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SURVEY

The thermodynamic approach to multifractal analysis

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Abstract. Most results in multifractal analysis are obtained using either a thermodynamic approach based on the existence and uniqueness of equilibrium states or an orbit-gluing approach based on some version of the specification property. A general framework incorporating the most important multifractal spectra was introduced by Barreira and Saussol, who used the thermodynamic approach to establish the multifractal formalism in the uniformly hyperbolic setting, unifying many existing results. We extend this framework to apply to a broad class of non-uniformly hyperbolic systems, including examples with phase transitions, and obtain new results for a number of examples that have already been studied using the orbit-gluing approach. We compare the thermodynamic and orbit-gluing approaches and give a survey of many of the multifractal results in the literature.

1. *Introduction*

1.1. *Overview and basic concepts.*

Many important characteristics of dynamical systems are given as local asymptotic quantities, such as Birkhoff averages, Lyapunov exponents, local entropies, and pointwise dimensions, which reveal information about a single point or trajectory. It is of interest to study the level sets for these quantities: when this is done using measure theory, one is led into ergodic theory; when it is done using dimension theory, one is led into multifractal analysis.

The purpose of this paper is two-fold. First, we consider multifractals from the point of view set out in [**9**] and prove results generalizing those in [**13**]; in particular, we give purely thermodynamic conditions for the multifractal formalism to hold, allowing multifractal analysis to be carried out for a large class of systems by studying the topological pressure. Roughly speaking, one may say that the results in [**13**] unify the menagerie of multifractal spectra in the uniformly hyperbolic case, and the results of the present paper extend this unification to many non-uniformly hyperbolic systems.

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Second, we give an account of the relationship between this thermodynamic approach to multifractals and other approaches in the literature, particularly the orbit-gluing approach, with the aim of elucidating the similarities and distinctions between these works†.

The basic scheme of multifractal analysis is as follows. Given a dynamical system $f\colon X \to X$, we begin with some local asymptotic quantity $\phi_\infty(x) := \lim_{n\to\infty} \phi_n(x)$, where $\phi_n(x)$ is some sequence of functions and the limit may not exist everywhere. Typically each ϕ_n is continuous, but ϕ_∞ is not; rather, the level sets $K_\alpha := \{x \in X \mid \phi_\infty(x) = \alpha\}$ are often dense (when they are non-empty). This *multifractal decomposition* leads to a *multifractal spectrum* $\mathcal{F}(\alpha) := \dim K_\alpha$, where dim is a Carathéodory dimension characteristic [**97**] such as Hausdorff dimension or topological entropy.

Given the definition of $\mathcal{F}(\alpha)$, it is *a priori* rather surprising to find that $\mathcal{F}$ is in many cases a real analytic and concave function of α. This so-called *multifractal miracle* is closely related to the fact that $\mathcal{F}$ is often related to the topological pressure function $P\colon C(X) \to \mathbb{R}$ via a Legendre transform. Pressure is convex and is often analytic on certain subspaces of $C(X)$.

The multifractal spectrum $\mathcal{F}$ depends on both the local and global quantities, and so by varying these, we may consider many different spectra. We follow Barreira and Saussol [**13**] and consider spectra defined by functions $\varphi, \psi, u \in C(X)$. The local quantity is given by $S_n\varphi(x)/S_n\psi(x)$, where S_n denotes the nth Birkhoff sum, $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k(x))$. The global quantity $\dim_u$ was introduced in [**15**] and is defined in §2.1.

A survey of multifractal spectra is found in [**9**]. In many important cases, these spectra fit into the above framework via appropriate choices of φ, ψ, and u. We describe these briefly now; precise statements will be given in §§2.5–2.7.

(1) If $u \equiv 1$ and $\psi \equiv 1$, then the u-dimension reduces to the topological entropy, while the ratio level sets K_α become the level sets for Birkhoff averages of φ. Thus, $\mathcal{F}(\alpha) = \mathcal{B}_E(\alpha)$ is the entropy spectrum for Birkhoff averages.

