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A stabilizing effect of vanishing diffusion in electrohydrodynamics

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We consider a model of the motion of a viscous dielectric liquid subjected to a DC electric field when the bulk conduction results from the presence of a dissociation-recombination process. It is shown that any weak solution approaches a neighbourhood of a spatially homogeneous steady state with radius r E $(d_+ + d_-)^{\frac{1}{2}}$, where d_+, d_- are the diffusion coefficients.

1 Problem description

In this paper, the issue of stability and asymptotic behaviour is examined for the equations of electrohydrodynamics. The results are somewhat unusual as we show that our information about the large time behaviour improves as the diffusion coefficients go to zero (see §3, Theorem 1).

Consider a bounded, connected domain Ω Z R^3 containing a viscous, incompressible dielectric liquid subjected to a DC electric field. A physically relevant simplification based on the theory of electrohydrodynamics, in which the effects of the magnetic field are neglected, allows to express the *electric field* $\vec{E} = \vec{E}(x,t)$ as

$$\vec{E} = -\nabla\Phi \quad \text{on } \Omega,$$

where Φ is the *electric potential* fixed on a part of the boundary Γ of Ω. To avoid regularity problems, we shall assume a specific shape for Ω, namely,

$$\Omega = \{(x_1, x_2, x_3) | (x_1, x_2) \grave{} \omega, \quad x_3 \grave{} (0, h)\},$$

where ω Z R^2 is a regular domain with smooth boundary. We assume that $\Gamma = \Gamma_1$ e Γ_2, where

$$\Gamma_1 = (\omega \times \{0\}) \text{ e } (\omega \times \{h\}), \quad \Gamma_2 = \partial\omega \times (0, h)$$

and prescribe

$$\Phi|_{\Gamma_1} = \Phi_b(x), \quad \nabla\Phi \cdot \vec{n}_\Omega|_{\Gamma_2} = 0, \tag{1.1}$$

$\vec{n}_\Omega$ being the outward unit normal vector to Γ.

Under the presence of only one ionic family in the fluid, the time evolution of the *cation concentration* $P = P(x,t)$ and the *anion concentration* $N = N(x,t)$ inside Ω is governed by the *Nernst–Planck equations*:

$$P_t - d_+ \Delta P + \operatorname{div}(-k_+ P\nabla\Phi + P\vec{v}) = M_0^2 - \alpha N P, \tag{1.2}$$

$$N_t - d_- \Delta N + \operatorname{div}(k_- N\nabla\Phi + N\vec{v}) = M_0^2 - \alpha N P, \tag{1.3}$$

coupled with the *Poisson equation*

$$-\epsilon\Delta\Phi = P - N. \tag{1.4}$$

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Here d_+, d_- are the *di usion coe cients*, k_+, k_- the *ionic mobilities*, and ϵ the *electric permeability* – all positive constants in what follows. The expression on the right-hand side of (1.2), (1.3) represents a recombination law with positive physical constants M_0, α.

The boundary value of P, N will be fixed on Γ_2

$$P|_{\Gamma_2} = P_b(x) \,\&\, 0, \quad N|_{\Gamma_2} = N_b(x) \,\&\, 0 \tag{1.5}$$

while on its complement we assume the equilibrium state

$$P|_{\Gamma_1} = N|_{\Gamma_1} = M_0/\sqrt{\alpha}. \tag{1.6}$$

The symbol $\vec{v} = \vec{v}(x,t)$ stands for the *velocity* of the fluid satisfying the incompressibility condition

$$\operatorname{div} \vec{v} = 0 \quad \text{in } \Omega. \tag{1.7}$$

We refer to *Rubinstein* [9] for a derivation of this model with $M_0 = \alpha = 0$ and to Atten [1], where the influence of the dissociation-recombination effects is taken into account. Further information may be found in Bradley [2] and Roberts [8]. At this stage, it seems interesting to note that the phenomenological background of (1.2)–(1.4), except for the convective terms $P\vec{v}$, $N\vec{v}$, is the same as in the semiconductor theory (cf. also the introductory section of Rubinstein [9]).

