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Existence of a measurable saturated compensation function between subshifts and its applications

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Abstract. We show the existence of a bounded Borel measurable saturated compensation function for any factor map between subshifts. As an application, we find the Hausdorff dimension and measures of full Hausdorff dimension for a compact invariant set of an expanding non-conformal map on the torus given by an integer-valued diagonal matrix. These problems were studied in [**23**] for a compact invariant set whose symbolic representation is a shift of finite type under the condition of the existence of a saturated compensation function. By using the ergodic equilibrium states of a constant multiple of a Borel measurable compensation function, we extend the results to the general case where this condition might not hold, presenting a formula for the Hausdorff dimension for a compact invariant set whose symbolic representation is a subshift and studying invariant ergodic measures of full dimension. We study uniqueness and properties of such measures for a compact invariant set whose symbolic representation is a topologically mixing shift of finite type.

1. *Introduction*

Let (X, σ_X) and (Y, σ_Y) be subshifts. Let $\pi : X \to Y$ be a factor map, i.e. π is a continuous and surjective function from X to Y that satisfies $\pi \circ \sigma_X = \sigma_Y \circ \pi$. A function $F \in C(X)$ is a *compensation function* for $(\sigma_X, \sigma_Y, \pi)$ if

$$\sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \int (F + \phi \circ \pi)\, d\mu \right\} = \sup_{m \in M(Y,\sigma_Y)} \left\{ h_m(\sigma_Y) + \int \phi\, dm \right\} \quad (1.1)$$

for all $\phi \in C(Y)$, where $M(X, \sigma_X)$ and $M(Y, \sigma_Y)$ are the sets of invariant probability measures on X and Y, respectively. If $F = G \circ \pi$ with $G \in C(Y)$, then $G \circ \pi$ is a *saturated compensation function*. Compensation functions were introduced by Boyle and Tuncel [**4**] and studied by Walters [**22**] in connection with relative pressure. If (X, σ_X), (Y, σ_Y) are shifts of finite type, then there always exists a compensation function [**22**]. An example of a factor map between shifts of finite type without a saturated compensation function was

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given by Shin [**19**] and necessary and sufficient conditions for the existence of a saturated compensation function were also studied by Shin [**18**]. In this paper, we first consider a *Borel measurable* function $F : X \to \mathbb{R}$ that satisfies equation (1.1). If there is such an F, we say that F is a *Borel measurable compensation function*. F is a *Borel measurable saturated compensation function* if $F = G \circ \pi$, where $G : Y \to \mathbb{R}$ is Borel measurable. In Theorem 3.2, we show that for (X, σ_X), (Y, σ_Y) subshifts there always exists a bounded Borel measurable saturated compensation function and characterize the equilibrium states. Replacing F in equation (1.1) by this Borel measurable compensation function, in Theorem 3.9 we also show that equation (1.1) holds for a larger class of potentials on Y called subadditive potentials (see §1 for the definition of subadditive potentials). Using Borel measurable saturated compensation functions, we then study for a fixed $\alpha > 0$ the measures that maximize the weighted entropy functional $\phi_\alpha(\mu) = h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)$. Finding such measures is useful in problems on Hausdorff dimension (see [**8**]). When there is a saturated compensation function between subshifts, the measures that maximize ϕ_α are, according to Shin [**20**], the equilibrium states of a constant multiple of a saturated compensation function. We extend the results for any subshifts without assuming the existence of a saturated compensation function (Proposition 3.14 and Theorem 3.15).

As an application, we study the problem on dimensions of compact invariant sets of non-conformal expanding maps; in particular, we consider the endomorphism of the torus given by $T(x, y) = (lx \bmod 1, my \bmod 1)$, $l > m \geq 2$, $l, m \in \mathbb{N}$. Throughout this paper, by a measure μ of full dimension for a compact T-invariant set K, we mean that μ is a Borel probability measure of full Hausdorff dimension for K and $\mu(K) = 1$.

In [**23**], saturated compensation functions were used to find the Hausdorff dimension and measures of full dimension of some compact T-invariant sets, giving a systematic way to approach the problem (see [**23**] for more details). As a result, existence of a saturated compensation function was one of the main difficulties. We circumvent this issue by using a *Borel measurable* saturated compensation function (which always exists between subshifts by Theorem 3.2) instead of a *continuous* saturated compensation function.

The compact invariant sets considered in this paper generalize the sets studied by Bedford [**3**] and McMullen [**13**], who independently answered the question on Hausdorff dimension for compact T-invariant subsets which we call *NC carpets* (see §4 for the definition). Uniqueness of the invariant ergodic measure of full dimension for an NC carpet was shown by Kenyon and Peres [**10**]. Using a coding map constructed by a Markov partition for T, one obtains a symbolic representation of the carpet which is a full shift on finitely many symbols. Kenyon and Peres [**9**] studied the Hausdorff dimension of a more general set whose symbolic representation is a shift of finite type (*SFT-NC carpet*) and a sofic shift (*sofic-NC carpet*). Using a different approach, SFT-NC carpets were also considered in [**23**]. The results in [**23**] gave a formula for the Hausdorff dimension for an SFT-NC carpet for which a saturated compensation function exists (when it is represented in symbolic dynamics), and uniqueness of T-invariant ergodic measures of full dimension was studied by using the ergodic equilibrium states of a constant multiple of a saturated compensation function.

In this paper, we extend the formula for a compact T-invariant set whose symbolic representation is a subshift (X, σ_X) (Theorem 4.4). We can do this because there always

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exists a Borel measurable saturated compensation function between subshifts. In order to find T-invariant ergodic measures of full dimension, we study the ergodic equilibrium states of a constant multiple of a Borel measurable saturated compensation function, which turns out to be a potential $-\Phi \circ \pi$ on (X, σ_X), where $\Phi = \{\phi_n\}_{n=1}^{\infty}$ is a sequence of real-valued bounded Borel measurable functions on (Y, σ_Y) that satisfies the subadditivity condition (Theorem 4.4, see also §2 for the definition of a sequence of real-valued functions). For a compact T-invariant set whose symbolic representation is a topologically mixing shift of finite type, we study uniqueness of the T-invariant ergodic measure of full dimension and properties of the unique measure (Theorem 4.14). In this case, a Borel measurable saturated compensation function can be replaced by a much nicer function (see Corollary 3.11). Using it, we study the equilibrium states for a potential $-\Phi \circ \pi$ on (X, σ_X), where $\Phi = \{\phi_n\}_{n=1}^{\infty}$ is a sequence of real-valued continuous functions on (Y, σ_Y) that satisfies the subadditivity condition (Theorem 4.1). Here Φ becomes an almost additive potential on (Y, σ_Y) under a certain condition (see §4). In order to study the equilibrium states for a potential $-\Phi \circ \pi$, we show that a constant multiple of $\Phi = \{\phi_n\}_{n=1}^{\infty}$ has a unique equilibrium state which is Gibbs regardless of the condition of almost additivity, generalizing the results in [**23**]. We use the work on almost additive potentials by Barreira [**2**] and Mummert [**14**] and on subadditive potentials by Cao *et al* [**5**].

After I submitted the paper, I was informed about the recent paper [**15**] by Olivier. He gave a formula (which is similar to the formula in [**23**]) for a subshift-NC carpet under certain conditions and studied a uniqueness property for sofic-NC carpets (Remarks 4.15). I also became aware of the recent preprints [**1, 7**]. Feng [**7**] studied a similar question for $\phi_{\alpha,\beta,f}(\mu) = \mu(f) + \alpha h_\mu(\sigma_X) + \beta h_{\pi\mu}(\sigma_Y)$, $f \in C(X)$, where (X, σ_X) is a subshift, and also higher-dimensional cases. In comparison, Barral and Feng [**1**] studied the case in which f is replaced by a more generalized potential Φ. A formula for Hausdorff dimension which is slightly different from our formula is deduced in [**1, 7**] and uniqueness for a subshift which satisfies weak specification is shown in [**7**].

2. *Background*

We give a brief overview of the recent results in pressure theory for almost additive potentials and subadditive potentials [**2, 5, 14**]. These generalize the work of Ruelle and Walters on the variational principle for continuous functions. For notation and terminology not explained here including the definitions of a full shift and a shift of finite type, see [**12, 21**]. Throughout this paper, we consider one-sided subshifts. (X, σ_X) is a subshift if X is a closed shift-invariant subset of $\{1, \ldots, k\}^{\mathbb{N}}$ for some $k \geq 1$, i.e. $\sigma(X) \subseteq X$, where the shift $\sigma_X : X \to X$ is defined by $\sigma_X(x) = x'$, for $x = (x_n)_{n=1}^{\infty}$, $x' = (x'_n)_{n=1}^{\infty} \in X$, $x'_n = x_{n+1}$ for all $n \in \mathbb{N}$. Define a metric d on X by $d(x, x') = 1/2^k$ if $x_i = x'_i$ for all $1 \leq i \leq k$ and $x_{k+1} \neq x'_{k+1}$, $d(x, x') = 1$ if $x_1 \neq x'_1$, and $d(x, x') = 0$ otherwise. For each $n \in \mathbb{N}$, denote by $B_n(X)$ the set of all n-blocks that occur in points in X. $x_1 \cdots x_n$ is an allowable word of length n if $x_1 \cdots x_n \in B_n(X)$. If $x_1 \cdots x_n \in B_n(X)$, then $[x_1 \cdots x_n]$ is a cylinder in X. Herein, we assume that any shift of finite type (X, σ_X) is one-step, i.e. there exists a set F of forbidden blocks of length less than or equal to 2 such that $X = \{x \in \{1, \ldots, k\}^{\mathbb{N}} : \omega \text{ does not appear in } x \text{ for any } \omega \in F\}$. Denote by $M(X, \sigma_X)$ the collection of all σ_X-invariant Borel probability measures on X and by $\mathrm{Erg}(X, \sigma_X)$ all ergodic members of $M(X, \sigma_X)$.

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Let (X, σ_X) be a subshift and $f : X \to \mathbb{R}$ a bounded Borel measurable function. A σ_X-invariant Borel probability measure $\bar{\mu}$ on X is an *equilibrium state* for f if

$$h_{\bar{\mu}}(\sigma_X) + \int f \, d\bar{\mu} = \sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \int f \, d\mu \right\}.$$

Denote by $M_f(X, \sigma_X)$ the collection of all equilibrium states for f. Let $\Phi = \{\phi_n\}_{n=1}^\infty$ be a sequence of real-valued continuous functions on (X, σ_X). Barreira [**2**] and Mummert [**14**] independently considered almost additive potentials. $\Phi = \{\phi_n\}_{n=1}^\infty$ is an *almost additive potential* if there exists a constant $C > 0$ such that for every $n, m \in \mathbb{N}$ and for every $x \in X$, $-C + \phi_n(x) + \phi_m(\sigma_X^n x) \le \phi_{n+m}(x) \le C + \phi_n(x) + \phi_m(\sigma_X^n x)$. Let

$$\gamma_n(\Phi) = \sup\{|\phi_n(x) - \phi_n(x')| : x_i = x_i' \text{ for all } 1 \le i \le n\}.$$

Then $\Phi = \{\phi_n\}_{n=1}^\infty$ has *bounded variation* if $\sup_{n \in \mathbb{N}} \gamma_n(\Phi) < \infty$. The notion of Gibbs measure for a continuous function is generalized (see [**2, 14**]).

Definition 2.1. Let $\Phi = \{\phi_n\}_{n=1}^\infty$ be an almost additive potential on a subshift (X, σ_X). A Borel probability measure μ on X is a *Gibbs measure* for Φ if there exist $C > 0$ and P such that

$$\frac{1}{C} < \frac{\mu([x_1 x_2 \cdots x_n])}{e^{-nP + \phi_n(x)}} < C \tag{2.1}$$

for every $x \in X$ and $n \in \mathbb{N}$.

In [**2, 14**], the topological pressure $P(\Phi)$ of an almost additive potential Φ satisfying a certain condition was defined and the variational principle and uniqueness of equilibrium states under some conditions were studied. We summarize the results that we shall need from [**2, 14**].

THEOREM 2.2. [**2, 14**] *Let (X, σ_X) be a topologically mixing shift of finite type and $\Phi = \{\phi_n\}_{n=1}^\infty$ an almost additive potential on (X, σ_X) with bounded variation. Then*

$$P(\Phi) = \sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \lim_{n \to \infty} \frac{1}{n} \int \phi_n \, d\mu \right\}, \tag{2.2}$$

where $P(\Phi) = \lim_{n\to\infty}(1/n) \log(\sum_{i_1 \cdots i_n \in B_n(X)} \sup e^{\phi_n(x)})$, where the supremum is taken over all $x \in [i_1 \cdots i_n]$, $n \in \mathbb{N}$. There exists a unique σ_X-invariant measure that attains the maximum in equation (2.2). It is Gibbs for Φ and mixing.

