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A variety of groups

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Abstract

The following theorem is proved. For any positive integers k and n there exists t depending only on k and n such that the class of all groups G having $\gamma_k(G)$ locally finite and satisfying the condition that the product of any t commutators of the form $[x_1, x_2, \dots, x_k]$ is of order dividing n is a variety.

1. *Introduction*

Following Zelmanov's solution of the Restricted Burnside Problem [**14**, **15**] some new interesting varieties of groups have been descovered. It is well known that the solution of the Restricted Burnside Problem is equivalent to each of the following statements.

1·1. *The class of locally finite groups of exponent n is a variety.*

1·2. *The class of locally nilpotent groups of exponent n is a variety.*

The equivalence of the above two results follows from the famous Hall–Higman reduction theorem [**4**].

A variety is a class of groups defined by equations. More precisely, if W is a set of words in $x_1, x_2, \dots$, the class of all groups G such that $W(G) = 1$ is called the variety determined by W. By a well-known theorem of Birkhoff, varieties are precisely classes of groups closed with respect to taking quotients, subgroups and cartesian products of their members. If w is a word in variables $x_1, \dots, x_d$ we think of it primarily as a function of d variables defined on any given group G. We denote by $w(G)$ the verbal subgroup of G generated by the values of w. The word w is commutator if the sum of the exponents of any variable involved in w is zero. In [**11**] we raised the following questions.

Problem 1·3. Let $n \geqslant 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally finite. Is that a variety?

Problem 1·4. Let $n \geqslant 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally nilpotent. Is that a variety?

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According to the solution of the Restricted Burnside Problem the answer to the above questions is positive if $w(x) = x$. In fact it is easy to see that the answer is positive whenever w is any non-commutator word. Indeed, suppose that $w(x_1, \dots, x_d)$ is such a word and the sum of the exponents of x_i is $r \neq 0$. Now, given any group G satisfying the identity $w^n \equiv 1$, substitute the unit for all the variables except x_i and an arbitrary element $g \in G$ for x_i. We see that g^r is a w-value for all $g \in G$. Hence G satisfies the identity $x^{nr} \equiv 1$, that is, G has finite exponent. It is now not difficult to see that the results (1·1) and (1·2) imply positive answers to (1·3) and (1·4).

Hence, Problems 1·3 and 1·4 are essentially about commutator words. It is not known whether Problems 1·3 and 1·4 are equivalent. In [**10**] Problem 1·4 was solved positively in the case that w is a lower central word $[x_1, x_2, \dots, x_k]$. In [**11**, **12**] we have presented positive solutions for Problem 1·4 for some other words w. This paper deals with Problem 1·3 in the case that w is a product of several commutators of the form $[x_1, x_2, \dots, x_k]$.

Having fixed a positive integer k, the symbol w_j will denote the word that is the product of j commutators of the form $[x_1, x_2, \dots, x_k]$, all of whose entries are independent. Thus, for example, w_2 will denote the word $[x_1, x_2, \dots, x_k][y_1, y_2, \dots, y_k]$. It is clear that if an element of a group is a w_i-value, then it is also a w_j-value for any $i \leqslant j$. Therefore, for any $i \leqslant j$, the identity $w_j^n \equiv 1$ in a group implies the identity $w_i^n \equiv 1$.

Our main result is the following theorem.

THEOREM 1·5. *For any positive integers k and n there exists t depending only on k and n such that the class of all groups G satisfying the identity $w_t^n \equiv 1$ and having $\gamma_k(G)$ locally finite is a variety.*

We record one immediate corollary of Theorem 1·5 that is related to the results obtained in [**9**].

COROLLARY 1·6. *Let k and n be positive integers, and let t be as in Theorem* 1·5. *Assume that G is a residually finite group satisfying the identity $w_t^n \equiv 1$. Then $\gamma_k(G)$ is locally finite.*

The above corollary is no longer valid if the assumption of residual finiteness of G is dropped. In [**1**] Deryabina and Kozhevnikov showed that if n is a sufficiently big odd integer, then, even in the case $k = 2$, a group can satisfy the identity $w_i^n \equiv 1$ for an arbitrarily large i and yet have the derived group non-periodic.

