A system of set constraints is a conjunction of inclusions of the form $e_l \subseteq e_r$ with some restrictions on the set expressions that can appear in the left or right hand sides.

*Co-definite* set constraints (Charatonik and Podelski 1998) is the class of set constraints where constraints are inclusions between *positive* set expressions and where the set expression in the left hand side is restricted to contain variables, constants, unary function symbols and the union operator.[12] Sets of co-definite constraints, if satisfiable, always have a greatest solution. The satisfiabilty problem for co-definite set constraints is DEXPTIME-complete and an algorithm is given in Charatonik and Podelski (1998).

The restriction to positive expressions (i.e. without the use of complementation) is essential for our purposes, as the complement of an ideal is not an ideal. The rest of operators are continuous, so we can translate existing results in set constraint theory to our domains.

The following definitions formalize these notions.

**Definition 23 (set expressions)** Given a typed alphabet $\Sigma$ with constants ($a, b, c, \dots$) and nonconstant function symbols ($f, g, h, \dots$) and a typed set **VS** of variable symbols ($u, v, x, y, \dots$), *set expressions* follow the syntax:[13]

$$e \quad ::= \quad x \mid a \mid f(u_1, \dots, u_n) \mid f^{-1}_{(k)}(u) \mid e_1 \cup e_2 \mid \emptyset.$$

This syntax represents finite and infinite trees and we will use the notation $T_\Sigma$ for the whole set of well-formed (w.r.t. types) trees.

**Definition 24 (valuation)** A *valuation* ($\sigma$) is a function from variable symbols to proper subsets of $T_\Sigma$ ($\sigma : \mathbf{VS} \to 2^{T_\Sigma}$).

**Definition 25 (interpretation of set expressions)** Given a valuation $\sigma$, the *standard interpretation* $I_\sigma$ of set expressions over $\Sigma$ and **VS** is defined as[14]

$$\begin{aligned}
I_\sigma(x) &= \sigma(x) \\
I_\sigma(a) &= \{a\} \\
I_\sigma(f(u_1, \dots, u_n)) &= \{f(t_i, \dots, t_n) \mid \forall i \in \{1, \dots, n\}.\ t_i \in I_\sigma(u_i)\} \\
I_\sigma(f^{-1}_{(k)}(u)) &= \{t \mid \exists t_1, \dots, t_n.\ t_k = t \ \wedge \ f(t_1, \dots, t_n) \in I_\sigma(u)\} \\
I_\sigma(e_1 \cup e_2) &= I_\sigma(e_1) \cup I_\sigma(e_2) \\
I_\sigma(\emptyset) &= \emptyset.
\end{aligned}$$

**Definition 26 (solution)** A valuation $\sigma$ is a *solution* of a set constraint $e_l \subseteq e_r$ iff

$$I_\sigma(e_l) \subseteq I_\sigma(e_r).$$

**Definition 27 (satisfaction)** A system of set constraints $S$ is *satisfiable* if there is some valuation $\sigma$ that is a solution of every constraint in $S$.

[12] In Definition 28 a restricted but equivalent characterisation is proposed.
[13] The symbol originally used in set constraint theory is $\perp$.
[14] Where symbols $\cup$ and $\emptyset$ are overloaded.

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**Definition 28 (co-definite set constraints)** A constraint $\varphi$ is a co-definite set constraint when it follows the syntax:

$$\begin{array}{rcl} \tau & ::= & x \mid f(u_1, \ldots, u_n) \mid \tau_1 \cup \tau_2 \mid \emptyset \\ \varphi & ::= & a \subseteq x \mid x \subseteq \tau \mid x \subseteq f^{-1}_{(k)}(u). \end{array}$$

We will use the notation $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ to refer to the system $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$.

### *5.2 Co-definite set constraints and partial predicates*

A partial predicate $\pi$ is interpreted as the set $\pi^{-1}(True) = \{x \in T_\Sigma \mid \pi(x) = True\}$ that is in the codomain of interpretations of set expressions. Conversely, if $S$ is an ideal, $\Pi(S)$ will denote its corresponding partial predicate, i.e. $\Pi(S)(x) = True$ if $x \in S$, otherwise $\Pi(S)(x) = \bot$. We will encode partial predicates as variables and the greatest solution of a system of co-definite set constraints.