(2) If μ is a Gibbs measure for φ and $u \equiv \psi \equiv 1$, then the Birkhoff averages of φ and the local entropies of μ have the same level sets, so $\mathcal{F}$ determines the entropy spectrum for local entropies $\mathcal{E}_E$.

(3) If f is conformal (Df is a scalar multiple of an isometry), μ is a Gibbs measure for ϕ, and $\psi = \log \|Df\|$, then writing $\varphi = P(\phi) - \phi$, the ratios $S_n\varphi/S_n\psi$ converge to the pointwise dimension of μ, so the ratio level sets K_α become the level sets for pointwise dimensions of μ. If, in addition, $u = \log \|Df\|$, then the u-dimension becomes the Hausdorff dimension and $\mathcal{F}(\alpha) = \mathcal{D}_D(\alpha)$ is the dimension spectrum for pointwise dimensions.

(4) If f is conformal, $\varphi = \log \|Df\|$, and $\psi \equiv 1$, then the ratio level sets become the level sets for Lyapunov exponents. If $u \equiv 1$, then $\mathcal{F}(\alpha)$ is the entropy spectrum for Lyapunov exponents $\mathcal{L}_E$; if $u = \log \|Df\|$, then it is the dimension spectrum for Lyapunov exponents $\mathcal{L}_D$.

When $u = \psi$, the spectrum $\mathcal{F}(\alpha)$ is the Legendre transform of a certain thermodynamic function. For $u = \varphi$, the spectrum itself is not a Legendre transform, but is closely related to one. For the more general case of *mixed multifractal spectra* where u differs from both

† In a previous version of this paper the orbit-gluing approach was referred to as the 'saturation approach', taking its name from the title of [**103**], which is representative of that approach.

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φ and ψ, Legendre transforms cannot be used directly, but we can still obtain the spectrum from thermodynamic considerations.

All this has been thoroughly discussed in [**13**]: let X be a compact metric space, $f: X \to X$ a continuous map, and φ, ψ, u continuous functions with $\psi, u > 0$. Denote by $\mathcal{M}^f(X)$ the space of f-invariant Borel probability measures on X. Suppose that the entropy map $h: \mathcal{M}^f(X) \to \mathbb{R}$ is upper semi-continuous and the pressure function $P: C(X) \to \mathbb{R}$ is differentiable on the span of $\{\varphi, \psi, u\}$; then it is shown in [**13**] that the multifractal formalism holds for $\mathcal{F} = \mathcal{F}_u^{\varphi,\psi}$. One goal of this paper is to show that these results (or suitably weakened versions) hold in a more general setting:

(1) the requirement that ψ and u be uniformly positive is replaced by weaker asymptotic positivity criteria, which in particular allow the possibility that $u = 0$ at a fixed point of f, see §4.2;

(2) the pressure function need not be differentiable on the entire subspace spanned by $\{\varphi, \psi, u\}$, but only on a subset thereof (with a concomitant weakening of the results), see Theorem B;

(3) even if the pressure function is not differentiable (there is a phase transition), thermodynamic considerations can still sometimes be used to guarantee that the formalism holds, see Corollary 2.5.

In addition to the spectra described in [**9**] and the mixed spectra studied in [**13**], many related objects are studied under the name 'multifractal analysis', including the dimension spectrum for pointwise dimensions (see §3.1), the irregular set (§3.5), various coarse spectra (§§2.5 and 3.6), and spectra for averages of multiple functions (§3.7).

1.2. *Description of approach.* The dimension spectrum for pointwise dimensions $\mathcal{D}_D$ was introduced in [**51**], where it is the Legendre transform of a function $T(q)$ related to the Hentschel–Procaccia spectrum:

$$\mathcal{D}_D(\alpha) = T^L(\alpha) := \inf_{q \in \mathbb{R}} (T(q) + q\alpha). \tag{1.1}$$

It follows from the general properties of Legendre transforms [**110**] that in this case $\mathcal{D}_D$ is concave. Furthermore, it is true in many cases that T is an analytic and strictly convex function of q, which implies that $\mathcal{D}_D$ is an analytic and strictly concave function of α.