The variational principle for subadditive potentials was studied by Cao *et al* [**5**]. Let $\Phi = \{\phi_n\}_{n=1}^\infty$ be a sequence of real-valued continuous functions on (X, σ_X). Suppose that Φ satisfies the subadditivity condition, i.e. for every $n, m \in \mathbb{N}$ and $x \in X$, $\phi_{n+m}(x) \le \phi_n(x) + \phi_m(\sigma_X^n x)$. Then Φ is a *subadditive potential*. We note that if $\phi \in C(X)$, defining $\phi_n(x) = \sum_{k=0}^{n-1} \phi \circ \sigma_X^k$, $\Phi = \{\phi_n\}_{n=1}^\infty$ is subadditive. We define the topological pressure of Φ using separated sets (see [**5**]). Let $\epsilon > 0$. A set E is called an (n, ϵ) separated subset of X with respect to σ_X if $\max_{0 \le i \le n-1} d(\sigma_X^i(x), \sigma_X^i(y)) > \epsilon$ for all $x, y \in E$, $x \ne y$. Define

$$P_n(\sigma_X, \Phi, \epsilon) = \sup\left\{ \sum_{x \in E} e^{\phi_n(x)} : E \text{ is an } (n, \epsilon) \text{ separated subset of } X \right\}$$

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and

$$P(\sigma_X, \Phi, \epsilon) = \limsup_{n\to\infty} (1/n) \log P_n(\sigma_X, \Phi, \epsilon).$$

Define the *subadditive topological pressure of* Φ with respect to σ_X by $P(\sigma_X, \Phi) = \lim_{\epsilon\to 0} P(\sigma_X, \Phi, \epsilon)$.

THEOREM 2.3. (Special case of Theorem 1.1 in [**5**]) *Let* $\Phi = \{\phi_n\}_{n=1}^{\infty}$ *be a subadditive potential on a subshift* (X, σ_X). *Then*

$$P(\sigma_X, \Phi) = \sup_{\mu\in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, d\mu \right\}.$$

$P(\sigma_X, \Phi) = -\infty$ *if and only if* $\lim_{n\to\infty}(1/n) \int \phi_n \, d\mu = -\infty$ *for all* $\mu \in M(X, \sigma_X)$.

A σ_X-invariant Borel probability measure $\bar{\mu}$ on X is an *equilibrium state* for Φ if

$$h_{\bar{\mu}}(\sigma_X) + \lim_{n\to\infty} (1/n) \int \phi_n \, d\bar{\mu} = \sup_{\mu\in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \lim_{n\to\infty} (1/n) \int \phi_n \, d\mu \right\}.$$

We denote the collection of all equilibrium states for Φ by $M_\Phi(X, \sigma_X)$. A Borel probability measure μ is a *Gibbs measure* for a subadditive potential Φ on (X, σ_X) if the inequalities (2.1) in Definition 2.1 are satisfied.

Next we summarize basic definitions. Let (X, σ_X) and (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. If the ith position of the image of x under π depends only on x_i, then π is a one-block factor map. Throughout the paper, we assume that π is a one-block factor map. For each $n \in \mathbb{N}$ and $y = (y_1 \cdots y_n \cdots) \in Y$, denote by $E_n(y)$ a set consisting of exactly one point from each cylinder $[x_1 \cdots x_n]$ in X such that $\pi([x_1 \cdots x_n]) \subseteq [y_1 \cdots y_n]$. We note that for $y, y' \in Y$ with $y_i = y'_i$ for all $1 \le i \le n$, the cardinality of $E_n(y)$ is equal to that of $E_n(y')$. Denote by $|\pi^{-1}[y_1 \cdots y_n]|$ the cardinality of the set $E_n(y)$. Denote by $D_n(y)$ a set consisting of one point from each non-empty set $\pi^{-1}(y) \cap [i_1 \cdots i_n]$ in X and by $|D_n(y)|$ the cardinality of the set $D_n(y)$. Using this notation, for $y = (y_1 \cdots y_n \cdots) \in Y$, $|D_n(y)| \le |\pi^{-1}[y_1 \cdots y_n]|$. If (X, σ_X) is an irreducible shift of finite type, $|\pi^{-1}[y_1 \cdots y_n]|$ is the number of blocks $x_1 \cdots x_n$ of length n in X that are mapped to the block $y_1 \cdots y_n$ in Y.

Using the above notation, we now review some results on relative pressure. Relative pressure was studied by Ledrappier and Walters [**11**].

THEOREM 2.4. [**11**] (Special case of the relative variational principle) *Let* (X, σ_X), (Y, σ_Y) *be subshifts and* $\pi : X \to Y$ *a factor map. Let* $f \in C(X)$ *and* $m \in M(Y, \sigma_Y)$. *Then there exists a Borel measurable function* $P(\sigma_X, \pi, f) : Y \to \mathbb{R}$ *such that*

$$\int P(\sigma_X, \pi, f) \, dm = \sup \left\{ h_\mu(\sigma_X) - h_m(\sigma_Y) + \int f \, d\mu : \mu \in M(X, \sigma_X) \text{ and } \pi\mu = m \right\}.$$

$\mu \in M(X, \sigma_X)$ is a *measure of maximal relative entropy* over $m \in M(Y, \sigma_Y)$ if μ attains the maximum in $\sup\{h_\mu(\sigma_X) : \mu \in M(X, \sigma_X), \pi\mu = m\}$. The properties of the Borel measurable function $P(\sigma_X, \pi, f)$ were studied by Walters [**22**].

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THEOREM 2.5. (Special case of Theorem 4.6 in [**22**]) *Let* (X, σ_X), (Y, σ_Y) *be subshifts and* $\pi : X \to Y$ *a factor map. For each* $y \in Y$,

$$P(\sigma_X, \pi, 0)(y) = \limsup_{n\to\infty} \frac{1}{n} \log |D_n(y)|.$$

It is not easy to calculate $|D_n(y)|$ in general. Some other simpler function can be substituted for $P(\sigma_X, \pi, 0)$ in some cases.

THEOREM 2.6. (Special case of corollary in [**17**]) *Let* (X, σ_X) *be an irreducible shift of finite type,* (Y, σ_Y) *a subshift and* $\pi : X \to Y$ *a factor map. Then*

$$P(\sigma_X, \pi, 0)(y) = \limsup_{n\to\infty} \frac{1}{n} \log |\pi^{-1}[y_1 \cdots y_n]|,$$

almost everywhere with respect to every σ_Y*-invariant Borel probability measure on* Y.

3. *Existence of a Borel measurable saturated compensation function*

We first study the existence of a bounded Borel measurable saturated compensation function for a factor map between subshifts. Theorem 3.2 extends some results of Walters [**22**] on (continuous) compensation functions.

LEMMA 3.1. *Let* (X, σ_X), (Y, σ_Y) *be subshifts and* $\pi : X \to Y$ *a factor map. For* $m \in M(Y, \sigma_Y)$*, there exists a measure* μ *of maximal relative entropy over* m.

Proof. This is clear from the upper semi-continuity of the entropy map. □

THEOREM 3.2. *Let* (X, σ_X), (Y, σ_Y) *be subshifts and* $\pi : X \to Y$ *a factor map. Define* $F : Y \to \mathbb{R}$ *by* $F(y) = P(\sigma_X, \pi, 0)(y)$. *Then* $-F \circ \pi : X \to \mathbb{R}$ *is a bounded Borel measurable saturated compensation function for* $(\sigma_X, \sigma_Y, \pi)$*, i.e.*

$$\sup_{\mu\in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \int (-F \circ \pi + \phi \circ \pi)\, d\mu \right\} \tag{3.1}$$

$$= \sup_{m\in M(Y,\sigma_Y)} \left\{ h_m(\sigma_Y) + \int \phi\, dm \right\} \tag{3.2}$$

for all $\phi \in C(Y)$. *Then* $M_{-F\circ\pi+\phi\circ\pi}(X, \sigma_X)$ *is non-empty.* μ *is an equilibrium state for* $-F \circ \pi + \phi \circ \pi$ *if and only if (i)* $\pi\mu$ *is an equilibrium state for* ϕ *and (ii)* μ *is a measure of maximal relative entropy over* $\pi\mu$.

Remark 3.3. We will see in §4 that if we let $\phi_n(y) = \log |D_n(y)|$, then $-\int F \circ \pi\, d\mu = -\lim_{n\to\infty}(1/n) \int \phi_n \circ \pi\, d\mu$ for all $\mu \in M(X, \sigma_X)$.

Proof of equality (3.2) in Theorem 3.2. By Theorem 2.4 with $f = 0$, if $m \in M(Y, \sigma_Y)$, then

$$\sup\{h_\mu(\sigma_X) : \mu \in M(X, \sigma_X), \pi\mu = m\} = h_m(\sigma_Y) + \int_Y P(\sigma_X, \pi, 0)\, dm. \tag{3.3}$$

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If $\phi \in C(Y)$, then, since $P(\sigma_X, \pi, 0)$ is a function on Y,

$$\begin{aligned}
&\sup\left\{h_\mu(\sigma_X) + \int (-P(\sigma_X, \pi, 0) \circ \pi + \phi \circ \pi)\, d\mu : \mu \in M(X, \sigma_X), \pi\mu = m\right\} \\
&\quad = \sup\{h_\mu(\sigma_X) : \mu \in M(X, \sigma_X), \pi\mu = m\} - \int P(\sigma_X, \pi, 0)\, dm + \int \phi\, dm \\
&\quad = h_m(\sigma_Y) + \int \phi\, dm.
\end{aligned}$$

By taking the supremum over $m \in M(Y, \sigma_Y)$, we obtain equality (3.2). Theorem 2.5 implies that $P(\sigma_X, \pi, 0)$ is bounded by $\log S$, where S is the number of symbols in X. □

The rest of Theorem 3.2 follows from the following three lemmas. In the lemmas, we assume that (X, σ_X) and (Y, σ_Y) are subshifts, $\pi : X \to Y$ is a factor map and $F(y) = P(\sigma_X, \pi, 0)(y)$ for each $y \in Y$.

LEMMA 3.4. *Suppose that m is an equilibrium state for $\phi \in C(Y)$. Let μ be a measure of maximal relative entropy over m. Then μ attains the maximum in the expression (3.1). Therefore, $M_{-F\circ\pi+\phi\circ\pi}(X, \sigma_X)$ is non-empty.*

Proof. Let m be an equilibrium state for $\phi \in C(Y)$. Since μ is a measure of maximal relative entropy over m,

$$\begin{aligned}
&h_\mu(\sigma_X) - \int P(\sigma_X, \pi, 0) \circ \pi\, d\mu + \int \phi \circ \pi\, d\mu \\
&\quad = h_\mu(\sigma_X) - \sup\{h_{\bar\mu}(\sigma_X) - h_{\pi\mu}(\sigma_Y) : \bar\mu \in M(X, \sigma_X), \pi\bar\mu = \pi\mu\} + \int \phi \circ \pi\, d\mu \\
&\quad = h_\mu(\sigma_X) - \sup\{h_{\bar\mu}(\sigma_X) : \bar\mu \in M(X, \sigma_X), \pi\bar\mu = \pi\mu\} + h_{\pi\mu}(\sigma_Y) + \int \phi \circ \pi\, d\mu \\
&\quad = h_{\pi\mu}(\sigma_Y) + \int \phi\, d\pi\mu = \sup_{\nu \in M(Y,\sigma_Y)} \left\{h_\nu(\sigma_Y) + \int \phi\, d\nu\right\}.
\end{aligned}$$

□

LEMMA 3.5. *Suppose that m is an equilibrium state for $\phi \in C(Y)$ and μ is a measure of maximal relative entropy over m. Then μ is an equilibrium state for $-F \circ \pi + \phi \circ \pi$.*

Proof. This is clear from the proof of Lemma 3.4. □

LEMMA 3.6. *Suppose that μ is an equilibrium state for $-F \circ \pi + \phi \circ \pi$, $\phi \in C(Y)$. Then $\pi\mu$ is an equilibrium state for ϕ and μ is a measure of maximal relative entropy over $\pi\mu$.*

Proof. We use similar arguments as in the proof of Corollary 3.4 [**22**], replacing $F \in C(X)$ in Corollaries 3.3 and 3.4 [**22**] by $-P(\sigma_X, \pi, 0) \circ \pi$ and using equation (3.3). □

Next we will show that Theorem 3.2 is still valid when we replace $\phi \in C(Y)$ in the expressions (3.1) and (3.2) by subadditive potentials on (Y, σ_Y).

We recently found the preprint [**6**] by Feng and Huang and the upper semi-continuity in the proof of Proposition 3.7(2) is shown in a more general setting in Proposition 3.1(2) [**6**] for a larger class of potentials that contains subadditive potentials.