**Example 29 (some basic partial predicates)** The following table shows how some partial predicates can be encoded ($z$, $s$, $n$ and $c$ refer, respectively, to constructor predicate transformers *zero*, *succ*, *nil* and *cons* as described in Definition 7):

| **partial predicate** | **system of set constraints** | **variable** |
|---|---|---|
| $hnfNat = z \sqcup s(any)$ | $\{hnf \subseteq x \cup y, x \subseteq Zero, y \subseteq Succ(_)\}$ | $hnf$ |
| $nfNat = z \sqcup s(nfNat)$ | $\{nf \subseteq x \cup y, x \subseteq Zero, y \subseteq Succ(nf)\}$ | $nf$ |
| $nfListNat = n \sqcup c(nfNat, nfListNat)$ | $\{nf \subseteq u \cup v, u \subseteq [], v \subseteq (nf' : nf),$ $nf' \subseteq x \cup y, x \subseteq Zero, y \subseteq Succ(nf')\}$ | $nf$ |
| $spine = n \sqcup c(any, spine)$ | $\{snf \subseteq u \cup v, u \subseteq [], v \subseteq (_ : snf)\}$ | $snf$ |

where _ represents fresh variables.

**Example 30 (dependency)** The intrinsic difficulty of working with dependency that was already patent in Mariño *et al*. (1993) can be put in relation with the syntax of set constraints. A typical property that can be represented with dependent demand patterns is that a pair is made of lists of the same length. For instance, a demand typing for function *zip*

```
zip :: ([a],[b]) -> [(a,b)]
zip ([],[]) = [];                    zip (x:xs,y:ys) = (x,y) : (zip (xs,ys))
```

is $zip$ : $spine \Leftarrow samelength$ where

```
samelength :: PP ([a],[b])
samelength ([],[]) = True;    samelength (x:xs,y:ys) = samelength (xs,ys)
```

The greatest solution to the following set constraint system for the variable $sl$ captures the dependent information of the partial predicate *samelength*:

$$\{sl \subseteq (l_1, l_2) \cup (l'_1, l'_2),\ l_1 \subseteq [],\ l_2 \subseteq [],\ l'_1 \subseteq x : xs,\ l'_2 \subseteq y : ys,\ (xs, ys) \subseteq sl\}.$$

But this system is not co-definite (last constraint has a binary function symbol in the left hand side). To get a co-definite set constraint system, the last constraint is substituted by two constraints: $xs \subseteq (,)^{-1}_{(1)}(sl)$ and $ys \subseteq (,)^{-1}_{(2)}(sl)$. With the substitution we have

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lost the dependency information. Nevertheless, the solution is correct with respect to the interpretation of the partial predicate in the following formal sense:

$$samelength^{-1}(True) \subseteq I_{\sigma}(sl)$$

where $\sigma$ is the greatest solution to the system of co-definite set constraints.

### *5.3 Generating co-definite set constraints*

DAC (*Demandedness Analysis for Curry*) is a tool that generates a system of co-definite set constraints from a given program. In this section we explain how DAC generates the system. Observe that the solving of a system of set constraints is completely independent of the application, i.e. the fact that we are encoding partial predicates is immaterial. Observe, as well, there could be other ways to generate correct systems of constraints.

First, we need to introduce the logic that relates the variables in the system of set constraints with the meaning of the program. This connection relies on the fact the partial predicates the user is interested in are defined as functions in the kernel language.

**Definition 31 (variable construction)** Given $f \in FS$ and $p \in FS$, with types $\tau_1 \to \tau_2$ and $PP\ \tau_2$, respectively, the infix operator ($\bullet$) is used to construct a new variable $p\bullet f \in \mathbf{VS}$ that represents the degree of evaluation demanded by $f$ to give a result as evaluated as $p$.

Auxiliary variables are introduced for different subterms in the program equations that define $f$:

- $p\bullet f.i$ refers to the demandedness information introduced by the $i$-th equation defining $f$
- $p\bullet f.i.pos$ refers to the subterm $lhs|_{pos}$ if $lhs$ is the left hand side of the $i$-th equation defining $f$.
- $p_f.i.pos$ refers to the subterm $rhs|_{pos}$ if $rhs$ is the right hand side of the $i$-th equation defining $f$.