To obtain a full system, we add the Navier–Stokes equations

$$\rho \vec{v}_t - \nu \Delta \vec{v} + D\vec{v} \cdot \vec{v} = -\nabla q - (P - N) \nabla \Phi \tag{1.8}$$

assumed to govern the mechanical motion of the fluid. Here $q = q(x,t)$ denotes the *pressure*, ρ the *density* and ν the *kinematic viscosity* of the liquid.

The original state of the system at the reference time $t = 0$ is determined by the initial conditions

$$P(x,0) = P_0(x) \,\&\, 0, \quad N(x,0) = N_0(x) \,\&\, 0, \tag{1.9}$$

$$\vec{v}(x,0) = \vec{v}_0(x). \tag{1.10}$$

2 Global existence – weak solutions

The mathematical theory of the problem is based on the crucial observation that the region

$$\mathcal{M} = \{(P,N) \,|\, 0 \,\%\, P, N \,\%\, m\}$$

is invariant under the flow generated by the system (1.2), (1.3) as soon as the constant m has been chosen large enough (see Cimatti [4]). Therefore, the hypothesis

$$0 \,\%\, P_b(x), N_b(x), P_0(x), N_0(x) \,\%\, C_d \tag{2.1}$$

yields *a priori* estimates

$$0 \,\%\, P(\cdot,t), \quad N(\cdot,t) \,\%\, C_d \tag{2.2}$$

for any solution of (1.2), (1.3). Here (and always) the symbol C_d stands for a generic positive constant depending on the data only. In particular, C_d is independent of the diffusion coefficients d_+, d_-.

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Although the boundary Γ is not smooth, the L^p-estimates for the Laplace operator are still available. Specifically, we claim boundedness of the linear operators

$$A_N^{-1}, A_D^{-1}: L^p(\Omega) \to W^{2,p}(\Omega) \tag{2.3}$$

for any $p \in (1, \infty)$, where

$$A_N v = f$$

iff $v \in V = \{w \mid w \in W^{1,2}(\Omega), w|_{\Gamma_1} = 0\}$ and the identity

$$\int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Omega f w \, dx$$

holds for all $w \in V$, A_D being defined in the same way with V replaced by $W_0^{1,2}(\Omega)$. Indeed, for the odd and $2h$-periodic prolongations of v, f with respect to x_3, the problem of boundary regularity near Γ_1 reduces to interior regularity on the domain $\omega \times R$. Here the symbols $W^{k,p}$ denote the standard Sobolev spaces of functions having generalized derivatives up to order k in L^p.

Consequently, the smoothness of the boundary data Φ_b, and P_b, N_b on Γ_1, and Γ_2, respectively, together with the compatibility conditions

$$P_b|_{\partial\omega\times\{0\}} = N_b|_{\partial\omega\times\{h\}} = M_0/\sqrt{\alpha}, \tag{2.4}$$

$$\nabla\Phi_b \cdot \vec{n}_\omega|_{\partial\omega\times\{0\}} = \nabla\Phi_b \cdot \vec{n}_\omega|_{\partial\omega\times\{h\}} = 0 \tag{2.5}$$

would guarantee global existence of classical solutions to the system (1.2)–(1.4) if the velocity $\vec{v}$ were a smooth function.

However, in the absence of any global existence theory of classical solutions to the Navier–Stokes equations, we have to consider the weak formulation of (1.7), (1.8) due to Leray–Hopf, yielding solutions within the class

$$\|\vec{v}(t)\|_{(L^2(\Omega))^3} \leqslant C_d \quad \text{for a.e. } t \geqslant 0, \tag{2.6}$$

$$\sup_{t\geqslant 0} \int_t^{t+1} \int_\Omega |\nabla\vec{v}|^2 \, dx \, d\tau \leqslant C_d \tag{2.7}$$

for square integrable data $\vec{v}_0$ (cf. Temam [10]). Note that (2.2) implies boundedness of the forcing term in (1.8).