PROPOSITION 3.7. *Let $\Phi = \{\phi_n\}_{n=1}^\infty$ be a subadditive potential on a subshift (Y, σ_Y). Then:*

(1) *$M_\Phi(Y, \sigma_Y)$ is convex;*

(2) *$M_\Phi(Y, \sigma_Y)$ is non-empty and compact;*

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(3) *the extreme points of* $M_\Phi(Y, \sigma_Y)$ *are precisely the ergodic members of* $M_\Phi(Y, \sigma_Y)$*;*
(4) $\sup_{\mu\in M(Y,\sigma_Y)}\{h_\mu(\sigma_Y) + \lim_{n\to\infty}(1/n)\int \phi_n\, d\mu\}$
$= \sup_{\mu\in Erg(Y,\sigma_Y)}\{h_\mu(\sigma_Y) + \lim_{n\to\infty}(1/n)\int \phi_n\, d\mu\}$*;*
(5) $M_\Phi(Y, \sigma_Y)$ *contains an ergodic measure.* □

Proof. By Theorem 2.3, if $P(\sigma_Y, \Phi) = -\infty$, then $M_\Phi(Y, \sigma_Y) = M(Y, \sigma_Y)$ and we obtain the results. Therefore, we consider the case when $P(\sigma_Y, \Phi) \neq -\infty$. Statement (1) is clear. For statement (2), we first notice that the subadditive ergodic theorem implies that $h_\mu(\sigma_Y) + \lim_{n\to\infty}(1/n)\int \phi_n\, d\mu$ takes a value in $[-\infty, \infty)$ for each $\mu \in M(Y, \sigma_Y)$. Let $\{\mu_m\}_{m=1}^\infty$ converge to $\mu \in M(Y, \sigma_Y)$ in the weak* topology. Since the entropy map is upper semi-continuous, in order to show that $M_\Phi(Y, \sigma_Y)$ is non-empty, it is enough to prove that

$$\limsup_{m\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu_m \le \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu.$$

Using subadditivity, we have for $\mu \in M(Y, \sigma_Y)$,

$$\lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu \le \frac{1}{n}\int \phi_n\, d\mu \quad \text{for all } n \in \mathbb{N}.$$

Let μ_m be fixed. Then

$$\lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu_m \le \frac{1}{n}\int \phi_n\, d\mu_m.$$

Let n be fixed and $m \to \infty$. Then

$$\limsup_{m\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu_m \le \limsup_{m\to\infty} \frac{1}{n}\int \phi_n\, d\mu_m = \frac{1}{n}\int \phi_n\, d\mu.$$

Letting $n \to \infty$,

$$\limsup_{m\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu_m \le \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu.$$

Compactness follows by using the upper semi-continuity of the map $\mu \to h_\mu(\sigma_Y) + \lim_{n\to\infty}(1/n)\int \phi_n\, d\mu$. For statements (3), (4) and (5), noticing that $P(\sigma_Y, \Phi) < \infty$ by statement (2), we make the standard arguments as in the proofs of Theorem 8.7 and Corollary 8.6.1(i) in [**21**], combining these with Proposition A.1(3) [**6**]. □

Let (X, σ_X) be a subshift. Let $f : X \to \mathbb{R}$ be a bounded Borel measurable function and $\Phi = \{\phi_n\}_{n=1}^\infty$ a subadditive potential on (X, σ_X). A σ_X-invariant Borel probability measure $\bar\mu$ on X is an *equilibrium state* for $f + \Phi$ if

$$h_{\bar\mu}(\sigma_X) + \int f\, d\bar\mu + \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\bar\mu = \sup_{\mu\in M(X,\sigma_X)} \left\{h_\mu(\sigma_X) + \int f\, d\mu + \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu\right\}.$$

Denote by $M_{f+\Phi}(X, \sigma_X)$ the collection of all equilibrium states for $f + \Phi$. It is easy to see that

$$\sup_{\mu\in M(X,\sigma_X)} \left\{h_\mu(\sigma_X) + \int f\, d\mu + \lim_{n\to\infty} \frac{1}{n}\int \phi_n\, d\mu\right\} = -\infty$$

if and only if $\lim_{n\to\infty}(1/n)\int \phi_n\, d\mu = -\infty$ for all $\mu \in M(X, \sigma_X)$.

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PROPOSITION 3.8. *Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. Let $\Phi = \{\phi_n\}_{n=1}^{\infty}$ be a subadditive potential on (Y, σ_Y). Define $\Phi \circ \pi = \{\phi_n \circ \pi\}_{n=1}^{\infty}$ and $F : Y \to \mathbb{R}$ as in Theorem 3.2. Then:*

(1) *$M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$ is convex;*
(2) *$M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$ is non-empty;*
(3) *the extreme points of $M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$ are precisely the ergodic members of $M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$.* □

Proof. We consider the case when

$$\sup_{\mu\in M(X,\sigma_X)}\left\{h_\mu(\sigma_X) - \int F\circ\pi\, d\mu + \lim_{n\to\infty}\frac{1}{n}\int \phi_n\circ\pi\, d\mu\right\} \neq -\infty.$$

Statement (1) is clear. For statement (2), we claim that equality (3.2) in Theorem 3.2 holds when we replace ϕ by Φ. To see this, replacing ϕ by Φ, we consider $\mu \in M(X, \sigma_X)$ such that $\lim_{n\to\infty}(1/n)\int \phi_n \circ \pi\, d\mu \neq -\infty$. Let $m = \pi\mu$. Taking the supremum over $m \in M(Y, \sigma_Y)$ such that $\lim_{n\to\infty}(1/n)\int \phi_n\, dm \neq -\infty$ in the proof of equality (3.2) in Theorem 3.2, we obtain the claim. Using Proposition 3.7(2), Lemma 3.4 holds when we replace ϕ by Φ. Therefore, we obtain statement (2). For statement (3), since

$$\sup_{\mu\in M(X,\sigma_X)}\left\{h_\mu(\sigma_X) - \int F\circ\pi\, d\mu + \lim_{n\to\infty}\frac{1}{n}\int \phi_n\circ\pi\, d\mu\right\} < \infty,$$

we use [**21**] as in the proof of Proposition 3.7. □

THEOREM 3.9. *Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. Let $\Phi = \{\phi_n\}_{n=1}^{\infty}$ be a subadditive potential on (Y, σ_Y). Define $\Phi \circ \pi = \{\phi_n \circ \pi\}_{n=1}^{\infty}$ and $F : Y \to \mathbb{R}$ as in Theorem 3.2. Then*

$$\begin{aligned}&\sup_{\mu\in M(X,\sigma_X)}\left\{h_\mu(\sigma_X) - \int F\circ\pi\, d\mu + \lim_{n\to\infty}\frac{1}{n}\int \phi_n\circ\pi\, d\mu\right\}\\ &\quad= \sup_{m\in M(Y,\sigma_Y)}\left\{h_m(\sigma_Y) + \lim_{n\to\infty}\frac{1}{n}\int \phi_n\, dm\right\}\end{aligned} \tag{3.4}$$

for all Φ. Then $M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$ is non-empty. If $P(\sigma_Y, \Phi) \neq -\infty$, then μ is an equilibrium state for $-F \circ \pi + \Phi \circ \pi$ if and only if (i) $\pi\mu$ is an equilibrium state for Φ and (ii) μ is a measure of maximal relative entropy over $\pi\mu$.

Remark 3.10. Let $S : X \to X$, $T : Y \to Y$ be continuous maps of compact metric spaces and π a factor map. Then Theorem 3.9 holds by setting $F = P(S, \pi, 0)$ if the entropy maps $\mu \to h_\mu(S)$, $m \to h_m(T)$ are upper semi-continuous and the topological entropy of S is finite (see the relative variational principle by Ledrappier and Walters in [**22**] for the notation of $P(S, \pi, 0)$).

Proof. If $P(\sigma_Y, \Phi) = -\infty$, then we clearly have the results. Suppose that $P(\sigma_Y, \Phi) \neq -\infty$. Equality (3.4) is shown in the first part of the proof of Proposition 3.8(2). For the if and only if part, we use Lemmas 3.5 and 3.6 with ϕ replaced by Φ. □

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COROLLARY 3.11. *Suppose that (X, σ_X) is an irreducible shift of finite type and (Y, σ_Y) is a subshift. Then we can replace $|D_n(y)|$ in the definition of F (see Theorem 2.5) by $|\pi^{-1}[y_1 \cdots y_n]|$ in Theorems 3.2 and 3.9.*

Proof. Apply Theorem 2.6. We remark that the above result holds for a subshift (X, σ_X) for which Theorem 2.6 holds. □

PROPOSITION 3.12. *Under the assumptions of Proposition 3.8, $M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$ is compact if $M_\Phi(Y, \sigma_Y)$ consists of one point.*

Proof. If $P(\sigma_Y, \Phi) = -\infty$, then $M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X) = M(X, \sigma_X)$ and so we obtain the result. If $P(\sigma_Y, \Phi) \neq -\infty$, then Proposition 3.7(2) implies that $P(\sigma_Y, \Phi) < \infty$. Let $\mu_n \in M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$ for all $n \in \mathbb{N}$. Since $\{\mu_n\}_{n=1}^\infty$ is a sequence in $M(X, \sigma_X)$, there exists a subsequence $\{\mu_{n_k}\}_{k=1}^\infty$ that converges to $\mu \in M(X, \sigma_X)$. We show that $\mu \in M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$. Clearly, $\{\pi\mu_{n_k}\}_{k=1}^\infty$ converges to $\pi\mu$ in the weak* topology. By Theorem 3.9, $\pi\mu_{n_k} \in M_\Phi(Y, \sigma_Y)$ for all $k \in \mathbb{N}$ and so $\pi\mu \in M_\Phi(Y, \sigma_Y)$ by Proposition 3.7(2). Now we show that μ is a measure of maximal relative entropy over $\pi\mu$. Since, by Theorem 3.9, μ_{n_k} is a measure of maximal relative entropy over $\pi\mu_{n_k}$,

$$\limsup_{k\to\infty}(h_{\pi\mu_{n_k}}(\sigma_Y) + \int P(\sigma_X, \pi, 0)\, d\pi\mu_{n_k}) \leq h_\mu(\sigma_X)$$
$$\leq h_{\pi\mu}(\sigma_Y) + \int P(\sigma_X, \pi, 0)\, d\pi\mu.$$

Since $M_\Phi(Y, \sigma_Y)$ consists of one point, if $M_\Phi(Y, \sigma_Y) = \{m\}$, then $\pi\mu_{n_k} = m$ for all $k \in \mathbb{N}$. Therefore, $h_\mu(\sigma_X) = h_{\pi\mu}(\sigma_Y) + \int P(\sigma_X, \pi, 0)\, d\pi\mu$. Now we apply Theorem 3.9 to obtain $\mu \in M_{-F\circ\pi+\Phi\circ\pi}(X, \sigma_X)$. □

In [**20**], for each $\alpha > 0$, the measures that maximize the weighted entropy functional $\phi_\alpha(\mu) = h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)$ were studied for the case when there is a *continuous* saturated compensation function between subshifts. We extend the results to arbitrary subshifts without the existence of a *continuous* saturated compensation function. Applying Theorem 3.9, we first study the case when (X, σ_X) is an irreducible shift of finite type.

Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. Let $\alpha > 0$. In the following proposition and theorems, let $K_\alpha = \{\bar{\mu} \in M(X, \sigma_X) : h_{\bar{\mu}}(\sigma_X) + \alpha h_{\pi\bar{\mu}}(\sigma_Y) = \sup_{\mu\in M(X,\sigma_X)} \phi_\alpha(\mu)\}$.

PROPOSITION 3.13. [**20**] *K_α contains a σ_X-invariant ergodic measure.*

PROPOSITION 3.14. *Let (X, σ_X) be an irreducible shift of finite type, (Y, σ_Y) a subshift and $\pi : X \to Y$ a factor map. Let $\alpha > 0$. For each $n \in \mathbb{N}$, define $\phi_n : Y \to \mathbb{R}$ by $\phi_n(y) = \log |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)}$ and let $\Phi = \{\phi_n\}_{n=1}^\infty$. Then*

$$\sup_{\mu\in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)\}$$
$$= (\alpha + 1) \sup_{m\in M(Y,\sigma_Y)} \left\{h_m(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n\, dm\right\} \tag{3.5}$$
$$= (\alpha + 1) \sup_{\mu\in M(X,\sigma_X)} \left\{h_\mu(\sigma_X) - \lim_{n\to\infty} \frac{1}{n} \int \alpha\phi_n \circ \pi\, d\mu\right\}. \tag{3.6}$$

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Furthermore, $\mu \in K_\alpha$ if and only if (i) $\pi\mu$ is an equilibrium state for Φ and (ii) μ is a measure of maximal relative entropy over $\pi\mu$.