Finally, the constraint generation algorithm generates variables of the form $(q\bullet_g)\bullet f$. Although these can be given a neat interpretation, the constraint solver will treat them as indivisible, so they will have to be transformed in some way in order to be useful.

The intuitive meaning of $d\bullet f$ is to denote (an approximation of) $d \circ f$.[15] This connection will be formalized below. The intuitive meaning of $p\bullet f.i.pos$ is the projection at position $pos$ of the set $p\bullet f.i$. Variables generated from positions in the right hand sides have a less evident meaning or, perhaps, more operational but they provide valuable information for compilation.

**Example 32** Figure 4 shows a system of constraints generated from function *plus* (Example 21) to give a result in normal form.

[15] Hence the choice of the symbol '$\bullet$'.

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$$\begin{aligned}
\text{nf}\bullet plus &\subseteq \text{nf}\bullet plus.1 \cup \text{nf}\bullet plus.2 && (1)\\
\text{nf}\bullet plus.1 &\subseteq (\text{nf}\bullet plus.1.1, \text{nf}\bullet plus.1.2) && (2)\\
\text{nf}\bullet plus.1.1 &\subseteq Zero && (3)\\
\text{nf}\bullet plus.1.2 &\subseteq \text{nf} && (4)\\
\text{nf}\bullet plus.2 &\subseteq (Succ(\text{nf}\bullet plus 2.1.1), \text{nf}\bullet plus.2.2) && (5)\\
Succ(\text{nf_}plus.2.1) &\subseteq \text{nf} && (6)\\
\text{nf}\bullet plus.2.1.1 &\subseteq \text{nf_}plus.2.1.1.1 && (7)\\
\text{nf}\bullet plus.2.2 &\subseteq \text{nf_}plus.2.1.1.2 && (8)\\
(\text{nf_}plus.2.1.1.1, \text{nf_}plus.2.1.1.2)) &\subseteq \text{nf}\bullet plus && (9)
\end{aligned}$$

Fig. 4. Set of constraints generated from program *plus*.

### *5.4 Generating systems of set constraints*

The generation scheme is presented here as a set of rules. These will be stated in a moderately informal way, in order to hide some of the details to the reader, especially those concerned with the handling of occurrence indices.

**Rule 1 (main function constraint)** Given a partial predicate symbol $p$ and function symbol $f$, with defining rules

$$f\ t_1 = b_1 \quad \ldots \quad f\ t_n = b_n$$

the following constraint is generated:

$$p\bullet f \subseteq p\bullet f.1 \cup \ldots \cup p\bullet f.n.$$

**Rule 2 (main rule constraint)** For every rule

$$f_i\ (t_1, \ldots, t_n) = b_i$$

the following constraint is added:

$$p\bullet f.i \subseteq (p\bullet f.i.1, \ldots, p\bullet f.i.n).$$

Notice that this is a *lossy* step, i.e. possible dependencies among the arguments through the body of the rule ($b_i$) can be lost. This means that an analyzer based on program transformation techniques can, at least theoretically, achieve a better accuracy.

**Rule 3 (head constraints)** For every rule

$$f_i\ (t_1, \ldots, t_n) = b_i$$

and for every $j \in \{1, \ldots, n\}$ the following constraint is added:

$$p\bullet f.i.j \subseteq \Delta(p, f, i.j, t_j),$$

where $\Delta$ is the function that constructs a set expression from a term by replacing every occurrence of a program variable with a demand variable decorated with its position, i.e.:

$$\begin{aligned}
\Delta(p, f, i, c(t_1, \ldots, t_m)) &= c(\Delta(p, f, i.1, t_1), \ldots, \Delta(p, f, i.m, t_m))\\
\Delta(p, f, i, x) &= p\bullet f.i.
\end{aligned}$$

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This step usually generates superfluous constraints of the form $v \subseteq v$ which can be discarded later.

