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Simple groups of dynamical origin

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Abstract. We associate with every étale groupoid $\mathfrak{G}$ two normal subgroups $\mathsf{S}(\mathfrak{G})$ and $\mathsf{A}(\mathfrak{G})$ of the topological full group of $\mathfrak{G}$, which are analogs of the symmetric and alternating groups. We prove that if $\mathfrak{G}$ is a minimal groupoid of germs (e.g., of a group action), then $\mathsf{A}(\mathfrak{G})$ is simple and is contained in every non-trivial normal subgroup of the full group. We show that if $\mathfrak{G}$ is expansive (e.g., is the groupoid of germs of an expansive action of a group), then $\mathsf{A}(\mathfrak{G})$ is finitely generated. We also show that $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ is a quotient of $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$.

1. *Introduction*

We associate with every group or semigroup of local homeomorphisms of the Cantor set (more generally with an étale groupoid with space of units homeomorphic to the Cantor set) groups $\mathsf{A}(\mathfrak{G})$ and $\mathsf{S}(\mathfrak{G})$ that are analogs of the alternating and symmetric groups. They are subgroups of the *topological full group* of the action. We prove that if the action is minimal, i.e., if all orbits are dense, then $\mathsf{A}(\mathfrak{G})$ is simple. This produces a very wide class of simple groups that includes some well-known examples: block-diagonal direct limits of direct products of finite alternating groups, derived subgroups of topological full groups of minimal homeomorphisms of the Cantor set, derived subgroups of the Higman–Thompson groups and full groups of one-sided shifts of finite type [**20**], and many interesting new examples. We show that if the action is expansive, then $\mathsf{A}(\mathfrak{G})$ is finitely generated. In a subsequent paper we use these results to construct first examples of simple groups of intermediate growth.

Topological full groups (of $\mathbb{Z}$-actions) were defined by Giordano, Putnam, and Skau in connection with the theory of orbit equivalence of minimal homeomorphisms of the Cantor set; see [**8**]. They also appeared earlier (for a special class of group actions) in the paper [**13**]. For a homeomorphism $\tau : \mathcal{X} \longrightarrow \mathcal{X}$, the topological full group $[[\tau]]$ is the group of all homeomorphisms $\phi : \mathcal{X} \longrightarrow \mathcal{X}$ such that for every $x \in \mathcal{X}$ there exist $n \in \mathbb{Z}$ and a neighborhood U of x such that $\phi|_U = \tau^n|_U$.

The definition is local in the sense that $[[\tau]]$ depends only on the germs of the iterations of τ. The notion of a topological full group has therefore a very

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natural generalization to arbitrary semigroups of local homeomorphisms, and a natural setting for such a generalization is the theory of étale groupoids. A group G of homeomorphisms of a topological space $\mathcal{X}$ is a full topological group of some semigroup of local homeomorphisms if the following condition is satisfied. If $\phi : \mathcal{X} \longrightarrow \mathcal{X}$ is a homeomorphism, and for every $x \in \mathcal{X}$ there exists a neighborhood U of x such that $\phi|_U$ coincides with the restriction of an element of G to U, then ϕ belongs to G. There are many examples of topological full groups in this sense: the Thompson groups [**4**], Monod's examples [**23**], a version of Lodha and Moore's example [**17**], groups associated with expanding maps [**25**], diagram groups [**9**], etc.

Topological full groups of étale groupoids were studied by Matui in a series of papers [**18**–**20**]. In particular, he showed that for some classes of groupoids the derived subgroup of the topological full group is simple, and that in some cases (e.g., for minimal subshifts) it is also finitely generated or even finitely presented (for groupoids associated with one-sided shifts of finite type). For a survey of results on full topological groups, see [**6**].

It was shown in [**5**] that if an action of the free abelian group $\mathbb{Z}^n$ on a Cantor set is minimal and expansive (i.e., conjugate to a subshift), then the derived subgroup of the topological full group is finitely generated.

We prove the following generalizations (see Theorems 4.1 and 5.10 below) of the above-mentioned results. They are valid for a much wider class of groupoids, though we leave open the question when the group $\mathsf{A}(\mathfrak{G})$ coincides with the derived subgroup of the topological full group.

THEOREM 1.1. *Let $\mathfrak{G}$ be a minimal groupoid of germs with the space of units homeomorphic to the Cantor set. Then $\mathsf{A}(\mathfrak{G})$ is simple and is contained in every non-trivial normal subgroup of the topological full group of $\mathfrak{G}$.*

THEOREM 1.2. *If $\mathfrak{G}$ is an expansive étale groupoid with the space of units homeomorphic to the Cantor set and every $\mathfrak{G}$-orbit contains at least five points, then $\mathsf{A}(\mathfrak{G})$ is finitely generated.*

The expansivity condition for groupoids is a natural generalization of the notion of an expansive dynamical system. In particular, the groupoid of an expansive action of a group on a Cantor set is expansive.

We illustrate these results by defining groups naturally associated with quasicrystals (e.g., Penrose tilings). We associate with every repetitive aperiodic Delaunay set with finite local complexity a finitely generated simple group; see Theorem 6.11. This is a generalization of a result of [**5**] about the Penrose tiling.

We also study the quotient $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$. We prove that it is an image of the zero homology group $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$ under a natural homomorphism.

In all known examples $\mathsf{A}(\mathfrak{G})$ coincides with the derived subgroup $\mathsf{D}(\mathfrak{G})$ of the topological full group, while $\mathsf{S}(\mathfrak{G})$ coincides with the kernel $\mathsf{K}(\mathfrak{G})$ of the natural *index map* from the topological full group to $H_1(\mathfrak{G}, \mathbb{Z})$. In particular, it follows from the results of Matui that this is true for *almost finite* and for *purely infinite* groupoids. If $\mathfrak{G}$ is such that $\mathsf{A}(\mathfrak{G}) = \mathsf{D}(\mathfrak{G})$ and $\mathsf{S}(\mathfrak{G}) = \mathsf{K}(\mathfrak{G})$, then our results confirm for $\mathfrak{G}$ a conjecture

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of Matui about the structure of the abelianization of the topological full group (see [**21**, Conjecture 2.9]). It would be interesting to understand when the equalities $\mathsf{A}(\mathfrak{G}) = \mathsf{D}(\mathfrak{G})$ and $\mathsf{S}(\mathfrak{G}) = \mathsf{K}(\mathfrak{G})$ hold.

2. *Étale groupoids*

2.1. *Basic definitions.* A *groupoid* is a small category of isomorphisms, i.e., a set $\mathfrak{G}$ with partially defined multiplication $\gamma_1\gamma_2$ and everywhere defined involutive operation of taking inverse γ^{-1} satisfying the following axioms.

(1) If $\gamma_1\gamma_2$ and $(\gamma_1\gamma_2)\gamma_3$ are defined, then $\gamma_2\gamma_3$ and $\gamma_1(\gamma_2\gamma_3)$ are defined and $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$.

(2) The products $\gamma\gamma^{-1}$ and $\gamma^{-1}\gamma$ are always defined. If $\gamma_1\gamma_2$ is defined, then $\gamma_1 = \gamma_1\gamma_2\gamma_2^{-1}$ and $\gamma_2 = \gamma_1^{-1}\gamma_1\gamma_2$.

The elements of the form $\gamma\gamma^{-1}$ are called *units*. We denote the set of units of a groupoid $\mathfrak{G}$ by $\mathfrak{G}^{(0)}$, the set of composable pairs by $\mathfrak{G}^{(2)}$, and the source and range maps by s and r, so that

$$\mathsf{s}(\gamma) = \gamma^{-1}\gamma, \quad \mathsf{r}(\gamma) = \gamma\gamma^{-1}$$

for all $\gamma \in \mathfrak{G}$.

A *topological groupoid* is a groupoid together with a topology on it such that the operations of multiplication and taking inverse are continuous, and the maps $\mathsf{s}, \mathsf{r} : \mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$ are open. We *do not* require that the groupoid is Hausdorff.

Units $x, y \in \mathfrak{G}^{(0)}$ belong to one $\mathfrak{G}$-*orbit* if there exists $\gamma \in \mathfrak{G}$ such that $\mathsf{s}(\gamma) = x$ and $\mathsf{r}(\gamma) = y$. A groupoid $\mathfrak{G}$ is said to be *minimal* if all its orbits are dense in $\mathfrak{G}^{(0)}$.

The *isotropy group* of a unit $x \in \mathfrak{G}^{(0)}$ is the group $\{\gamma \in \mathfrak{G} : \mathsf{s}(\gamma) = \mathsf{r}(\gamma) = x\}$. The groupoid $\mathfrak{G}$ is said to be *principal* if all its isotropy groups are trivial, i.e., if $\mathsf{s}(\gamma) = \mathsf{r}(\gamma) = x$ implies that $\gamma = x$.

The groupoid is said to be *essentially principal* if the set of points with trivial isotropy group is dense in $\mathfrak{G}^{(0)}$.

For $U \subset \mathfrak{G}^{(0)}$, we denote by $\mathfrak{G}|_U$ the subgroupoid $\{\gamma \in \mathfrak{G} : \mathsf{s}(\gamma), \mathsf{r}(\gamma) \in U\}$, called the *restriction* of $\mathfrak{G}$ to U. Note that the orbits of $\mathfrak{G}_U$ are equal to the intersections of the orbits of $\mathfrak{G}$ with U.

Example 2.1. Let G be a group acting by homeomorphisms on a topological space $\mathcal{X}$. Then $G \times \mathcal{X}$ is a groupoid in a natural way. Its elements are multiplied according to the rule

$$(g_1, x_1)(g_2, x_2) = (g_1g_2, x_2),$$

where the product is defined if and only if $g_2(x_2) = x_1$. We have $\mathsf{s}(g, x) = x$ and $\mathsf{r}(g, x) = g(x)$, where a point $x \in \mathcal{X}$ is identified with $(1, x)$. This groupoid is called the *groupoid of the action*.

Example 2.2. Let G be a discrete group acting by homeomorphisms on $\mathcal{X}$. Introduce an equivalence relation on $G \times \mathcal{X}$ by the rule: $(g_1, x_1) \sim (g_2, x_2)$ if $x_1 = x_2$ and there exists a neighborhood U of x_1 such that $g_1|_U = g_2|_U$. The equivalence relation $\sim$ agrees with the multiplication in the groupoid of the action, so that the quotient $G \times \mathcal{X}/\sim$ is also a groupoid. We call it the *groupoid of germs* of the action. The $\sim$-equivalence classes are called *germs*. For every open set U, the set of germs (g, y) for $y \in U$ is considered to be

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an open set of $\mathfrak{G}$, and the collection of all such open sets is a basis of topology on the groupoid of germs.

We define in the same way the *groupoid of germs* of a semigroup of local homeomorphisms of a topological space $\mathcal{X}$.

Standing assumptions. All groupoids in this paper are étale (see below) and the unit space $\mathfrak{G}^{(0)}$ is homeomorphic to the Cantor set. Some of our definitions (e.g., bisection and compact generation) are adapted specifically to the case of a Cantor set space of units, and should be changed in the general case.

Definition 2.3. A $\mathfrak{G}$-*bisection* is a compact open subset $F \subset \mathfrak{G}$ such that $\mathsf{s} : F \longrightarrow \mathsf{s}(F)$ and $\mathsf{r} : F \longrightarrow \mathsf{r}(F)$ are homeomorphisms.

A topological groupoid $\mathfrak{G}$ is said to be *étale* if it has a basis of topology consisting of $\mathfrak{G}$-bisections.

Every bisection $F \subset \mathfrak{G}$ defines the *associated homeomorphism* $\mathsf{s}(F) \longrightarrow \mathsf{r}(F)$ given by the formula $x \mapsto \mathsf{r}(Fx)$ for $x \in \mathsf{s}(F)$ or, equivalently, by $\mathsf{s}(\gamma) \mapsto \mathsf{r}(\gamma)$ for $\gamma \in F$. It is easy to see that the set of homeomorphisms associated with bisections is a semigroup: the homeomorphism associated with the product F_1F_2 of two bisections is equal to the composition of the homeomorphisms associated with F_i.

The groupoid of an action (see Example 2.1) and the groupoid of germs of an action (see Example 2.2) are étale.

Definition 2.4. An étale groupoid $\mathfrak{G}$ is said to be a *groupoid of germs* (or *effective*) if for every non-unit element $\gamma \in \mathfrak{G}$ and every bisection F containing γ, there exists $\delta \in F$ such that $\mathsf{s}(\delta) \neq \mathsf{r}(\delta)$.

If $\mathfrak{G}$ is a groupoid of germs, then for every element γ and for every bisection $F \ni \gamma$, the element γ is uniquely determined by $\mathsf{s}(\gamma)$ and the homeomorphism associated with F.

For every étale groupoid $\mathfrak{G}$, the groupoid of germs of the semigroup of local homeomorphisms of $\mathfrak{G}^{(0)}$ associated with bisections is a groupoid of germs which is a quotient of $\mathfrak{G}$. We call this quotient *the groupoid of germs associated with* $\mathfrak{G}$.

There is a close connection between groupoids of germs and essentially principal groupoids.

PROPOSITION 2.5. *If $\mathfrak{G}$ is essentially principal and Hausdorff, then it is a groupoid of germs.*

Every second countable groupoid of germs is essentially principal.

Proof. Assume that $\mathfrak{G}$ is essentially principal and Hausdorff. Suppose that $\mathfrak{G}$ is not a groupoid of germs. Then there exist $\gamma \in \mathfrak{G} \setminus \mathfrak{G}^{(0)}$ and a bisection $F \ni \gamma$ such that for every $\delta \in F$, we have $\mathsf{s}(\delta) = \mathsf{r}(\delta)$. Since $\mathfrak{G}$ is essentially principal, for every bisection $F' \subset F$, the set F' contains a unit. It follows that the unit $\mathsf{s}(\gamma)$ and the element γ do not have disjoint neighborhoods, which contradicts Hausdorffness of $\mathfrak{G}$.

Suppose that $\mathfrak{G}$ is a second countable groupoid of germs and let $F \subset \mathfrak{G}$ be a bisection. Consider the set $A = \{\gamma \in F \,:\, \mathsf{s}(\gamma) = \mathsf{r}(\gamma)\}$. The set A is closed in F. According to the definition of a groupoid of germs, if γ belongs to the interior of A, then γ is a unit. It follows that the set of non-unit elements of F is nowhere dense. Consequently, by the

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Baire category theorem, the set of units with trivial isotropy group is dense in $\mathfrak{G}^{(0)}$ if $\mathfrak{G}$ is second countable. □

Definition 2.6. Let $\mathfrak{G}$ be an étale groupoid. Its *full group* (*topological full group*), denoted $\mathsf{F}(\mathfrak{G})$, is the set of bisections $F \subset \mathfrak{G}$ such that $\mathsf{s}(F) = \mathsf{r}(F) = \mathfrak{G}^{(0)}$ with respect to multiplication of bisections.