Thus, the maximal regularity of global solutions of (1.2), (1.3) is given by (2.2) and

$$d_+ \sup_{t\geqslant 0} \int_t^{t+1} \int_\Omega |\nabla P|^2 \, dx \, d\tau + d_- \sup_{t\geqslant 0} \int_t^{t+1} \int_\Omega |\nabla N|^2 \, dx \, d\tau \leqslant C_d, \tag{2.8}$$

where the latter follows from the standard energy estimates. The system (1.2), (1.3) may be interpreted as an evolution equation in the dual space $(W^{-1,2}(\Omega))^2$ by means of the identification

$$W_0^{1,2} \subset L^2 \subset (L^2)^* \subset W^{-1,2}$$

(see Lions & Magenes [6] for more information about weak solutions defined this way).

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Now, we may use (2.2) along with

$$P_t, N_t \in L^2_{\rm loc}(R^+, W^{-1,2}(\Omega)),$$

to conclude $$P, N \in C(R^+, L^p(\Omega)) \quad \text{for all} \quad p < \infty,$$

which makes meaningful the initial conditions (1.9) (cf. [6]).

Finally, by virtue of (2.3), (2.5), the potential Φ may be computed from (1.4) to obtain

$$\Phi \in C(R^+, W^{2,p}(\Omega)) \quad \text{for all} \quad p < \infty;$$

hence, by virtue of the Sobolev embedding theorem,

$$\nabla\Phi(\cdot, t) \in C^{0,\mu}(\bar{\Omega}), \quad \mu < 1 \tag{2.9}$$

uniformly with respect to t.

The question of existence of weak solutions is addressed in Cimatti [4].

3 Main result – discussion

The following theorem demonstrates that any weak solution to the Nernst–Planck system (1.2), (1.3) approaches a spatially homogeneous state as time goes to infinity when the diffusion coefficients are negligible:

Theorem 1 *Let P, N be any weak solutions of the equations* (1.2), (1.3) *with the boundary data* (1.5), (1.6) *and the initial condition* (1.9). *Moreover, let $P, N, \Phi, \vec{v}$ belong to the regularity class given by* (2.2), (2.6)–(2.9).

Then there exists a positive constant β depending only on $(k_+, k_-, \epsilon, \alpha)$ such that

$$\|P(\cdot, t) - M_0/\sqrt{\alpha}\|_{L^1(\Omega)} + \|N(\cdot, t) - M_0/\sqrt{\alpha}\|_{L_1(\Omega)} \leq C_d(\exp(-\beta t) + (d_+ + d_-)^{\frac{1}{4}}). \tag{3.1}$$

The result is seemingly at odds with traditional stability analyses, as it improves with vanishing diffusion. This is, of course, due to the boundary conditions (1.5), which are arbitrary and thus create a boundary layer preventing uniform stability in Ω unless

$$P_b = N_b = M_0/\sqrt{\alpha} \quad \text{on} \quad \Gamma_2. \tag{3.2}$$

For the same reason, the integral norm in (3.1) is optimal in general (cf. Ni & Takagi [7]). On the other hand, the presence of a boundary layer in the distribution of the electric potential Φ may speed up the stabilization process *inside* Ω (cf. Choi & Lui [3]).

On the other hand, it will be clear from the proof of Theorem 1 that condition (3.2) implies global exponential stability of the stationary state $(M_0/\sqrt{\alpha}, M_0/\sqrt{\alpha})$ regardless the values of d_+, d_-.

In fact, the motivation for the present study comes from the fact that the corresponding stationary problem for the equations without diffusion ($d_+ = d_- = 0$) is uniquely solvable (see Atten [1]), while such a problem containing the diffusion terms is not in the sense that some physical considerations show that in general there is non-uniqueness of steady states (cf. also Lin *et al.* [5]).

It is important to realize that Theorem 1 holds for *any* weak solution in the aforementioned class. The issue of uniqueness of such solutions is somewhat delicate because of the well-known problems related to the Navier–Stokes equations.

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Finally, the above result suggests (and formally proves) global asymptotic stability for the limiting hyperbolic system when $d_+ = d_- = 0$. This sheds some light on the local results stated in Atten [1].