Proof. Define $F : Y \to \mathbb{R}$ as in Theorem 3.2. Then

$$\begin{aligned}
&\sup_{\mu\in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)\} \\
&\quad = \sup_{\mu\in M(X,\sigma_X)} \{h_\mu(\sigma_X) - h_{\pi\mu}(\sigma_Y) + (1+\alpha)h_{\pi\mu}(\sigma_Y)\} \\
&\quad = \sup_{\mu\in M(X,\sigma_X)} \left\{(\alpha+1)h_{\pi\mu}(\sigma_X) + \int F\, d\pi\mu\right\} \quad \text{(by Lemma 3.1 and equation (3.3))} \\
&\quad = (\alpha+1) \sup_{m\in M(Y,\sigma_Y)} \left\{h_m(\sigma_Y) + \frac{1}{\alpha+1}\int F\, dm\right\}.
\end{aligned}$$

Clearly, Φ is a subadditive potential on (Y, σ_Y). Therefore, using Theorem 2.6 and Kingman's subadditive ergodic theorem [**21**, Theorem 10.1], we get equality (3.5). For equality (3.6), we apply Corollary 3.11 (set $\phi_n(y) = \log |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)}$). Now suppose that $\mu \in K_\alpha$. Since μ is clearly a measure of maximal relative entropy over $\pi\mu$, we conclude that $\pi\mu$ is an equilibrium state for Φ by using the proof of equation (3.5). Next suppose that m is an equilibrium state for Φ and μ is a measure of maximal relative entropy over m. Then

$$\begin{aligned}
&(\alpha+1)\left(h_m(\sigma_Y) + \lim_{n\to\infty}\frac{1}{n}\int \phi_n\, dm\right) \\
&\quad = (\alpha+1)h_{\pi\mu}(\sigma_Y) + \lim_{n\to\infty}\frac{1}{n}\int (\alpha+1)\phi_n\, d\pi\mu \\
&\quad = \sup\{h_{\bar\mu}(\sigma_X) - h_{\pi\mu}(\sigma_Y) : \bar\mu \in M(X,\sigma_X), \pi\bar\mu = \pi\mu\} + (\alpha+1)h_{\pi\mu}(\sigma_Y) \\
&\quad = h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y).
\end{aligned}$$

Using equality (3.5), we conclude that $\mu \in K_\alpha$. □

THEOREM 3.15. *Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. Let $\alpha > 0$ and $F : Y \to \mathbb{R}$ be defined as in Theorem 3.2. Then*

$$\begin{aligned}
&\sup_{\mu\in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)\} \\
&\quad = (\alpha+1) \sup_{m\in M(Y,\sigma_Y)} \left\{h_m(\sigma_Y) + \frac{1}{\alpha+1}\int F\, dm\right\} \qquad (3.7) \\
&\quad = (\alpha+1) \sup_{\mu\in M(X,\sigma_X)} \left\{h_\mu(\sigma_X) - \frac{\alpha}{\alpha+1}\int F\circ\pi\, d\mu\right\}. \qquad (3.8)
\end{aligned}$$

Furthermore, $K_\alpha = M_{-(\alpha/(\alpha+1))F\circ\pi}(X, \sigma_X)$. Also, $\mu \in K_\alpha$ if and only if (i) $\pi\mu$ is an equilibrium state for $(1/(\alpha+1))F$ and (ii) μ is a measure of maximal relative entropy over $\pi\mu$.

Remark 3.16. Walters [**22**, Theorem 3.4] showed that if there is a saturated compensation function $G \circ \pi$, $G \in C(Y)$, then $\int G\, dm = -\int F\, dm$ for all $m \in M(Y, \sigma_Y)$. Therefore, Theorem 3.15 extends the results of Shin [**19**, Theorem 1.1].

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Proof. We get equality (3.7) by the same proof as in Proposition 3.14. Equality (3.8) follows from the proof of equality (3.2) in Theorem 3.2 (with $\phi = (1/(\alpha+1))F$). For the if and only if part, we use the proof of Proposition 3.14, replacing Φ by $(1/(\alpha+1))F$. We observe that we cannot apply Theorem 3.9 in order to show that $K_\alpha = M_{-(\alpha/(\alpha+1))F\circ\pi}(X, \sigma_X)$ because $\Psi = \{\log|D_n(y)|\}_{n=1}^\infty$ is not always a sequence of continuous functions on (Y, σ_Y). Suppose that $\mu \in K_\alpha$. Since $\pi\mu$ is an equilibrium state for $(1/(\alpha+1))F$ and μ is a measure of maximal relative entropy over $\pi\mu$, we conclude that $\mu \in M_{-(\alpha/(\alpha+1))F\circ\pi}(X, \sigma_X)$ by using the proof of Lemma 3.5 (with $\phi = (1/(\alpha+1))F$). Now suppose that $\mu \in M_{-(\alpha/(\alpha+1))F\circ\pi}(X, \sigma_X)$ (such a μ exists from the above). Since Lemma 3.6 holds for $\phi = (1/(\alpha+1))F$, we obtain $\mu \in K_\alpha$. □

4. *Applications—dimensions of non-conformal expanding maps*

We will apply the results from §3 to find the Hausdorff dimension of a compact invariant set of a non-conformal expanding map and measures of full dimension. In particular, we consider a compact invariant set of a non-conformal expanding map of the torus represented by a diagonal matrix. For more details and background material on this section, see [**23**]. We now review the definitions of an NC carpet and an SFT-NC carpet. Fix two positive integers l and m, $l > m \geq 2$. Throughout this paper, we let T be the endomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $T(x, y) = (lx \bmod 1, my \bmod 1)$. Let

$$\mathcal{P} = \left\{ \left[\frac{i}{l}, \frac{i+1}{l}\right] \times \left[\frac{j}{m}, \frac{j+1}{m}\right] : 0 \leq i \leq l-1, 0 \leq j \leq m-1 \right\}$$

be the natural Markov partition for T. Label $[i/l, (i+1)/l] \times [j/m, (j+1)/m]$, $0 \leq i \leq l-1$, $0 \leq j \leq m-1$, by the symbol (i, j). Define (Σ_{lm}^+, σ) to be the full shift on these lm symbols. Consider the coding map $\chi : \Sigma_{lm}^+ \to \mathbb{T}^2$ defined by

$$\chi(\{(x_k, y_k)\}_{k=1}^\infty) = \left(\sum_{k=1}^\infty \frac{x_k}{l^k}, \sum_{k=1}^\infty \frac{y_k}{m^k} \right).$$

Let $R = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$ be a subalphabet of the symbols of $\mathcal{P}$. The non-conformal carpet (*NC carpet*) $K(T, R)$ is defined by

$$K(T, R) = \left\{ \left(\sum_{k=1}^\infty \frac{x_k}{l^k}, \sum_{k=1}^\infty \frac{y_k}{m^k} \right) : (x_k, y_k) \in R \text{ for all } k \in \mathbb{N} \right\}.$$

It is a compact T-invariant subset of the torus. Denote by A a transition matrix among the members of R. The *SFT-NC carpet* $K(T, R, A)$ is defined by

$$K(T, R, A) = \left\{ \left(\sum_{k=1}^\infty \frac{x_k}{l^k}, \sum_{k=1}^\infty \frac{y_k}{m^k} \right) : (x_k, y_k) \in R,\ A_{(x_k,y_k)(x_{k+1},y_{k+1})} = 1 \text{ for all } k \in \mathbb{N} \right\}.$$

Now let (X, σ_X) be the shift of finite type with the transition matrix A as above. Let $\pi : X \to Y$ be the factor map determined by the one-block map $\pi((a_k, b_k)) = b_k$ and let $Y = \pi(X)$. In [**23**], the Hausdorff dimension for $K(T, R, A)$ was studied. If there exists a saturated compensation function $G \circ \pi$, $G \in C(Y)$, then the Hausdorff dimension is given by

$$\dim_H K(T, R, A) = \frac{1}{\log m} \sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) + \frac{\alpha}{\alpha+1} \int G \circ \pi \, d\mu \right\}.$$

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This formula extends the formula for the Hausdorff dimension of $K(T, R)$ given by McMullen [**13**].

Given a shift of finite type (X, σ_X), a subshift (Y, σ_Y) and a factor map $\pi : X \to Y$, we can construct an SFT-NC carpet by defining T and choosing R appropriately. The construction is not unique. We call such a carpet an SFT-NC carpet corresponding to (X, Y, π).

We will extend the formula above to the general case by considering a compact T-invariant set whose symbolic representation is a subshift. Let $S \subset \Sigma_r^+ \subseteq \Sigma_{lm}^+$ be a subshift on the members of R. Then $\chi(S)$ is a *subshift-NC carpet*. Given any subshifts (X, σ_X), (Y, σ_Y) and a factor map $\pi : X \to Y$, we define a subshift-NC carpet corresponding to (X, Y, π) in the same manner as we did for an SFT-NC carpet. For the symbolic representation of such a carpet, there does not always exist a saturated compensation function. In Theorem 4.4, we will give a formula for the Hausdorff dimension of a subshift-NC carpet by using a bounded Borel measurable compensation function.

Recall that for $y \in Y$, $|\pi^{-1}[y_1 \cdots y_n]|$ denotes the cardinality of $E_n(y)$ (see §2) and, in particular, it is the number of blocks of $x_1 \cdots x_n$ of length n in X that are mapped to the block $y_1 \cdots y_n$ in Y if (X, σ_X) is an irreducible shift of finite type. Let $\Phi = \{\phi_n\}_{n=1}^{\infty}$ be a subadditive potential on (X, σ_X). Then $-\Phi = \{-\phi_n\}_{n=1}^{\infty}$ is a superadditive potential on (X, σ_X). A measure $\bar{\mu} \in M(X, \sigma_X)$ is an *equilibrium state* for $-\Phi$ if

$$h_{\bar{\mu}}(\sigma_X) - \lim_{n\to\infty} (1/n) \int \phi_n \, d\bar{\mu} = \sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) - \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, d\mu \right\}.$$

We note that $\sup_{\mu \in M(X,\sigma_X)} \{h_\mu(\sigma_X) - \lim_{n\to\infty}(1/n) \int \phi_n \, d\mu\}$ takes a value in $(-\infty, \infty]$.

THEOREM 4.1. *Let (X, σ_X) be an irreducible shift of finite type, (Y, σ_Y) a subshift and $\pi : X \to Y$ a factor map. For $l, m \in \mathbb{N}$, $l > m \geq 2$, set $\alpha = \log_m l - 1$. Define $\phi_n : Y \to \mathbb{R}$ for each $n \in \mathbb{N}$ by $\phi_n(y) = \log |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)}$. Let $\Phi = \{\phi_n\}_{n=1}^{\infty}$ and $-\alpha\Phi \circ \pi = \{-\alpha\phi_n \circ \pi\}_{n=1}^{\infty}$. Then the Hausdorff dimension of an SFT-NC carpet K corresponding to (X, Y, π) is given by*

$$\dim_H K = \frac{1}{\log m} \sup_{\nu \in M(Y,\sigma_Y)} \left\{ h_\nu(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, d\nu \right\} \tag{4.1}$$

$$= \frac{1}{\log m} \sup_{\mu \in M(X,\sigma_X)} \left\{ h_\mu(\sigma_X) - \lim_{n\to\infty} \frac{1}{n} \int \alpha\phi_n \circ \pi \, d\mu \right\} \tag{4.2}$$

$$= \frac{1}{\log m} \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{y_1 \cdots y_n \in B_n(Y)} |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)} \right). \tag{4.3}$$

The images under the coding map χ of the ergodic equilibrium states for $-\alpha\Phi \circ \pi$ are T-invariant ergodic measures of full dimension for K.

Proof. Using [**23**, proof of Corollary 2.7] and [**8**, Theorem 3], we obtain

$$\begin{aligned} \dim_H K &= \sup\{\dim_H \mu : \mu(K) = 1, \, \mu \text{ is } T - \text{invariant and ergodic}\} \\ &= \frac{1}{\log l} \sup_{\mu \in M(X,\sigma_X)} \{h_\mu(\sigma_X) + (\log_m l - 1) h_{\pi\mu}(\sigma_Y)\}. \end{aligned}$$

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Setting $\alpha = \log_m l - 1$ and applying Proposition 3.14 establishes equalities (4.1) and (4.2). For equality (4.3), we use Theorem 2.3 and the subadditivity property (see the inequalities (4.8) and (4.9) in Lemma 4.6, below). For the last part, we use the proof of Corollary 2.7 [**23**]. □

LEMMA 4.2. *Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. Let $\alpha > 0$. For all $n \in \mathbb{N}$, define $\phi'_n : Y \to \mathbb{R}$ by $\phi'_n(y) = \log |D_n(y)|^{1/(\alpha+1)}$. Then, for $n, m \in \mathbb{N}$, $y \in Y$, $\phi'_{n+m}(y) \le \phi'_n(y) + \phi'_m(\sigma_Y^m(y))$.*

Proof. It is enough to show that for $y \in Y$, $m, n \in \mathbb{N}$,

$$|D_{n+m}(y)| \le |D_n(y)||D_m(\sigma_Y^n(y))|. \tag{4.4}$$

Let $z = (z_1 \cdots z_n z_{n+1} \cdots z_{n+m} \cdots) \in D_{n+m}(y)$. Then the number of points in $D_{n+m}(y)$ starting with $z_1 \cdots z_n$ is less than or equal to $|D_m(\sigma_Y^n(y))|$. Since the number of all possible distinct $z_1 \cdots z_n$ of the points in $D_{n+m}(y)$ is equal to $|D_n(y)|$, we obtain inequality (4.4). □

LEMMA 4.3. *Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. For all $n \in \mathbb{N}$, define ϕ'_n as in Lemma 4.2 and let $\Phi' = \{\phi'_n\}_{n=1}^{\infty}$. Let S be the number of symbols in X. For $0 < \epsilon < 1/2$, let $k \ge 3$ be such that $1/2^{k-1} \le \epsilon < 1/2^{k-2}$. If ϕ'_n is continuous for all $n \in \mathbb{N}$, then*

$$\sum_{y_1 \cdots y_{n+k-2}} \left(\frac{|\pi^{-1}[y_1 \cdots y_{n+k-2}]|}{S} \right)^{1/(\alpha+1)} \le P_n(\sigma_Y, \Phi', \epsilon) \le \sum_{y_1 \cdots y_{n+k-2}} |\pi^{-1}[y_1 \cdots y_{n+k-2}]|^{1/(\alpha+1)},$$

where the summations are taken over all allowable words $y_1 \cdots y_{n+k-2}$ of length $(n + k - 2)$ in Y.