The topological full group is often denoted in the literature by $[[\mathfrak{G}]]$, but we changed the notation for consistency with our notations for various normal subgroups of the full group.

Our definition is slightly different from the definition due to Matui [**19**, **20**], where the topological full group is defined as the topological full group of the groupoid of germs associated with $\mathfrak{G}$, so that it is a homeomorphism group of $\mathfrak{G}^{(0)}$. But if $\mathfrak{G}$ is a groupoid of germs, then our definition agrees with Matui's definition. Note that it is often assumed in some papers that $\mathfrak{G}$ is essentially principal and Hausdorff, which implies that $\mathfrak{G}$ is a groupoid of germs; see Proposition 2.5.

2.2. *Homology of étale groupoids.* Following Matui, let us define, for an étale groupoid $\mathfrak{G}$ and an abelian group A, the homology groups $H_k(\mathfrak{G}, A)$.

Denote by $\mathfrak{G}^{(n)}$ the space of sequences $(\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathfrak{G}^n$ such that the product $\gamma_1\gamma_2 \cdots \gamma_n$ is defined. In particular, $\mathfrak{G}^{(1)} = \mathfrak{G}$. We keep the notation $\mathfrak{G}^{(0)}$ for the space of units.

For every compact open subset $F \subset \mathfrak{G}^{(n)}$ and $a \in A$, denote by a_F the function $\mathfrak{G}^{(n)} \longrightarrow A$ equal to a on F and to 0 on $\mathfrak{G}^{(n)} \setminus F$. Denote by $C_c(\mathfrak{G}^{(n)}, A)$ the subgroup of the abelian group $A^{\mathfrak{G}^{(n)}}$ generated by the functions of the form a_F. If $\mathfrak{G}$ is Hausdorff, then $C_c(\mathfrak{G}^{(n)}, A)$ coincides with the group of continuous A-valued compactly supported functions on $\mathfrak{G}^{(n)}$, where A has discrete topology.

Consider the maps $d_i : \mathfrak{G}^{(n)} \longrightarrow \mathfrak{G}^{(n-1)}$, for $i = 0, 1, \dots, n$, given by

$$d_i(\gamma_1, \gamma_2, \dots, \gamma_n) = \begin{cases} (\gamma_2, \gamma_3, \dots, \gamma_n) & \text{for } i = 0, \\ (\gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_n) & \text{for } 1 \le i \le n-1, \\ (\gamma_1, \gamma_2, \dots, \gamma_{n-1}) & \text{for } i = n. \end{cases}$$

If $n = 1$, then we set $d_0(\gamma) = \mathsf{s}(\gamma)$ and $d_1(\gamma) = \mathsf{r}(\gamma)$.

Define $d_{i*} : C_c(\mathfrak{G}^{(n)}, A) \longrightarrow C_c(\mathfrak{G}^{(n-1)}, A)$ by

$$d_{i*}(f)(x) = \sum_{y \in d_i^{-1}(x)} f(y)$$

and

$$\delta_n = \sum_{i=0}^{n} (-1)^i d_{i*}.$$

Then

$$0 \xleftarrow{\delta_0} C_c(\mathfrak{G}^{(0)}, A) \xleftarrow{\delta_1} C_c(\mathfrak{G}^{(1)}, A) \xleftarrow{\delta_2} C_c(\mathfrak{G}^{(2)}, A) \cdots$$

is a chain complex. We denote by $H_n(\mathfrak{G}, A) = \mathrm{Ker}\delta_n / \mathrm{Im}\delta_{n+1}$ its homology.

Definition 2.7. The *index map* is the homomorphism $I : \mathsf{F}(\mathfrak{G}) \longrightarrow H_1(\mathfrak{G}, \mathbb{Z})$ mapping a bisection $F \in \mathsf{F}(\mathfrak{G})$ to the class $[1_F]$ of its indicator.

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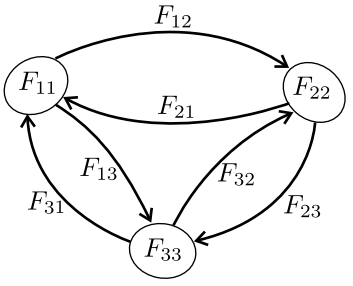


FIGURE 1. A multisection.

It is easy to check that the index map is a well-defined homomorphism; see [**19**, **20**]. We denote by $\mathsf{K}(\mathfrak{G})$ the kernel of the index map and by $\mathsf{D}(\mathfrak{G})$ the derived subgroup $[\mathsf{F}(\mathfrak{G}), \mathsf{F}(\mathfrak{G})]$ of the full group. We obviously have $\mathsf{D}(\mathfrak{G}) \leq \mathsf{K}(\mathfrak{G})$.

3. *Groups* $\mathsf{S}(\mathfrak{G})$ *and* $\mathsf{A}(\mathfrak{G})$

3.1. *Multisections.*

Definition 3.1. A *multisection of degree d* is a collection of d^2 bisections $\{F_{i,j}\}_{i,j=1}^{d}$ such that:

(1) $F_{i_2,i_3}F_{i_1,i_2} = F_{i_1,i_3}$ for all $1 \leq i_1, i_2, i_3 \leq d$;
(2) the bisections $F_{i,i}$ are disjoint subsets of $\mathfrak{G}^{(0)}$.

We call $\bigcup_{i=1}^{d} F_{i,i}$ the *domain* of the multisection and the sets $F_{i,i}$ the *components of the domain*.

See Figure 1 for a schematic illustration of a degree-3 multisection. In semigroup-theoretic terms we can define multisections as homomorphisms from a finite symmetric inverse monoid to the inverse monoid of bisections of $\mathfrak{G}$.

LEMMA 3.2. *Let $X = \{x_1, x_2, \ldots, x_d\} \subset \mathfrak{G}^{(0)}$ be a set contained in one $\mathfrak{G}$-orbit and let $U \subset \mathfrak{G}^{(0)}$ be a neighborhood of X. Then there exists a multisection $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^{d}$ such that $x_i \in F_{i,i}$, $x_j = \mathsf{r}(F_{i,j}x_i)$, and the domain of $\mathcal{F}$ is contained in U.*

Proof. Find for every $i = 2, \ldots, d$ an element $\gamma_i \in \mathfrak{G}$ such that $\mathsf{s}(\gamma_i) = x_1$ and $\mathsf{r}(\gamma_i) = x_i$. Set $\gamma_1 = x_1$. Find $\mathfrak{G}$-bisections H_i such that $\gamma_i \in H_i$. We can find a clopen neighborhood W of x_1 such that $W \subset \mathsf{s}(H_i) \cap U$ for every i, and the sets $\mathsf{r}(H_i W)$ are disjoint subsets of U. We also assume that $H_1 W = W$. Define then $F_{i,j} = (H_j W)(H_i W)^{-1}$. □

Let $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^{d}$ be a multisection with domain U, and let $\pi \in \mathsf{S}_d$ be a permutation, where S_d denotes the symmetric group of degree d. Then $\bigcup_{i=1}^{d} F_{i,\pi(i)} \cup (\mathfrak{G}^{(0)} \setminus U)$ is an element of $\mathsf{F}(\mathfrak{G})$, which we will denote $\mathcal{F}_\pi$. It is easy to see that $\pi \mapsto \mathcal{F}_\pi$ is an embedding of S_d into $\mathsf{F}(\mathfrak{G})$. Let us denote by $\mathsf{S}(\mathcal{F})$ the image of this embedding, and denote by $\mathsf{A}(\mathcal{F})$ the image of the alternating subgroup $\mathsf{A}_d < \mathsf{S}_d$.

PROPOSITION 3.3. *Let $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^{d}$ be a multisection of degree $d \geq 5$. Let $\mathcal{F}^{(k)} = \{F_{i,j}^{(k)}\}_{i,j=1}^{d}$, for $k = 1, 2, \ldots, n$, be multisections such that $\bigcup_{k=1}^{n} F_{i,j}^{(k)} = F_{i,j}$ for all $1 \leq i, j \leq d$. Then $\mathsf{A}(\mathcal{F})$ is contained in the group generated by $\bigcup_{k=1}^{n} \mathsf{A}(\mathcal{F}^{(k)})$.*

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Proof. Let us prove at first the following lemma.

LEMMA 3.4. *Let $d \geq 5$. Then the set $A = \{(\pi, \pi, 1) : \pi \in \mathsf{A}_d\} \cup \{(1, \pi, \pi) : \pi \in \mathsf{A}_d\}$ generates $\mathsf{A}_d \times \mathsf{A}_d \times \mathsf{A}_d$.*

Proof. We have $[(\pi_1, \pi_1, 1), (1, \pi_2, \pi_2)] = (1, [\pi_1, \pi_2], 1)$. Since A_d is equal to its commutator subgroup, every element $(1, \pi, 1)$, $\pi \in \mathsf{A}_d$, belongs to $\langle A \rangle$. It follows that every element of the form $(\pi, 1, 1)$ or $(1, 1, \pi)$ belongs to $\langle A \rangle$. □

It is enough to prove Proposition 3.3 for $n = 2$.

Denote $P_{i,j} = F^{(1)}_{i,j} \cap F^{(2)}_{i,j}$, $D^{(1)}_{i,j} = F^{(1)}_{i,j} \setminus F^{(2)}_{i,j}$, and $D^{(2)}_{i,j} = F^{(2)}_{i,j} \setminus F^{(1)}_{i,j}$. Then $\mathcal{P} = \{P_{i,j}\}^d_{i,j=1}$, $\mathcal{D}^{(1)} = \{D^{(1)}_{i,j}\}^d_{i,j=1}$, and $\mathcal{D}^{(2)} = \{D^{(2)}_{i,j}\}^d_{i,j=1}$ are multisections. It is obvious that $\mathsf{A}(\mathcal{F}) < \langle \mathsf{A}(\mathcal{P}) \cup \mathsf{A}(\mathcal{D}^{(1)}) \cup \mathsf{A}(\mathcal{D}^{(2)}) \rangle$; hence, it is enough to prove that $\mathsf{A}(\mathcal{P})$, $\mathsf{A}(\mathcal{D}^{(1)})$, and $\mathsf{A}(\mathcal{D}^{(2)})$ are subgroups of $\langle \mathsf{A}(\mathcal{F}^{(1)}) \cup \mathsf{A}(\mathcal{F}^{(2)}) \rangle$. The elements of $\mathsf{A}(\mathcal{F}^{(i)})$ leave the domains of $\mathcal{P}$ and $\mathcal{D}^{(i)}$ invariant. For every $\pi \in \mathsf{A}_d$, the restrictions of $\mathcal{F}^{(1)}_\pi$ to the domains of $\mathcal{P}$ and $\mathcal{D}^{(1)}$ are equal to $\mathcal{P}_\pi$ and $\mathcal{D}^{(1)}_\pi$, respectively, while its restriction to the domain of $\mathcal{D}^{(2)}$ is trivial. Similarly, the restrictions of $\mathcal{F}^{(2)}_\pi$ to the domains of $\mathcal{P}$ and $\mathcal{D}^{(2)}$ are equal to $\mathcal{P}_\pi$ and $\mathcal{D}^{(2)}_\pi$, respectively, while its restriction to the domain of $\mathcal{D}^{(1)}$ is trivial. It follows from Lemma 3.4 that $\mathsf{A}(\mathcal{P})$ and $\mathsf{A}(\mathcal{D}^{(i)})$, $i = 1, 2$, are contained in $\langle \mathsf{A}(\mathcal{F}^{(1)}) \cup \mathsf{A}(\mathcal{F}^{(2)}) \rangle$. □

Let $\mathcal{F} = \{F_{i,j}\}^d_{i,j=1}$ be a multisection and let $U \subset F_{i,i}$ be a clopen set for some $i \in \{1, 2, \ldots, d\}$. Denote, for every $j = 1, 2, \ldots, d$, by U_j the set $\mathsf{r}(F_{i,j}U) \subset F_{j,j}$ and by $F'_{i,j}$ the restriction $F_{i,j}U_i$. Then $\mathcal{F}' = \{F'_{i,j}\}^d_{i,j=1}$ is a multisection, which we call the *restriction* of $\mathcal{F}$ to U.

PROPOSITION 3.5. *Let $\mathcal{G} = \{G_{i,j}\}^{d_1}_{i,j=1}$ and $\mathcal{H} = \{H_{i,j}\}^{d_2}_{i,j=1}$ be multisections such that $d_i \geq 3$. Suppose that the intersection U of the domains of $\mathcal{G}$ and $\mathcal{H}$ is equal to the intersection $G_{1,1} \cap H_{1,1}$. Denote by $\mathcal{G}'$ and $\mathcal{H}'$ the restrictions to U of $\mathcal{G}$ and $\mathcal{H}$, respectively. Let $\mathcal{F}$ be the multisection with the domain equal to the union of the domains of $\mathcal{G}'$ and $\mathcal{H}'$ and consisting of all possible compositions of the elements of $\mathcal{G}' \cup \mathcal{H}'$. We have then $\mathsf{A}(\mathcal{F}) \leq \langle \mathsf{A}(\mathcal{G}) \cup \mathsf{A}(\mathcal{H}) \rangle$.*

Proof. Let us prove at first the following lemma.

LEMMA 3.6. *Let X, Y be finite sets such that $|X|, |Y| \geq 3$, and $|X \cap Y| = 1$. Then the normal closure in the group $\langle \mathsf{A}_X \cup \mathsf{A}_Y \rangle$ of the set $[\mathsf{A}_X, \mathsf{A}_Y]$ is the whole alternating group $\mathsf{A}_{X \cup Y}$.*

Proof. Denote by t the unique element of $X \cap Y$. Let us show at first that $\mathsf{A}_X \cup \mathsf{A}_Y$ generates $\mathsf{A}_{X \cup Y}$. It is known that $\mathsf{A}_{X \cup Y}$ is generated by cycles of length three. If a cycle (x, y, z) is such that $\{x, y, z\} \subset X$ or $\{x, y, z\} \subset Y$, then it already belongs to $\mathsf{A}_X \cup \mathsf{A}_Y$. If the cycle contains t, then it also belongs to $\langle \mathsf{A}_X \cup \mathsf{A}_Y \rangle$, since a cycle (y_1, x_1, t) for $x_1 \in X$ and $y_1 \in Y$ is equal to the commutator of the cycles $(x_1, x_2, t) \in \mathsf{A}_X$ and $(y_1, y_2, t) \in \mathsf{A}_Y$.

Suppose that $\{x, y, z\}$ is such that $x, y \in X \setminus Y$ and $z \in Y \setminus X$. Then (x, y, z) is equal to the cycle obtained by conjugating $(x, y, t) \in \mathsf{A}_X$ by a cycle of the form $(t, z, t') \in \mathsf{A}_Y$ such that $\{t, z, t'\} \subset Y$.