A similar approach is used for more general spectra $\mathcal{F}(\alpha)$. A family of functions T_α is defined using the topological pressure; this family determines a 'predicted' multifractal spectrum. For 'non-mixed' spectra, this expected spectrum is the Legendre transform of T_0. For mixed spectra, the relationship is more subtle: details are given in §2.2. To keep our notation uniform, we denote the expected spectrum by $\mathcal{S}(\alpha)$ in both cases.

Now the multifractal formalism consists of showing that $\mathcal{F}(\alpha) = \mathcal{S}(\alpha)$. In the Barreira–Saussol setting, this means showing that $\dim_u K_\alpha = \mathcal{S}(\alpha)$. As often happens in dimension theory, the upper bound is easier to obtain than the lower bound. Indeed, the inequality $\mathcal{F}(\alpha) \le \mathcal{S}(\alpha)$ holds in almost complete generality; this is well known for a number of spectra [**91**, **123**, **127**], and we show in Theorem A that it holds for general spectra under minimal hypotheses.

Further hypotheses are necessary to obtain the lower bound $\mathcal{F}(\alpha) \ge \mathcal{S}(\alpha)$. On the interior of its domain (modulo a small technical condition), we will see in Theorem C that

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the predicted spectrum $\mathcal{S}(\alpha)$ can be written in terms of a conditional variational principle:

$$\mathcal{S}(\alpha) = \sup\left\{ \frac{h_\mu(f)}{\int u\, d\mu} \;\middle|\; \mu \in \mathcal{M}^f(X), \int (\varphi - \alpha\psi)\, d\mu = 0, \int u\, d\mu > 0 \right\}.$$

Given $\mu \in \mathcal{M}^f(X)$, we denote the set of μ-generic points by

$$G_\mu := \left\{ x \in X \;\middle|\; \frac{1}{n} S_n \phi(x) \to \int \phi\, d\mu \ \forall \phi \in C(X) \right\}. \tag{1.2}$$

If $\int (\varphi - \alpha\psi)\, d\mu = 0$, then $G_\mu \subset K_\alpha$, so $\dim_u G_\mu \le \dim_u K_\alpha$. Writing

$$\mathcal{M}^f_\alpha(X) := \left\{ \mu \in \mathcal{M}^f(X) \;\middle|\; \int (\varphi - \alpha\psi)\, d\mu = 0, \int u\, d\mu > 0 \right\}, \tag{1.3}$$

it follows that

$$\sup_{\mu \in \mathcal{M}^f_\alpha(X)} (\dim_u G_\mu) \le \mathcal{F}(\alpha) \le \sup_{\mu \in \mathcal{M}^f_\alpha(X)} \left(\frac{h_\mu(f)}{\int u\, d\mu} \right). \tag{1.4}$$

There are two rather different approaches to proving equality in (1.4): the *thermodynamic* approach and the *orbit-gluing* approach.

1.2.1. *The thermodynamic approach.* It was shown by Bowen [**19**] that $h_{\text{top}} G_\mu = h_\mu(f)$ when μ is ergodic, and it is not hard to derive as a corollary that if $\int u\, d\mu > 0$, then

$$\dim_u G_\mu = \frac{h_\mu(f)}{\int u\, d\mu}. \tag{1.5}$$

Thus, equality holds in (1.4) provided the second supremum does not change its value when we restrict our attention to ergodic measures.

An ergodic measure μ that achieves the second supremum in (1.4) is called a *full measure* for the spectrum $\mathcal{F}$ at the value α. In its simplest incarnation, the thermodynamic approach produces a full measure as an ergodic equilibrium state μ for a certain potential function ϕ. The fact that μ has the appropriate u-dimension and integrates $\varphi - \alpha\psi$ to zero is guaranteed by using Bowen's equation (for u-dimension) and Ruelle's formula (for the derivative of pressure), under the assumption that the pressure function satisfies appropriate differentiability criteria at ϕ. This is the approach followed in [**13**] and is also our primary strategy here.