**Rule 4 (body constraints)** We can distinguish several cases here:

1. (*The body is a variable*) If the rule is of the form

$$f_i\ (t_1, \ldots, t_n) = x$$

$x$ being a program variable, the constraint

$$p{\bullet}f.i.1.pos \subseteq p$$

where $pos$ is the position where $x$ occurs in the left hand side, is added to the system.
2. (*The body is a constant*) If the rule is of the form

$$f_i\ (t_1, \ldots, t_n) = k$$

$k$ being a constant, the constraint

$$k \subseteq p$$

is added to the system. This constraint will often be trivial.
3. (*The body is a function application*) This is the clumsiest case. To simplify the presentation, let us assume, without loss of generality, that the form of the rule is the following:

$$f_i\ (\bar{t}) = g\ (h_1(\bar{t}), \ldots, h_m(\bar{t})).$$

The constraints

$$\begin{aligned} p_f.i.1 &\subseteq (,)^{-1}_{(1)}(p{\bullet}g) \\ &\vdots \\ p_f.i.m &\subseteq (,)^{-1}_{(m)}(p{\bullet}g) \end{aligned}$$

are added to the system, and also the constraints :

$$\begin{aligned} p{\bullet}f.i &\subseteq p_f.i.1{\bullet}h_1 \\ &\vdots \\ p{\bullet}f.i &\subseteq p_f.i.m{\bullet}h_m. \end{aligned}$$

Notice that this step is responsible for the appearance of 'nested' demand variables.

**Rule 5 (simplification)** In this step two kind of actions are performed: the shortcut of transitive chains and the simplification of nested demand variables.

If a variable of the form $(p{\bullet}g){\bullet}f$ is found, and a concrete representation $p'$ for $(p{\bullet}g)$ is known – which is usually the case when $g$ is a data constructor – then it is replaced by $p'{\bullet}f$. This step is necessary when standard (i.e. *problem independent*) techniques for solving the constraint systems are going to be used.

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**Rule 6 (weakening)** Sometimes it is not easy to compute the $(p \bullet g)$ of the previous step, so some sort of approximation is necessary, i.e. using any $p''$ satisfying $p'' \sqsupseteq p'$ instead of $p'$. This is often possible. In the worst case *any* can be used.

The following result states the soundness of the analysis based on the solution of this set of constraints.

**Theorem 33** *(Soundness of the Analysis)* Let $S$ denote the system of constraints generated from a given program applying the rules above. Let $\sigma$ be a solution of $S$. For every variable $d \bullet f$ occurring in $S$ the following must hold:

$$\Pi[\![\sigma(d \bullet f)]\!] \sqsupseteq d \circ f.$$

We will just sketch the proof here. A more detailed explanation can be found in the electronic version. The idea is to apply a set of program transformation rules to $d \circ f$, for every possible combination of $d$ and $f$, so that the resulting program is structurally similar to $S$.

## 6 Application to code generation

For the sake of brevity, we will not develop the issues related to code generation in full here. A detailed discussion can be found in Mariño (2002), first in an abstract fashion – by means of an operational semantics driven by degrees of definiteness – and then in the context of a stack-based machine. Anyway, code generation from the demand information (represented by partial predicates) is a challenging task on its own and some of the details of a full compiler are still open.

The basic idea is that different, specialized versions of a given function can be compiled for different degrees of evaluation demanded on its result. For instance, if the main goal of a certain program is

```
?- mergeSort (f x)
```

the result must be shown in normal form, so a special version *mergeSort_nf* will be generated. In order to give a result in normal form, the argument to *mergeSort* must also be a total value, which implies that $f$ can also be replaced by an specialized version – $f_nf$ – and so on, i.e. demand is back-propagated from the result to the argument expressions.

*Some Experiments* The following example programs were executed on a stack-based narrowing machine (Moreno-Navarro *et al*. 1990). The narrowing machine is an extension of a purely functional machine enriched by mechanisms for unification andt.

Based on the implementation of the presented ideas on the stack-based narrowing machine, we have tried some example programs and measured their runtimes with the naive lazy approach and with our new approach. Additionally, we have measured the runtimes for eager narrowing.