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We have shown that $\langle \mathsf{A}_X \cup \mathsf{A}_Y \rangle = \mathsf{A}_{X\cup Y}$. As we have mentioned before, any cycle of the form (y_1, x_1, t) for $x_1 \in X$ and $y_1 \in Y$ belongs to $[\mathsf{A}_X, \mathsf{A}_Y]$. Conjugating it by elements of $\mathsf{A}_{X\cup Y}$, we obtain all cycles of length three, thus generating $\mathsf{A}_{X\cup Y}$ by conjugates of a single element of $[\mathsf{A}_X, \mathsf{A}_Y]$. □

It is easy to see that for any $\pi_i \in \mathsf{A}_{d_i}$ we have $[\mathcal{G}_{\pi_1}, \mathcal{H}_{\pi_2}] \in \mathsf{A}(\mathcal{F})$. If we conjugate an element of $\mathsf{A}(\mathcal{F})$ by an element of the set $\mathsf{A}(\mathcal{G}) \cup \mathsf{A}(\mathcal{H})$, then we get an element of $\mathsf{A}(\mathcal{F})$. It follows that the normal closure of the set $[\mathsf{A}(\mathcal{G}), \mathsf{A}(\mathcal{H})]$ in the group $\langle \mathsf{A}(\mathcal{G}) \cup \mathsf{A}(\mathcal{H}) \rangle$ is contained in $\mathsf{A}(\mathcal{F})$. The above lemma then finishes the proof. □

3.2. *Definition of* $\mathsf{A}(\mathfrak{G})$ *and* $\mathsf{S}(\mathfrak{G})$.

Definition 3.7. Denote by $\mathsf{S}_d(\mathfrak{G})$ (respectively $\mathsf{A}_d(\mathfrak{G})$) the subgroup of $\mathsf{F}(\mathfrak{G})$ generated by the union of the subgroups $\mathsf{S}(\mathcal{F})$ (respectively $\mathsf{A}(\mathcal{F})$) for all multisections $\mathcal{F}$ of degree d.

Denote $\mathsf{S}(\mathfrak{G}) = \mathsf{S}_2(\mathfrak{G})$ and $\mathsf{A}(\mathfrak{G}) = \mathsf{A}_3(\mathfrak{G})$.

Note that for every multisection $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^{d}$ and every $g \in \mathsf{F}(\mathfrak{G})$, the set $\mathcal{F}^g = \{g^{-1}F_{i,j}g\}_{i,j=1}^{d}$ is also a multisection, and $\mathsf{S}(\mathcal{F}^g) = g^{-1}\mathsf{S}(\mathcal{F})g$, $\mathsf{A}(\mathcal{F}^g) = g^{-1}\mathsf{A}(\mathcal{F})g$. It follows that all subgroups $\mathsf{A}_d(\mathfrak{G})$ and $\mathsf{S}_d(\mathfrak{G})$ are normal in $\mathsf{F}(\mathfrak{G})$.

We obviously have $\mathsf{S}_d(\mathfrak{G}) \geq \mathsf{S}_{d+1}(\mathfrak{G})$ for every $d \geq 2$, and $\mathsf{A}_d(\mathfrak{G}) \geq \mathsf{A}_{d+1}(\mathfrak{G})$ for every $d \geq 3$. Namely, for every multisection $\mathcal{F}$ of degree $d+1$ the group $\mathsf{S}(\mathcal{F})$ (respectively the group $\mathsf{A}(\mathcal{F})$) is generated by the groups $\mathsf{S}(\mathcal{F}')$ (respectively $\mathsf{A}(\mathcal{F}')$), where $\mathcal{F}'$ runs through submultisections of $\mathcal{F}$ of degree d.

In particular, $\mathsf{A}(G) \geq \mathsf{A}_d(\mathfrak{G})$ for all $d \geq 3$, and $\mathsf{S}(\mathfrak{G}) \geq \mathsf{S}_d(\mathfrak{G})$ for all $d \geq 2$.

Note that if there exists a $\mathfrak{G}$-orbit of size $d \geq 3$, then $\mathsf{A}_d(\mathfrak{G})$ and $\mathsf{S}_d(\mathfrak{G})$ act on it transitively, while $\mathsf{A}_{d+1}(\mathfrak{G})$ and $\mathsf{S}_{d+1}(\mathfrak{G})$ act on it trivially; hence, $\mathsf{A}_d(\mathfrak{G}) \neq \mathsf{A}_{d+1}(\mathfrak{G})$ and $\mathsf{S}_d(\mathfrak{G}) \neq \mathsf{S}_{d+1}(\mathfrak{G})$. It follows that $\mathsf{A}_{d+1}(\mathfrak{G}) \neq \mathsf{A}(\mathfrak{G})$ and $\mathsf{S}_{d+1}(\mathfrak{G}) \neq \mathsf{S}(\mathfrak{G})$ in this case.

PROPOSITION 3.8. *Suppose that all orbits of* $\mathfrak{G}$ *have at least* n *points. Then* $\mathsf{A}_d(\mathfrak{G}) = \mathsf{A}(\mathfrak{G})$ *for all* $3 \leq d \leq n$, *and* $\mathsf{S}_d(\mathfrak{G}) = \mathsf{S}(\mathfrak{G})$ *for all* $2 \leq d \leq n$.

Proof. Let us prove the proposition for $\mathsf{S}(\mathfrak{G})$; the proof for $\mathsf{A}(\mathfrak{G})$ is analogous. We know that $\mathsf{S}_k(\mathfrak{G}) \geq \mathsf{S}_{k+1}(\mathfrak{G})$ for all $2 \leq k \leq n-1$. We have to prove that $\mathsf{S}_k(\mathfrak{G}) \leq \mathsf{S}_{k+1}(\mathfrak{G})$. Let $\mathcal{F}$ be a multisection of degree k. It is shown in the same way as in the proof of Lemma 3.2 that for any sufficiently small subset U of a component of the domain of $\mathcal{F}$ there exists a multisection $\mathcal{F}'$ of degree $k+1$ containing the restriction of $\mathcal{F}$ to U. It follows that we can split a component $F_{1,1}$ of the domain of $\mathcal{F}$ into a finite disjoint union of sets U_i such that the restriction of $\mathcal{F}$ to every U_i is contained in a multisection of degree $k+1$. It follows that $\mathsf{S}(\mathcal{F}) \subset \mathsf{S}_{k+1}(\mathfrak{G})$. □

COROLLARY 3.9. *Suppose that every orbit of* $\mathfrak{G}$ *is infinite. Then* $\mathsf{A}_d(\mathfrak{G}) = \mathsf{A}(\mathfrak{G})$ *for all* $d \geq 3$, *and* $\mathsf{S}_d(\mathfrak{G}) = \mathsf{S}(\mathfrak{G})$ *for all* $d \geq 2$.

It is easy to see that $\mathsf{S}(\mathfrak{G}) \leq \mathsf{K}(\mathfrak{G})$ and $\mathsf{A}(\mathfrak{G}) \leq \mathsf{D}(\mathfrak{G})$. We have a diagram of subgroups of $\mathsf{F}(\mathfrak{G})$ shown in Figure 2.

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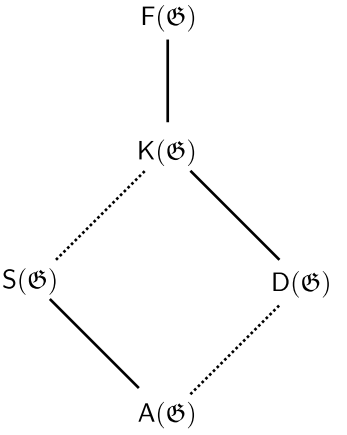


FIGURE 2. Subgroups of $\mathsf{F}(\mathfrak{G})$.

3.3. *Rigidity.* The next theorem follows directly from [**28**, Theorems 0.2 and 3.3].

Let G be a group acting by homeomorphisms of a topological space $\mathcal{X}$. Denote, for $U \subset \mathcal{X}$, by $G_{(U)}$ the subgroup consisting of elements $g \in G$ acting trivially on $\mathcal{X} \setminus U$. We say that G is *locally minimal* if there exists a basis of open sets $\mathcal{U}$ such that for every $U \in \mathcal{U}$ and $x \in U$ the $G_{(U)}$-orbit of x is dense in U.

THEOREM 3.10. *If G_i are locally minimal groups of homeomorphisms of locally compact Hausdorff spaces $\mathcal{X}_i$, then for every isomorphism $\phi : G_1 \longrightarrow G_2$ there exists a homeomorphism $F : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ such that $\phi(g) = F \circ g \circ F^{-1}$ for all $g \in G_1$.*

As a corollary, we get the following.

THEOREM 3.11. *Let $\mathfrak{G}_i$, for $i = 1, 2$, be minimal groupoids of germs. Then the following conditions are equivalent.*

(1) *The groupoids $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are isomorphic as topological groupoids.*
(2) *The groups $\mathsf{F}(\mathfrak{G}_1)$ and $\mathsf{F}(\mathfrak{G}_2)$ are isomorphic.*
(3) *The groups $\mathsf{D}(\mathfrak{G}_1)$ and $\mathsf{D}(\mathfrak{G}_2)$ are isomorphic.*
(4) *The groups $\mathsf{K}(\mathfrak{G}_1)$ and $\mathsf{K}(\mathfrak{G}_2)$ are isomorphic.*
(5) *The groups $\mathsf{S}(\mathfrak{G}_1)$ and $\mathsf{S}(\mathfrak{G}_2)$ are isomorphic.*
(6) *The groups $\mathsf{A}(\mathfrak{G}_1)$ and $\mathsf{A}(\mathfrak{G}_2)$ are isomorphic.*

Equivalence of statements (1)–(4) was proved in [**20**]. For methods of proving similar statements without using Rubin's theorem, see [**20**, **22**].

4. *Simplicity of* $\mathsf{A}(\mathfrak{G})$

THEOREM 4.1. *Suppose that $\mathfrak{G}$ is a minimal groupoid of germs. Then every non-trivial subgroup of $\mathsf{F}(\mathfrak{G})$ normalized by $\mathsf{A}(\mathfrak{G})$ contains $\mathsf{A}(\mathfrak{G})$. In particular, $\mathsf{A}(\mathfrak{G})$ is simple and is contained in every non-trivial normal subgroup of $\mathsf{F}(\mathfrak{G})$.*

Proof. Let $U \subset \mathfrak{G}^{(0)}$ be a clopen subset. We will naturally identify $\mathsf{A}(\mathfrak{G}|_U)$ with a subgroup of $\mathsf{A}(\mathfrak{G})$, extending elements of $\mathsf{A}(\mathfrak{G}|_U)$ identically on $\mathfrak{G}^{(0)} \setminus U$.

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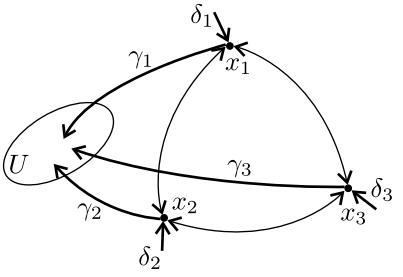


FIGURE 3. Proof of simplicity of $\mathsf{A}(\mathfrak{G})$.

Let $N \leq \mathsf{F}(\mathfrak{G})$ be a subgroup normalized by $\mathsf{A}(\mathfrak{G})$ and let $g \in N$ be a non-trivial element. Since $\mathfrak{G}$ is a groupoid of germs, there exists an element $\gamma \in g$ such that $\mathsf{s}(\gamma) \neq \mathsf{r}(\gamma)$. There exists a clopen neighborhood U of $\mathsf{s}(\gamma)$ such that $U \cap g(U) = \emptyset$. Let $\mathcal{F}$ be a multisection of degree ≥ 5 whose domain is a subset of U. Let $h_1, h_2 \in \mathsf{A}(\mathcal{F})$. Then $g\mathcal{F}g^{-1}$ is a multisection with domain in $g(U)$, and $gh_1g^{-1} \in \mathsf{A}(g\mathcal{F}g^{-1})$. It follows that $[g^{-1}, h_1] = gh_1^{-1}g^{-1}h_1$ acts trivially outside $U \cup g(U)$, as h_1 on U, and as $gh_1^{-1}g^{-1}$ on $g(U)$. Consequently, $[[g^{-1}, h_1], h_2] = [h_1, h_2]$. But $[[g^{-1}, h_1], h_2] \in N$ since N is normalized by $\mathsf{A}(\mathfrak{G})$, $h_1, h_2 \in \mathsf{A}(\mathcal{F}) \subset \mathsf{A}(\mathfrak{G})$ and $g \in N$. It follows that the derived subgroup of $\mathsf{A}(\mathcal{F})$ is contained in N. But every element of $\mathsf{A}(\mathcal{F})$ is a commutator of elements of $\mathsf{A}(\mathcal{F})$; hence, the derived subgroup of $\mathsf{A}(\mathcal{F})$ is $\mathsf{A}(\mathcal{F})$, so that we have $\mathsf{A}(\mathcal{F}) \leq N$.

We have proved that there exists a non-empty clopen subset U of $\mathfrak{G}^{(0)}$ such that $\mathsf{A}(\mathfrak{G}|_U) \leq N$.

It remains to prove that the normal closure of $\mathsf{A}(\mathfrak{G}|_U)$ in $\mathsf{A}(\mathfrak{G})$ is equal to $\mathsf{A}(\mathfrak{G})$. It is enough to show that for every multisection $\mathcal{F}$ of degree 5, the group $\mathsf{A}(\mathcal{F})$ is contained in the normal closure of $\mathsf{A}(\mathfrak{G}|_U)$ in $\mathsf{A}(\mathfrak{G})$; see Proposition 3.8.

Let $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^5$ be a multisection. Let $x_1 \in F_{1,1}$ be arbitrary, and denote $x_i = \mathsf{r}(F_{1,i}x_1)$. Then all $x_i \in F_{i,i}$ belong to one $\mathfrak{G}$-orbit. Since $\mathfrak{G}$ is minimal, all its orbits are infinite and dense, and we can find elements $\gamma_i, \delta_i \in \mathfrak{G}$ such that $\mathsf{s}(\gamma_i) = \mathsf{r}(\delta_i) = x_i$, the units $\mathsf{r}(\gamma_i)$ belong to U, and all the units $x_1, \dots, x_d, \mathsf{r}(\gamma_1), \dots, \mathsf{r}(\gamma_d), \mathsf{s}(\delta_1), \dots, \mathsf{s}(\delta_d)$ are pairwise different, see Figure 3.