4 Proof of Theorem 1

Rescaling

$$t \mathrel{E} \frac{\sqrt{\alpha}}{M_0} t, \quad \vec{v} \mathrel{E} \frac{\sqrt{\alpha}}{M_0} \vec{v}$$

and writing

$$p = \frac{\sqrt{\alpha}}{M_0} P - 1, \quad p = \frac{\sqrt{\alpha}}{M_0} N - 1, \quad \phi = \epsilon \frac{\sqrt{\alpha}}{M_0} \Phi,$$

we may rewrite the equations (1.2)–(1.4), (1.7) in the form

$$p_t - D_+ \Delta p - \frac{k_+}{\epsilon} \nabla p \cdot \nabla \phi - \frac{k_+}{\epsilon} (1+p)(n-p) + \operatorname{div}(p\vec{v}) + \alpha(p+n+np) = 0, \tag{4.1}$$

$$n_t - D_- \Delta n + \frac{k_-}{\epsilon} \nabla n \cdot \nabla \phi + \frac{k_-}{\epsilon} (1+n)(n-p) + \operatorname{div}(n\vec{v}) + \alpha(p+n+np) = 0, \tag{4.2}$$

$$-\Delta \phi = p - n, \tag{4.3}$$

$$\operatorname{div} \vec{v} = 0, \tag{4.4}$$

where $D_\pm = \frac{\sqrt{\alpha}}{M_0} d_\pm$. The relevant boundary conditions are

$$p|_{\Gamma_1} = n|_{\Gamma_1} = 0, \quad \phi|_{\Gamma_1} = \phi_b, \tag{4.5}$$

$$p|_{\Gamma_2} = p_b, \quad n|_{\Gamma_2} = n_b, \quad \nabla \phi \cdot \vec{n}_\Omega|_{\Gamma_2} = 0, \tag{4.6}$$

$$\vec{v}|_\Gamma = 0, \tag{4.7}$$

and the relation (2.2) reads

$$-1 \,\%\, p(\cdot, t), n(\cdot, t) \,\%\, C_d \quad \text{for a.e. } t \,\&\, 0. \tag{4.8}$$

As the domain ω is regular, we can find a function $\chi \grave{} C^\infty(\bar{\omega})$ such that

$$\lim_{y \to \partial\omega} \frac{\chi(y)}{\operatorname{dist}(y, \partial\omega)} = 1, \quad \chi \text{ strictly positive in } \operatorname{int}(\omega). \tag{4.9}$$

The main idea of the proof of Theorem 1 is to multiply (4.1), (4.2) by the expression $C_1 \operatorname{sgn}(p) \chi_\delta$, $C_2 \operatorname{sgn}(n) \chi_\delta$, respectively, with suitable constants C_1, C_2, where

$$\chi_\delta = \frac{1}{\delta} \min\{\chi, \delta\}, \quad \delta \text{ " } 0.$$

To justify this step, we approximate the function $\operatorname{sgn}(u) \mathrel{E} s_n(u)$, where s_n are smooth, monotone functions, $s_n(0) = 0$, coinciding with sgn outside the interval $\left(-\frac{1}{n}, \frac{1}{n}\right)$. To keep the notation clear, we drop the subscript n in what follows.

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Now, we can multiply (4.1) by the expression $s(p)\,\chi_\delta \in L^2_{\rm loc}(R^+, W^{1,2}_0(\Omega))$ to obtain

$$\frac{\rm d}{{\rm d}t}\int_\Omega S(p)\,\chi_\delta\,{\rm d}x + D_+\int_\Omega s'(p)\,|\nabla p|^2\,\chi_\delta\,{\rm d}x + D_+\int_\Omega s(p)\,\nabla p\cdot\nabla\chi_\delta\,{\rm d}x$$
$$-\frac{k_+}{\epsilon}\int_\Omega \nabla p\cdot\nabla\phi\,s(p)\,\chi_\delta\,{\rm d}x - \frac{k_+}{\epsilon}\int_\Omega (1+p)(n-p)\,s(p)\,\chi_\delta\,{\rm d}x$$
$$+\int_\Omega \alpha(p+n+np)\,s(p)\,\chi_\delta\,{\rm d}x + \int_\Omega {\rm div}\,(p\vec{v})\,s(p)\,\chi_\delta\,{\rm d}x = 0, \tag{4.10}$$

