IRREGULAR THREEFOLDS WHICH POSSESS ANTICANONICAL SYSTEMS

By L. ROTH

Received 2 February 1956

Introduction. The present paper is a sequel to a previous study (7) of the completely regular threefolds which possess anticanonical systems, i.e. for which the virtual canonical system, reversed in sign, is effective of positive order. On any such threefold the process of successive adjunction, applied to any linear system of surfaces, must terminate; we have thus to deal with a special case of the adjunction problem for the regular threefolds. By making certain simplifying hypotheses (such as irreducibility and absence of base elements) concerning the anticanonical systems, one can classify the threefolds in broad outline and show that, provided the anticanonical systems are sufficiently ample, the corresponding threefolds are either unirational or birational.

In this work we consider the irregular threefolds containing anticanonical systems, with regard to which we make the same assumptions as before. The discussion is based on certain properties of irregular surfaces and threefolds which are outlined in §§ 2 and 3. From these we deduce the following results:

If an irregular threefold V contains an irreducible anticanonical system $|A|$, ∞^2 at least, and free from base points, then the generic A is a Picard surface. V contains a rational congruence of elliptic curves, to which $|A|$ belongs, and an elliptic pencil of rational surfaces which are unisecant to the congruence. The geometric genus and plurigenera of V are all zero, and the arithmetic genus and superficial irregularity are equal to unity. If the elliptic pencil is free from fundamental surfaces, then V is an elliptic threefold of determinant unity.

1. *Anticanonical systems in general.* Let V_d be a non-singular algebraic variety of any dimension $d \geqslant 2$, and let $|K|$ denote its virtual impure canonical system; then, if the system $|-K|$ is effective and of positive order, we say that V_d possesses an *anticanonical system** and we denote the latter by $|A|$.

It follows at once that the geometric genus P_g and the plurigenera P_i of V_d are all zero. And since the adjoint C' of any hypersurface C is given by $C' \equiv C + K \equiv C - A$, it is clear that the adjunction process, applied to the system $|C|$, must terminate. Again, since $A' \equiv 0$, the hypersurface A, if irreducible, possesses an effective canonical system of order zero and is free from exceptional subvarieties†.

We denote by $a-1$ the freedom of the complete system $|A|$, and call a the *antigenus* of V_d; it is easily shown that this is a relative invariant of V_d. A study of the cases $d = 2, 3$ suggests that the antigenus is intimately connected with problems of unirationality and birationality.

* Such systems were first considered in the cases $d = 2, 3$ by Castelnuovo (1) and Fano (6) respectively. The term anticanonical is due to Severi.

† For if such subvarieties existed, A' would be effective of positive order.

https://doi.org/10.1017/S0305004100031674 Published online by Cambridge University Press

Consider first the case $d = 2$; here, according to a classical result of Castelnuovo and Enriques (3), V_2 must be rational or scrollar. Since the curve A is virtually elliptic, it follows that the only surfaces which can *a priori* carry anticanonical systems are either rational or elliptic scrollar; in the first case $a \leqslant 10$, the maximum value of a being attained on the plane; in the second, we may show (8) that, if the surface is *abstract*, i.e. with no reducible generators, then $a = 0$. Severi (13) has remarked that, for any elliptic scrollar surface, $a \leqslant 2$, but it is not known whether there actually exist any such surfaces with $a > 0$. In any case we may conclude that, for $a \geqslant 3$, the surface is rational.

The completely regular threefolds which possess anticanonical systems $|A|$ have been examined in (7), subject to the proviso that $|A|$ is irreducible, ∞^2 at least, and free from base points; in that case A is a regular surface of genera and plurigenera unity. It may be proved that, if $a > 7$, such threefolds are unirational; that, if $a > 15$, they are birational; and that, in any case, $a \leqslant 39$.

2. *Some properties of irregular surfaces.* The ensuing discussion depends essentially on certain properties of irregular surfaces which for convenience are assembled here; proofs of most of the results are given in (4).

To begin with, any surface which carries a pencil of genus $q\,(>0)$ of curves has irregularity q at least; and a surface of irregularity q which carries such a pencil cannot contain a second irrational pencil. Again, any surface which carries an involution of irregularity q has irregularity q at least.

Next, let F be a surface of irregularity $q = 1$; in this case its Picard variety (in the original sense of Castelnuovo*) is an elliptic curve, so that any such surface contains a (unique) elliptic pencil of curves. It follows from this that, on a threefold, any congruence of curves of irregularity 1 contains a unique elliptic pencil of surfaces.