Proof. Using Lemma 4.2, Φ' is a subadditive potential on (Y, σ_Y) if ϕ'_n is continuous for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be fixed. Take an (n, ϵ) separated subset A of Y. If $x, y \in A$, then there exists $1 \le i \le n + k - 2$ such that $x_i \ne y_i$. Thus,

$$\sum_{y \in A} |D_{n+k-2}(y)|^{1/(\alpha+1)} \le \sum_{y_1 \cdots y_{n+k-2} \in B_{n+k-2}(Y)} |\pi^{-1}[y_1 \cdots y_{n+k-2}]|^{1/(\alpha+1)}.$$

Let $[i_1 \cdots i_{n+k-2}]$ be a cylinder in Y and let $y \in [i_1 \cdots i_{n+k-2}]$. Then the number of possible choices of symbols in the $(n + k - 2)$th position of a point in $E_{n+k-2}(y)$ is less than or equal to S. Therefore, there exists a symbol a in the $(n + k - 2)$th position of a point in $E_{n+k-2}(y)$ such that

$$\frac{|\pi^{-1}[i_1 \cdots i_{n+k-2}]|}{S} \le \text{the total number of } a \text{ that appears in the } (n+k-2)\text{th position of points in } E_{n+k-2}(y).$$

Take $x \in E_{n+k-2}(y)$ such that $x_{n+k-2} = a$. Let $\pi(x) = z$. Then

$$\frac{|\pi^{-1}[i_1 \cdots i_{n+k-2}]|}{S} \le |D_{n+k-2}(z)|.$$

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Now let E be the set obtained by taking one such z from each distinct cylinder $[i_1 \cdots i_{n+k-2}]$ in Y. Then E is an (n, ϵ) separated subset of Y and

$$\sum_{y_1 \cdots y_{n+k-2} \in B_{n+k-2}(Y)} \left(\frac{|\pi^{-1}[y_1 \cdots y_{n+k-2}]|}{S} \right)^{1/(\alpha+1)} \leq \sum_{y \in E} |D_{n+k-2}(y)|^{1/(\alpha+1)}.$$

Now Lemma 4.3 is proved. □

Now we consider a sequence of real-valued functions $\Phi = \{\phi_n\}_{n=1}^{\infty}$ on (X, σ_X) such that Φ satisfies the subadditivity condition but ϕ_n is merely Borel measurable on X. Then $-\Phi$ is a sequence of Borel measurable functions on (X, σ_X) such that $-\Phi$ satisfies the superadditivity condition. We define the equilibrium states for Φ and $-\Phi$ in the same manner as we did for subadditive potentials and superadditive potentials.

THEOREM 4.4. *Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. For $l, m \in \mathbb{N}$, $l > m \geq 2$, set $\alpha = \log_m l - 1$. For all $n \in \mathbb{N}$, define ϕ_n and Φ as in Theorem 4.1 and ϕ'_n and Φ' as in Lemma 4.3. Then the Hausdorff dimension of a subshift-NC carpet K corresponding to (X, Y, π) is given by*

$$\dim_H K = \frac{1}{\log m} \sup_{\nu \in M(Y, \sigma_Y)} \left\{ h_\nu(\sigma_Y) + \lim_{n \to \infty} \frac{1}{n} \int \phi'_n \, d\nu \right\} \tag{4.5}$$

$$= \frac{1}{\log m} \sup_{\mu \in M(X, \sigma_X)} \left\{ h_\mu(\sigma_X) - \lim_{n \to \infty} \frac{1}{n} \int \alpha \phi'_n \circ \pi \, d\mu \right\}. \tag{4.6}$$

The images under the coding map χ of the ergodic equilibrium states for $-\alpha\Phi' \circ \pi$ are T-invariant ergodic measures of full dimension for K. In addition, if ϕ'_n is continuous for all $n \in \mathbb{N}$, then:

(1) $\dim_H K = (1/\log m) \lim_{n \to \infty} (1/n) \log(\sum_{y_1 \cdots y_n \in B_n(Y)} |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)})$;

(2) $M_{\Phi'}(Y, \sigma_Y) \subseteq M_\Phi(Y, \sigma_Y)$. *In particular, if $M_\Phi(Y, \sigma_Y)$ consists of one point, then $M_{\Phi'}(Y, \sigma_Y) = M_\Phi(Y, \sigma_Y)$;*

(3) *if $M_\Phi(Y, \sigma_Y)$ consists of one point, then the images under the coding map χ of the ergodic equilibrium states for $-\alpha\Phi \circ \pi$ are T-invariant ergodic measures of full dimension for K.*

Remarks 4.5.

(1) A formula for a subshift-NC carpet which is slightly different from the above is deduced in [**1, 7**].

(2) Clearly, statements (1), (2) and (3) in the above theorem also hold if there exists $N \in \mathbb{N}$ such that ϕ'_n is continuous for all $n \geq N$.

Proof. We first notice that the map $y \to \log |D_n(y)|$ is Borel measurable for each $n \geq 1$, by the proof of Lemma 3.3 (with $f = 0$) in [**11**]. Equality (4.5) is clear from the proof of equation (4.1) in Theorem 4.1. Equality (4.6) follows by combining Theorem 3.15, Lemma 4.2 and the subadditive ergodic theorem. By the same proof used to show Theorem 4.1, the images under χ of the ergodic equilibrium states for $-\alpha\Phi' \circ \pi$ are

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T-invariant ergodic measures of full dimension for K. By Lemma 4.3,

$$P(\sigma_Y, \Phi', \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(\sigma_Y, \Phi', \epsilon)$$
$$= \limsup_{n\to\infty} \frac{1}{n} \log\Bigg(\sum_{y_1\cdots y_n \in B_n(Y)} |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)} \Bigg).$$

Using the variational principle and the proof of equation (4.3) in Theorem 4.1, we obtain statement (1). For statement (2), we notice by statement (1) that if ϕ_n' is continuous for all $n \in \mathbb{N}$, then

$$\sup_{\nu\in M(Y,\sigma_Y)} \left\{ h_\nu(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n' \, d\nu \right\} = \sup_{\nu\in M(Y,\sigma_Y)} \left\{ h_\nu(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, d\nu \right\}.$$

Using $\phi_n' \le \phi_n$ for all $n \in \mathbb{N}$, we obtain $M_{\Phi'}(Y, \sigma_Y) \subseteq M_\Phi(Y, \sigma_Y)$. Since $M_{\Phi'}(Y, \sigma_Y) \neq \emptyset$, if $M_\Phi(Y, \sigma_Y)$ consists of one point, then $M_{\Phi'}(Y, \sigma_Y) = M_\Phi(Y, \sigma_Y)$. Statement (3) follows by combining Theorems 3.9 and 3.15. □

Next we want to know whether or not there is a unique invariant ergodic measure of full dimension and to study the properties of the unique measures. In [**23**], uniqueness and the properties of the unique measure were studied for some SFT-NC carpets for which (continuous) saturated compensation functions exist. In order to generalize these results, we use the approach in [**2**], which was originally developed to show uniqueness of the equilibrium state for an almost subadditive potential.

Throughout the rest of this section, we assume that (X, σ_X) is a topologically mixing shift of finite type and (Y, σ_Y) is a subshift. Let A be the transition matrix for (X, σ_X) and M the smallest integer such that $A^M > 0$. Let $\alpha > 0$. Define

$$K = \sum_{i_1\cdots i_M \in B_M(Y)} |\pi^{-1}[i_1 \cdots i_M]|^{1/(\alpha+1)}.$$

For $n \in \mathbb{N}$, define

$$S_n = \sum_{i_1\cdots i_n \in B_n(Y)} |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)}.$$

Define $\phi_n : Y \to \mathbb{R}$ by $\phi_n(y) = \log |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)}$ for all $n \in \mathbb{N}$ and $\Phi = \{\phi_n\}_{n=1}^\infty$. Then Φ is a subadditive potential on (Y, σ_Y) with bounded variation. Φ is almost additive if and only if there exist $K_1, K_2 > 0$ such that for any allowable word $i_1 \cdots i_n j_1 \cdots j_l$ of length $(n + l)$ in Y, $n, l \in \mathbb{N}$,

$$K_1 \le \frac{|\pi^{-1}[i_1 \cdots i_n j_1 \cdots j_l|]}{|\pi^{-1}[i_1 \cdots i_n]||\pi^{-1}[j_1 \cdots j_l]|} \le K_2. \tag{4.7}$$

We notice that Φ is not always almost additive. Therefore, the hypothesis of Theorem 2.2 is not satisfied. Nevertheless, we will show that there is a unique equilibrium state for Φ which is Gibbs and mixing. We prove this by showing that Lemmas 1 and 2, and therefore [**2**, Theorem 5], still hold in our setting. For all $n \in \mathbb{N}$, let A_n be a set consisting of exactly one point from each cylinder of length n in Y. Define the Borel probability measure ν_n on Y concentrated on A_n by

$$\nu_n = \frac{\sum_{y\in A_n} e^{\phi_n(y)} \delta_y}{\sum_{y\in A_n} e^{\phi_n(y)}},$$

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where δ_y is the Dirac measure at y. Then, for each cylinder $[i_1 \cdots i_n]$ of length n in Y,

$$\nu_n([i_1 \cdots i_n]) = \frac{|\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)}}{\sum_{j_1 \cdots j_n \in B_n(Y)} |\pi^{-1}[j_1 \cdots j_n]|^{1/(\alpha+1)}}.$$

Since ν_n is a Borel probability measure on Y for all $n \in \mathbb{N}$, there exists a subsequence $\{\nu_{n_k}\}_{k=1}^{\infty}$ that converges to a Borel probability measure ν on Y in the weak* topology. In the following lemmas, let K, S_n, Φ and ν_n be defined as above.

LEMMA 4.6. *There exist K_1, $K_2 > 0$ such that $K_1 \le e^{nP(\sigma_Y,\Phi)}/S_n \le K_2$ for all $n \in \mathbb{N}$.*

Proof. Notice that $P(\sigma_Y, \Phi) = \limsup_{n\to\infty}(1/n)\log S_n$ by the subadditive topological pressure of Φ. We will first find $K_2 > 0$. Since

$$\sum_{i_1 \cdots i_n j_1 \cdots j_l \in B_{n+l}(Y)} |\pi^{-1}[i_1 \cdots i_n j_1 \cdots j_l]|^{1/(\alpha+1)}$$
$$\le \sum_{i_1 \cdots i_n \in B_n(Y)} |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)} \sum_{j_1 \cdots j_l \in B_l(Y)} |\pi^{-1}[j_1 \cdots j_l]|^{1/(\alpha+1)}, \quad (4.8)$$

$\{\log S_n\}_{n=1}^{\infty}$ is subadditive. Therefore,

$$P(\sigma_Y, \Phi) = \lim_{n\to\infty} \frac{\log S_n}{n} \le \frac{\log S_n}{n} \quad \text{for all } n \ge 1, \quad (4.9)$$

and so $e^{nP(\sigma_Y,\Phi)}/S_n \le 1$. Set $K_2 = 1$. Now let $l > M$. To find $K_1 > 0$, we first show that $S_{l+n} \ge S_n S_{l-M}$. Given any two allowable words in Y, $i_1 \cdots i_n$ of length n and $j_1 \cdots j_{l-M}$ of length $(l-M)$, there exists an allowable word $a_1 \cdots a_M$ of length M in Y such that $i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M}$ is an allowable word of length $(l+n)$ in Y. For fixed allowable words $i_1 \cdots i_n$ of length n and $j_1 \cdots j_{l-M}$ of length $(l-M)$,

$$\sum_{i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M} \in B_{n+l}(Y)} |\pi^{-1}[i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M}]|$$
$$\ge |\pi^{-1}[i_1 \cdots i_n]||\pi^{-1}[j_1 \cdots j_{l-M}]|.$$