where we have used the relation

$$\int_\Omega s(p)\,p_t\,\chi_\delta\,{\rm d}x = \frac{\rm d}{{\rm d}t}\int_\Omega S(p)\,\chi_\delta\,{\rm d}x, \quad S(p) = \int_0^p s(z)\,{\rm d}z, \quad \text{for a.e. } t$$

which may be proved following the arguments of Temam [10, chap. III.1, lemma 1.2].

Observe that $\nabla p s(p) = \nabla S(p)$; hence

$$\int_\Omega \nabla p\cdot\nabla\phi\,s(p)\,\chi_\delta\,{\rm d}x = \int_\Omega \nabla S(p)\cdot\nabla\phi\,\chi_\delta\,{\rm d}x$$
$$= -\int_\Omega S(p)\,\Delta\phi\,\chi_\delta\,{\rm d}x - \int_\Omega S(p)\,\nabla\phi\cdot\nabla\chi_\delta\,{\rm d}x$$
$$= \int_\Omega S(p)\,(p-n)\,\chi_\delta\,{\rm d}x - \int_\Omega S(p)\,\nabla\phi\cdot\nabla\chi_\delta\,{\rm d}x \tag{4.11}$$

and, by virtue of (4.4), (4.7),

$$\int_\Omega {\rm div}\,(p\vec{v})\,s(p)\,\chi_\delta\,{\rm d}x = \int_\Omega \nabla S(p)\cdot\vec{v}\,\chi_\delta\,{\rm d}x = -\int_\Omega S(p)\,\vec{v}\cdot\nabla\chi_\delta\,{\rm d}x. \tag{4.12}$$

Consequently, (4.10) together with (4.11), (4.12), yield the estimate

$$\frac{\rm d}{{\rm d}t}\int_\Omega S(p)\,\chi_\delta\,{\rm d}x + \int_\Omega\left[\alpha(p+n+np)\,s(p) + \frac{k_+}{\epsilon}((1+p)\,s(p)-S(p))(p-n)\right]\chi_\delta\,{\rm d}x$$
$$\% \; C_d\left(d_+\int_\Omega |\nabla p\cdot\nabla\chi_\delta|\,{\rm d}x + \int_\Omega (|\nabla\phi\cdot\nabla\chi_\delta| + |\vec{v}\cdot\nabla\chi_\delta|\,{\rm d}x\right). \tag{4.13}$$

Repeating the same procedure for equation (4.2) and adding the result to (4.13) we get

$$\frac{\rm d}{{\rm d}t}\int_\Omega (C_1\,S(p) + C_2\,S(n))\,\chi_\delta\,{\rm d}x + \int_\Omega \alpha(p+n+np)(C_1\,s(p) + C_2\,s(n))\,\chi_\delta\,{\rm d}x$$
$$+\int_\Omega\left[C_1\frac{k_+}{\epsilon}((1+p)\,s(p)-S(p))(p-n) - C_2\frac{k_-}{\epsilon}((1+n)\,s(n)-S(n))(p-n)\right]\chi_\delta\,{\rm d}x$$
$$\% \; C_d\left(d_+\int_\Omega |\nabla p\cdot\nabla\chi_\delta|\,{\rm d}x + d_-\int_\Omega |\nabla n\cdot\nabla\chi_\delta|\,{\rm d}x + \int_\Omega (|\nabla\phi\cdot\nabla\chi_\delta| + |\vec{v}\cdot\nabla\chi_\delta|\,{\rm d}x\right) \tag{4.14}$$

with the constants C_1, C_2 to be chosen below.