We have investigated the following example programs: 1) the computation of all the sublists of *reverse* $[1, \ldots, n]$, and 2) the $n$-queens problem (using a simple generate and test approach). Both examples have the property that a lot of reevaluations are needed since demanded arguments are not evaluated in advance. Due to space limitation we omit

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Table 1. *Runtimes of the example programs in seconds*

| *example* | *n* | *eager* | *naive lazy* | *demand driven* |
|---|---|---|---|---|
| *sublists* | 10 | 0.47 | 1.00 | 0.62 |
| | 11 | 0.91 | 1.98 | 1.24 |
| | 12 | 1.86 | 3.96 | 2.42 |
| | 13 | 3.68 | 7.94 | 4.80 |
| | 14 | 7.42 | 15.82 | 9.60 |
| | 15 | 15.05 | 31.75 | 19.23 |
| *n queens* | 4 | 1.06 | 0.89 | 0.21 |
| | 5 | 14.50 | 10.55 | 2.60 |
| | 6 | 247.03 | 174.30 | 39.30 |

Table 2. *Results in a lazy producer-consumer scheme*

| *sublists* ∘ *reverse* $[1,\ldots,n]$ | *n* | *without demand anal.* | *with demand anal.* | *ratio* |
|---|---|---|---|---|
| | 3 | 41 | 36 | 1.13 |
| | 4 | 88 | 72 | 1.22 |
| | 5 | 183 | 141 | 1.29 |
| | 6 | 374 | 275 | 1.36 |
| | 7 | 757 | 538 | 1.40 |
| | 8 | 1524 | 1058 | 1.44 |

the code. The runtimes are depicted in Table 1. The examples show that the runtimes can be considerably improved, if the demanded arguments are evaluated in advance. Notice that in examples like *n*-queens, the lazy strategy is even better than the eager one. Moreover, bigger real examples have a lot of nested function calls, which implies a considerable risk of reevaluation.

Table 2 shows the results obtained with the *sublists* ∘ *reverse* example using the translation into Prolog, with and without code optimization based on demand analysis. The measures have been taken in discrete resolution steps.

## 7 Related work

Our original work on demand analysis (Mariño and Moreno-Navarro 1992; Moreno-Navarro *et al*. 1993; Mariño and Herranz 1993; Mariño *et al*. 1993) was based on the generation and solution of a set of *demand equations* that were solved in a domain of regular trees (*demand patterns*). Similar, in spirit, to the techniques presented in Section 5, a semantic justification was missing and the solving method was ad-hoc. Partial predicates provide the necessary semantic ground and the advances in set constraint resolution makes unnecessary to reinvent the wheel.

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With respect to partial predicates, the most striking similarity is with *projection analysis* (Wadler and Hughes 1993). However, the rationale and meaning for these two formalisms differ in some key aspects. A *projection*, in a domain-theoretic sense, is an idempotent approximation to the identity (in a given type), i.e. $\alpha :: \tau \to \tau$ is a projection (in $\tau$) iff $\alpha \sqsubseteq id$ and $\alpha \circ \alpha = \alpha$.

While partial predicates try to be an extension of classic strictness analysis, projection analysis are designed to capture the property that a given function is invariant under certain program transformations. The typical example is *head-strictness*, the property that a function on lists gives the same results when the list constructor in its argument is replaced by a version strict in its first argument. Mathematically, this transformation is a projection $\mathsf{H} :: [a] \to [a]$, so the property of $f$ being head-strict is expressed as $f = f \circ \mathsf{H}$. In general, projection analysis studies properties of the form $\alpha \circ f = \alpha \circ f \circ \beta$, where $\alpha$ and $\beta$ are projections. These are abbreviated as $f : \alpha \Rightarrow \beta$.

Properties expressible in both formalisms are different. First, let us show that head strictness cannot be represented by a partial predicate typing.

**Theorem 34** There is no pair of partial predicates $\pi_1, \pi_2$ such that the set of functions $\{f | f : \pi_1 \Leftarrow \pi_2\}$ coincides with that of the head-strict ones.