Every irregular surface F with an effective canonical curve of order zero is Picardian ($p_g = 1, q = 2$). Any involution of irregularity 2 carried by F is itself Picardian, i.e. mapped by a Picard surface; and on any non-singular model of F such an involution is free from coincidences (5). If F has general moduli, it contains no irrational pencils of curves; if, however, it contains such a pencil, this is elliptic, consisting of birationally equivalent elliptic curves, and in that case there is a complementary elliptic pencil of birationally equivalent elliptic curves: we then say that F is *special*. A Picard involution carried by a Picard surface is special if and only if the surface is special.

Suppose that a surface F carries a pencil $\{\mathscr{C}\}$ of genus $\rho > 1$ of elliptic curves $\mathscr{C}$, such that the irreducible members of the pencil have the same modulus, while the reducible members (if any) are all multiple elliptic curves; in that case F is elliptic†, and contains a second irrational pencil, which is elliptic and which consists of birationally equivalent curves. The geometric genus of F is equal to ρ; and if there are no multiple elliptic curves on F a canonical curve $\mathscr{K}$ is given by the equivalence $\mathscr{K} \equiv (2\rho - 2)\mathscr{C}$, where the right-hand side represents a set of curves mapped by a canonical set of points on the curve image of $\{\mathscr{C}\}$.

* The use of a Picard variety to map continuous systems of a surface (or manifold) should properly be ascribed to Castelnuovo (2). Severi has proposed calling it the 'Picard variety of the first kind' associated with the given manifold. The Picard variety of the second kind, which is commonly attributed to Albanese, was introduced by Severi.

† There are, of course, elliptic surfaces having $\rho \leqslant 1$; but we are not concerned with them here.

https://doi.org/10.1017/S0305004100031674 Published online by Cambridge University Press

3. *On irregular threefolds.* We shall also require a number of concepts and results concerning threefolds, most of which are due to Severi; these are stated briefly in the present section—for detailed information we may refer to (10).

Let V be a threefold which, without loss of generality, we may assume to be non-singular (15). We denote by Ω_0, Ω_1, Ω_2 respectively the virtual grade, curvilinear genus and arithmetic genus of the virtual canonical system $|K|$; then the arithmetic genus P_a of V is given by Severi's relation (11)

$$2P_a = \Omega_0 - \Omega_1 + \Omega_2 + 4.$$

The number $P_g - P_a$ is called the *tridimensional irregularity* of V; this may be positive, negative or zero—in this last case V is said to be tridimensionally regular.

We next introduce the *superficial irregularity* $q\,(\geqslant 0)$ of V; this is the (constant) irregularity of a *generic* surface, i.e. any surface which can vary in an ∞^2 linear system with irreducible free characteristic curve. (If a surface is non-generic, the latter curve must belong to a congruence.) Evidently q is an absolute birational invariant of V. It may also be proved (11) that the number $P_g - P_a + q$ is likewise an absolute invariant whose value is non-negative. V is said to be *completely regular* if and only if $P_g = P_a$ and $q = 0$.

In the case where $P_g = 0$, $q > 0$, Severi (12) has further proved that V contains either a pencil of genus q of surfaces, or a congruence of irregularity q of curves; the pencil, being irrational, is of course free from base points, but the method of proof shows in addition that the congruence, when it exists, is free from base points and also from singular points, i.e. points through which pass ∞^1 curves of the system. We note that, in the former case, the pencil in question is the sole irrational pencil on V; for if there were a second such pencil, the irregularity of the generic surface would exceed q (see § 2).

4. *Irregular threefolds which possess anticanonical systems.* Suppose now that V is irregular (tridimensionally or superficially or both) and endowed with an anticanonical system $|A|$, ∞^2 at least, irreducible and free from base points. The generic surface A can *a priori* be either regular or Picardian (§ 2); in either case it is free from exceptional curves (§ 1). We first prove that A cannot be regular. For when A is regular its arithmetic genus is unity; also the virtual grade n and curvilinear genus p of $|A|$ must satisfy the relation $n = 2p - 2$. We now calculate the virtual characters of the system $|-A|$ in terms of p (cf. (10), Chapter I), thereby obtaining for the invariants Ω_0, Ω_1, Ω_2 the values $\Omega_0 = -2p+2$, $\Omega_1 = -3p+4$, $\Omega_2 = -p-2$. Inserting these in Severi's relation, we find that $P_a = 0$.

Now since $P_g = 0$ (§ 1), it follows that V has superficial irregularity $q > 0$. And since A is by hypothesis regular, the surface cannot be generic, which means that the characteristic curve of $|A|$ is compounded of a congruence Γ of curves (which, by reason of the relation $n = 2p - 2$, are elliptic). Moreover, since Γ may be considered as generated by a rational pencil of surfaces A, each of which contains a rational pencil of these curves, it follows that Γ is rational.