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Finally, we integrate (4.14) with respect to t and then pass to the limit in the sequence $s_n(u) \to \operatorname{sgn}(u)$, $S_n(u) \to |u|$ to obtain

$$\int_\Omega G(p,n)(t+\xi)\,\chi_\delta\,dx - \int_\Omega G(p,n)(t)\,\chi_\delta\,dx + \int_t^{t+\xi}\int_\Omega F(p,n)\,\chi_\delta\,dx$$
$$\leqslant C_d\left(d_+\int_t^{t+1}\int_\Omega |\nabla p\cdot\nabla\chi_\delta|\,dx\,d\tau + d_-\int_t^{t+1}\int_\Omega |\nabla n\cdot\nabla\chi_\delta|\,dx\,d\tau\right.$$
$$\left. + \int_t^{t+1}\int_\Omega |\nabla\phi\cdot\nabla\chi_\delta| + |\vec{v}\cdot\nabla\chi_\delta|\,dx\,d\tau\right), \quad \xi \in [0,1], \tag{4.15}$$

where we have written

$$G(p,n) = C_1|p| + C_2|n|,$$
$$F(p,n) = \left(C_1\frac{k_+}{\epsilon}\operatorname{sgn}(p) - C_2\frac{k_-}{\epsilon}\operatorname{sgn}(n)\right)(p-n) + \alpha(p+n+np)(C_1\operatorname{sgn}(p) + C_2\operatorname{sgn}(n)).$$

Our present goal is to estimate the terms on the right-hand side of (4.15). Using (2.8) and the properties of χ, we get

$$\int_t^{t+1}\int_\Omega |\nabla p\cdot\nabla\chi_\delta|\,dx\,d\tau$$
$$\leqslant \left(\int_t^{t+1}\int_\Omega |\nabla p|^2\,dx\,d\tau\right)^{\frac12}\left(\int_t^{t+1}\int_\Omega |\nabla\chi_\delta|^2\,dx\,d\tau\right)^{\frac12} \leqslant \frac{C_d}{\delta\sqrt{d_+}}\left(\int_{\mathscr{U}(\delta,\Gamma_2)} |\nabla\chi|^2\,dx\right)^{\frac12},$$

where $$\mathscr{U}(\delta,\Gamma_2) = \{x \in \Omega \mid \operatorname{dist}(x,\Gamma_2) \leqslant 2\delta\}$$

provided that δ is small enough. Since $\nabla\chi$ is bounded and

$$\operatorname{meas}(\mathscr{U}(\delta,\Gamma_2)) \leqslant \operatorname{const}\delta,$$

we get
$$d_+\int_t^{t+1}\int_\Omega |\nabla p\cdot\nabla\chi_\delta|\,dx\,d\tau \leqslant C_d\sqrt{\left(\frac{d_+}{\delta}\right)} \tag{4.16}$$

and, similarly,

$$d_-\int_t^{t+1}\int_\Omega |\nabla n\cdot\nabla\chi_\delta|\,dx\,d\tau \leqslant C_d\sqrt{\left(\frac{d_-}{\delta}\right)}. \tag{4.17}$$

Analogously, we have

$$\int_\Omega |\nabla\phi\cdot\nabla\chi_\delta|\,dx \leqslant \frac{1}{\delta}\int_{\mathscr{U}(\delta,\Gamma_2)} |\nabla\phi\cdot\nabla\chi|\,dx$$
$$\leqslant \frac{1}{\delta}\left(\int_{\mathscr{U}(\delta,\Gamma_2)}\frac{|\nabla\phi\cdot\nabla\chi|^2}{\chi^2}dx\right)^{\frac12}\left(\int_{\mathscr{U}(\delta,\Gamma_2)}\chi^2\,dx\right)^{\frac12} \leqslant \sqrt{\delta}\left(\int_\Omega\frac{|\nabla\phi\cdot\nabla\chi|^2}{\chi^2}dx\right)^{\frac12},$$

and, similarly,

$$\int_\Omega |\vec{v}\cdot\nabla\chi_\delta|\,dx \leqslant \sqrt{\delta}\left(\int_\Omega\frac{|\vec{v}\cdot\nabla\chi|^2}{\chi^2}dx\right)^{\frac12}.$$