*Proof*
Let us note that the property of being head-strict is just 'too polymorphic' as it does not take into account the type of the result, so equivalence just makes sense fixing a particular type, i.e. considering just the head-strict functions for a given type $[\sigma] \to \tau$. This makes the proof shorter, as we can restrict ourselves to the type $[Bool] \to Bool$. There will be just five possibilities for $\pi_2$: *nothing*, *true*, *false*, *hnf* and *any*. The key to the proof is in considering the functions *any*, *nothing* (which are head-strict) and *spine* (which is not). Any combination of partial predicates which would hold for both *any* and *nothing* would also hold for *spine*, contradiction. □

Projections, on the other hand, are able to express partial predicate typings, but only if a tricky artifact is added to the formalism: assuming the existence in the domain of a new element ($\lightning$) less defined than $\bot$. This had to be introduced by Wadler and Hughes in order to capture classic strictness with projections, but complicates the formalism in several ways. The following result holds assuming programs are $\lightning$-strict:

**Theorem 35** For every pair of partial predicates $\pi_1, \pi_2$ there is a pair of projections $\alpha, \beta$ such that the set $\{f | f : \pi_1 \Leftarrow \pi_2\}$ coincides with $\{f | f : \alpha \Rightarrow \beta\}$ – under reasonable type restrictions.

*Proof*
The proof is constructive. Define $\alpha$ and $\beta$ in the following way:

$$\begin{array}{llll} \alpha\ x = x & \text{if } \pi_2\ (f\ x) = true & \beta\ x = x & \text{if } \pi_1\ x = true \\ \alpha\ x = \lightning & \text{otherwise} & \beta\ x = \lightning & \text{otherwise.} \end{array}$$

Let us examine both implications:

(i) $(f : \pi_1 \Leftarrow \pi_2 \implies f : \alpha \Rightarrow \beta)$
There are two possibilities for any argument $x$ to $f$:

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a) $(x \in \pi_1)$ Trivial: $(\alpha \circ f \circ \beta)\ x = (\alpha \circ f)(\beta\ x) = (\alpha \circ f)\ x$.

b) $(x \notin \pi_1)$ In this case we know that $f\ x \notin \pi_2$. So, in one hand we have:

$$(\alpha \circ f \circ \beta)\ x = \alpha(f(\beta\ x)) = \alpha(f\ \lightning) = \alpha\ \lightning = \lightning.$$

On the other hand, using the fact that $f\ x \notin \pi_2$, $\alpha(f\ x) = \lightning$.

(ii) $(f : \alpha \Rightarrow \beta \implies f : \pi_1 \Leftarrow \pi_2)$

Using *reductio ad absurdum*: suppose there is some $z$ s.t. $z \notin \pi_1$ and $f\ z \in \pi_2$. Then it is trivial to show that $(\alpha \circ f \circ \beta)\ z = \lightning$ and $(\alpha \circ f)\ z = f\ z$, contradiction. □

Projections in the lifted domain are no longer expressible in source code, precluding the possibility of using program transformation or the other techniques that are applicable to partial predicate typings.

The use of program transformation techniques for program analysis is used in other approaches, like *abstract compilation*, but to our best knowledge, the application of the fold/unfold method of program transformation for program analysis is a novel idea. The only similar approach appears in Gallagher and Peralta (2000) to develop a type inference system for Prolog, and Comini *et al*. (2000) to verify program properties.

There is also some existing work on using constraint generation for this kind of problems. In Sekar and Ramakrishnan (1995), two degrees of definiteness are defined: normal form and head normal form, which leads to the notion of *e*-demand (normal form needed) and *d*-demand (head normal form needed). Recursive equations for computing how these degrees of demand are propagated are generated for a given program, based on an operational semantics. The authors also mention the possibility of using demand analysis for sequentiality recovery, although the idea is not developed there.

## 8 Open issues and future work

The importance of demandedness analysis goes beyond functional-logic languages. In Falaschi *et al*. (2000) dependent demand patterns are used for the analysis of concurrent (constraint) logic languages.

The question whether program transformation tools can be used for program analysis following the techniques presented here is still open, although several problems appear. In the first place, deciding on the equality of functions is harder, in general, than checking the equality of set expressions. Second, although the theory behind program transformation is well developed, practical implementations are scarce. However, this is an active area and we plan to study the possibility of adapting the tools by Vidal's group (Ramos *et al*. 2005) to serve this purpose.