On the other hand, since $P_g = 0$, $q > 0$, V must contain either an irrational pencil of surfaces or an irregular congruence of curves (§ 3). Now, if V carried an irrational

https://doi.org/10.1017/S0305004100031674 Published online by Cambridge University Press

pencil, this could not belong to Γ, and so would cut an irrational pencil of curves on A. And if V carried an irregular congruence, necessarily distinct from Γ, it would cut an irregular involution on A. Thus neither possibility can be realized.*

5. *Anticanonical systems of Picard surfaces.* Supposing then that A is a Picard surface, there subsists as before the relation $n = 2p-2$ between the grade and curvilinear genus of $|A|$, but the arithmetic genus of A is now -1. Calculating the invariants Ω_i by means of the system $|-A|$ we find that $\Omega_0 = -2p+2$, $\Omega_1 = -3p+4$, $\Omega_2 = -p$, whence, by Severi's relation, $P_a = 1$. So once more we must have $q > 0$. There are now two possibilities.

First, suppose that the characteristic system of $|A|$ is simple; this implies that $|A|$ is ∞^3 at least. Since, by hypothesis, A is generic, we must have $q = 2$. And since the only irrational pencils that A can carry are elliptic (§ 2), it follows that V must contain a congruence Γ of curves $\mathscr{C}$, to which $|A|$ does not belong. Moreover, since the congruence has irregularity 2, it must be Picardian (§ 2).

We now prove that the curves $\mathscr{C}$ are rational and are bisecant to $|A|$. For suppose that $\mathscr{C}$ has genus π; and consider any surface C belonging to Γ, and therefore generated by a pencil, without base points, of curves $\mathscr{C}$. From the equivalence $A \equiv C - C'$, we have $[A\mathscr{C}] = -[C'\mathscr{C}] = -(2\pi - 2)$. And since the intersection number $[A\mathscr{C}]$ is positive, we conclude that $\pi = 0$, $[A\mathscr{C}] = 2$.

On the other hand, as stated in § 2, the involution cut by Γ on any non-singular surface A must be free from coincidences; hence no such surface can touch any curve $\mathscr{C}$. But since $|A|$ is bisecant to Γ, we can certainly find non-singular members of $|A|$ to touch the generic $\mathscr{C}$ at a general point. It follows that a system $|A|$ of the kind we have considered cannot be anticanonical.

6. *Case of a compound characteristic system.* Suppose, in the second place, that the characteristic system of $|A|$ is compounded of a congruence Γ, necessarily consisting of elliptic curves $\mathscr{C}$. Since by hypothesis $|A|$ is free from base points, so also is Γ. Evidently the generic A is a special Picard surface (§ 2), so that the pencil $\{\mathscr{C}\}$ carried by it is elliptic, consisting of birationally equivalent curves and free from reducible members. And since A contains no exceptional curves (§ 1), the pencil $\{\mathscr{C}\}$ has no fundamental curves. We may thus construct infinitely many surfaces in Γ, each containing a pencil of curves $\mathscr{C}$ without fundamental curves or reducible members; and all such surfaces are elliptic of the type described in § 2.

Let F denote a non-singular unexceptional birational model of Γ; then, to the pencil $\{\mathscr{C}\}$ on A, there corresponds an elliptic curve $\mathscr{C}_1$, say, on F. Now, as stated in § 2, on any (elliptic) surface containing a pencil of genus $\rho > 1$ of curves $\mathscr{C}$, a canonical curve consists of $2\rho - 2$ curves $\mathscr{C}$ corresponding to a canonical set on the image of the pencil on F. It follows from this that the process of successive adjunction applied to surfaces in Γ is paralleled by that of successive adjunction on F. So to the anticanonical system $|A|$ there corresponds an anticanonical system on F, evidently constituted by curves $\mathscr{C}_1$. And since there are at least ∞^2 curves $\mathscr{C}_1$, F must be rational (§ 1); whence Γ must be rational.

* This result, as well as the following, is readily deducible from the theory of differential forms (**14**).

https://doi.org/10.1017/S0305004100031674 Published online by Cambridge University Press

It now follows that the irrational pencil or irregular congruence (as the case may be) which V must carry cannot be contained in or coincident with Γ. The irrational pencil, if such exists, is necessarily elliptic, since it must cut an irrational pencil on A. If instead V contains an irregular congruence Γ', this must have irregularity $q \leqslant 2$, since A has irregularity 2.