Since $\mathfrak{G}$ is étale, we can find bisections $G_i \ni \gamma_i$, $D_i \ni \delta_i$, and $F'_{i,j} \subset F_{i,j}$ such that $\mathcal{F}' = \{F'_{i,j}\}_{i,j=1}^d$ is a multisection, $F'_{i,i} = \mathsf{s}(G_i) = \mathsf{r}(D_i)$, $\mathsf{r}(G_i) \subset U$, and all the sets $\mathsf{s}(G_1), \dots, \mathsf{s}(G_d), \mathsf{r}(G_1), \dots, \mathsf{r}(G_d), \mathsf{s}(D_1), \dots, \mathsf{s}(D_d)$ are pairwise disjoint.

Then $\mathcal{H} = \{G_jF'_{i,j}G_i^{-1}\}_{i,j=1}^d$ is a multisection with domain in U. For every i, denote $C_i = G_i \cup D_i \cup D_i^{-1}G_i^{-1}$. Then $\mathsf{s}(C_i) = \mathsf{r}(C_i)$, the sets $\mathsf{s}(C_i)$, $i = 1, \dots, d$, are disjoint, and $g = \bigcup_{i=1}^d C_i \cup \left(\mathfrak{G}^{(0)} \setminus \bigcup_{i=1}^d \mathsf{s}(C_i)\right)$ is an element of $\mathsf{A}(\mathfrak{G})$. It satisfies $G_jF'_{i,j}G_i^{-1} = gF_{i,j}g^{-1}$ and $g\mathsf{A}(\mathcal{F}')g^{-1} = \mathsf{A}(\mathcal{H}) \leq \mathsf{A}(\mathfrak{G}|_U)$. It follows that $\mathsf{A}(\mathcal{F}') \leq g^{-1}\mathsf{A}(\mathfrak{G}|_U)g$.

We have shown that for every point $x_1 \in F_{1,1}$, there exist a restriction $\mathcal{F}'$ of $\mathcal{F}$ and an element $g \in \mathsf{A}(\mathfrak{G})$ such that x_1 belongs to the domain of $\mathcal{F}'$ and $\mathsf{A}(\mathcal{F}') \leq g^{-1}\mathsf{A}(\mathfrak{G}|_U)g$. It follows then from compactness of $F_{1,1}$ and Proposition 3.3 that $\mathsf{A}(\mathcal{F})$ is contained in the normal closure of $\mathsf{A}(\mathfrak{G}|_U)$ in $\mathsf{A}(\mathfrak{G})$. □

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It was proved by Matui (see [**20**, Theorems 4.7 and 4.16]) that if $\mathfrak{G}$ is a minimal groupoid of germs belonging either to the class of *almost finite* or to the class of *purely infinite* groupoids (see [**20**] for the definitions), then any non-trivial subgroup of $\mathsf{F}(\mathfrak{G})$ normalized by the commutator subgroup $\mathsf{D}(\mathfrak{G})$ contains $\mathsf{D}(\mathfrak{G})$. Theorem 4.1 implies that we have $\mathsf{D}(\mathfrak{G}) = \mathsf{A}(\mathfrak{G})$ for such groupoids. It would be interesting to find an example of a minimal groupoid of germs $\mathfrak{G}$ such that $\mathsf{A}(\mathfrak{G}) \neq \mathsf{D}(\mathfrak{G})$.

4.1. *Example: AF groupoids and LDA groups.* The groups $\mathsf{A}(\mathfrak{G})$ for a special class of *AF groupoids* are well known, though they are usually defined without using terminology of groupoids.

A *Bratteli diagram* B consists of two sequences $V_1, V_2, \dots$ and $E_1, E_2, \dots$ of finite sets together with maps $\mathsf{s}_n : E_n \longrightarrow V_n$ and $\mathsf{r}_n : E_n \longrightarrow V_{n+1}$. We interpret $e \in E_n$ as an edge connecting the vertices $\mathsf{s}_n(e) \in V_n$ and $\mathsf{r}_n(e) \in V_{n+1}$. We assume that the maps s_n and r_n are surjective.

The associated *space of paths* $\mathcal{P}$ of the diagram B is the space of sequences $(e_1, e_2, \dots) \in E_1 \times E_2 \times \cdots$ such that $\mathsf{r}_n(e_n) = \mathsf{s}_{n+1}(e_{n+1})$ for every $n \geq 1$. The topology is induced by the direct product topology on $E_1 \times E_2 \times \cdots$, where E_n are discrete.

A *finite path* is a sequence $(e_1, e_2, \dots, e_n)$ such that $\mathsf{r}_i(e_i) = \mathsf{s}_{i+1}(e_{i+1})$ for all $i = 1, 2, \dots, n-1$. The vertices $\mathsf{s}(e_1)$ and $\mathsf{r}_n(e_n)$ are called the *beginning* and the *end* of the path, respectively.

Suppose that $(e_1, e_2, \dots, e_n)$, $(f_1, f_2, \dots, f_n)$ is a pair of finite paths with coinciding ends. This pair defines a homeomorphism between clopen subsets of $\mathcal{P}$ mapping every path of the form $(e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots)$ to the path $(f_1, f_2, \dots, f_n, e_{n+1}, e_{n+2}, \dots)$. Let $\mathfrak{B}$ be the groupoid of germs of the semigroup generated by such local homeomorphisms. Such groupoids, i.e., groupoids associated in the described way with Bratteli diagrams, are called *AF groupoids*; see [**19**].

For $v \in V_n$, denote by $d(v)$ the number of finite paths $(e_1, e_2, \dots, e_n)$ ending in v. It is easy to see that $\mathsf{A}(\mathfrak{B})$ is isomorphic to a direct limit of the direct products $G_n = \prod_{v \in V_n} \mathsf{A}_{d(v)}$. The embeddings $G_n \longrightarrow G_{n+1}$ are block diagonal. Namely, if $\alpha \in \mathsf{A}_{d(v)}$ is a permutation of the set of paths ending in v, then it acts on all continuations of the paths only changing the beginnings by the permutation α, so that α is copied $|\mathsf{s}_{n+1}^{-1}(v)|$ times in the direct product G_{n+1}. See more details in [**14**].

The groupoid $\mathfrak{B}$ is minimal if and only if for every two vertices $v_1, v_2 \in \bigcup_{n \geq 1} V_n$ there exist finite paths starting in v_1 and v_2 and ending in one vertex. If this condition is satisfied, then the group $\mathsf{A}(\mathfrak{B})$ is simple. Such simple block-diagonal limits of direct products of alternating groups form an important class of simple locally finite groups (LDA groups); see [**15**, **16**].

5. *Expansive groupoids and finite generation of* $\mathsf{A}(\mathfrak{G})$

5.1. *Compactly generated and expansive groupoids.*

Definition 5.1. A groupoid $\mathfrak{G}$ is said to be *compactly generated* if there exists a compact set $S \subset \mathfrak{G}$ (called a *generating set*) such that $\mathfrak{G} = \bigcup_{n \geq 1} (S \cup S^{-1})^n$.

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Here $(S \cup S^{-1})^n$ is the set of products $\gamma_1\gamma_2\cdots\gamma_n$ such that $\gamma_i \in S \cup S^{-1}$.

The general definition of a compactly generated groupoid (in the case when $\mathfrak{G}^{(0)}$ is not a Cantor set) is slightly different; see [**10**, **26**].

Suppose that S is a compact generating set of $\mathfrak{G}$. Then any set $S_1 \supseteq S$ is also a generating set. Consequently, there exists a compact open generating set of $\mathfrak{G}$.

We will need the following technical lemma later.

LEMMA 5.2. *If $\mathfrak{G}$ is compactly generated and every orbit has more than one point, then there exists a compact open generating set S of $\mathfrak{G}$ such that for every $\gamma \in S$ we have $\mathsf{s}(\gamma) \neq \mathsf{r}(\gamma)$.*

Proof. Let S be an arbitrary compact open generating set of $\mathfrak{G}$. Assume that $\mathfrak{G}^{(0)} \subset S$ (otherwise, replace S by $\mathfrak{G}^{(0)} \cup S$).

Let $\mathcal{S}$ be a finite cover of S by bisections. Suppose that $F \in \mathcal{S}$ and $\gamma \in F$ are such that $\mathsf{s}(\gamma) = \mathsf{r}(\gamma)$. There exists $\gamma' \in \mathfrak{G}$ such that $\mathsf{s}(\gamma') = \mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma') \neq \mathsf{s}(\gamma')$. We can choose a sufficiently small bisection U containing γ' such that $\mathsf{s}(U) \cap \mathsf{r}(U) = \emptyset$ and $\mathsf{s}(UF) \cap \mathsf{r}(UF) = \emptyset$. Then $U_\gamma = U^{-1}UF$ is a neighborhood of γ. Denote $W_\gamma = U \cup UF$. Then W_γ does not contain elements of isotropy groups, and U_γ is contained in the groupoid generated by W_γ.

If $\gamma \in F$ is such that $\mathsf{s}(\gamma) \neq \mathsf{r}(\gamma)$, then choose $U_\gamma \subset F$ such that $\mathsf{s}(U_\gamma) \cap \mathsf{r}(U_\gamma) = \emptyset$, and denote $W_\gamma = U_\gamma$. The collection $\{U_\gamma\}_{\gamma\in F}$ is an open covering of F; hence, there exists a finite subcovering $\{U_{\gamma_i}\}$. Let F' be the union of the corresponding sets W_γ. Then F' is compact, F is contained in the groupoid generated by F', and F' does not contain elements of isotropy groups. □

Definition 5.3. Let S be an open compact generating set of $\mathfrak{G}$. A finite cover $\mathcal{S}$ of S by bisections is said to be *expansive* if the set $\bigcup_{n\geq 1}\left(\mathcal{S} \cup \mathcal{S}^{-1}\right)^n$ is a basis of topology of $\mathfrak{G}$. A groupoid $\mathfrak{G}$ is said to be *expansive* if it admits an expansive cover of a compact generating set.

PROPOSITION 5.4. *Suppose that there exists an expansive cover of a compact generating set of $\mathfrak{G}$. Then for every compact generating set S of $\mathfrak{G}$ any finite cover $\mathcal{S}$ of S by sufficiently small bisections is expansive.*

Proof. Note that if $\mathcal{S}$ is an expansive set of bisections, then any finite set of bisections containing $\mathcal{S}$ is also expansive. If $\mathcal{S}_1$ is a cover subordinate to an expansive cover, then $\mathcal{S}_1$ is also expansive.

Let S be a compact generating set such that there exists an expansive cover $\mathcal{S}$ of S. Let S_1 be another compact generating set. Let $\mathcal{S}_1$ be an arbitrary cover of S_1 by bisections. For every $F \in \mathcal{S}$ and every $\gamma \in F$, there exists a product of elements of $\mathcal{S}_1$ containing γ. It follows that F can be covered by a finite number of products of elements of $\mathcal{S}_1$. Replacing $\mathcal{S}_1$ by a refinement, we can find for every $F \in \mathcal{S}$ and every $\gamma \in F$ a product F_1 of a finite number of elements of $\mathcal{S}_1$ such that $\gamma \in F_1 \subset F$. Then expansivity of $\mathcal{S}_1$ will follow from expansivity of $\mathcal{S}$. Since any cover of S_1 subordinate to $\mathcal{S}_1$ is also expansive, any cover of S_1 by sufficiently small bisections is expansive, by Lebesgue's covering lemma. □

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PROPOSITION 5.5. *Suppose that $\mathcal{S}$ is a finite cover by bisections of a compact generating set S. Denote, for $x \in \mathfrak{G}^{(0)}$ and $n \geq 1$, by $U_n(x)$ the intersection of all sets of the form $\mathsf{s}(F)$ containing x, for $F \in (\mathcal{S} \cup \mathcal{S}^{-1})^k$ and $k \leq n$. Then the following conditions are equivalent.*

(1) *The cover $\mathcal{S}$ is expansive.*
(2) *The sets $\mathsf{s}(F)$ for $F \in \bigcup_{n\geq 1}(\mathcal{S} \cup \mathcal{S}^{-1})^n$ form a basis of topology on $\mathfrak{G}^{(0)}$.*
(3) *For every two different units x, $y \in \mathfrak{G}^{(0)}$, there exist A_1, $A_2 \in \bigcup_{n\geq 1}(\mathcal{S} \cup \mathcal{S}^{-1})^n$ such that $x \in \mathsf{s}(A_1)$, $y \in \mathsf{s}(A_2)$, and $\mathsf{s}(A_1) \cap \mathsf{s}(A_2) = \emptyset$.*
(4) *For every $x \in \mathfrak{G}^{(0)}$, we have $\bigcap_{n\geq 1} U_n(x) = \{x\}$.*

Proof. The implication (1) $\Longrightarrow$ (2) follows directly from the definition of an étale groupoid.

For every bisection $F \subset \mathfrak{G}$, we have $\mathsf{s}(F) = F^{-1}F$ and hence for every two products $F_1F_2\cdots F_n$ and $G_1G_2\cdots G_m$ of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ the restriction

$$G_1G_2\cdots G_m\mathsf{s}(F_1F_2\cdots F_n)$$

of $G_1G_2\cdots G_m$ to $\mathsf{s}(F_1F_2\cdots F_n)$ is equal to a product of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$. This shows that (2) implies (1).

It also shows that for any two products $F_1F_2\cdots F_n$ and $G_1G_2\cdots G_m$ of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ the set $\mathsf{s}(F_1F_2\cdots F_n) \cap \mathsf{s}(G_1G_2\cdots G_m)$ is a product of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ (recall that if U, $V \subset \mathfrak{G}^{(0)}$, then $U \cap V = UV$).

The implication (2) $\Longrightarrow$ (3) is obvious. Let us show the converse implication. Let $x \in \mathfrak{G}^{(0)}$ and let U be an open neighborhood of x. For every unit $y \in \mathfrak{G}^{(0)} \setminus U$, there exist products U_y and V_y of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ such that $\mathsf{s}(U_y) \cap \mathsf{s}(V_y) = \emptyset$, $x \in \mathsf{s}(U_y)$, and $y \in \mathsf{s}(V_y)$. By compactness of $\mathfrak{G}^{(0)} \setminus U$, we can find a finite collection $U_{y_1}, U_{y_2}, \ldots, U_{y_k}$, $V_{y_1}, V_{y_2}, \ldots, U_{y_k}$ such that $x \in \bigcap_{1\leq i\leq k} \mathsf{s}(U_{y_i})$, $\mathsf{s}(U_{y_i}) \cap \mathsf{s}(V_{y_i}) = \emptyset$ for all $1 \leq i \leq k$, and $\mathfrak{G}^{(0)} \setminus U \subset \bigcup_{1\leq i\leq k} \mathsf{s}(V_{y_i})$. By the above argument, the intersection $\bigcap_{1\leq i\leq k} \mathsf{s}(U_{y_i})$ is a product of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$. It is an open neighborhood of x contained in U.