It may happen that a full measure cannot be found as a unique equilibrium state; this is often the case when a phase transition occurs (see §4.2). Nevertheless, it is often still possible to find a family of ergodic measures whose u-dimensions approach the second supremum in (1.4). Two different approaches to this are given in Corollary 2.5 and in [**32**].

In addition to the equality between $\mathcal{F}(\alpha)$ and $\mathcal{S}(\alpha)$, the thermodynamic approach also establishes the stronger conditional variational principle

$$\dim_u K_\alpha = \sup\{\dim_u \mu \mid \mu \in \mathcal{M}^f_E(K_\alpha)\}, \tag{1.6}$$

where $\mathcal{M}^f_E := \{\mu \in \mathcal{M}^f \mid \mu \text{ is ergodic}\}$.

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1.2.2. *The orbit-gluing approach.* One can also approach (1.4) by extending (1.5) to non-ergodic measures. In general, G_μ need not even be non-empty when μ is not ergodic, and so extending (1.5) requires further hypotheses on the map f. As always, the difficult step is proving the lower bound $\dim_u G_\mu \geq h_\mu(f)/\int u\, d\mu$; the other bound is easy.

Most applications of the orbit-gluing approach assume that f satisfies some version of the specification property and use this to show that every invariant measure has $h_{\text{top}} G_\mu = h_\mu(f)$ (see [**103**, **117**]). The lower bound $h_{\text{top}} G_\mu \geq h_\mu(f)$ is obtained by gluing together orbit segments that 'are almost in G_μ' to build a measure ν supported on G_μ such that $h_{\text{top}}\nu := \inf\{h_{\text{top}} Z \mid \nu(Z) = 1\} = h_\mu(f)$. Because G_μ does not support any invariant measures, the measure ν cannot be obtained via the usual thermodynamic methods, which only yield invariant measures as equilibrium states.

1.2.3. *Comparison of approaches and open problems.* The thermodynamic approach does not require any knowledge of the system beyond its thermodynamic properties, thus one need not assume a version of specification. This approach establishes the strengthened conditional variational principle equation (1.6), which the orbit-gluing approach does not prove, and which does not follow from equality in (1.4). This suggests the following problem.

Problem 1.1. Does there exist a dynamical system $f : X \to X$ and functions φ, ψ, u such that the associated multifractal spectrum $\mathcal{F}(\alpha)$ satisfies the multifractal formalism, but the conditional variational principle equation (1.6) fails and there exists α such that

$$\sup\{\dim_u \mu \mid \mu \in \mathcal{M}_E^f(K_\alpha)\} < \dim_u K_\alpha = \sup\{\dim_u \mu \mid \mu \in \mathcal{M}_\alpha^f(X)\}?$$

On the other hand, the thermodynamic approach requires careful application in the presence of phase transitions (points where pressure is non-differentiable), whereas the orbit-gluing approach works without modification. Indeed, in the case $u = 1$ (where $\dim_u = h_{\text{top}}$), any system with

$$h_{\text{top}}(G_\mu) = h_\mu(f) \quad \text{for all } \mu \in \mathcal{M}^f(X) \tag{1.7}$$

satisfies the multifractal formalism for every pair of potentials φ, $\psi \in C(X)$ with $\psi > 0$.

Problem 1.2. Is (1.7) a necessary property for the full multifractal formalism to hold? That is, suppose there is some $\mu \in \mathcal{M}^f(X)$ such that $h_{\text{top}}(G_\mu) < h_\mu(f)$; is it necessarily the case that there exist φ, ψ, u for which the formalism does not hold?

Finally, the orbit-gluing approach can be used to establish detailed multifractal results about the irregular set (see §3.5), which cannot be studied using the usual thermodynamic approach, as it does not support any invariant measures.