Therefore,

$$\sum_{i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M} \in B_{n+l}(Y)} |\pi^{-1}[i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M}]|^{1/(\alpha+1)}$$
$$\ge \left(\sum_{i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M} \in B_{n+l}(Y)} |\pi^{-1}[i_1 \cdots i_n a_1 \cdots a_M j_1 \cdots j_{l-M}]| \right)^{1/(\alpha+1)}$$
$$\ge |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)} |\pi^{-1}[j_1 \cdots j_{l-M}]|^{1/(\alpha+1)}.$$

Summing over all allowable words $i_1 \cdots i_n$ of length n in Y and $j_1 \cdots j_{l-M}$ of length $(l-M)$ in Y, we obtain

$$S_{l+n} \ge S_n S_{l-M}. \quad (4.10)$$

Now we show that $S_l \le K S_{l-M}$. For any allowable word $i_{M+1} \cdots i_l$ in Y, there exists $a_1 \cdots a_M$ such that $a_1 \cdots a_M i_{M+1} \cdots i_l$ is an allowable word of length l in Y. Let $i_{M+1} \cdots i_l$ be fixed. Then

$$\sum_{i_1 \cdots i_M i_{M+1} \cdots i_l \in B_l(Y)} |\pi^{-1}[i_1 \cdots i_M i_{M+1} \cdots i_l]|^{1/(\alpha+1)} \le K |\pi^{-1}[i_{M+1} \cdots i_l]|^{1/(\alpha+1)}.$$

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Summing over all allowable words $i_{M+1}\cdots i_l$ of length $(l-M)$ in Y, we get

$$S_l \le K S_{l-M}. \tag{4.11}$$

By inequalities (4.10) and (4.11), $S_{l+n} \ge S_l S_n/K$ for $l > M$, $n \ge 1$. For $l+n \le 2M$, we can find $K' \in \mathbb{N}$ such that $S_{l+n} \ge S_l S_n/K'$, because there are only finitely many choices of (l, n) such that $l+n \le 2M$. Set $\widetilde{K} = \max\{K, K'\}$. Then $\{\log(S_n/\widetilde{K})\}_{n=1}^{\infty}$ is superadditive. Thus,

$$\lim_{n\to\infty}\frac{1}{n}\log S_n = \lim_{n\to\infty}\frac{1}{n}\log\frac{S_n}{\widetilde{K}} \ge \frac{1}{n}\log\frac{S_n}{\widetilde{K}} \quad \text{for all } n \ge 1.$$

Using $P(\sigma_Y, \Phi) = \lim_{n\to\infty}(1/n)\log S_n$, we get $1/\widetilde{K} \le e^{nP(\sigma_Y,\Phi)}/S_n$. Set $K_1 = 1/\widetilde{K}$. □

LEMMA 4.7. *There exist $C_1, C_2 > 0$ such that*

$$C_1 \le \frac{\nu_l([i_1\cdots i_n])}{e^{-nP(\sigma_Y,\Phi)}|\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}} \le C_2,$$

for all $l, n \in \mathbb{N}$, $l > n+M$, and cylinders $[i_1\cdots i_n]$ in Y.

Proof. Let $[i_1\cdots i_n]$ be a fixed cylinder of length n in Y. By the definition of ν_l, for $n < l$,

$$\nu_l([i_1\cdots i_n]) = \frac{\sum_{i_1\cdots i_n j_1\cdots j_{l-n}\in B_l(Y)}|\pi^{-1}[i_1\cdots i_n j_1\cdots j_{l-n}]|^{1/(\alpha+1)}}{\sum_{i_1\cdots i_l\in B_l(Y)}|\pi^{-1}[i_1\cdots i_l]|^{1/(\alpha+1)}}. \tag{4.12}$$

We first find an upper bound $C_2 > 0$. For $l > n+M$, using the property of topologically mixing,

$$\begin{aligned}&\sum_{i_1\cdots i_n j_1\cdots j_{l-n}\in B_l(Y)}|\pi^{-1}[i_1\cdots i_n j_1\cdots j_{l-n}]|^{1/(\alpha+1)}\\ &\quad\le K|\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}\sum_{j_{M+1}\cdots j_{l-n}\in B_{l-n-M}(Y)}|\pi^{-1}[j_{M+1}\cdots j_{l-n}]|^{1/(\alpha+1)}.\end{aligned} \tag{4.13}$$

Therefore, using equation (4.12), the inequality (4.13) and Lemma 4.6 (with $K_1 = 1/\widetilde{K}$ and $K_2 = 1$),

$$\begin{aligned}&\frac{\nu_l([i_1\cdots i_n])}{e^{-nP(\sigma_Y,\Phi)}|\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}}\\ &\quad\le \frac{K S_{l-n-M}}{S_l}e^{nP(\sigma_Y,\Phi)} \le K\cdot\widetilde{K}e^{-MP(\sigma_Y,\Phi)} \quad \text{(by Lemma 4.6).}\end{aligned}$$

Next we will find a lower bound $C_1 > 0$. For two fixed allowable words $i_1\cdots i_n$ of length n and $j_{M+1}\cdots j_{l-n}$ of length $(l-n-M)$ in Y, there exists $a_1\cdots a_M$ such that $i_1\cdots i_n a_1\cdots a_M j_{M+1}\cdots j_{l-n}$ is an allowable word of length l in Y. Then

$$\begin{aligned}&\sum_{i_1\cdots i_n a_1\cdots a_M j_{M+1}\cdots j_{l-n}\in B_l(Y)}|\pi^{-1}[i_1\cdots i_n a_1\cdots a_M j_{M+1}\cdots j_{l-n}]|^{1/(\alpha+1)}\\ &\quad\ge |\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}|\pi^{-1}[j_{M+1}\cdots j_{l-n}]|^{1/(\alpha+1)}.\end{aligned}$$

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Summing over all allowable words $j_{M+1}\cdots j_{l-n}$ of length $(l-n-M)$ in Y, we get

$$\sum_{i_1\cdots i_n j_1\cdots j_{l-n}\in B_l(Y)} |\pi^{-1}[i_1\cdots i_n j_1\cdots j_{l-n}]|^{1/(\alpha+1)} \geq |\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)} \sum_{j_{M+1}\cdots j_{l-n}\in B_{l-n-M}(Y)} |\pi^{-1}[j_{M+1}\cdots j_{l-n}]|^{1/(\alpha+1)}. \tag{4.14}$$

Therefore, using inequality (4.14) and Lemma 4.6,

$$\frac{\nu_l([i_1\cdots i_n])}{e^{-nP(\sigma_Y,\Phi)}|\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}} \geq \frac{S_{l-n-M}}{S_l} e^{nP(\sigma_Y,\Phi)} \geq \frac{S_{l-n-M}S_n}{\widetilde{K}S_l} \geq \frac{e^{-MP(\sigma_Y,\Phi)}}{\widetilde{K}^2}. \qquad \square$$

By Lemma 4.7, if a subsequence $\{\nu_{n_k}\}_{k=1}^{\infty}$ of $\{\nu_n\}_{n=1}^{\infty}$ converges to a Borel probability measure ν on Y in the weak* topology, then

$$C_1 \leq \frac{\nu([i_1\cdots i_n])}{e^{-nP(\sigma_Y,\Phi)}|\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}} \leq C_2 \quad \text{for all } n\in\mathbb{N}. \tag{4.15}$$

LEMMA 4.8. *Let ν be the limit point of a convergent subsequence $\{\nu_{n_k}\}_{k=1}^{\infty}$ of $\{\nu_n\}_{n=1}^{\infty}$. Let $\mu_n = (1/n)\sum_{i=0}^{n-1}\sigma_Y^i\nu$. Then any weak limit point μ of $\{\mu_n\}_{n=1}^{\infty}$ is a σ_Y-invariant Gibbs measure for Φ.*

Proof. We follow the arguments in the first part of the proof of Theorem 1 [**2**]. Suppose that $\{\mu_{n_k}\}_{k=1}^{\infty}$ converges to μ in the weak* topology. Then μ is a σ_Y-invariant Borel probability measure on Y [**21**, Theorem 6.9]. We want to show that μ is Gibbs. Let $i_1\cdots i_n$ be a fixed allowable word of length n in Y. For each l, $n\in\mathbb{N}$, $l>M$,

$$\begin{aligned}(\sigma_Y^l\nu)([i_1\cdots i_n]) &= \sum_{j_1\cdots j_l i_1\cdots i_n\in B_{l+n}(Y)} \nu([j_1\cdots j_l i_1\cdots i_n])\\ &\geq \sum_{j_1\cdots j_l i_1\cdots i_n\in B_{l+n}(Y)} C_1 e^{-(l+n)P(\sigma_Y,\Phi)}|\pi^{-1}[j_1\cdots j_l i_1\cdots i_n]|^{1/(\alpha+1)}\\ &\geq C_1|\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)}e^{-(l+n)P(\sigma_Y,\Phi)} \sum_{j_1\cdots j_{l-M}\in B_{l-M}(Y)} |\pi^{-1}[j_1\cdots j_{l-M}]|^{1/(\alpha+1)} \qquad (4.16)\\ &\geq e^{-MP(\sigma_Y,\Phi)}\frac{C_1}{C_2}\nu([i_1\cdots i_n]) \quad \text{(by Lemma 4.6 and the inequalities (4.15))}.\end{aligned}$$

For the inequality (4.16), we use the following inequality (4.17), which is easy to show.

For a fixed allowable word $i_1\cdots i_n$ of length n, $l>M$,

$$\sum_{j_1\cdots j_l i_1\cdots i_n\in B_{l+n}(Y)} |\pi^{-1}[j_1\cdots j_l i_1\cdots i_n]|^{1/(\alpha+1)} \geq |\pi^{-1}[i_1\cdots i_n]|^{1/(\alpha+1)} \sum_{j_1\cdots j_{l-M}\in B_{l-M}(Y)} |\pi^{-1}[j_1\cdots j_{l-M}]|^{1/(\alpha+1)}. \tag{4.17}$$

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For a fixed allowable word $i_1 \cdots i_n$ of length n in Y,

$$\begin{aligned}(\sigma_Y^l \nu)([i_1 \cdots i_n]) &\le \sum_{j_1 \cdots j_l i_1 \cdots i_n \in B_{l+n}(Y)} C_2 e^{-(n+l)P(\sigma_Y,\Phi)} |\pi^{-1}[j_1 \cdots j_l i_1 \cdots i_n]|^{1/(\alpha+1)} \\ &\le C_2 \frac{\sum_{j_1 \cdots j_l \in B_l(Y)} |\pi^{-1}[j_1 \cdots j_l]|^{1/(\alpha+1)}}{e^{(l+n)P(\sigma_Y,\Phi)}} |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)} \\ &\le \frac{\widetilde{K} C_2}{C_1} \nu([i_1 \cdots i_n]) \quad \text{(by Lemma 4.6 and the inequalities (4.15)).}\end{aligned}$$

Therefore, for all $m > M+1$, using the inequalities (4.15), we obtain

$$\frac{m-M-1}{m}\left(\frac{C_1^2}{C_2 e^{(M+n)P(\sigma_Y,\Phi)}} |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)}\right) \le \frac{1}{m}\sum_{l=0}^{m-1} (\sigma_Y^l \nu)([i_1 \cdots i_n])$$

and

$$\frac{1}{m}\sum_{l=0}^{m-1} (\sigma_Y^l \nu)([i_1 \cdots i_n]) \le \frac{m-M-1}{m}\left(\frac{\widetilde{K} C_2{}^2}{C_1 e^{nP(\sigma_Y,\Phi)}} |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)}\right) + \frac{M+1}{m}.$$

Since $\{\mu_{n_k}\}_{k=1}^{\infty}$ converges to μ in the weak* topology, replacing m by n_k and letting $k \to \infty$,

$$\frac{C_1^2}{C_2 e^{MP(\sigma_Y,\Phi)}} \le \frac{\mu[i_1 \cdots i_n]}{e^{-nP(\sigma_Y,\Phi)} |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)}} \le \frac{\widetilde{K} C_2{}^2}{C_1}.$$

Therefore, μ is Gibbs for Φ. □

By Lemma 4.8, it is easy to see that μ is an equilibrium state for Φ by using the same arguments as in the proof of Lemma 17 in [**14**].

