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Separate approximate continuity implies measurability

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1. *Introduction.* It is known that a real-valued function f of two real variables which is continuous in each variable separately need not be continuous in (x, y), but must be in the first Baire class (1). Moreover if f is continuous in x for each y and merely measurable in y for each x then it must be Lebesgue-measurable (7), and this result can be extended to more general product spaces (2). However, the continuum hypothesis implies that this result fails if continuity is replaced by approximate continuity, as can be seen from the proof of Theorem 2 of (2). This makes Mišik's question (5) very natural: is a function which is separately approximately continuous in both variables necessarily Lebesgue-measurable? Our main aim is to establish an affirmative answer. It will be shown that such a function must in fact be in the second Baire class, although not necessarily in the first Baire class (unlike approximately continuous functions of one variable (3)). Finally, we show that the existence of a measurable cardinal would imply that a separately continuous real function on a product of two topological finite complete measure spaces need not be product-measurable.

2. *Main theorem.*

THEOREM 1. *If $f: \mathbb{R}^2 \to \mathbb{R}$ is separately approximately continuous then it is Lebesgue-measurable.*

This is an immediate consequence of Lemmas 2 and 4 below (obviously we may consider I^2 in place of $\mathbb{R}^2$). No doubt Lemmas 1 and 2 are well known; the short proofs are included for completeness. Lemma 3 is a kind of Baire theorem for approximate continuity.

LEMMA 1. *Let $(X, \mathbf{X}, \mu)$ be a finite measure space, and $\mathbf{D}$ a subclass of X such that whenever $E \in \mathbf{X}$ and $\mu(E) > 0$ there exists $D \subset E$ with $D \in \mathbf{D}$* and $\mu(D) > 0$. *Then there exists a sequence of disjoint sets $D_n \in \mathbf{D}$ with $\mu[X \setminus \cup_n D_n] = 0$.*

Proof. Define sets D_n inductively as follows: let $D_0 = \phi$; for $n = 1, 2, \ldots$, let

$$\mathbf{D}_n = \{D \in \mathbf{D} : D \subset X \setminus \cup_{\nu<n} D_\nu\}, \quad \lambda_n = \sup\{\mu(D) : D \in \mathbf{D}_n\},$$

and select $D_n \in \mathbf{X}_n$ with $\mu(D_n) \geqslant \frac{1}{2}\lambda_n$ (unless $\mathbf{D}_n$ is empty or $\lambda_n = 0$, in which cases the process terminates). Suppose if possible that $\mu[X \setminus \cup_n D_n] > 0$. Then this set contains a subset $D \in \mathbf{D}$ with $\mu(D) > 0$. Since $D \in \mathbf{D}_n$ for all n, we obtain $\lambda_n \geqslant \mu(D)$ for all n, giving the contradiction

$$\infty > \mu(X) \geqslant \Sigma_n \mu(D_n) \geqslant \Sigma_n \tfrac{1}{2}\mu(D) = \infty.$$

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LEMMA 2. *Let $(X, \mathbf{X}, \mu)$ be a finite measure space, and let $f: X \to \mathbb{R}$ be such that for every $\epsilon > 0$ the class*

$$\mathbf{D}_\epsilon = \{D \in \mathbf{X} : \operatorname{osc} f \leqslant \epsilon \quad \text{on} \quad D\}$$

satisfies the conditions of Lemma 1. Then f is $\bar{\mu}$-measurable.

Proof. By Lemma 1, for each $m = 1, 2, \ldots$, we can select a sequence of disjoint sets $D_{mn} \in \mathbf{X}$ such that $\mu[X \setminus \cup_n D_{mn}] = 0$ and $\operatorname{osc} f \leqslant m^{-1}$ on D_{mn}. Let $D = \cap_m (\cup_n D_{mn})$; then $\mu[X \setminus D] = 0$. Define $f_m: X \to \mathbb{R}$ by setting

$$f_m(x) = \sup\{f(\zeta) : \zeta \in D_{mn}\} \quad \text{for} \quad x \in D_{mn} \cap D \quad (n = 1, 2, \ldots), \quad \text{and} \quad f_m(x) = 0$$

otherwise. Since clearly f_m is measurable and $f_m \to f$ on D, the conclusion follows.

LEMMA 3. *If g is approximately continuous on $I = (0, 1)$ and F is a measurable subset of I of positive Lebesgue measure then given $\epsilon > 0$ there exists an open interval $J \subset I$ and a set Z of measure zero such that $\operatorname{osc} g \leqslant \epsilon$ on $(J \cap F) \setminus Z$ and $J \cap F$ is of positive measure.*