2. *Proofs*

Recall that the Fitting subgroup $F(G)$ of a group G is the product of all normal nilpotent subgroups of G. The Fitting series of G can be defined by the rules: $F_0(G) = 1$ and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ for $i = 1, 2, \dots$. If G is a finite soluble group, then the minimal number $h = h(G)$ such that $F_h(G) = G$ is called the Fitting height of G. In [**13**] Turull considers finite soluble groups acted on by an operator group A. He introduces the notion of irreducible A-tower of height h. We need not concern ourselves with the precise definition of these subgroups. We just mention that these are A-invariant subgroups of Fitting height h. In the particular case that $A = 1$ any irreducible A-tower of height h can be generated by at most h elements. Turull proves

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in [**13**, lemma 1·4] that if G has Fitting height h, then G possesses an irreducible A-tower of height h. Therefore we have the following lemma.

LEMMA 2·1. *Let G be a finite soluble group of Fitting height h. Then one can choose h elements $g_1, \dots, g_h \in G$ such that the subgroup $\langle g_1, \dots, g_h \rangle$ has Fitting height precisely h.*

We need the following result, due to Segal [**8**].

LEMMA 2·2. *There exists a number $f = f(k, m)$, depending only on k and m, such that if G is a finite m-generated soluble group then any element $x \in \gamma_k(G)$ is a product of at most f commutators of the form $[x_1, x_2, \dots, x_k]$.*

Throughout the rest of the paper $f(k, m)$ will stand for the Segal function as in Lemma 2·2.

A well-known corollary of the Hall–Higman theory [**4**] says that the Fitting height of any finite soluble group of exponent n is bounded by the number of prime divisors of n, counting multiplicities. We will denote the number by $h(n)$.

LEMMA 2·3. *Let j, n be positive integers satisfying the inequality $j \geqslant f(k, h(n) + 2)$. Let G be a finite soluble group satisfying the identity $w_j^n \equiv 1$. Then $h(G) \leqslant h(n) + 1$.*

Proof. Assume the lemma is false. Set $h = h(n)$. By Lemma 2·1 we can choose $h + 2$ elements $g_1, \dots, g_{h+2} \in G$ with the property that the subgroup $H = \langle g_1, \dots, g_{h+2} \rangle$ has Fitting height at least $h + 2$. Segal's theorem tells us that any element of $\gamma_k(H)$ can be written as a product of at most $f(k, h + 2)$ commutators of the form w_1. By the hypothesis any such product has order dividing n. So $\gamma_k(H)$ is of exponent n, whence $h(\gamma_k(H)) \leqslant h(n)$. Therefore $h(H) \leqslant h(n) + 1$, a contradiction.

Following [**4**] we call a group G monolithic if it has a unique minimal normal subgroup which is non-abelian simple. In other terminology G can also be called almost simple.

PROPOSITION 2·4. *Let G be a finite group satisfying the identity $w_j^n \equiv 1$ for some positive integers j and n, and let j_1 be any positive integer such that $j \geqslant j_1 n + 1$. Assume that G has no non-trivial normal soluble subgroups. Then G possesses a normal subgroup L such that L is residually monolithic and G/L residually belongs to the class of finite groups satisfying an identity $w_{j_1}^{n/p} \equiv 1$ for some prime divisor p of n.*

Proof. Let M be a minimal normal subgroup of G. Then $M \cong S_1 \times \cdots \times S_r$, where $S_1, \dots, S_r$ are isomorphic simple groups. The group G acts on M by permuting the simple factors so we obtain a representation of G by permutations of the set $\{S_1, \dots, S_r\}$. Let L_M be the kernel of the representation. Choose in S_1 a non-trivial element b of the form w_1, and let p be a prime divisor of the order of b. We want to show that G/L_M satisfies the identity $w_{j_1}^{n/p} \equiv 1$. Suppose this is not true. Let $q = p^\alpha$ be the largest power of p dividing n. Since the identity $w_{j_1}^{n/p} \equiv 1$ does not hold in G/L_M, there exists an element $a \in G$ of the form w_{j_1} such that q divides the order of a modulo L_M. Write $n = n_1 q$ and $a_1 = a^{n_1}$. Then a_1 is an element of the form $w_{j_1 n_1}$ that has order q and permutes regularly some q factors in $\{S_1, \dots, S_r\}$. Without any loss of generality we will assume that S_1 is one of those factors. Write