The implication (3) $\Longrightarrow$ (4) is obvious. Suppose that (3) does not hold for some points x, $y \in \mathfrak{G}^{(0)}$. As we have seen above, for every $n \geq 1$ and $x \in \mathfrak{G}^{(0)}$, the set $U_n(x)$ is a product of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$. Then for every $n \geq 1$ the intersection $U_n(x) \cap U_n(y)$ is non-empty. But $U_n(x) \cap U_n(y)$, for $n \geq 1$, is a descending sequence of closed sets and hence its intersection $\bigcap_{n\geq 1} U_n(x) \cap \bigcap_{n\geq 1} U_n(y)$ is non-empty, so that (4) does not hold. This proves the implication (4) $\Longrightarrow$ (3). □

PROPOSITION 5.6. *Let $U \subset \mathfrak{G}^{(0)}$ be a clopen $\mathfrak{G}$-transversal (i.e., a set intersecting every $\mathfrak{G}$-orbit). Then the groupoid $\mathfrak{G}|_U$ is expansive if and only if $\mathfrak{G}$ is expansive.*

Proof. Suppose that $\mathfrak{G}$ is expansive. Let S be a compact generating set of $\mathfrak{G}$ and let $\mathcal{S}$ be a finite expansive cover of S by bisections. For every $x \in \mathfrak{G}^{(0)}$, there exists a bisection F such that $x \in \mathsf{s}(F)$ and $\mathsf{r}(F) \subset U$. It follows that there exists a finite set of bisections $\mathcal{T}$ such that the sets $\mathsf{s}(F)$, for $F \in \mathcal{T}$, cover $\mathfrak{G}^{(0)}$ and $\mathsf{r}(F) \subset U$ for every $F \in \mathcal{T}$. We will assume that $U \in \mathcal{T}$.

Consider the set $\mathcal{T}\mathcal{S}\mathcal{T}^{-1}$ of all possible products $T_1FT_0^{-1}$, $T_i \in \mathcal{T}$, $F \in \mathcal{S}$. Note that $H \subset \mathfrak{G}|_U$ for every $H \in \mathcal{T}\mathcal{S}\mathcal{T}^{-1}$. For every $\gamma \in \mathfrak{G}|_U$, there is a product $F = F_n\cdots F_2F_1$ of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ such that $\gamma \in F$, and the set of such products

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forms a basis of neighborhoods of γ. Find elements $T_i \in \mathcal{T}$, $i = 1, \dots, n-1$, such that $\mathsf{r}(F_i F_{i-1} \cdots F_1 \mathsf{s}(\gamma)) \in \mathsf{s}(T_i)$. Then

$$\gamma \in U F_n T_{n-1}^{-1} \cdot T_{n-1} F_{n-1} T_{n-2}^{-1} \cdots T_1 F_1 U^{-1} \subset F.$$

It follows that $\mathcal{T}\mathcal{S}\mathcal{T}^{-1}$ is an expansive cover of a compact generating set of $\mathfrak{G}|_U$.

Suppose now that $\mathfrak{G}|_U$ is expansive, and let $\mathcal{S}$ be an expansive cover by bisections of a compact generating set of $\mathfrak{G}|_U$. Let $\mathcal{T}$ be as above. Then for every $\gamma \in \mathfrak{G}$ there exist $T_0, T_1 \in \mathcal{T}$ such that $\mathsf{s}(\gamma) \in \mathsf{s}(T_0)$ and $\mathsf{r}(\gamma) \in \mathsf{s}(T_1)$. Then $T_1 \gamma T_0^{-1} \in \mathfrak{G}|_U$. If $F_n F_{n-1} \cdots F_2 F_1$ is a product of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ containing $T_1 \gamma T_0^{-1}$, then $\gamma \in T_1^{-1} F_1 F_{n-1} \cdots F_2 F_1 T_0$. Similarly to the above, this implies that the set $\mathcal{T}^{-1}\mathcal{S}\mathcal{T}$ is an expansive cover of a compact generating set of $\mathfrak{G}$. □

PROPOSITION 5.7. *Let G be a finitely generated group acting by homeomorphisms on a Cantor set $\mathcal{X}$. Let $G \times \mathcal{X}$ be the groupoid of the action and let $\mathfrak{G}$ be the groupoid of germs of the action. Then the following conditions are equivalent.*

(1) *The groupoid $G \times \mathcal{X}$ is expansive.*
(2) *The groupoid $\mathfrak{G}$ is expansive.*
(3) *The action of G on $\mathcal{X}$ is expansive, i.e., there exists a neighborhood $W \subset \mathcal{X} \times \mathcal{X}$ of the diagonal such that if $x, y \in \mathcal{X}$ are such that $(g(x), g(y)) \in W$ for all $g \in G$, then $x = y$.*
(4) *The dynamical system $(G, \mathcal{X})$ is a subshift, i.e., there exist a finite set X and a G-equivariant homeomorphism between $\mathcal{X}$ and a G-invariant closed subset of X^G (with the natural action of G on X^G).*

Proof. Let us show that each condition (1) and (2) implies (3). If A is a finite generating set of G, then $S \times \mathcal{X}$ and the set of germs of elements of S are compact open generating sets of the groupoids $G \times \mathcal{X}$ and $\mathfrak{G}$, respectively. Suppose that one of the groupoids $G \times \mathcal{X}$ or $\mathfrak{G}$ is expansive. Then there exists an expansive cover $\mathcal{S}$ of the above generating sets by bisections of the form $(s, U) = \{(s, x) : x \in U\}$, where $s \in S$ and U is a clopen subset of $\mathcal{X}$. Moreover, we may assume that U belongs to a finite partition $\mathcal{P}$ of $\mathcal{X}$ into disjoint clopen subsets. Let $W \subset \mathcal{X} \times \mathcal{X}$ be the set of pairs (x, y) such that x and y belong to the same element of $\mathcal{P}$.

Let $x, y \in \mathcal{X}$ and $x \neq y$. Suppose, by contradiction, that $(g(x), g(y)) \in W$ for all $g \in G$. For every element $A = (s, U) \in \mathcal{S}$, we have $\mathsf{s}(A) = U \in \mathcal{P}$. It follows that $x \in \mathsf{s}(A)$ is equivalent to $y \in \mathsf{s}(A)$. The elements $(s(x), s(y))$ again belong to one element of $\mathcal{P}$; hence, by induction on n, we get that for every sequence $A_1, A_2, \dots, A_n$ of elements of $\mathcal{S}$ the condition $x \in \mathsf{s}(A_1 A_2 \cdots A_n)$ is equivalent to the condition $y \in \mathsf{s}(A_1 A_2 \cdots A_n)$. But it is a contradiction with Proposition 5.5.

Implication (3)$\Longrightarrow$(4) is classical. Let us repeat the proof for completeness. Let W be the neighborhood of the diagonal satisfying the definition of expansivity of the action of G on $\mathcal{X}$. We can always take W smaller, so we may assume that there exists a finite partition $\mathcal{P}$ of $\mathcal{X}$ into clopen sets such that $(x, y) \in W$ if and only if x and y belong to the same element of $\mathcal{P}$. Consider now the following map $\phi : \mathcal{X} \longrightarrow \mathcal{P}^G$:

$$\phi(\xi)(g) = U \Longleftrightarrow g(\xi) \in U,$$

for $\xi \in \mathcal{X}$, $g \in G$, $U \in \mathcal{P}$.

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It is easy to see that the map ϕ is continuous and G-equivariant. Since $\mathcal{X}$ and $\mathcal{P}^G$ are compact Hausdorff, in order to prove that ϕ is a homeomorphism of $\mathcal{X}$ with a closed subset of $\mathcal{P}^G$, it is enough to show that ϕ is injective. But this follows directly from the definition of expansivity.

Finally, let us prove that (4) implies (1) and (2). Let $\mathcal{X}$ be a G-invariant closed subset of X^G, where X is a finite alphabet. Let A be a finite symmetric generating set of G, and let $S = \{(s, x) : s \in A, x \in \mathcal{X}\}$ be the corresponding generating set of $G \times \mathcal{X}$ or $\mathfrak{G}$. Consider the following cover of S:

$$\mathcal{S} = \{(s, C_x) : x \in X\},$$

where C_x is the cylindrical set of elements $w : G \longrightarrow X$ of $\mathcal{X}$ such that $w(1) = x$.

Then $w \in \mathsf{s}((s_n, C_{x_n})(s_{n-1}, C_{x_{n-1}}) \cdots (s_1, C_{x_1}))$ implies that $w(1) = x_1$, $w(s_1) = x_2$, $w(s_2 s_1) = x_3, \ldots, w(s_{n-1} s_{n_2} \cdots s_1) = x_n$. Consequently, by taking a long enough product $B_1 B_2 \cdots B_n$ of elements $B_i \in \mathcal{S}$, we can specify an arbitrarily large number of coordinates of w by the condition $w \in \mathsf{s}(B_1 B_2 \cdots B_n)$. This implies that $G \times \mathcal{X}$ and $\mathfrak{G}$ are expansive, by Proposition 5.5. □

Example 5.8. Consider an arbitrary group G acting by automorphisms on a locally finite rooted tree T. Consider the corresponding action by homeomorphisms on the boundary ∂T of the tree. Let $\mathfrak{G}$ be the groupoid of the action of G on ∂T (or the groupoid of germs of the action). Then for every compact open subset $S \subset \mathfrak{G}$ and for every finite cover $\mathcal{S}$ of S there exist a level L_n of the tree T and a refinement $\mathcal{S}'$ of $\mathcal{S}$ such that every element of $\mathcal{S}'$ is of the form $\{g\} \times \partial T_v$, where T_v is the rooted subtree of T with the root $v \in L_n$. The group G acts on L_n by permutations and hence any product of elements of $\mathcal{S}'$ is also of this form. It follows that $\mathfrak{G}$ is not expansive.

Example 5.9. The AF groupoids associated with Bratteli diagrams (see Example 4.1) are also never expansive. For every compact subset C of such a groupoid there exists n such that every element of C is a germ of a transformation changing beginnings of paths of length at most n. The set $\mathfrak{G}_n$ of all such germs is equal to the union of a finite number of multisections. It follows that for any finite set $\mathcal{S}$ of open compact bisections the set of all products of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ is finite.

5.2. *Finite generation of* $\mathsf{A}(\mathfrak{G})$.

THEOREM 5.10. *Suppose that $\mathfrak{G}$ is expansive and every $\mathfrak{G}$-orbit has at least five points. Then the group $\mathsf{A}(\mathfrak{G})$ is finitely generated.*

Compare this result with [**27**, Proposition 4.4]. Note that we do not require $\mathfrak{G}$ to be a groupoid of germs.

Proof. If all $\mathfrak{G}$-orbits have at least five points, then $\mathsf{A}(\mathfrak{G}) = \mathsf{A}_5(\mathfrak{G})$; see Proposition 3.8. Let us prove that $\mathsf{A}_5(\mathfrak{G})$ is finitely generated.

Let S be a compact open generating set of an expansive groupoid $\mathfrak{G}$ such that $S = S^{-1}$ and S does not contain elements γ such that $\mathsf{s}(\gamma) = \mathsf{r}(\gamma)$; see Lemma 5.2. We may also assume that $|Sx| \geq 4$ for every $x \in \mathfrak{G}^{(0)}$, after possibly increasing S by adding to it bisections with disjoint source and target.

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LEMMA 5.11. *There exists a decomposition $\mathcal{P} = \{U_i\}$ of $\mathfrak{G}^{(0)}$ into a finite disjoint union of clopen sets such that every $\mathfrak{G}$-orbit intersects at least five elements of $\mathcal{P}$.*

Proof. Let d be a metric on $\mathfrak{G}^{(0)}$. For every $x \in \mathfrak{G}^{(0)}$, there exist four elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathfrak{G}$ such that $\mathsf{s}(\gamma_i) = x$, and the units $\mathsf{s}(\gamma_1), \mathsf{r}(\gamma_1), \mathsf{r}(\gamma_2), \mathsf{r}(\gamma_3), \mathsf{r}(\gamma_4)$ are pairwise different. Hence, there exist bisections $H_1, H_2, \dots, H_4$ such that $\gamma_i \in H_i$, $\mathsf{s}(H_1) = \mathsf{s}(H_2) = \cdots = \mathsf{s}(H_4)$, and the sets $\mathsf{s}(H_1), \mathsf{r}(H_1), \dots, \mathsf{r}(H_4)$ are pairwise disjoint. It follows that there exists $\epsilon > 0$ such that for every unit y from the ϵ-neighborhood of x the orbit of y contains at least five points (including y) that are of distance more than ϵ from each other. It follows from the compactness of $\mathfrak{G}^{(0)}$ that there exists $\epsilon > 0$ such that for every $x \in \mathfrak{G}^{(0)}$ there exist at least five elements of the orbit of x that are of distance more than ϵ from each other. Any partition $\mathcal{P}$ of $\mathfrak{G}^{(0)}$ into clopen sets of diameters less than ϵ will satisfy the conditions of the lemma. □

Let $\mathcal{P}$ be a partition of $\mathfrak{G}^{(0)}$ satisfying the conditions of the above lemma. Since S is compact and does not contain elements of isotropy groups, there exists $\epsilon > 0$ such that for every $\gamma \in S$ the distance between $\mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma)$ is greater than ϵ. Consequently, replacing the partition $\mathcal{P}$ by a finite refinement, we may assume that for every $\gamma \in S$ the units $\mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma)$ belong to different elements of $\mathcal{P}$. We also may assume that for every $x \in \mathfrak{G}^{(0)}$ there exist elements $s_1, s_2, s_3, s_4 \in S$ such that $\mathsf{s}(s_i) = x$ and the points $x, \mathsf{r}(s_1), \mathsf{r}(s_2), \mathsf{r}(s_3), \mathsf{r}(s_4)$ belong to pairwise different elements of $\mathcal{P}$. (Recall that $|Sx| \geq 4$ for every $x \in \mathfrak{G}^{(0)}$, by the choice of S.)

Let $\mathcal{S}$ be an expansive finite cover of S by bisections such that $\mathcal{S}^{-1} = \mathcal{S}$ and $\bigcup_{F \in \mathcal{S}} F = S$. Taking elements of $\mathcal{S}$ small enough, we may assume that for every $F \in \mathcal{S}$ the sets $\mathsf{s}(F)$ and $\mathsf{r}(F)$ are subsets of different elements of $\mathcal{P}$.

Consider the set T of elements $\gamma \in \bigcup_{k=1}^{3} S^k$ such that $\mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma)$ belong to different elements of $\mathcal{P}$. Note that $S \subset T$ and T is compact.