1.3. *Outline of the paper.* In §2.1, we define the notions of topological pressure and u-dimension. Section 2.2 introduces the multifractal spectrum $\mathcal{F}(\alpha)$, the predicted spectrum $\mathcal{S}(\alpha)$, and gives our main results relating the two. Section 2.3 shows that the predicted spectrum $\mathcal{S}(\alpha)$ can also be given in terms of a conditional variational principle. Sections 2.4–2.7 examine various spectra that can be obtained as specific instances of the main results.

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Section 3 surveys some of the history of multifractal analysis and describes various results in the literature from both the thermodynamic and orbit-gluing approaches. Section 4 gives applications of our main results to various classes of systems, including some non-uniformly hyperbolic ones. Section 5 contains all of the proofs, and §6 contains background material on Carathéodory dimension characteristics from [**97**].

2. *Definitions and results*

2.1. *Definition of pressure and u-dimension.*

Definition 2.1. Let X be a compact metric space and $f\colon X \to X$ be continuous. Fix $\phi \in C(X)$ and $Z \subset X$; denote by $S_n\phi(x)$ the Birkhoff sum $\phi(x) + \phi(f(x)) + \cdots + \phi(f^{n-1}(x))$. For a given $\delta > 0$ and $N \in \mathbb{N}$, let $\mathcal{P}(Z, N, \delta)$ be the collection of countable sets $\{(x_i, n_i)\} \subset Z \times \{N, N+1, \dots\}$ such that $Z \subset \bigcup_i B(x_i, n_i, \delta)$. For each $s \in \mathbb{R}$, consider the set functions

$$\begin{aligned} m_P(Z, s, \phi, N, \delta) &= \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} \exp(-n_i s + S_{n_i}\phi(x_i)), \\ m_P(Z, s, \phi, \delta) &= \lim_{N\to\infty} m_P(Z, s, \phi, N, \delta). \end{aligned} \tag{2.1}$$

This function is non-increasing in s, and takes values ∞ and 0 at all but at most one value of s. Denoting the critical value of s by

$$\begin{aligned} P_Z(\phi, \delta) &= \inf\{s \in \mathbb{R} \mid m_P(Z, s, \phi, \delta) = 0\} \\ &= \sup\{s \in \mathbb{R} \mid m_P(Z, s, \phi, \delta) = \infty\}, \end{aligned}$$

we get $m_P(Z, s, \phi, \delta) = \infty$ when $s < P_Z(\phi, \delta)$, and 0 when $s > P_Z(\phi, \delta)$.

The *topological pressure* of ϕ on Z is $P_Z(\phi) = \lim_{\delta\to 0} P_Z(\phi, \delta)$; the limit exists because given $\delta_1 < \delta_2$, we have $\mathcal{P}(Z, N, \delta_1) \subset \mathcal{P}(Z, N, \delta_2)$, and hence $m_P(Z, s, \phi, \delta_1) \geq m_P(Z, s, \phi, \delta_2)$, so $P_Z(\phi, \delta_1) \geq P_Z(\phi, \delta_2)$.

In the particular case $\phi = 0$, this definition yields the topological entropy $h_{\text{top}}(Z) = P_Z(0)$, as defined by Bowen for non-compact sets [**19**].

Remark. This definition differs from that given by Pesin and Pitskel' [**95**] (see also [**97**]), but yields the same value [**31**, Proposition 5.2].

When $Z = X$, we can also compute the pressure as follows. For each $n \in \mathbb{N}$ and $\delta > 0$, let

$$\mathcal{Z}_n(\phi, \delta) = \inf\left\{ \sum_{x\in E} \exp\left(\sup_{y\in B(x,n,\delta)} S_n\phi(y) \right) \,\middle|\, B(E, n, \delta) = X \right\},$$

where we write $B(E, n, \delta) := \bigcup_{x\in E} B(x, n, \delta)$; then the pressure is given by

$$P(\phi) = \lim_{\delta\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(\phi, \delta).$$

We always assume that $h_{\text{top}}(X) < \infty$, since otherwise the pressure function is infinite everywhere and does not contain the information we need.