Proof. Suppose not. Let x_1 be a density point of F and let I_1 (with $\bar{I}_1 \subset I$) be an open interval containing x_1 such that $|g(x) - g(x_1)| < \frac{1}{8}\epsilon$ on a subset of $I_1 \cap F$ of mean density exceeding $\frac{1}{2}$ in I_1. Since we cannot take $J = I_1$ there exists a subset of $I_1 \cap F$ of positive measure in which $|g(x) - g(x_1)| \geqslant \frac{1}{2}\epsilon$; let x_2 be a density point of this subset, and observe that $|g(x_2) - g(x_1)| \geqslant \frac{1}{2}\epsilon$. Let I_2 (with $\bar{I}_2 \subset I_1$) be an open interval containing x_2 such that $|g(x) - g(x_2)| < \frac{1}{8}\epsilon$ on a subset of $I_2 \cap F$ of mean density exceeding $\frac{1}{2}$ in I_2. There exists a subset of $I_2 \cup F$ of positive measure in which $|g(x) - g(x_2)| \geqslant \frac{1}{2}\epsilon$; let x_3 be a density point of this subset, and observe that $|g(x_3) - g(x_2)| \geqslant \frac{1}{2}\epsilon$; and so on.

We may suppose that $m(I_n) \to ;0$ then $\cap_n I_n$ consists of a single point, say x_0. Since g is approximately continuous at x_0 there exists N such that for $n \geqslant N$ the mean density in I_n of the set of points at which $|g(x) - g(x_0)| < \frac{1}{8}\epsilon$ is greater than $\frac{1}{2}$. On the other hand, the mean density in I_n of the set of points at which $|g(x) - g(x_n)| < \frac{1}{8}\epsilon$ is also greater than $\frac{1}{2}$, we can conclude that the two sets have a non-empty intersection and therefore $|g(x_0) - g(x_n)| < \frac{1}{4}\epsilon$ for $n \geqslant N$. This is in contradiction with the fact that $|g(x_{N+1}) - g(x_N)| \geqslant \frac{1}{2}\epsilon$.

COROLLARY. *Under the above conditions,* $\operatorname{osc} g \leqslant \epsilon$ *on the set of density points of F in J.*

LEMMA 4. *Let $f: I^2 \to \mathbb{R}$ be separately approximately continuous, and let E be a measurable subset of I^2 of positive Lebesgue plane measure. Then given $\epsilon > 0$ there exists a measurable set $H \subset E$ of positive measure, on which* $\operatorname{osc} f \leqslant \epsilon$.

Proof. Let Q denote the set of points $x \in I$ such that the section E_x is measurable and of positive linear measure; then Q is a measurable set of positive linear measure. Let (J_n) be an enumeration of all open subintervals of I with rational end-points, and let (K_n) be an enumeration of all closed intervals in $\mathbb{R}$ with rational end-points and of length not exceeding ϵ. In view of Lemma 3 and the Corollary (applied with ϵ replaced by $\frac{1}{2}\epsilon$), we have $Q = \cup_{r,s} Q_{r,s}$, where $Q_{r,s}$ is the set of points $x \in Q$ such that $m(J_r \cap E_x) > 0$

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and $f(x,y) \in K_s$ for every density point y of E_x in J_r. Therefore $Q_{r,s}$ has positive outer measure for some pair of indices r, s (henceforth fixed).

Let P denote the set of points $x \in I$ at which $Q_{r,s}$ has density 1 (with respect to Lebesgue outer measure); then P is a measurable set of positive measure. Let F denote the set $E \cap (P \times J_r)$. This is a measurable set, and it has positive plane measure because F_x has positive linear measure for all $x \in Q_{r,s}$. As is known (compare(6), pp. 130–131), the set G of points $(x,y) \in F$ such that F_x is measurable and y is a density point of F_x is measurable and consists of almost all points of F, and the set H of points $(x,y) \in G$ such that G^y is measurable and x is a density point of G^y is also measurable and consists of almost all points of G. In particular, H is a measurable subset of E of positive measure, and it will therefore be sufficient to prove that $f(\zeta,\eta) \in K_s$ for every point $(\zeta,\eta) \in H$.

Given $\delta > 0$ there exists an open interval L containing ζ such that the mean density of the set G^η in L exceeds $\frac{3}{4}$, the mean density of the set $\{x: |f(x,\eta) - f(\zeta,\eta)| < \delta\}$ in L exceeds $\frac{3}{4}$, and the mean density with respect to outer measure of the set $Q_{r,s}$ in L exceeds $\frac{3}{4}$. Consequently these three sets have a common point, say x. Now $(x,\eta) \in G$, and therefore F_x is measurable and η is a density point of F_x, and *a fortiori* of E_x. Moreover $\eta \in J_r$ and $x \in Q_{r,s}$, and therefore $f(x,\eta) \in K_s$. Thus $f(\zeta,\eta)$ is within δ of a value in K_s. Since δ was arbitrarily small, it follows that $f(\zeta,\eta) \in K_s$, as required.