For every $\gamma \in T$ and every $H \in \bigcup_{k=1}^{3} \mathcal{S}^k$ such that $\gamma \in H$, we can find a degree-5 multisection $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^{5}$ such that the sets $F_{i,i}$ are contained in pairwise different elements of $\mathcal{P}$, and $\gamma \in F_{1,2} \subset H$. By compactness, we can find a finite set M of multisections of this form such that the corresponding sets $F_{1,2}$ cover every element of T.

Let us denote by A the group generated by the set $\bigcup_{\mathcal{F} \in M} \mathsf{A}(\mathcal{F})$. It is enough to prove that $\mathsf{A}(\mathfrak{G}) = \mathsf{A}$.

LEMMA 5.12. *Suppose that $\gamma \in \mathfrak{G}$ is such that $\mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma)$ belong to different elements of $\mathcal{P}$. Then for every bisection $F \ni \gamma$ there exists a degree-5 multisection $\mathcal{H} = \{H_{i,j}\}$ such that $\mathsf{A}(\mathcal{H}) \leq \mathsf{A}$, $\gamma \in H_{1,2} \subset F$, and the sets $H_{i,i}$ are contained in pairwise different elements of $\mathcal{P}$.*

Proof. It is enough to prove it for bisections F of the form $G_1 G_2 \cdots G_n$, where $G_i \in \mathcal{S}$. The lemma is true for $n \leq 3$ by the choice of M. Suppose that it is true for $n \geq 3$; let us show that it is true for $F = G_0 G_1 \cdots G_n$ and $\gamma \in F$ such that $\mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma)$ belong to different elements of $\mathcal{P}$.

Denote by $\gamma_i \in G_i$ the elements such that $\gamma = \gamma_0 \gamma_1 \cdots \gamma_n$. There exists $\gamma' \in S$ such that $\mathsf{s}(\gamma') = \mathsf{s}(\gamma_1)$ and the points $\mathsf{r}(\gamma')$, $\mathsf{s}(\gamma)$, $\mathsf{r}(\gamma)$ belong to pairwise different elements of $\mathcal{P}$.

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We have $\gamma = (\gamma_0') \cdot (\gamma' \gamma_2 \cdots \gamma_n)$ for $\gamma_0' = \gamma_0 \gamma_1 (\gamma')^{-1} \in S^3$. Let $G' \in \mathcal{S}$ be such that $\gamma' \in G'$, and denote $G_0' = G_0 G_1 (G')^{-1} \in \mathcal{S}^3$. Then $F = G_0 G_1 \cdots G_n \supseteq G_0' G' G_2 \cdots G_n \ni \gamma$.

By the inductive hypothesis, there exists a degree-5 bisection $\mathcal{H}_0 = \{H_{i,j}\}_{i,j=1}^5$ such that all the components $H_{i,i}$ are contained in pairwise different elements of $\mathcal{P}$, the element $\gamma' \gamma_2 \cdots \gamma_n$ belongs to $H_{1,2} \subset G' G_2 \cdots G_n$, and $\mathsf{A}(\mathcal{H}_1) \subset \mathsf{A}$. Assume that $H_{3,3}$ and $\mathsf{r}(\gamma)$ are contained in different elements of $\mathcal{P}$, and let $\mathcal{H}_1 = \{H_{i,j}\}_{i,j=1}^3$ be the corresponding degree-3 multisection.

By the choice of M, there exists a multisection $\mathcal{F}_1 = \{F_{i,j}\}_{i,j=1}^3$ which is a subset of an element of M, and such that $\gamma_0' \in F_{1,2} \subset G_0'$, and the elements of $\mathcal{P}$ containing $F_{2,2}$ and $F_{3,3}$ do not contain any component of the domain of $\mathcal{H}_1$. Then, applying Proposition 3.5 to $\mathcal{H}_1$ and $\mathcal{F}_1$, we will find a degree-5 multisection $\mathcal{H} = \{T_{i,j}\}$ such that all components $T_{i,i}$ are contained in pairwise different elements of $\mathcal{P}$, and one of the elements $T_{i,j} \in \mathcal{H}$ satisfies $\gamma \in T_{i,j} \subset G_0' G' G_2 \cdots G_n \subset F$. □

LEMMA 5.13. *Suppose that $\gamma \in \mathfrak{G}$ is such that $\mathsf{s}(\gamma) \neq \mathsf{r}(\gamma)$, and $\{\mathsf{s}(\gamma), \mathsf{r}(\gamma)\}$ is contained in one element of $\mathcal{P}$. Then for every bisection $F \ni \gamma$ there exists a degree-5 multisection $\mathcal{H} = \{H_{i,j}\}_{i,j=1}^5$ such that $\mathsf{A}(\mathcal{H}) \leq \mathsf{A}$, $\gamma \in H_{1,2} \subset F$, and the sets $H_{1,1} \cup H_{2,2}$, $H_{3,3}$, $H_{4,4}$, $H_{5,5}$ are contained in pairwise different elements of $\mathcal{P}$.*

Proof. Let U_0 be the element of $\mathcal{P}$ containing $\mathsf{s}(\gamma)$ and $\mathsf{r}(\gamma)$. Let $\gamma' \in \mathfrak{G}$ be such that $\mathsf{s}(\gamma') = \mathsf{s}(\gamma)$, and $\mathsf{r}(\gamma') \notin U_0$. Then by Lemma 5.12 there exists a degree-3 multisection $\mathcal{H}' = \{H_{i,j}'\}_{i,j=1}^3$ such that $\mathsf{A}(\mathcal{H}') \leq \mathsf{A}$, $\gamma' \in H_{1,2}'$, $H_{i,i}'$ are contained in pairwise different elements of $\mathcal{P}$. Then, again by Lemma 5.12, there exists a degree-3 multisection $\mathcal{H}'' = \{H_{i,j}''\}_{i,j=1}^3$ such that $\mathsf{A}(\mathcal{H}'') \leq \mathsf{A}$, the bisection $H_{1,2}''$ is an arbitrarily small neighborhood of $\gamma' \gamma^{-1}$, the sets $H_{i,i}''$ are contained in pairwise different elements of $\mathcal{P}$, and the element of $\mathcal{P}$ containing $H_{3,3}''$ is different from the element containing $H_{3,3}'$. Then applying Proposition 3.5 to $\mathcal{H}'$ and $\mathcal{H}''$, we obtain a degree-5 multisection satisfying the conditions of the lemma. □

Let now $\gamma_1, \gamma_2 \in \mathfrak{G}$ be such that $\mathsf{s}(\gamma_1) = \mathsf{s}(\gamma_2)$, and the points $\mathsf{s}(\gamma_1)$, $\mathsf{r}(\gamma_1)$, $\mathsf{r}(\gamma_2)$ are pairwise different. Let $\mathcal{H}_1$ be a multisection satisfying the conditions of Lemma 5.12 or 5.13 for $\gamma = \gamma_1$.

Note that the bisection $H_{1,2} \in \mathcal{H}_1$ containing γ_1 can be made arbitrarily small and hence we may assume that its domain and range do not contain $\mathsf{r}(\gamma_2)$. There are at most three elements of $\mathcal{P}$ intersecting $\{\mathsf{s}(\gamma_1), \mathsf{r}(\gamma_1), \mathsf{r}(\gamma_2)\}$. Consequently, we can find a component $H_{i,i}$ of $\mathcal{H}_1$ different from $H_{1,1} = \mathsf{s}(H_{1,2})$ and $H_{2,2} = \mathsf{r}(H_{1,2})$ and not containing $\mathsf{r}(\gamma_2)$. Let $\mathcal{H}_1' \subset \mathcal{H}_1$ be the degree-3 submultisection with the components of the domain $H_{1,1}$, $H_{2,2}$, $H_{i,i}$. Then the domain of $\mathcal{H}_1'$ does not contain $\mathsf{r}(\gamma_2)$.

Let $\mathcal{H}_2$ be the multisection satisfying the conditions of Lemma 5.12 or 5.13 for $\gamma = \gamma_2$. Since the element of $\mathcal{H}_2$ containing γ_2 can be made arbitrarily small, we may assume that the component of the domain of $\mathcal{H}_2$ containing $\mathsf{r}(\gamma_2)$ does not intersect the domain of $\mathcal{H}_1'$.

One of the components of $\mathcal{H}_2$ is contained in an element of $\mathcal{P}$ not containing $\mathsf{r}(\gamma_2)$ and not intersecting the domain of $\mathcal{H}_1$. It follows that a subset of $\mathcal{H}_2$ is a degree-3 multisection $\mathcal{H}_2'$ such that the intersection of the domains of $\mathcal{H}_1'$ and $\mathcal{H}_2'$ is contained in one component

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of $\mathcal{H}'_1$ and one component of $\mathcal{H}'_2$. Then applying Proposition 3.5, we get a degree-3 multisection $\mathcal{H}$ such that two of its elements are arbitrarily small neighborhoods of γ_1 and γ_2 and $\mathsf{A}(\mathcal{H}) \leq \mathsf{A}$. Consequently, for every degree-3 multisection $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^3$ there exists a finite covering $F_{i,j} = \bigcup_{k=1}^m F_{i,j}^{(k)}$ such that $\mathcal{F}^{(k)} = \{F_{i,j}^{(k)}\}_{i,j=1}^3$ are multisections and $\mathsf{A}(\mathcal{F}^{(k)}) \leq \mathsf{A}$.

It follows that the same statement is also true for degree-5 multisections (by Proposition 3.5). Proposition 3.3 implies then that $\mathsf{A}_5(\mathfrak{G}) = \mathsf{A}$. □

6. *Cayley graphs of groupoids and expansivity*

Proposition 5.7 is usually a convenient criterion for deciding if a groupoid coming from a group action is expansive. It is harder to use it in the cases when an étale groupoid is defined without reference to a group action. Here we analyze the expansivity condition for general étale groupoids.

6.1. *General case.* Let $\mathfrak{G}$ be an étale groupoid. We say that a set of bisections $\mathcal{S}$ is a *labeled generating set* of $\mathfrak{G}$ if the union of its elements is a generating set of $\mathfrak{G}$ (according to Definition 5.1). The groupoid has a finite labeled generating set of bisections if and only if it is compactly generated.

If S is a compact generating set of $\mathfrak{G}$, then the *Cayley graph* $\mathfrak{G}(x, S)$ is the graph with the set of vertices $\mathsf{s}^{-1}(x)$ in which there is an arrow from γ_1 to γ_2 if and only if there is $s \in S$ such that $\gamma_2 = s\gamma_1$.

For a given finite labeled generating $\mathcal{S}$ of $\mathfrak{G}$, we consider the *labeled Cayley graph* $\mathfrak{G}(x, \mathcal{S})$ with the same set of vertices $\mathsf{s}^{-1}(x)$ in which there is an arrow from γ_1 to γ_2 labeled by an element $F \in \mathcal{S}$ if and only if there exists $s \in F$ such that $\gamma_2 = s\gamma_1$.

We denote by $\widehat{\mathfrak{G}}(x, S)$ and $\widehat{\mathfrak{G}}(x, \mathcal{S})$ the universal coverings of the graphs $\mathfrak{G}(x, S)$ and $\mathfrak{G}(x, \mathcal{S})$, respectively. The orientation and labeling of the universal coverings are lifted from the orientation and labeling of $\mathfrak{G}(x, S)$ and $\mathfrak{G}(x, \mathcal{S})$.

Note that for every vertex v of the graph $\mathfrak{G}(x, \mathcal{S})$ and for every label $F \in \mathcal{S}$, there exist at most one outgoing arrow labeled by F and at most one incoming arrow labeled by F. It follows that any morphism $\phi : \mathfrak{G}(x, \mathcal{S}) \longrightarrow \mathfrak{G}(y, \mathcal{S})$ of oriented labeled graphs is uniquely determined by the pair $(x, \phi(x))$. The same is true for the (oriented labeled) universal coverings $\widehat{\mathfrak{G}}(x, \mathcal{S})$ of the Cayley graphs.

The following is proved in [**26**, Corollary 2.3.4].

PROPOSITION 6.1. *Let $\mathfrak{G}$ be a groupoid and let $U \subset \mathfrak{G}^{(0)}$ be a clopen transversal. Let S and S_1 be compact generating sets of $\mathfrak{G}$ and $\mathfrak{G}|_U$, respectively. Then for every $x \in U$ the identical embedding of the set of vertices of $\mathfrak{G}|_U(x, S_1)$ into the set of vertices of $\mathfrak{G}(x, S)$ is a quasi-isometry of graphs.*

A *rooted graph* is a graph with a marked vertex. A morphism of rooted graphs must map the root to the root. Every Cayley graph $\mathfrak{G}(x, \mathcal{S})$ is considered to be an oriented labeled rooted graph with the root x.

If (Γ, v) is a rooted connected graph, then its rooted universal covering is the usual universal covering of Γ in which the root is any preimage of v under the covering map.

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Note that it is unique up to an isomorphism of rooted graphs. The universal covering can be defined as the space of homotopy classes of paths in Γ starting at v, where the root is the trivial path at v. If Γ is labeled and oriented, then the universal covering is also labeled and oriented in the natural way.

PROPOSITION 6.2. *Let $\mathcal{S}$ be a finite labeled generating set. Then $\mathcal{S}$ is non-expansive if and only if there exist $x, y \in \mathfrak{G}^{(0)}$ such that $x \neq y$ and there exists a morphism of rooted labeled graphs $\widehat{\mathfrak{G}}(x, \mathcal{S}) \longrightarrow \widehat{\mathfrak{G}}(y, \mathcal{S})$.*

Proof. Every rooted path in $\mathfrak{G}(x, \mathcal{S})$ corresponds to a product of the elements of $\mathcal{S}$ and their inverses (one just has to read the labels). Such a path exists if and only if x belongs to the set of sources of the corresponding product. It follows that the universal covering is precisely the tree of all possible products F of the elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ such that $x \in \mathtt{s}(F)$. The proof is then finished by using condition (4) of Proposition 5.5. □

THEOREM 6.3. *A compactly generated groupoid $\mathfrak{G}$ is expansive if and only if there exists a finite labeled generating set $\mathcal{S}$ of bisections such that for every pair of different points $x, y \in \mathfrak{G}^{(0)}$ the universal coverings $\widehat{\mathfrak{G}}(x, \mathcal{S})$ and $\widehat{\mathfrak{G}}(y, \mathcal{S})$ are non-isomorphic as rooted oriented labeled graphs.*

Proof. Every expansive generating set $\mathcal{S}$ satisfies Proposition 6.2 and hence has pairwise non-isomorphic rooted graphs $\widehat{\mathfrak{G}}(x, \mathcal{S})$.