We will consider multifractal spectra described by the following dimensional quantity, which was introduced by Barreira and Schmeling in [**15**].

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Definition 2.2. With X, f as above, fix a continuous function $u\colon X \to \mathbb{R}$ satisfying the following condition†.

(P) For all $\delta > 0$ there are covers $E_N \in \mathcal{P}(X, N, \delta)$ with $\inf_{(x,n)\in E_N} S_n u(x) \to \infty$ as $N \to \infty$.

Given a set $Z \subset X$, consider the set functions

$$\begin{aligned} m_u(Z, s, N, \delta) &= \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} e^{-sS_{n_i}u(x)}, \\ m_u(Z, s, \delta) &= \lim_{N\to\infty} m_u(Z, s, N, \delta). \end{aligned} \tag{2.2}$$

Once again, this function is non-increasing in s, and takes values ∞ and 0 at all but at most one value of s. Denoting the critical value of s by

$$\dim_u(Z, \delta) = \inf\{s \in \mathbb{R} \mid m_u(Z, s, \delta) = 0\},$$

we get $m_u(Z, s, \delta) = \infty$ when $s < \dim_u(Z, \delta)$, and 0 when $s > \dim_u(Z, \delta)$.

The *u-dimension* of Z is $\dim_u Z = \lim_{\delta\to 0} \dim_u(Z, \delta)$; the limit exists for the same reason as in the previous definition. In the particular case $u = 1$, this definition once again yields the topological entropy $h_{\text{top}}(Z)$.

The function u is allowed to take negative values, but **(P)** guarantees some statistical non-negativity, as the following proposition shows.

PROPOSITION 2.1. *If $u \in C(X)$ satisfies **(P)**, then $\int u\, d\mu \geq 0$ for every $\mu \in \mathcal{M}^f(X)$.*

Topological pressure and u-dimension are both examples of Carathéodory dimension characteristics (see §6). Thus, they both satisfy *countable stability*: for any countable collection of arbitrary sets $Z_i \subset X$,

$$P_{\bigcup_i Z_i}(\phi) = \sup_i P_{Z_i}(\phi), \tag{2.3}$$

$$\dim_u\left(\bigcup_i Z_i\right) = \sup_i \dim_u(Z_i). \tag{2.4}$$

An important tool for computing u-dimension is Bowen's equation.

PROPOSITION 2.2. *With X, f as above, let $u \in C(X)$ satisfy **(P)**. Suppose that $Z \subset X$ has the property that $\underline{\lim}_{n\to\infty}(1/n)S_n u(x) > 0$ for all $x \in Z$. Then*

$$\dim_u Z = \inf\{t \in \mathbb{R} \mid P_Z(-tu) \leq 0\}. \tag{2.5}$$

The most effective way to obtain lower bounds on $\dim_u Z$ is to study measures supported on Z.

Definition 2.3. The u-dimension of a probability measure $\mu \in \mathcal{M}(X)$ is

$$\dim_u \mu := \inf\{\dim_u Z \mid \mu(Z) = 1\}. \tag{2.6}$$

† In [**15**] it is assumed that $u > 0$, which implies **(P)**. However, there are examples where u is not strictly positive but **(P)** is still satisfied (see §4.2). We see from Proposition 2.1 that **(P)** implies some non-negativity properties for u.

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PROPOSITION 2.3. *Let u satisfy **(P)** and let $\mu \in \mathcal{M}_E^f(X)$ have $\int u\, d\mu > 0$. Then the u-dimension of μ is given by*

$$\dim_u \mu = \frac{h_\mu(f)}{\int_X u\, d\mu}. \tag{2.7}$$

Remark. Note that (2.7) fails for non-ergodic measures; indeed, the functions $\mu \mapsto h_\mu(f)$ and $\mu \mapsto \int u\, d\mu$ are both affine, while it is an easy consequence of equation (2.6) that $\dim_u(t\mu_1 + (1-t)\mu_2) = \max(\dim_u \mu_1, \dim_u \mu_2)$ for $0 < t < 1$.