Another possible extension of this work is to study the application of partial predicates to other analysis problems, like groundness, etc.

The extension of the analysis when higher order functions are used deserves an additional discussion. In fact, some complications do appear if higher order definitions are introduced. Let us consider an example involving curried definitions.

Take, for instance, the standard definition of the addition of Peano naturals of Example 21.

An interesting property that we would like to express is the fact that the first argument must be evaluated to head normal form in order to get a result in head normal form. This

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is very simple for the noncurried form: $plus \;:\; (hnfNat \times any) \Leftarrow hnfNat$, but it is not clear at all how to express that for the curried version. In first place, $(+)$ maps naturals to *a new function*, and it is not this function we are interested in, but the result of applying it to any other natural number. To grab the problem more formally, we will make use of the following lemma.

**Lemma 36 (currying lemma)** Let $(f\ \circ)$ denote $\lambda\, x.\ f \circ x$. Then, the following holds:

$$\text{curry}\ (f \circ g) \;=\; (f\ \circ) \circ (\text{curry}\ g).$$

What we are looking for is a property of the form: $(+) : hnfNat \Leftarrow \pi$ and what we actually have is $plus : (hnfNat \times any) \Leftarrow hnfNat$. This is equivalent to:

$$hnfNat \circ plus \sqsubseteq (hnfNat \times any).$$

As *curry* is continuous, we can curry both sides of the inequality:

$$curry\ (hnfNat \circ plus) \sqsubseteq curry\ (hnfNat \times any)$$

and using the lemma above:

$$(hnfNat\ \circ) \circ (+) \sqsubseteq curry\ (hnfNat \times any)$$

or equivalently

$$(+) : curry\ (hnfNat \times any) \Leftarrow (hnfNat\ \circ).$$

Well, this gives *essentially* the same information as the demand typing for the noncurried version, but there is a problem: the functions in the typing are no longer partial predicates, i.e. they are not in the domain $\tau \to Two$.

There are essentially two ways of dealing with these problems in practice. One possibility is to avoid higher order definitions as much as possible, translating curried versions to noncurried ones – and vice-versa with the results of the analysis.

The other possibility is to generalize demand types to pairs of functions in the domain $\tau \to \tau'$. Although the theoretical interest of this lifting to higher order domains is apparent, and a higher order metalanguage seems feasible, the practical use will be very restricted, as the techniques in Section 4.2 will not be applicable.

## 9 Conclusion

We have presented a semantic framework for the denotation of demand properties, decoupling it from its abstraction or any implementation detail of the analysis. In spite of defining a new language and an associated demand logic (cf. the strictness logic in Benton (1992)), properties are expressed in the language under study and problem independent techniques are used as proof methods: equational reasoning or set constraint solving.

The collection of analysis that can be modelled includes classic strictness analysis, Wadler's four point domain, etc. In particular it allows to describe demand analysis that is very important for the efficient implementation of several aspects of functional-logic languages: efficient implementation of lazy narrowing (Moreno-Navarro *et al*. 1993); Mariño *et al*. 1993), compilation of nonsequential programs (Mariño and Moreno-Navarro 2000) or the lazy management of default rules (Moreno-Navarro 1996). The formalism has been

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used to prove the correctness of a method based on set constraint solving, in a constructive way. It can also be used to generate domains suitable for abstract interpretation (uniform predicates).

Furthermore, the formalism is quite intuitive as it resembles the language to reason about and uses program transformation techniques. We have also shown how polymorphism and higher-order properties can be managed – at least theoretically – in this framework, although the extension for the analysis of higher-order program presents some challenges.

An original feature of this research is that it is completely based on a declarative (model theoretically) denotational semantics of the language, rather than on an operational one.[16] From a practical point of view, performing the analysis on operational data is often easier, but considering that the domain of properties can be understood in a purely declarative setting, we wanted to explore the possibility of performing the analysis without using a particular operational semantics.

We also felt that the applications of set constraints to program analysis have been biased towards problems stated in an operational fashion, and that deriving set constraints from semantic equations was an original approach and a challenge worth taking up.

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[16] We include here denotational presentations of essentially operational semantics like Zartmann (1997).

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