3. *Baire class.*

THEOREM 2. *If $f: \mathbb{R} \to \mathbb{R}$ is separately approximately continuous then it is in the second Baire class.*

Proof. Without loss of generality, we may suppose that the range of f is included in I. Then it follows at once from the approximate continuity in x that $f = \lim_n f_n$, where $f_n(x,y) = n \int f(\zeta,y)\, d\zeta$, the integration being over the interval

$$x - (2n)^{-1} \leqslant \zeta \leqslant x + (2n)^{-1}.$$

On the other hand, it is easy to see from the approximate continuity in y, together with the measurability of f (guaranteed by Theorem 1), that $f_n = \lim_m f_{nm}$, where

$$f_{nm}(x,y) = m \int f_n(x,\eta)\, d\eta,$$

the integration being over the interval $y - (2m)^{-1} \leqslant \eta \leqslant y + (2m)^{-1}$. Since f_{nm} is obviously continuous (its value at (x,y) is the integral of f over the rectangle centred at (x,y) with breadth n^{-1} and height m^{-1}), the theorem is proved.

REMARK. The more precise result, Theorem 2, has been obtained as a consequence of the weaker Theorem 1; it would be of interest to find a direct proof.

In the proof of Theorem 3 we repeatedly use the following result, closely related to the complete regularity property of the density topology on $\mathbb{R}$ (see(4)).

LEMMA 5. *If x_0 is a dispersion point of the closed linear set Q, then there exists an approximately continuous function $g: \mathbb{R} \to [0,1]$ such that $g(x_0) = 1$, and $Q \backslash \{x_0\}$ does not intersect the closure of the set $\{x: g(x) \neq 0\}$.*

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Proof. It is easy to see that we can enclose $Q\backslash\{x_0\}$ in an open set G whose closure has density zero at x_0. The sets $E_1 = \bar{G}\backslash\{x_0\}$, $E_2 = \{x_0\}$, $H = \mathbb{R}\backslash(E_1 \cup E_2)$ then satisfy the conditions of Lemma 12 of Zahorski (8) (also quoted on page 501 of (4)), and this yields an approximately continuous function $g: \mathbb{R} \to [0, 1]$ such that $g(x) = 0$ on E_1,

$$0 < g(x) < 1$$

on H, and $g(x_0) = 1$. This is the required function.

THEOREM 3. *There exists a separately approximately continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ that is not in the first Baire class.*

Proof. Let P be a perfect linear set of measure zero and let $D = \{x_n\}$ be a denumerable dense subset of P. We shall construct a separately approximately continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, x) = 1$ for $x \in D$ and $f(x, x) = 0$ for $x \in P\backslash D$; it follows from Baire's famous criterion that f is not in the first Baire class.

By Lemma 5 (applied with $Q = P$) there is an approximately continuous function $g_1: \mathbb{R} \to [0, 1]$ such that $g_1(x_1) = 1$, and $P\backslash\{x_1\}$ does not intersect the closure Q_1 of the set $\{x: g_1(x) \neq 0\}$. By Lemma 5 again (applied with $Q = P \cup Q_1$), there is an approximately continuous function $g_2: \mathbb{R} \to [0, 1]$ such that $g_2(x_2) = 1$, and $(P \cup Q_1)\backslash\{x_2\}$ does not intersect the closure Q_2 of the set $\{x: g_2(x) \neq 0\}$. Similarly, there is an approximately continuous function $g_3: \mathbb{R} \to [0, 1]$ such that $g_3(x_3) = 1$, and $(P \cup Q_1 \cup Q_2)\backslash\{x_3\}$ does not intersect the closure Q_3 of the set $\{x: g_3(x) \neq 0\}$; and so on.

Consider the function f on $\mathbb{R}^2$ defined by

$$f(x, y) = \sum_{n=1}^{\infty} g_n(x)\, g_n(y).$$

It is immediate from the construction that $f(x, x) = 1$ for $x \in D$ and $f(x, x) = 0$ for $x \in P\backslash D$. Moreover, for any fixed value of (say) y, we have $g_n(y) \neq 0$ for at most a single value of n, and if there is such a value, say N, then $f^y(x) = \text{const}\, g_N(x)$ is approximately continuous; otherwise $f^y(x) \equiv 0$. Thus f is separately approximately continuous.

4. *A general problem.* Let (X, μ) and (Y, ν) be topological sigma-finite measure spaces. Is a separately continuous function from $X \times Y$ to $\mathbb{R}$ necessarily $\overline{\mu \times \nu}$-measurable? This is known (2) to be true when one of X, Y is (topologically) second countable, and Theorem 1 above provides a specific case where this is true in the absence of second countability. But it is certainly not always true if the existence of a measurable cardinal is assumed. Namely, let X be a set whose cardinality is the smallest measurable cardinal, equipped with the discrete topology; let μ be a non-trivial $\{0, 1\}$-measure on the sigma-algebra of all subsets of X. It is easy to see that a set in $X \times X$ is $\overline{\mu \times \mu}$-measurable if and only if it coincides with ϕ or $X \times X$ to within a subset of a set of the form $(Z \times X) \cup (X \times Z)$, where $\mu(Z) = 0$. Therefore the diagonal $\{(x, x): x \in X\}$ is not $\overline{\mu \times \mu}$-measurable; and neither is its characteristic function, in spite of being separately continuous. It would be of interest to construct an example without assuming the existence of a measurable cardinal.

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