We emphasize that by the appropriate choice of the function u, the u-dimension $\dim_u$ can be made to equal either the topological entropy (for any map) or the Hausdorff dimension (for conformal maps), and so all of our results involving $\dim_u$ apply to both of these quantities.

2.2. *Mixed multifractal spectra and main results.* We now fix functions $\varphi, \psi, u \in C(X)$ such that u satisfies **(P)**. Given $\alpha \in \mathbb{R}$, consider the level set

$$K_\alpha = K_\alpha(\varphi, \psi) = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\psi(x)} = \alpha \right\}. \tag{2.8}$$

The *multifractal spectrum* associated to the triple (φ, ψ, u) is the function $\mathcal{F}\colon \mathbb{R} \to \mathbb{R}$ given by

$$\mathcal{F}(\alpha) = \mathcal{F}_u^{\varphi,\psi}(\alpha) := \dim_u K_\alpha(\varphi, \psi),$$

where we adopt the convention that $\dim_u \emptyset = -\infty$, and so $\mathcal{F}(\alpha) = -\infty$ for those values of α corresponding to empty level sets.

Consider the set of points

$$\hat{X} := \left\{ x \in X \;\middle|\; \lim_{n\to\infty} S_n\psi(x) > 0 \text{ and } \lim_{n\to\infty} \frac{1}{n} \log S_n u(x) > 0 \right\}, \tag{2.9}$$

and the level sets $\hat{K}_\alpha := K_\alpha \cap \hat{X}$. We will obtain estimates on the spectrum

$$\hat{\mathcal{F}}(\alpha) = \dim_u \hat{K}_\alpha(\varphi, \psi).$$

In our applications, we will find that $\hat{\mathcal{F}} = \mathcal{F}$ on the domain of interest, and thus our estimates apply directly to the original spectrum.

For every $\alpha \in \mathbb{R}$, define a function $R_\alpha\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$R_\alpha(q, t) := P(q(\varphi - \alpha\psi) - tu), \tag{2.10}$$

and define $T_\alpha\colon \mathbb{R} \to \mathbb{R}$ implicitly by

$$T_\alpha(q) := \inf\{t \in \mathbb{R} \mid R_\alpha(q, t) \le 0\}. \tag{2.11}$$

The *thermodynamically predicted spectrum* $\mathcal{S}\colon \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{S}(\alpha) := \inf\{T_\alpha(q) \mid q \in \mathbb{R}\}. \tag{2.12}$$

The predicted domain of the spectrum is

$$I := \{\alpha \in \mathbb{R} \mid \mathcal{S}(\alpha) > -\infty\}. \tag{2.13}$$

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PROPOSITION 2.4. *We have $\mathcal{S}(\alpha) \geq 0$ for every $\alpha \in I$.*

In §2.3, we will relate the predicted spectrum to a conditional variational principle, which is the form that many results in the literature take. In §2.4, we will examine the particular case when $u = \psi$, which covers many of the most important spectra, and where $\mathcal{S}$ can be obtained as a Legendre transform.

Our first main result holds in great generality, and establishes one half of the multifractal formalism.

THEOREM A. *Fix φ, ψ, $u \in C(X)$ such that u satisfies **(P)**. Then*

$$\hat{\mathcal{F}}(\alpha) \leq \mathcal{S}(\alpha) \tag{2.14}$$

for all $\alpha \in \mathbb{R}$. In particular, $\hat{K}_\alpha = \emptyset$ for all $\alpha \notin I$.

Once again, we emphasize that in all our examples we have $\hat{\mathcal{F}} = \mathcal{F}$, so that Theorem A applies directly to the spectrum of interest. We also point out that (2.14) holds for $\alpha \in \delta I$, but not for $\alpha = \infty$.