Suppose that $\mathcal{S}$ is a finite generating set of bisections such that the rooted graphs $\widehat{\mathfrak{G}}(x, \mathcal{S})$ are pairwise non-isomorphic. Let $U_n(x)$ be the sets defined in Proposition 5.5. The number of possible sets of the form $U_1(x)$ is finite. Therefore, there exists a finite labeled generating set $\mathcal{S}_1$ such that each element of $\mathcal{S}$ is a disjoint union of elements of $\mathcal{S}_1$ and, for every $F \in \mathcal{S}_1$ and $x \in \mathtt{s}(F)$, we have $U_1(x) \supset \mathtt{s}(F)$. The Cayley graph $\mathfrak{G}(x, \mathcal{S})$ is determined by the Cayley graph $\mathfrak{G}(x, \mathcal{S}_1)$ and is obtained from it just by relabeling of the edges.

By the definition of $\mathcal{S}_1$, the label of every edge incident with a vertex $v \in \mathfrak{G}(x, \mathcal{S}_1)$ uniquely determines the labels of the edges adjacent to v in $\mathfrak{G}(x, \mathcal{S})$. In particular, the number of vertices adjacent to v in $\mathfrak{G}(x, \mathcal{S}_1)$ (which is the same as in $\mathfrak{G}(x, \mathcal{S})$) is uniquely determined by the label of any edge starting at v.

This implies that every morphism $\widehat{\mathfrak{G}}(x, \mathcal{S}_1) \longrightarrow \widehat{\mathfrak{G}}(y, \mathcal{S}_1)$ is an isomorphism that induces an isomorphism $\widehat{\mathfrak{G}}(x, \mathcal{S}) \longrightarrow \widehat{\mathfrak{G}}(y, \mathcal{S})$. It follows, by Proposition 6.2, that $\mathcal{S}_1$ is expansive. □

6.2. *Compactly presented groupoids.* It is often easier to prove that the Cayley graphs $\mathfrak{G}(x, \mathcal{S})$ based at different points x are different than to prove that their universal coverings are different.

In general, it is possible that all Cayley graphs $\mathfrak{G}(x, \mathcal{S})$ are pairwise non-isomorphic (for different x), but the groupoid is not expansive. For example, there are examples of finitely generated groups acting by automorphisms of a rooted tree such that the Cayley graphs of the associated groupoid of germs of the action on the boundary of the tree are pairwise non-isomorphic (see, for instance, [**7**]). But such groupoids can not be expansive; see Example 5.8.

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Note, however, that we have the following reformulation of Proposition 5.7, since any labeling of the Cayley graphs of the groupoid of the action can be encoded by a labeling of its vertices, and *vice versa*.

PROPOSITION 6.4. *Let G be a finitely generated group acting on a Cantor set $\mathcal{X}$. Let $\mathfrak{G} = G \times \mathcal{X}$ be the groupoid of the action. Then $\mathfrak{G}$ is expansive if and only if there exists a finite labeled generating set $\mathcal{S}$ such that for every $x, y \in \mathcal{X}$, $x \neq y$, the Cayley graphs $\mathfrak{G}(x, \mathcal{S})$ and $\mathfrak{G}(y, \mathcal{S})$ are non-isomorphic as rooted labeled directed graphs.*

Another class of groupoids for which it is enough to consider the Cayley graphs to prove expansivity are *compactly presented* groupoids.

For a graph Γ and a positive integer R, the *Rips complex* $\Delta_R\Gamma$ is the simplicial complex with the set of vertices equal to the set of vertices of Γ in which a subset A of the set of vertices is a simplex if and only if its diameter is not more than R.

Definition 6.5. Let $\mathfrak{G}$ be an étale groupoid and let S be a compact open generating set. The groupoid $\mathfrak{G}$ is said to be *compactly presented* if there exists $R > 0$ such that the Rips complex $\Delta_R\mathfrak{G}(x, S)$ is simply connected for every $x \in \mathfrak{G}^{(0)}$.

It is not hard to show that the definition does not depend on the choice of the generating set.

THEOREM 6.6. *Suppose that $\mathfrak{G}$ is a compactly presented Hausdorff étale groupoid. It is expansive if and only if there exists a finite generating set $\mathcal{S}$ of bisections such that the graphs $\mathfrak{G}(x, \mathcal{S})$ and $\mathfrak{G}(y, \mathcal{S})$ are non-isomorphic as rooted oriented labeled graphs for every pair $x \neq y$.*

Proof. If $\mathfrak{G}$ is expansive, then a generating set $\mathcal{S}$ satisfying the conditions of the theorem exists by Theorem 6.3.

Let us prove the converse statement. Let $\mathcal{S}$ be a generating set satisfying the conditions of the theorem.

If $\mathfrak{G}$ is a Hausdorff étale groupoid, then for every $\gamma \in \mathfrak{G} \setminus \mathfrak{G}^{(0)}$ there exists a bisection U such that $\gamma \in U$ and $U \cap \mathfrak{G}^{(0)} = \emptyset$. Otherwise, γ and $\mathtt{s}(\gamma)$ do not have disjoint neighborhoods. In other words, $\mathfrak{G}^{(0)}$ is closed. Note that it is open by definition of an étale groupoid.

Denote by $N_R(x, \mathcal{S})$, for $x \in \mathfrak{G}^{(0)}$ and $R \geq 0$, the set of points $y \in \mathfrak{G}^{(0)}$ such that the balls of radius R with centers x and y in the Cayley graphs $\mathfrak{G}(x, \mathcal{S})$ and $\mathfrak{G}(y, \mathcal{S})$ are isomorphic as rooted (with roots equal to the centers x and y) labeled oriented graphs. By the above, the sets $N_R(x, \mathcal{S})$ are clopen.

Replacing $\mathcal{S}$ by $(\mathcal{S} \cup \mathcal{S}^{-1})^n$ for some n, if necessary, we may assume that the Rips complexes $\Delta_1\mathfrak{G}(x, \mathcal{S})$ are simply connected. We may also assume that the sets $\mathtt{s}(F)$, for $F \in \mathcal{S}$, cover $\mathfrak{G}^{(0)}$, so that every vertex of every Cayley graph has at least one outgoing edge.

Let $\mathcal{S}_1$ be a finite set of bisections such that every element of $\mathcal{S}$ is a disjoint union of elements of $\mathcal{S}_1$ and, for every $F \in \mathcal{S}_1$ and $x \in \mathtt{s}(F)$, we have $\mathtt{s}(F) \subset N_1(x, \mathcal{S})$. Then the Cayley graphs $\mathfrak{G}(x, \mathcal{S})$ are obtained from $\mathfrak{G}(x, \mathcal{S}_1)$ by relabeling edges, and the label of

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any edge e of $\mathfrak{G}(x, \mathcal{S}_1)$ uniquely determines the 1-ball in $\mathfrak{G}(x, \mathcal{S})$ of the source of e. It also follows that if $x \neq y$, then the rooted graphs $\mathfrak{G}(x, \mathcal{S}_1)$ and $\mathfrak{G}(y, \mathcal{S}_1)$ are non-isomorphic.

Suppose that $\phi : \widehat{\mathfrak{G}}(x, \mathcal{S}_1) \longrightarrow \widehat{\mathfrak{G}}(y, \mathcal{S}_1)$ is a morphism of rooted graphs. Note that by the same argument as in the proof of Theorem 6.3, ϕ is an isomorphism.

The morphism ϕ induces a covering map $\phi' : \widehat{\mathfrak{G}}(x, \mathcal{S}_1) \longrightarrow \mathfrak{G}(y, \mathcal{S}_1)$, obtained by composing ϕ with the universal covering map $\widehat{\mathfrak{G}}(y, \mathcal{S}_1) \longrightarrow \mathfrak{G}(y, \mathcal{S}_1)$.

Let us introduce an equivalence relation on the sets of vertices and edges of $\widehat{\mathfrak{G}}(x, \mathcal{S}_1)$ by the rule that $a_1 \sim a_2$ if and only if $\phi'(a_1) = \phi'(a_2)$ and the images of a_1 and a_2 in $\mathfrak{G}(x, \mathcal{S}_1)$ coincide. Let Γ be the quotient of $\tilde{\mathfrak{G}}(x, \mathcal{S}_1)$ by this equivalence relation. We have natural maps $\psi : \Gamma \longrightarrow \mathfrak{G}(y, \mathcal{S}_1)$ and $\pi : \Gamma \longrightarrow \mathfrak{G}(x, \mathcal{S}_1)$ induced by ϕ' and the universal covering map, respectively.

Let $\Sigma = \{v_1, v_2, v_3\}$ be a face of $\Delta_1\mathfrak{G}(x, \mathcal{S}_1)$. Let e be an edge in Σ, and let $v_1 = \mathtt{s}(e)$ and $v_2 = \mathtt{r}(e)$. Then the label of e in $\mathfrak{G}(x, \mathcal{S}_1)$ uniquely determines the labels and orientations of the edges of Σ in $\mathfrak{G}(x, \mathcal{S})$. Moreover, any edge f of $\mathfrak{G}(x, \mathcal{S}_1)$ (or $\mathfrak{G}(y, \mathcal{S}_1)$) with the same label as e belongs in $\mathfrak{G}(x, \mathcal{S})$ (respectively in $\mathfrak{G}(y, \mathcal{S})$) to a face with the same labels of sides as in Σ. It follows that the maps $\psi : \Gamma \longrightarrow \mathfrak{G}(y, \mathcal{S}_1)$ and $\pi : \Gamma \longrightarrow \mathfrak{G}(x, \mathcal{S}_1)$ induce coverings of the respective Rips complexes (as the labels of the edges remember the unit balls). But $\Delta_1\mathfrak{G}(x, \mathcal{S}_1)$ and $\Delta_1\mathfrak{G}(y, \mathcal{S}_1)$ are simply connected and hence ψ and π are isomorphisms. □

6.3. *Quasicrystals.* Consider the following class of groupoids, as an example of an application of Theorem 6.6.

Definition 6.7. We say that a subset Q of the vector space $\mathbb{R}^n$ is a *quasicrystal* if it satisfies the following conditions.

(1) It is a *Delaunay set*:
   - there exists $R > 0$ such that for every $x \in \mathbb{R}^n$ there exists $q \in Q$ such that $\|x - q\| < R$;
   - there exists $\delta > 0$ such that for any $q_1, q_2 \in Q$, $q_1 \neq q_2$, we have $\|q_1 - q_2\| > \delta$.

(2) It has *finite local complexity*: for every $R > 0$ the set $\mathcal{B}_R(Q)$ of sets of the form $B_R(q) \cap Q - q$ for $q \in Q$ is finite.

(3) It is *repetitive*: for every $R > 0$ and $q \in Q$ there exists $D > 0$ such that for every $x \in \mathbb{R}^n$ there exists $p \in Q$ such that $\|x - p\| < D$ and $B_R(p) \cap Q - p = B_R(q) \cap Q - q$.

Here $B_R(x) = \{y \in \mathbb{R}^n \,:\, \|x - y\| < R\}$ is the open ball of radius R with center in x.

We say that a quasicrystal $Q \subset \mathbb{R}^n$ is *aperiodic* if $Q + v = Q$ for $v \in \mathbb{R}^n$ implies that $v = \vec{0}$.

Many examples of quasicrystals can be constructed using the *cut-and-project* method; see [**3**].

Definition 6.8. The *hull* $\mathcal{Q}$ of a quasicrystal Q is the set of all subsets $Q' \subset \mathbb{R}^n$ such that for any $R > 0$ there exists $q \in Q$ such that $B_R(\vec{0}) \cap Q' = B_R(q) \cap Q - q$; see [**1**].

One can show that every element of the hull is a quasicrystal. Note that the zero vector $\vec{0}$ belongs to every element of the hull.

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Let us introduce a metric on $\mathcal{Q}$ setting $d(Q_1, Q_2)$ equal to R^{-1}, where R is the supremum of radii r such that $B_r(\vec{0}) \cap Q_1 = B_r(\vec{0}) \cap Q_2$. It is easy to see that $\mathcal{Q}$ is compact and totally disconnected.

If Q is aperiodic, then the space $\mathcal{Q}$ has no isolated points and hence is homeomorphic to the Cantor set.

Groupoids and inverse semigroups are appropriate tools for the study of symmetries of quasiperiodic structures; see [**2**, **11**, **12**]. Let us describe a natural étale groupoid associated with a quasicrystal. Fix an aperiodic quasicrystal Q. Denote by $\mathfrak{Q}$ the following groupoid. Its elements are pairs (P, q), where $P \in \mathcal{Q}$ and $q \in P$. They are multiplied by the rule

$$(P_1, q_1)(P_2, q_2) = (P_2, q_1 + q_2),$$

where the product is defined if and only if $P_2 - q_2 = P_1$. The set of units is equal to the set of elements of the form $(P, \vec{0})$ and hence it is naturally identified with the hull $\mathcal{Q}$. We have then $\mathsf{s}(P, q) = P$ and $\mathsf{r}(P, q) = P - q$. In other words, the element (P, q) of $\mathfrak{Q}$ corresponds to the change of the origin in P from $\vec{0}$ to q. For $P \in \mathcal{Q}$, the $\mathfrak{Q}$-orbit of P is the set of quasicrystals of the form $P - p$ for $p \in P$.

The natural topology on $\mathfrak{Q}$ is defined in the same way as for $\mathcal{Q}$. The distance from (P_1, q_1) to (P_2, q_2) is equal to 1 if $q_1 \neq q_2$, and to R^{-1}, where R is the supremum of radii r such that $B_r(\vec{0}) \cap P_1 = B_r(\vec{0}) \cap P_2$, if $q_1 = q_2$. It is easy to show that $\mathfrak{Q}$ is étale, Hausdorff, principal, and minimal.

Denote by $\mathcal{B}_R$ the set of sets of the form $B_R(q) \cap Q - q$.

Definition 6.9. We say that a permutation $\alpha : Q \longrightarrow Q$ is *defined by local rules* if there exist $R > 0$ and a map $W : \mathcal{B}_R \longrightarrow \mathbb{R}^n$ such that for every $q \in Q$ we have

$$\alpha(q) - q = W((B_R(q) \cap Q) - q).$$

In other words, α is defined by local rules if $\alpha(q) - q$ is uniquely determined by the R-neighborhood of q in Q.

The following is straightforward.

PROPOSITION 6.10. *The group $\mathsf{F}(Q)$ is isomorphic to the topological full group $\mathsf{F}(\mathfrak{Q})$. Moreover, the action of $\mathsf{F}(Q)$ coincides with the action of $\mathsf{F}(\mathfrak{Q})$ on the $\mathfrak{Q}$-orbit of $Q \in \mathcal{Q}$.*

Let us denote by $\mathsf{A}(Q)$ the group $\mathsf{A}(\mathfrak{Q})$. It can be defined in terms of Q as the subgroup of $\mathsf{F}(Q)$ generated by permutations α such that the orbits of $\langle\alpha\rangle$ are of lengths 1 and 3 only.