$$(ba_1)^q = bb^{a_1^{-1}} b^{a_1^{-2}} \cdots b^{a_1}.$$

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Since each of the elements $b^{a_1^{-i}}$ belongs to a different subgroup S_i, the product $bb^{a_1^{-1}}b^{a_1^{-2}}\cdots b^{a_1}$ has the same order as b. Thus, $(ba_1)^q$ has order divisible by p so the order of ba_1 is divisible by $p^{\alpha+1}$. However ba_1 is of the form w_j so its order must divide n. This contradiction shows that indeed G/L_M satisfies the identity $w_{j_1}^{n/p}\equiv 1$.

Let now L be the intersection of all the subgroups L_M, where M ranges through the minimal normal subgroups of G. The previous paragraph implies that the proof of the proposition will be completed once it is shown that L is residually monolithic. If R is the product of the minimal normal subgroups of G, it is clear that R is the product of pairwise commuting simple groups $S_1,\dots,S_t$ and that L is the intersection of the normalizers of S_i. Since G has no non-trivial normal soluble subgroups, it follows that $C_G(R)=1$ and therefore any element of L induces a non-trivial automorphism of some of the S_i. Let ρ_i be the natural homomorphism of L into the group of automorphisms of S_i. It is easy to see that the image of ρ_i is monolithic and that the intersection of the kernels of all ρ_i is trivial. Hence L is residually monolithic.

Let us now choose a function $t(x,y)$ defined for any positive integers x,y with the following two properties.

(1) $t(x,y)\geqslant f(x,h(y)+2)$ for all x,y.

(2) $t(x,y)\geqslant y\cdot t(x,z)+1$ for all x,y,z such that z is a proper divisor of y.

Such a function can be constructed using induction on y. Indeed, fix a positive number x and define $t(x,1)=f(x,2)$. Now suppose $t(x,y)$ is defined for all $y\leqslant n-1$. Let M be the maximum of $f(x,h(n)+2)$ and $n\cdot t(x,z)+1$, where z ranges through the set of all proper divisors of n. Put $t(x,n)=M$. This can be performed for any x, thus establishing the existence of a function with the desired properties. Eventually, it will be shown that the chosen function satisfies the hypothesis of Theorem 1·5.

PROPOSITION 2·5. *Let k and n be positive integers and $t=t(k,n)$. Let G be a finite group satisfying the identity $w_t^n\equiv 1$. Assume that G can be generated by m elements, each of order dividing l. Then the order of G is bounded by a function depending only on k,m,n,l.*

Proof. If $n=1$, the group G is nilpotent of class $k-1$ and the proposition holds true. So we assume that $n\geqslant 2$ and that the proposition is true for groups satisfying an identity $w_{t(k,n/p)}^{n/p}\equiv 1$ for a prime divisor p of n. In other words, the induction hypothesis is that there exists a $\{k,m,n,l\}$-bounded number N_0 such that if G is a finite group satisfying an identity $w_{t(k,n/p)}^{n/p}\equiv 1$, and if G can be generated by m elements of order dividing l, then $|G|\leqslant N_0$.

Suppose first that G has no non-trivial normal soluble subgroups. Since $t\geqslant n\cdot t(k,n/p)+1$, Proposition 2·4 tells us that G possesses a normal subgroup L such that L is residually monolithic and G/L residually belongs to the class of finite groups satisfying an identity $w_{t(k,n/p)}^{n/p}\equiv 1$ for some prime divisor p of n. It follows that G/L is residually of order at most N_0. Since G/L is m-generated, the number of distinct normal subgroups of index at most N_0 in G/L is $\{m,N_0\}$-bounded [**3**, theorem 7·2·9]. Therefore L is of $\{k,m,n,l\}$-bounded index in G. In particular, it follows that L can be generated by r elements for some $\{k,m,n,l\}$-bounded number r.

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A result of Jones [**6**] says that any infinite family of finite simple groups generates the variety of all groups. It follows that up to isomorphism there exist only finitely many monolithic groups satisfying the identity $w_t^n \equiv 1$. Let N_1 be the maximum of their orders. Then L is residually of order at most N_1. Since L is r-generated, the number of distinct normal subgroups of index at most N_1 in L is $\{k, m, n, l\}$-bounded. Therefore L has $\{k, m, n, l\}$-bounded order. We conclude that $|G|$ has $\{k, m, n, l\}$-bounded order.