LEMMA 4.9. *For fixed allowable words $i_1 \cdots i_n$, $j_1 \cdots j_l$ in Y, $k > n+2M$,*

$$\begin{aligned}&\sum_{i_1 \cdots i_n a_1 \cdots a_{k-n} j_1 \cdots j_l \in B_{k+l}(Y)} |\pi^{-1}[i_1 \cdots i_n a_1 \cdots a_{k-n} j_1 \cdots j_l]|^{1/(\alpha+1)} \\ &\ge (|\pi^{-1}[i_1 \cdots i_n]||\pi^{-1}[j_1 \cdots j_l]|)^{1/(\alpha+1)} \\ &\quad \times \sum_{b_1 \cdots b_{k-n-2M} \in B_{k-n-2M}(Y)} |\pi^{-1}[b_1 \cdots b_{k-n-2M}]|^{1/(\alpha+1)}.\end{aligned}$$

Proof. Let $a_{M+1} \cdots a_{k-n-M}$ be an allowable word of length $(k-n-2M)$ in Y. Call it c. Then there exist $\overline{a_1 \cdots a_M}$, $\overline{a_{k-n-M+1} \cdots a_{k-n}}$ such that $i_1 \cdots i_n \overline{a_1 \cdots a_M}\, c \overline{a_{k-n-M+1} \cdots a_{k-n}} j_1 \cdots j_l$ is allowable in Y. Denote $\overline{a_1 \cdots a_M}$ by u and $\overline{a_{k-n-M+1} \cdots a_{k-n}}$ by v. Fix c and v. Then

$$\sum_{i_1 \cdots i_n ucv j_1 \cdots j_l \in B_{k+l}(Y)} |\pi^{-1}[i_1 \cdots i_n ucv j_1 \cdots j_l]| \ge |\pi^{-1}[i_1 \cdots i_n]||\pi^{-1}[cv j_1 \cdots j_l]|.$$

Therefore,

$$\begin{aligned}&\sum_{i_1 \cdots i_n ucv j_1 \cdots j_l \in B_{k+l}(Y)} |\pi^{-1}[i_1 \cdots i_n ucv j_1 \cdots j_l]|^{1/(\alpha+1)} \\ &\ge (|\pi^{-1}[i_1 \cdots i_n]||\pi^{-1}[cv j_1 \cdots j_l]|)^{1/(\alpha+1)}.\end{aligned}$$

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Now, for fixed $i_1 \cdots i_n$, $j_1 \cdots j_l$ and c, summing over all allowable words u, v in Y such that $i_1 \cdots i_n u c v j_1 \cdots j_l$ is allowable,

$$\begin{aligned}
&\sum_{i_1 \cdots i_n u c v j_1 \cdots j_l \in B_{k+l}(Y)} |\pi^{-1}[i_1 \cdots i_n u c v j_1 \cdots j_l]|^{1/(\alpha+1)} \\
&\quad \geq |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)} \sum_{c v j_1 \cdots j_l \in B_{k+l-n-M}(Y)} |\pi^{-1}[c v j_1 \cdots j_l]|^{1/(\alpha+1)} \\
&\quad \geq |\pi^{-1}[i_1 \cdots i_n]|^{1/(\alpha+1)} |\pi^{-1}[c]|^{1/(\alpha+1)} |\pi^{-1}[j_1 \cdots j_l]|^{1/(\alpha+1)}.
\end{aligned}$$

Finally, summing over all allowable words u, c, v in Y such that $i_1 \cdots i_n u c v j_1 \cdots j_l$ is allowable,

$$\begin{aligned}
&\sum_{i_1 \cdots i_n u c v j_1 \cdots j_l \in B_{k+l}(Y)} |\pi^{-1}[i_1 \cdots i_n u c v j_1 \cdots j_l]|^{1/(\alpha+1)} \\
&\quad \geq (|\pi^{-1}[i_1 \cdots i_n]||\pi^{-1}[j_1 \cdots j_l]|)^{1/(\alpha+1)} \\
&\qquad \times \sum_{a_{M+1} \cdots a_{k-n-M} \in B_{k-n-2M}(Y)} |\pi^{-1}[a_{M+1} \cdots a_{k-n-M}]|^{1/(\alpha+1)}.
\end{aligned}$$

□

LEMMA 4.10. *Let ν be a σ_Y-invariant Gibbs measure for Φ. Then ν is ergodic.*

Proof. We use the same arguments as in the proof of Lemma 2 in [**2**]. We show that there exists $C > 0$ such that for each $k > n + 2M$ and any two cylinders $[i_1 \cdots i_n]$, $[j_1 \cdots j_l]$, $\nu([i_1 \cdots i_n] \cap \sigma_Y^{-k}([j_1 \cdots j_l])) \geq C\nu([i_1 \cdots i_n])\nu([j_1 \cdots j_l])$. This implies that $\liminf_{k\to\infty} \nu(A \cap \sigma_Y^{-k}(B)) \geq C\nu(A)\nu(B)$ for any Borel measurable subsets A, B of Y. Suppose that ν is a σ_Y-invariant Gibbs measure for Φ which satisfies the inequalities (4.15). We denote $i_1 \cdots i_n$ by u_1 and $j_1 \cdots j_l$ by u_2. Using Lemmas 4.6 and 4.9,

$$\begin{aligned}
\nu([u_1] \cap \sigma_Y^{-k}([u_2])) &= \sum_{u_1 a_1 \cdots a_{k-n} u_2 \in B_{k+l}(Y)} \nu([u_1 a_1 \cdots a_{k-n} u_2]) \\
&\geq \sum_{u_1 a_1 \cdots a_{k-n} u_2 \in B_{k+l}(Y)} C_1 |\pi^{-1}[u_1 a_1 \cdots a_{k-n} u_2]|^{1/(\alpha+1)} e^{-(k+l)P(\sigma_Y,\Phi)} \\
&\geq C_1 \frac{(|\pi^{-1}[u_1]||\pi^{-1}[u_2]|)^{1/(\alpha+1)} S_{k-n-2M} e^{(n-k)P(\sigma_Y,\Phi)}}{e^{(n+l)P(\sigma_Y,\Phi)}} \quad \text{(by Lemma 4.9)} \\
&\geq C_1 C_2{}^{-2} \nu[u_1]\nu[u_2] \frac{S_{k-n-2M}}{e^{(k-n)P(\sigma_Y,\Phi)}} \geq \frac{C_1 \nu[u_1]\nu[u_2]}{C_2{}^2 e^{2MP(\sigma_Y,\Phi)}}.
\end{aligned}$$

□

PROPOSITION 4.11. *Let (X, σ_X) be a topologically mixing shift of finite type, (Y, σ_Y) a subshift and $\pi : X \to Y$ a factor map. Let $\alpha > 0$. For all $n \in \mathbb{N}$, define $\phi_n : Y \to \mathbb{R}$ by $\phi_n(y) = \log |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)}$ and $\Phi = \{\phi_n\}_{n=1}^{\infty}$. Then, for each $\alpha > 0$, there is a unique equilibrium state for Φ. The unique measure is Gibbs for Φ and mixing.*

Remarks 4.12. We have the same results for $\alpha = 0$. While every unique equilibrium state μ in the above is Gibbs for Φ, there is no guarantee that each μ is the unique Gibbs equilibrium state for a continuous function (see Example 5.6).

Proof. We follow the proof of Theorem 5 in [**2**] and so we only give an outline of the proof. In Lemma 4.8, we construct an equilibrium state μ for Φ which is Gibbs. If μ and μ' are two distinct σ_Y-invariant ergodic Borel probability measures on Y, then μ and μ' must be

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mutually singular. Therefore, using Lemma 4.10, μ in Lemma 4.8 is unique and it is the unique ergodic invariant Borel probability measure on Y that satisfies the Gibbs property for Φ. The same arguments as in the proof of Theorem 5 in [**2**] show that μ is the unique equilibrium state for Φ and that it is mixing. □

In order to study uniqueness of the equilibrium state for $-\alpha\Phi \circ \pi = \{-\alpha\phi_n \circ \pi\}_{n=1}^{\infty}$, we use the following results by Petersen *et al* [**16**].

THEOREM 4.13. [**16**] *Let (X, σ_X) be an irreducible shift of finite type, (Y, σ_Y) a subshift and $\pi : X \to Y$ a factor map. Suppose that π has a singleton clump a, i.e. there is a symbol $\{a\}$ of Y such that the number of preimages of $\{a\}$ under π is 1 in Y. Then every σ_Y-invariant ergodic Borel probability measure on Y which assigns positive measure to $[a]$ has a unique preimage of maximal relative entropy.*

The following results generalize the results on uniqueness of the invariant ergodic measure of full dimension in [**23**].

THEOREM 4.14. *Let (X, σ_X) be a topologically mixing shift of finite type, (Y, σ_Y) a subshift and $\pi : X \to Y$ a factor map. Let K be an SFT-NC carpet corresponding to (X, Y, π). If π has a singleton clump, then there is a unique T-invariant ergodic measure of full dimension for K. Define $\Phi = \{\phi_n\}_{n=1}^{\infty}$ as in Proposition 4.11. If Φ is an almost additive potential on (Y, σ_Y), then there is a unique T-invariant ergodic measure of full dimension for K and it is mixing. Furthermore, it is the image under the coding map χ of the unique equilibrium state for $-\alpha\Phi \circ \pi$, which is also Gibbs for $-\alpha\Phi \circ \pi$.*

Remarks 4.15.

(1) In [**15**], uniqueness of the measure of full dimension for topologically mixing SFT-NC carpets is shown by using a different approach.
(2) In [**7**], uniqueness for a subshift-NC carpet with the weak specification property (the case when S in the definition of a subshift-NC carpet satisfies weak specification) is shown.
(3) Theorem 4.14 can be extended for a sofic-NC carpet with the specification property for which ϕ_n' in Theorem 4.4 is continuous for all $n \geq N$ for some $N \in \mathbb{N}$ or Theorem 2.6 is valid.
(4) Φ is not always almost additive even when π has a singleton clump (Example 5.1). Φ can be almost additive even when π has no singleton clump (Example 5.3).

Proof. By Theorem 4.1, we study the ergodic equilibrium states for $-\alpha\Phi \circ \pi$. Since the coding map χ is bounded to one, it is enough to show uniqueness of an equilibrium state for $-\alpha\Phi \circ \pi$ (see [**23**, Corollary 2.7]). If Φ is almost additive, then $-\alpha\Phi \circ \pi$ is almost additive. Therefore, in this case, by Theorem 2.2, there is a unique equilibrium state and the unique measure is Gibbs and mixing. If π has a singleton clump, first notice by Proposition 3.14 that the equilibrium states are the measures of maximal relative entropy over the unique measure ν for Φ. Since by Proposition 4.11 the unique equilibrium state ν is Gibbs, if 1 is the singleton clump, then $\nu([1]) > 0$. Applying Theorem 4.13, we conclude that there is a unique equilibrium state for $-\alpha\Phi \circ \pi$. Therefore, there is a unique T-invariant ergodic measure of full dimension. □

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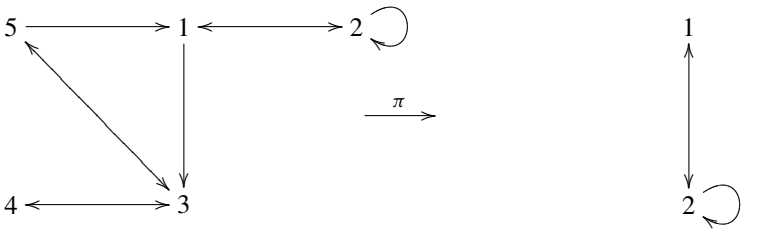


FIGURE 1. X, Y and π in Example 5.1.

5. *Examples*

In this section, we give examples that illustrate the results in §§3 and 4. Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ a factor map. Let $\alpha > 0$. For all $n \in \mathbb{N}$, define $\phi_n : Y \to \mathbb{R}$ by $\phi_n(y) = \log |\pi^{-1}[y_1 \cdots y_n]|^{1/(\alpha+1)}$, $\phi_n^0 : Y \to \mathbb{R}$ by $\phi_n^0(y) = \log |\pi^{-1}[y_1 y_2 \cdots y_n]|$ and $\psi_n : Y \to \mathbb{R}$ by $\psi_n(y) = \log |D_n(y)|$. Let $\Phi = \{\phi_n\}_{n=1}^{\infty}$ and $\Phi_0 = \{\phi_n^0\}_{n=1}^{\infty}$. Define $F : Y \to \mathbb{R}$ by $F(y) = P(\sigma_X, \pi, 0)(y)$ for all $y \in Y$.

Example 5.1. [**18**] (Singleton clump case without Φ_0 being almost additive)

This example appeared in [**18**, Example 3.1] and it was shown that there is no *continuous* saturated compensation function. Applying Corollary 3.11, we find a Borel measurable saturated compensation function $-F \circ \pi$, where $F = \limsup_{n\to\infty}(1/n)\phi_n^0$ almost everywhere with respect to every σ_Y-invariant Borel probability measure on Y. We claim that Φ_0 is not almost additive on (Y, σ_Y). Let $X \subset \{1, 2, 3, 4, 5\}^{\mathbb{N}}$ and $Y \subset \{1, 2\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given by Figure 1. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = \pi(4) = \pi(5) = 2$.

In [**18**], it was shown that for n odd $|\pi^{-1}[12^n 1]| = 1$ and for $n = 2k$ even $|\pi^{-1}[12^{2k}1]| = 2^{k-1} + 1$. For n odd, $|\pi^{-1}[12^n 1]|/(|\pi^{-1}[12]||\pi^{-1}[2^{n-1}1]|)$ is clearly not bounded below by a positive constant. Therefore, Φ_0 is not almost additive. Applying Theorem 4.14, an SFT-NC carpet corresponding to (X, Y, π) has a unique T-invariant ergodic measure of full dimension. The Hausdorff dimension of the set is given by the formulas in Theorem 4.1.