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Now we use that fact that both $\vec{v}\cdot\nabla\chi$ and $\nabla\phi\cdot\nabla\chi$ belong to $W^{1,2}(\Omega)$ and vanish on Γ_2 by virtue of (4.6), (4.7). In view of the result of Lions & Magenes [6, chap. I, theorem 11.3], we can estimate

$$\int_\Omega \frac{|\nabla\phi\cdot\nabla\chi|^2}{\chi^2}\,\mathrm{d}x \leqslant \text{const}\,\|\nabla\phi\|^2_{(W^{1,2})^3},$$

$$\int_\Omega \frac{|\vec{v}\cdot\nabla\chi|^2}{\chi^2}\,\mathrm{d}x \leqslant \text{const}\,\|v\|^2_{(W^{1,2})^3}.$$

Consequently, the estimate (4.15) may be rewritten in the form

$$\int_\Omega G(p,n)(t+\xi)\,\chi_\delta\,\mathrm{d}x - \int_\Omega G(p,n)(t)\,\chi_\delta\,\mathrm{d}x + \int_t^{t+\xi}\int_\Omega F(p,n)\,\chi_\delta\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C_d\left(\sqrt{\frac{d_+}{\delta}} + \sqrt{\frac{d_-}{\delta}} + \sqrt{\delta}\right). \quad (4.18)$$

At this stage, we use the following lemma, proved in §5:

Lemma 1 *Suppose that the constants C_1, C_2 satisfy*

$$\frac{\left|\frac{k_-}{\epsilon}-\alpha\right|}{\frac{k_+}{\epsilon}+\alpha} < \frac{C_1}{C_2} < \frac{\frac{k_-}{\epsilon}+\alpha}{\left|\frac{k_+}{\epsilon}-\alpha\right|}. \quad (4.19)$$

Then there exists $r > 0$ such that

$$F(p,n) \geqslant rG(p,n) \quad \textit{for all } p,n \geqslant -1. \quad (4.20)$$

Taking Lemma 1 for granted, observe that

$$\left|\int_\Omega G(p,n)\,\mathrm{d}x - \int_\Omega G(p,n)\,\chi_\delta\,\mathrm{d}x\right| \leqslant \left|\int_\Omega G(p,n)(1-\chi_\delta)\,\mathrm{d}x\right| \leqslant C_d\,\mathrm{meas}\,(\mathscr{U}(\delta,\Gamma_2)) \leqslant C_d\,\delta$$

and, consequently, the relation (4.18) may be rewritten as follows:

$$\int_\Omega G(p,n)(t+\xi)\,\mathrm{d}x - \int_\Omega G(p,n)(t)\,\mathrm{d}x + r\int_t^{t+\xi}\int_\Omega G(p,n)\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C_d\left(\sqrt{\left(\frac{d_+}{\delta}\right)} + \sqrt{\left(\frac{d_-}{\delta}\right)} + \sqrt{\delta} + \delta\right) \leqslant C_d(d_+ + d_-)^{\frac{1}{4}} \quad (4.21)$$

for the choice $\delta = \max\{\sqrt{d_+}, \sqrt{d_-}\}$.

Since (4.21) holds for *all* $t \geqslant 0$, $\xi \in [0,1]$, it is a matter of routine to check that (3.1) follows. Theorem 1 has been proved.

5 Proof of Lemma 1

The proof will be carried out in several steps.