Let us now drop the assumption that G has no non-trivial normal soluble subgroups. Let S be the product of all normal soluble subgroups of G. The above paragraph shows that the index $|G : S|$ is $\{k, m, n, l\}$-bounded. Since $t \geqslant f(k, h(n) + 2)$, according to Lemma 2·3 the Fitting height of S is $\{k, m, n, l\}$-bounded as well. Using induction on the Fitting height of S, we assume that if R is the Fitting subgroup of G, then the index $|G : R|$ is $\{k, m, n, l\}$-bounded.

Suppose for a minute that R is central. Then $G/Z(G)$ has $\{k, m, n, l\}$-bounded order and Schur's theorem [**7**, p. 102] guarantees that so does G'. Since G can be generated by m elements of order dividing l, it follows that G has $\{k, m, n, l\}$-bounded order.

Now let us see what happens if R is nilpotent of class at most $k-1$. Applying the results of the previous paragraph to the quotient $G/[R, G]$, it follows that $[R, G]$ has $\{k, m, n, l\}$-bounded index in G. An easy induction on s now shows that $[R, \underbrace{G, \ldots, G}_{s}]$ has $\{k, m, n, s, l\}$-bounded index in G for any positive integer s. Set $T = [R, \underbrace{G, \ldots, G}_{k}]$. It follows that T can be generated by $\{k, m, n, l\}$-boundedly many elements. Since T is a group of class $k-1$ generated by commutators of order dividing n, it is clear that T has exponent bounded in terms of k and n. We now derive that the order of T, and hence the order of G, is $\{k, m, n, l\}$-bounded.

Consider now the general case, that is, we do not assume anymore that R is nilpotent of class at most $k-1$. Applying the results of the previous paragraph to the quotient $G/\gamma_k(R)$, it follows that the order of $G/\gamma_k(R)$ is $\{k, m, n, l\}$-bounded. So we derive that $\gamma_k(R)$ can be generated by $\{k, m, n, l\}$-boundedly many elements. A general property of finite nilpotent groups that follows from the Burnside Basis Theorem [**5**, III·3·15] is that if such a group is d-generated, then any generating set contains a generating subset of precisely d elements. So we deduce that $\gamma_k(R)$ can be generated by $\{k, m, n, l\}$-boundedly many commutators of the form w_1, each of order at most n. Now Theorem 7 of [**10**] tells us that the order of $\gamma_k(R)$ is $\{k, m, n, l\}$-bounded. We already know that the order of $G/\gamma_k(R)$ is $\{k, m, n, l\}$-bounded, whence we conclude that so is the order of G. This completes the proof.

THEOREM 2·6. *Let k, n be positive integers and $t = t(k, n)$. Let $\mathfrak{X}$ denote the class of all groups G having $\gamma_k(G)$ locally finite and any w_t-value of order dividing n. Then $\mathfrak{X}$ is a variety.*

Proof. Clearly, the class $\mathfrak{X}$ is closed with respect to taking quotients and subgroups of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{X}$ then $D \in \mathfrak{X}$. Of course, the identity $w_t^n \equiv 1$ holds in D so it remains only to show that $\gamma_k(D)$ is locally finite. Let T be any finite subset of $\gamma_k(D)$. Clearly, there exist finitely many commutators $h_1, \ldots, h_m$ of the form w_1 in D such that

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$T \leqslant \langle h_1, \dots, h_m \rangle$. Thus, it is sufficient to show that the group $H = \langle h_1, \dots, h_m \rangle$ is finite. Note that $\gamma_k(D)$ is residually locally finite. By Proposition 2·5 any locally finite image of H has finite $\{k, m, n\}$-bounded order. We conclude that H is finite, as required.

Theorem 1·5 is immediate from Theorem 2·6. Let us deduce Corollary 1·6. Suppose G is a residually finite group satisfying the identity $w_t^n \equiv 1$. Any finite quotient of G belongs to the variety of groups H with the above identity and $\gamma_k(H)$ locally finite. However, it is clear that if a group residually belongs to a certain variety, then it actually belongs to the variety. Thus, it follows that $\gamma_k(G)$ is locally finite.

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