Example 5.2. (Singleton clump case with Φ_0 being almost additive)

Let $X \subset \{1, 2, 3, 4\}^{\mathbb{N}}$ and $Y = \{1, 2\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given by Figure 2. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = \pi(4) = 2$. We cannot apply theorems in [**23**] to find a saturated compensation function $G \circ \pi$, $G \in (Y)$, because the transition matrix among symbols of the preimages of $\{2\}$ is not primitive and so G is not defined. Applying Corollary 3.11, we find a Borel measurable saturated compensation function $-F \circ \pi$, where $F = \limsup_{n\to\infty}(1/n)\phi_n^0$ almost everywhere with respect to every σ_Y-invariant Borel probability measure on Y. We claim that Φ_0 is almost additive on (Y, σ_Y).

Let a be the largest eigenvalue of the irreducible matrix among symbols of the preimages of $\{2\}$. Notice that $|\pi^{-1}[1^{l_1}2^{n_1}1^{l_2}2^{n_2}1^{l_3}]| = |\pi^{-1}[12^{n_1}1]||\pi^{-1}[12^{n_2}1]|$. Define

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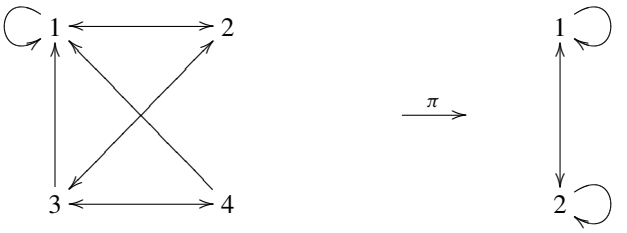


FIGURE 2. X, Y and π in Example 5.2.

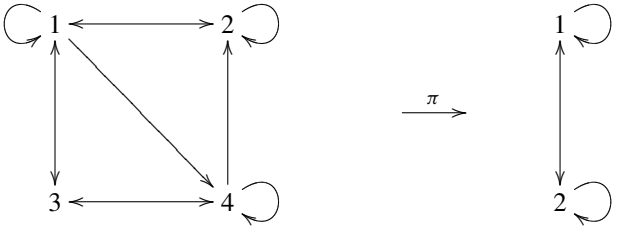


FIGURE 3. X, Y and π in Example 5.3.

$1^0 2^n = 2^n 1^0 = 2^n$. It is enough to show that there exist C_1, $C_2 > 0$ such that

$$C_1 \leq \frac{|\pi^{-1}[1^i 2^n 1^j]|}{|\pi^{-1}[1^i 2^{k_1}]||\pi^{-1}[2^{k_2} 1^j]|}, \frac{|\pi^{-1}[12^n 1^j]|}{|\pi^{-1}[1]||\pi^{-1}[2^n 1^j]|}, \frac{|\pi^{-1}[1^i 2^n 1]|}{|\pi^{-1}[1^i 2^n]||\pi^{-1}[1]|} \leq C_2,$$

for $n = k_1 + k_2$, $k_1, k_2 \geq 1$, $0 \leq i, j \leq 1$. Applying the Perron–Frobenius theorem, we can find $\widetilde{C_1}$, $\widetilde{C_2} > 0$ such that

$$\widetilde{C_1} a^{n-1} \leq |\pi^{-1}[12^n 1]|, |\pi^{-1}[2^n]|, |\pi^{-1}[12^n]|, |\pi^{-1}[2^n 1]| \leq \widetilde{C_2} a^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, Φ_0 is almost additive. By Theorem 4.14, an SFT-NC carpet corresponding to (X, Y, π) has a unique T-invariant ergodic measure of full dimension and it is mixing. It is the image under the coding map χ of the unique equilibrium state for $-\alpha\Phi \circ \pi$, which is also Gibbs for $-\alpha\Phi \circ \pi$. The Hausdorff dimension of the set is given by the formulas in Theorem 4.1.

Example 5.3. (No singleton clump case)

We construct (X, Y, π) such that Φ is almost additive on (Y, σ_Y). Let $X \subset \{1, 2, 3, 4\}^{\mathbb{N}}$ and $Y = \{1, 2\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given by Figure 3. Define π by $\pi(1) = \pi(2) = 1$, $\pi(3) = \pi(4) = 2$. Let A_1 be the transition matrix of symbols of 1 and 2 in X and A_2 be the transition matrix of symbols of 3 and 4 in X. Then the largest positive eigenvalue of A_1 is 2 and the largest positive eigenvalue of A_2 is $(1 + \sqrt{5})/2$. Let $a = (1 + \sqrt{5})/2$. Clearly, we have $|\pi^{-1}[12^n 1]| = |\pi^{-1}[2^n]|$ for all $n \in \mathbb{N}$ and $|\pi^{-1}[12]| = |\pi^{-1}[21]| = |\pi^{-1}[2]|$. It is easy to see that

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for $t \geq 1$, $k_1, \ldots, k_t, l_1, \ldots, l_t \geq 1$,

$$\begin{aligned}|\pi^{-1}[2^{k_1}1^{l_1}\cdots 2^{k_t}1^{l_t}]| &= |\pi^{-1}[1^{l_1}2^{k_1}\cdots 1^{l_t}2^{k_t}]| \\ &= 2^{l_1+\cdots+l_t-2(t-1)-1}|\pi^{-1}[2^{k_1}]|\cdots|\pi^{-1}[2^{k_t}]|\end{aligned}$$

and, for $t \geq 2$,

$$\begin{aligned}|\pi^{-1}[2^{k_1}1^{l_1}\cdots 1^{l_{t-1}}2^{k_t}]| &= 2^{l_1+\cdots+l_{t-1}-2(t-1)}|\pi^{-1}[2^{k_1}]|\cdots|\pi^{-1}[2^{k_t}]|, \\ |\pi^{-1}[1^{l_1}2^{k_1}\cdots 2^{k_{t-1}}1^{l_t}]| &= 2^{l_1+\cdots+l_t-2(t-2)-2}|\pi^{-1}[2^{k_1}]|\cdots|\pi^{-1}[2^{k_{t-1}}]|.\end{aligned}$$

Applying the Perron–Frobenius theorem, there exist $C_1, C_2 > 0$ such that $C_1 a^{k-1} \leq |\pi^{-1}[2^k]| \leq C_2 a^{k-1}$ for $k \in \mathbb{N}$. Then calculations show the inequalities (4.7) and so Φ is almost additive. Applying Corollary 3.11, $-F \circ \pi$, where $F = \limsup_{n\to\infty}(1/n)\phi_n^0$ almost everywhere with respect to every σ_Y-invariant Borel probability measure on Y, is a Borel measurable saturated compensation function for $(\sigma_X, \sigma_Y, \pi)$ and Φ_0 is almost additive on (Y, σ_Y). By Theorem 4.14, an SFT-NC carpet corresponding to (X, Y, π) has a unique T-invariant ergodic measure of full dimension and it is mixing. It is the image under the coding map χ of the unique equilibrium state for $-\alpha\Phi \circ \pi$, which is also Gibbs for $-\alpha\Phi \circ \pi$. The Hausdorff dimension of the set is given by the formulas in Theorem 4.1.

Example 5.4. (A subshift with ψ_n being continuous for all $n \in \mathbb{N}$)

Let $X \subset \{1, 2, 3, 4\}^{\mathbb{N}}$ be the subshift with the following forbidden blocks: 23, 24, 32, 42, 12^n1 for $n \geq 4$ and $1C_n1$, where C_n is an allowable word of the full shift of two symbols $\{3, 4\}$ which has length $n \geq 4$. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = \pi(4) = 2$. Then $Y \subset \{1, 2\}^{\mathbb{N}}$ is the subshift with the forbidden blocks 12^n1 for $n \geq 4$. It is easy to see that ψ_n is continuous for all $n \in \mathbb{N}$. Applying Theorem 3.2, $-F \circ \pi$ is a Borel measurable saturated compensation function for $(\sigma_X, \sigma_Y, \pi)$. The Hausdorff dimension of a subshift-NC carpet corresponding to (X, Y, π) is given by any formula in Theorem 4.4. The images under the coding map χ of the ergodic equilibrium states for $-\alpha\Phi' \circ \pi$, where Φ' is defined in Theorem 4.4, are T-invariant ergodic measures of full dimension.

Example 5.5. (A subshift with ψ_n being discontinuous for all $n \in \mathbb{N}$)

Let $X \subset \{1, 2, 3, 4\}^{\mathbb{N}}$ be the subshift with the following forbidden blocks: 23, 24, 32, 42, $12^{2n+1}1$ for $n \geq 0$ and $1C_{2n}1$, where C_{2n} is an allowable word of the full shift of two symbols $\{3, 4\}$ which has length $2n$, $n \geq 1$. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = \pi(4) = 2$. Then $Y = \{1, 2\}^{\mathbb{N}}$. We show that ψ_n is not continuous for all $n \in \mathbb{N}$. For $n \geq 3$, let $y = (y_1 \cdots y_i \cdots) \in Y$, where $y_1 = 2$ for $1 \leq i \leq n-2$ and $y_{n-1} = 1$, $y_i = 2$ for $i \geq n$. Then $|D_n(y)| = 3|\pi^{-1}[2^{n-2}1]|$. For $k \in \mathbb{N}$, let $z_k \in [2^{n-2}12^{2k}1] \subset Y$ and $w_k \in [2^{n-2}12^{2k+1}1] \subset Y$. Then $|D_n(z_k)| = |\pi^{-1}[2^{n-2}1]|$ and $|D_n(w_k)| = 2|\pi^{-1}[2^{n-2}1]|$ for all $k \in \mathbb{N}$. Hence, ψ_n is not continuous at y for $n \geq 3$. Similarly, for $n = 1, 2$, ψ_n is not continuous at $y = (y_1 \cdots y_i \cdots) \in Y$, where $y_1 = 1$ and $y_i = 2$ for $i \geq 2$. Applying Theorem 3.2, $-F \circ \pi$ is a Borel measurable saturated compensation function for $(\sigma_X, \sigma_Y, \pi)$. The Hausdorff dimension of a subshift-NC carpet corresponding to (X, Y, π) is given by the formulas (4.5) and (4.6). The images under the coding map χ of the ergodic equilibrium states for $-\alpha\Phi' \circ \pi$, where Φ' is defined in Theorem 4.4, are T-invariant ergodic measures of full dimension.

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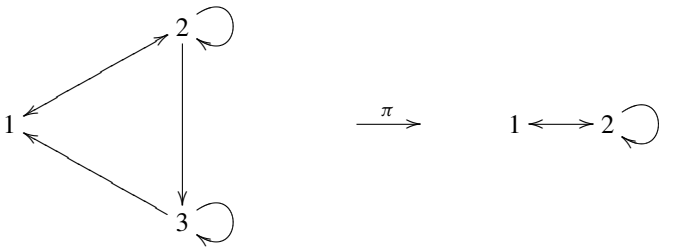


FIGURE 4. X, Y, and π in Example 5.6.

Example 5.6. (On Remark 4.12)

In this example, we find (X, Y, π) such that there is a unique equilibrium state ν for $-G \in C(Y)$ which is not Gibbs for $-G$, where $G \circ \pi$ is a saturated compensation function. We will see that ν is the unique equilibrium state for Φ_0 which is also Gibbs for Φ_0.

Let $X \subset \{1, 2, 3\}^{\mathbb{N}}$ and $Y \subset \{1, 2\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given by Figure 4. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$.

By [**23**, Theorem 3.1], there is a saturated compensation function $G \circ \pi$, where $G \in C(Y)$ is defined by

$$G(y) = \begin{cases} \log((n-1)/n) & \text{if } y \in [2^n 1],\ n \geq 2, \\ 0 & \text{if } y \in [1] \cup [21] \cup \{2^\infty\}, \end{cases}$$

and $-G$ has a unique equilibrium state ν which is not Gibbs. Since, by [**22**, Theorem 3.4],

$$-\int G\, dm = \int F\, dm \quad \text{for all } m \in M(Y, \sigma_Y),$$

we obtain

$$\sup_{m \in M(Y,\sigma_Y)} \left\{ h_m(\sigma_Y) - \int G\, dm \right\} = \sup_{m \in M(Y,\sigma_Y)} \left\{ h_m(\sigma_Y) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n^0\, dm \right\}.$$

A simple calculation shows that Φ_0 is not almost additive on (Y, σ_Y). By the first part of Remark 4.12, there is a unique equilibrium state for Φ_0 and it is Gibbs for Φ_0. It coincides with the unique equilibrium state ν for $-G$.

6. *Problems*

We finish by mentioning a question related to the results in §4. When is ϕ_n' in Theorem 4.4 continuous for all $n \geq N$ for some $N \in \mathbb{N}$ (see Examples 5.4 and 5.5)?

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