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(i) $-1 \% p, n \ ! \ 0$

We have

$$F(p,n) = |p|\left(\alpha C_1 + \alpha C_2 + \frac{k_+}{\epsilon} C_1 - \frac{k_-}{\epsilon} C_2\right) + |n|\left(\alpha C_1 + \alpha C_2 + \frac{k_-}{\epsilon} C_2 - \frac{k_+}{\epsilon} C_1\right) - np(\alpha C_1 + \alpha C_2).$$

It is easy to check that (4.19) implies

$$\alpha C_1 + \alpha C_2 - \left|\frac{k_+}{\epsilon} C_1 - \frac{k_-}{\epsilon} C_2\right| \ " \ 0. \tag{5.1}$$

Consequently, if $\frac{k_+}{\epsilon} C_1 - \frac{k_-}{\epsilon} C_2 = 0$, we get

$$F(p,n) \ \& \ (|p| + |n|)(\alpha C_1 + \alpha C_2) - \frac{\alpha C_1 + \alpha C_2}{2}(n^2 + p^2) \ \& \ \frac{\alpha C_1 + \alpha C_2}{2}(|p| + |n|)$$

as $n^2 \% |n|$, $p^2 \% |p|$ for $-1 \% p, n \ ! \ 0$.

If $\frac{k_+}{\epsilon} C_1 - \frac{k_-}{\epsilon} C_2 \ " \ 0$, then

$$F(p,n) \ \& \ |p|\left(\frac{k_+}{\epsilon} C_1 - \frac{k_-}{\epsilon} C_2\right) + |n|\left(\alpha C_1 + \alpha C_2 + \frac{k_-}{\epsilon} C_2 - \frac{k_+}{\epsilon} C_1\right)$$

as $|n| \% 1$. The case $\frac{k_-}{\epsilon} C_2 - \frac{k_+}{\epsilon} C_1 \ " \ 0$ is treated in a similar way.

(ii) $p = 0$ or $n = 0$

The former case leads to

$$F(p,n) = |n|\left(\frac{k_-}{\epsilon} C_2 + \alpha C_2\right)$$

while in the latter we get

$$F(p,n) = |p|\left(\frac{k_+}{\epsilon} C_1 + \alpha C_1\right).$$

(iii) $p, n \ " \ 0$

This hypothesis yields

$$F(p,n) \ \& \ |p|\left(\frac{k_+}{\epsilon} C_1 - \frac{k_-}{\epsilon} C_2 + \alpha C_1 + \alpha C_2\right) + |n|\left(\frac{k_-}{\epsilon} C_2 - \frac{k_+}{\epsilon} C_1 - \frac{k_+}{\epsilon} C_1 + \alpha C_1 + \alpha C_2\right)$$

and, again, condition (5.1) guarantees the desired result.

(iv) $p \ " \ 0, -1 \% n \ ! \ 0$

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In this situation, we arrive at the formula

$$F(p,n) = |p|\left(\frac{k_+}{\epsilon}C_1+\frac{k_-}{\epsilon}C_2+\alpha C_1-\alpha C_2\right)+|n|\left(\frac{k_+}{\epsilon}C_1+\frac{k_-}{\epsilon}C_2-\alpha C_1+\alpha C_2\right)+np(\alpha C_1-\alpha C_2).$$

Following the arguments of (i), we obtain the sufficient condition

$$\frac{k_+}{\epsilon}C_1+\frac{k_-}{\epsilon}C_2-|\alpha C_1-\alpha C_2| \ " \ 0 \tag{5.2}$$

that may be deduced, similarly as (5.1), from (4.19).

(v) $-1 \% p \ ! \ 0, n \ " \ 0$

Finally, we get

$$F(p,n) = |p|\left(\frac{k_+}{\epsilon}C_1+\frac{k_-}{\epsilon}C_2-\alpha C_2+\alpha C_1\right)+|n|\left(\frac{k_+}{\epsilon}C_1+\frac{k_-}{\epsilon}C_2+\alpha C_2-\alpha C_1\right)+np(\alpha C_2-\alpha C_1)$$

which gives the desired estimate (4.20) provided (5.2) holds.

Lemma 1 has been proved.

6 Conclusion

The present example shows the ambiguous role played by the diffusion coefficients in the stabilization problems for the reaction-diffusion equations. On the one hand, the diffusion has a stabilizing effect, smoothing out the solution *inside* the domain Ω and thus accelerating the stabilization process. Close to the boundary, however, the presence of diffusion can create a thin boundary layer preventing rapid uniform stabilization unless the boundary conditions correspond to the spatially homogeneous stationary state.

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