If $q = 2$, Γ' must be Picardian (§ 2); and we can show as in § 5 that this case cannot arise. This can also be proved as follows. Since A is special, Γ' also is special (§ 2), containing an elliptic pencil of surfaces and hence a complementary elliptic pencil, both without base points. Now from the equivalence $A \equiv C - C'$, applied to the surfaces C of either of these pencils, we have $(AC) \equiv -(C'C)$; thus C possesses an anticanonical system of freedom $\geqslant 2$, and hence is rational. But this is impossible since the pencil in Γ' which is complementary to $\{C\}$ cuts an elliptic pencil of curves on each C.

We conclude that Γ', if it exists, has irregularity 1, and so contains an elliptic pencil of surfaces (§ 2). Thus in any case V must carry an elliptic pencil $\{C\}$, say, of surfaces C; and, as above, we see from the equivalence $A \equiv C - C'$ that C is rational. Evidently $\{C\}$ cuts on the generic A the pencil of elliptic curves complementary to $\{\mathscr{C}\}$.

We remark that the congruence Γ may contain rational curves or reducible curves all of whose components are rational. But since a rational curve cannot carry an elliptic involution, it follows that all such curves must lie on one or more surfaces which are fundamental for $\{C\}$.

Now every irreducible elliptic curve $\mathscr{C}$ of Γ cuts each irreducible surface C in the same number d ($\geqslant 1$) of points; and these are necessarily distinct. On the other hand, the congruence Γ cuts a rational involution of order d on the rational surface C; and, if $d > 1$, this must possess a coincidence curve. And since no irreducible member of Γ can touch C, such a coincidence curve must be the section by C of one or more surfaces B_s, say, each of which is generated by ∞^1 elliptic curves $\mathscr{C}_s$ such that $\mathscr{C}_s$, counted s ($\geqslant 2$) times, is a member of Γ; the number s may *a priori* be any divisor of d, including d itself.

The situation is similar to that which arises in the theory of the elliptic threefolds (9); and, as in that theory, we may prove that the virtual canonical system of V must contain each surface B_s as a fixed $(s-1)$-fold component. It follows that, in order that V may possess an anticanonical system of the kind postulated, it is necessary that the surfaces B_s should be absent. Hence Γ must be unisecant to $\{C\}$.

It is clear that, when this condition is satisfied, V is in fact endowed with an anticanonical system whose dimension equals that of the anticanonical system of the image F of Γ.

If there are no fundamental surfaces in $\{C\}$, every curve $\mathscr{C}$ is elliptic; in that case V can be mapped unexceptionally on the product of a rational surface F and an elliptic curve which is birationally equivalent to $\{C\}$. V is then a particular kind of elliptic threefold of determinant unity. If, however, fundamental surfaces are present, it seems likely that these are exceptional, i.e. birationally transformable into neighbourhoods of simple points or simple curves of some birational model of V.

Thus all the results enunciated in the Introduction have been established.

https://doi.org/10.1017/S0305004100031674 Published online by Cambridge University Press

REFERENCES

(1) CASTELNUOVO, G. *R. C. Accad. Lincei* (5), 6 (1897), 372 and 406.
(2) CASTELNUOVO, G. *R. C. Accad. Lincei* (5), 14 (1905), 545, 593 and 655.
(3) CASTELNUOVO, G. and ENRIQUES, F. *Ann. Mat. pura appl.* (3), 6 (1901), 165.
(4) ENRIQUES, F. *Le superficie algebriche* (Bologna, 1949).
(5) ENRIQUES, F. and SEVERI, F. *Acta Math., Stockh.*, 32 (1909), 283.
(6) FANO, G. *Atti Congr. Int.* 1928 (Bologna, 1931), 4, 115.
(7) ROTH, L. *Proc. Camb. Phil. Soc.* 48 (1952), 233.
(8) ROTH, L. *Atti IV Congr. U.M.I.* (Roma, 1953), 2, 434.
(9) ROTH, L. *R. C. Circ. Mat. Palermo* (2), 2 (1953), 141.
(10) ROTH, L. *Algebraic threefolds (with special regard to problems of rationality)* (Berlin, 1955).
(11) SEVERI, F. *R. C. Circ. Mat. Palermo*, 28 (1909), 33.
(12) SEVERI, F. *R. C. Accad. Lincei* (5), 20 (1911), 537.
(13) SEVERI, F. *R. C. Accad. Lincei* (8), 18 (1955), 131.
(14) SEVERI, F. *R. C. Accad. Lincei* (8), 20 (1956), 7.
(15) ZARISKI, O. *Ann. Math., Princeton* (2), 45 (1944), 472.

IMPERIAL COLLEGE
LONDON

https://doi.org/10.1017/S0305004100031674 Published online by Cambridge University Press