THEOREM 6.11. *Let Q be a quasicrystal. Then $\mathsf{A}(Q)$ is a simple finitely generated group.*

Proof. For $P \in \mathcal{Q}$, denote by $\Delta_R P$ the Rips complex of P, i.e., the simplicial complex with the set of vertices P in which a set is a simplex if and only if its diameter is less than or equal to R. It follows from the definition of a quasicrystal that there exists R such that $\Delta_R P$ is quasi-isometric to $\mathbb{R}^n$ and simply connected for every $P \in \mathcal{Q}$.

Denote by T_v, for $v \in \mathbb{R}^n$, the set of elements of $\mathfrak{Q}$ of the form (P, v) for $P \in \mathcal{Q}$. The set T_v is a (possibly empty) $\mathfrak{Q}$-bisection. Let $\mathcal{S}$ be the set of all non-empty bisections T_v for $\|v\| \leq R$. It is finite and satisfies the conditions of Theorem 6.6, since the complexes $\Delta_R P$ are simply connected for all $P \in \mathcal{Q}$ □

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7. *The quotient* $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$

The following proposition follows directly from the definition of the homology groups $H_0(\mathfrak{G}, A)$.

PROPOSITION 7.1. *The group* $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$ *is isomorphic to the abelian group given by the following presentation.*

- *Its generators* 1_U *are labeled by clopen subsets* $U \subset \mathfrak{G}^{(0)}$.
- *For every clopen set* U, *we have* $1_U + 1_U = 0$.
- *For every decomposition* $U = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n$ *of a clopen set into a disjoint union of clopen subsets, we have* $1_U = 1_{U_1} + 1_{U_2} + \cdots + 1_{U_n}$.
- *For every bisection* $F \subset \mathfrak{G}$, *we have* $1_{\mathsf{s}(F)} = 1_{\mathsf{r}(F)}$.

Denote $\tau_F = F \cup F^{-1} \cup (\mathfrak{G}^{(0)} \setminus (\mathsf{s}(F) \cup \mathsf{r}(F))$, where F is a bisection such that $\mathsf{s}(F) \cap \mathsf{r}(F) = \emptyset$.

THEOREM 7.2. *Suppose that every* $\mathfrak{G}$*-orbit has at least three elements. Then the correspondence* $1_{\mathsf{s}(F)} \mapsto \tau_F$ *induces a well-defined epimorphism from* $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$ *to* $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$.

Proof. Note that for every $x \in \mathfrak{G}^{(0)}$ and for every sufficiently small clopen neighborhood $U \subset \mathfrak{G}^{(0)}$ of x, there exists a bisection F such that $\mathsf{s}(F) = U$ and $\mathsf{r}(F) \cap \mathsf{s}(F) = \emptyset$. It follows that elements of the form $1_{\mathsf{s}(F)}$ for bisections $F \subset \mathfrak{G}$ such that $\mathsf{s}(F) \cap \mathsf{r}(F) = \emptyset$ generate $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$.

Let us denote by t_F the image of τ_F in $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$. We have to show that the generators t_F of $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ depend only on $\mathsf{s}(F)$, and satisfy the defining relations of $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$.

The generators t_F of $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ are obviously of order 2 and, if $F = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n$ is a decomposition of F into a disjoint union of bisections, then $t_F = t_{F_1} t_{F_2} \cdots t_{F_n}$.

LEMMA 7.3. *If bisections* F_1, F_2 *are such that* $\mathsf{s}(F_1) = \mathsf{s}(F_2)$, *and the sets* $\mathsf{s}(F_1)$, $\mathsf{r}(F_1)$, $\mathsf{r}(F_2)$, *and* $\mathsf{s}(F_1)$ *are pairwise disjoint, then* $t_{F_1} = t_{F_2}$.

Proof. The element $\tau_{F_1} \tau_{F_2}^{-1}$ belongs to $\mathsf{A}(\mathfrak{G})$. □

LEMMA 7.4. *The element* t_F *depends only on* $\mathsf{s}(F)$.

Proof. Suppose that bisections F_1 and F_2 are such that $\mathsf{s}(F_1) = \mathsf{s}(F_2)$ and $\mathsf{s}(F_1) \cap \mathsf{r}(F_1) = \mathsf{s}(F_2) \cap \mathsf{r}(F_2) = \emptyset$. Let $x \in \mathsf{s}(F_i)$. If $\mathsf{r}(F_1 x) \neq \mathsf{r}(F_2 x)$, then for all sufficiently small clopen neighborhoods U of x the bisections $F_1 U$ and $F_2 U$ will satisfy the conditions of Lemma 7.3 and hence $t_{F_1 U} = t_{F_2 U}$. Suppose that $\mathsf{r}(F_1 x) = \mathsf{r}(F_2 x)$. Then there exists y in the orbit of x different from x and from $\mathsf{r}(F_i x)$. Let H be a bisection such that $x \in \mathsf{s}(H)$ and $y \in \mathsf{r}(H)$. Then for all sufficiently small clopen neighborhoods U of x, by Lemma 7.3, we have $t_{HU} = t_{F_1 U}$ and $t_{HU} = t_{F_2 U}$, which implies that $t_{F_1 U} = t_{F_2 U}$.

We have proved that for every $x \in \mathsf{s}(F_i)$ and all sufficiently small clopen neighborhoods U of x, we have $t_{F_1 U} = t_{F_2 U}$. It follows that we can decompose $\mathsf{s}(F_i)$ into a finite disjoint

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union $\mathsf{s}(F_i) = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n$ of clopen sets such that $t_{F_1 U_i} = t_{F_2 U_i}$ for all i. But then $t_{F_1} = t_{F_1 U_1} t_{F_1 U_2} \cdots t_{F_1 U_n} = t_{F_2 U_1} t_{F_2 U_2} \cdots t_{F_2 U_n} = t_{F_2}$. □

It remains to show that the elements t_F commute. In view of Lemma 7.4, by a slight abuse of notation we will denote by t_U any element t_F such that $U = \mathsf{s}(F)$. The element t_U is defined for all sufficiently small clopen sets U.

Let $U_1, U_2 \subset \mathfrak{G}^{(0)}$ be clopen sets such that t_{U_1} and t_{U_2} are defined. Then $t_{U_1} = t_{U_1 \setminus U_2} t_{U_1 \cap U_2}$ and $t_{U_2} = t_{U_2 \setminus U_1} t_{U_1 \cap U_2}$. It follows that it is sufficient to prove that t_{U_1} and t_{U_2} commute if U_1 and U_2 are disjoint (since then we will conclude that $t_{U_1 \setminus U_2}$, $t_{U_2 \setminus U_1}$, and $t_{U_1 \cap U_2}$ pairwise commute).

Suppose that U_1 and U_2 are disjoint. Let $x_i \in U_i$. If x_1 and x_2 belong to one $\mathfrak{G}$-orbit, then there exists a bisection F such that $x_1 \in \mathsf{s}(F)$, $x_2 \in \mathsf{r}(F)$, and $\mathsf{s}(F) \cap \mathsf{r}(F) = \emptyset$. Then for any clopen sets $V_1 \subset \mathsf{s}(F)$ and $V_2 \subset \mathsf{r}(F)$, the elements t_{FV_1} and $t_{V_2 F}$ commute. But $t_{FV_1} = t_{V_1}$ and $t_{V_2 F} = t_{V_2}$ by Lemma 7.4. We have shown that there exist neighborhoods $W_1 = \mathsf{s}(F)$ and $W_2 = \mathsf{r}(F)$ of x_1 and x_2, respectively, such that for all clopen sets $V_i \subset W_i$ the elements t_{V_1} and t_{V_2} commute.

Suppose now that x_1 and x_2 belong to different $\mathfrak{G}$-orbits. Then there exist elements $\gamma_i \in \mathfrak{G}$ such that $\mathsf{s}(\gamma_i) = x_i$, and points x_1, x_2, $\mathsf{r}(\gamma_1)$, $\mathsf{r}(\gamma_2)$ are pairwise different. It follows that there exist bisections F_1, F_2 such that $\gamma_i \in F_i$, and the sets $\mathsf{s}(F_1)$, $\mathsf{r}(F_1)$, $\mathsf{s}(F_2)$, $\mathsf{r}(F_2)$ are pairwise disjoint. Then for any clopen subsets $V_i \in \mathsf{s}(F_i)$, the elements $\tau_{F_1 V_1}$ and $\tau_{F_2 V_2}$ commute in $\mathsf{S}(\mathfrak{G})$ and hence t_{V_1} and t_{V_2} commute in $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$. We have shown again that there exist neighborhoods $W_i = \mathsf{s}(F_i)$ of x_i such that for any clopen subsets $V_i \subset W_i$ the elements t_{V_1} and t_{V_2} commute.

It follows then from the compactness of U_1 and U_2 that we can decompose U_1 and U_2 into finite disjoint unions of clopen sets $U_1 = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ and $U_2 = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m$ such that t_{A_i} and t_{B_j} commute for all i and j. But this implies that $t_{U_1} = t_{A_1} t_{A_2} \cdots t_{A_n}$ and $t_{U_2} = t_{B_1} t_{B_2} \cdots t_{B_m}$ commute. □

It was shown by Matui (see [**19**, Theorems 7.5 and 7.13]) that for almost finite groupoids of germs the index map $I : \mathsf{F}(\mathfrak{G}) \longrightarrow H_1(\mathfrak{G}, \mathbb{Z})$ is surjective, and its kernel $\mathsf{K}(\mathfrak{G})$ is equal to $\mathsf{S}(\mathfrak{G})$. Recall that if $\mathfrak{G}$ is in addition minimal, then $\mathsf{A}(\mathfrak{G}) = \mathsf{D}(\mathfrak{G})$. For groupoids in this class we get from Theorem 7.2 the following description of the abelianization of the topological full group.

PROPOSITION 7.5. *Let $\mathfrak{G}$ be an almost finite minimal groupoid of germs. Then the abelianization $\mathsf{F}(\mathfrak{G})/\mathsf{D}(\mathfrak{G})$ is an extension of a quotient of $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$ by $H_1(\mathfrak{G}, \mathbb{Z})$, i.e., there exists an exact sequence*

$$H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathsf{F}(\mathfrak{G})/\mathsf{D}(\mathfrak{G}) \longrightarrow H_1(\mathfrak{G}, \mathbb{Z}) \longrightarrow 0.$$

If, additionally, $\mathfrak{G}$ is expansive, and the groups $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z})$ and $H_1(\mathfrak{G}, \mathbb{Z})$ are finitely generated, then $\mathsf{F}(\mathfrak{G})$ is finitely generated.

See more on abelianization of the full group in [**21**].

Example 7.6. The epimorphism $H_0(\mathfrak{G}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ is not necessarily one-to-one. Consider the groupoid $\mathfrak{G}$ with the space of units $\mathfrak{G}^{(0)}$ equal to the space $\{1, 2, 3\}^{\mathbb{N}}$ of

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infinite sequences $x_1x_2\ldots$ of elements of the alphabet $X=\{1,2,3\}$. Consider the inverse semigroup generated by the transformations $T_x: x_1x_2\ldots\mapsto xx_1x_2\ldots$, for $x\in X$, and the transformation a given by the recursive rules:

$$a(1x_2x_3\ldots)=2x_2x_3\ldots,\quad a(2x_2x_3\ldots)=1a(x_2x_3\ldots),\quad a(3x_2x_3\ldots)=3x_2x_3\ldots.$$

Let $\mathfrak{G}$ be the groupoid of germs of this semigroup.

The full group $\mathsf{F}(\mathfrak{G})$ consists of transformations of the following type. Let $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ be two sequences of finite words such that $X^{\mathbb{N}}$ is equal to the disjoint unions $v_1X^{\mathbb{N}}\cup v_2X^{\mathbb{N}}\cup\cdots\cup v_nX^{\mathbb{N}}$ and $u_1X^{\mathbb{N}}\cup u_2X^{\mathbb{N}}\cup\cdots\cup u_nX^{\mathbb{N}}$. Let $a^{m_1}, a^{m_2}, \ldots, a^{m_n}$ be arbitrary elements of the infinite cyclic group generated by a. Then the corresponding element of $\mathsf{F}(\mathfrak{G})$ is (the set of germs of) the homeomorphism $v_iw\mapsto u_ia^{m_i}(w)$. We will denote such an element by the table $\left(\begin{smallmatrix} v_1 & v_2 & \cdots & v_n\\ a^{m_1} & a^{m_2} & \cdots & a^{m_n}\\ u_1 & u_2 & \cdots & u_n\end{smallmatrix}\right)$. See more for the groups of this type in [**24**, **25**]. It is not hard to show that $H_0(\mathfrak{G},\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$, where the image of $1_{vX^{\mathcal{N}}}$ is the generator of $\mathbb{Z}/2\mathbb{Z}$ for every finite word v.

The element $\left(\begin{smallmatrix} 1 & 2 & 3\\ a & a^{-1} & 1\\ 2 & 1 & 3\end{smallmatrix}\right)$ belongs to $\mathsf{S}(\mathfrak{G})$, and its image in $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ is equal to the image of the generator of $H_0(\mathfrak{G},\mathbb{Z}/2\mathbb{Z})$. The same is true for the element $\left(\begin{smallmatrix} 1 & 2 & 3\\ 1 & 1 & 1\\ 2 & 1 & 3\end{smallmatrix}\right)$. It follows that $g=\left(\begin{smallmatrix} 1 & 2 & 3\\ a & a^{-1} & 1\\ 1 & 2 & 3\end{smallmatrix}\right)$ belongs to $\mathsf{A}(\mathfrak{G})$. The element g is conjugate in $\mathsf{S}(\mathfrak{G})$ to the element

$$h=\begin{pmatrix} 11 & 12 & 13 & 2 & 3\\ 1 & a & 1 & a^{-1} & 1\\ 11 & 12 & 13 & 2 & 3\end{pmatrix}.$$

According to the recursive definition of a, the element g can be represented by the table

$$g=\begin{pmatrix} 11 & 12 & 13 & 2 & 3\\ 1 & a & 1 & a^{-1} & 1\\ 12 & 11 & 13 & 2 & 3\end{pmatrix}.$$

It follows that

$$gh=\begin{pmatrix} 11 & 12 & 13 & 2 & 3\\ 1 & 1 & 1 & 1 & 1\\ 12 & 11 & 13 & 2 & 3\end{pmatrix}$$

belongs to $\mathsf{A}(\mathfrak{G})$. But its image in $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ must be equal to the image of the generator of $H_0(\mathfrak{G},\mathbb{Z}/2\mathbb{Z})$. It follows that $\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ is trivial.

Other examples of the cases when the epimorphism $H_0(\mathfrak{G},\mathbb{Z}/2\mathbb{Z})\longrightarrow\mathsf{S}(\mathfrak{G})/\mathsf{A}(\mathfrak{G})$ is not an isomorphism are described in [**21**].

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