Remark. The infimum in (2.12) may not be achieved by any finite value of q; Theorem C gives conditions under which it is achieved. Even if it is achieved, however, the minimizing value of q may not be unique.

A measure ν is a *full measure* for the spectrum $\hat{\mathcal{F}}$ at α if ν is ergodic and invariant, $\nu(\hat{K}_\alpha) = 1$, and $\dim_u \nu = \mathcal{S}(\alpha)$.

THEOREM B. *Let φ, ψ, u be as in Theorem A. Fix $\alpha \in \mathbb{R}$, and suppose that $q = q(\alpha) \in \mathbb{R}$ realizes the infimum in (2.12), so that $\mathcal{S}(\alpha) = T_\alpha(q)$. Suppose also that:*

(1) *the potential $q(\alpha)(\varphi - \alpha\psi) - \mathcal{S}(\alpha)u$ has an equilibrium state ν_α such that $\int \psi \, d\nu_\alpha > 0$ and $\int u \, d\nu_\alpha > 0$; and*

(2) *the function R_α is differentiable at $(q(\alpha), \mathcal{S}(\alpha))$.*

Then ν_α is a full measure for $\hat{\mathcal{F}}$ at α; in particular, $\hat{\mathcal{F}}(\alpha) = \mathcal{S}(\alpha)$.

Note that if $\hat{\mathcal{F}}(\alpha) = \mathcal{F}(\alpha)$ and Theorem B applies, then ν_α is full for $\mathcal{F}$ as well, and $\mathcal{F}(\alpha) = \mathcal{S}(\alpha)$, so the multifractal formalism holds at α. This extends results of Barreira and Saussol [**13**], which apply when the entropy map is upper semi-continuous, ψ and u are uniformly positive, and R_α is differentiable for all values of α, q, t.

In §4.2, we consider systems and spectra for which certain values of α do not correspond to points of differentiability of P, but the formalism still holds and $\mathcal{F} \equiv \mathcal{S}$. For such systems†, we use the following easy corollary of Theorem B.

COROLLARY 2.5. *Let φ, ψ, u be as in Theorem A. Fix $\alpha \in \mathbb{R}$, and suppose that there exist compact invariant sets $X_n \subset X$ such that the hypotheses of Theorem B hold at α for the restriction of the system to X_n. Then equality holds in (2.16); in particular, $\hat{\mathcal{F}}(\alpha) = \mathcal{S}(\alpha)$.*

† An alternate approach to this situation, under the hypothesis that pressure is differentiable on a dense vector subspace of $C(X)$, is given in [**32**], using ideas of Hofbauer [**54**, **55**].

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2.3. *Conditional variational principle.* As mentioned earlier, the predicted spectrum can be given in terms of a conditional variational principle for nearly every value of α. Given $\alpha \in \mathbb{R}$, let

$$\mathcal{K}(\alpha) := \left\{ \int (\varphi - \alpha\psi)\, d\mu \;\middle|\; \mu \in \mathcal{M}^f(X) \right\},$$

and let $\hat{I} := \{\alpha \mid 0 \in \operatorname{int} \mathcal{K}(\alpha)\}$. Let $\mathcal{M}^f_\alpha(X)$ be as in (1.3).

THEOREM C. *Let φ, ψ, u be as in Theorem A. Then for every $\alpha \in \hat{I}$, the infimum in equation (2.12) is achieved at a finite value of q, and the predicted spectrum $\mathcal{S}(\alpha)$ satisfies*

$$\mathcal{S}(\alpha) = \sup_{\mu \in \mathcal{M}^f_\alpha(X)} \left(\frac{h_\mu(f)}{\int u\, d\mu} \right). \tag{2.15}$$

Remark. The equivalence between the two characterizations of $\mathcal{S}(\alpha)$ is well known in many cases where the formalism holds (see [**13**], for example). Theorem C establishes this even when the formalism does not hold.

PROPOSITION 2.6. *If there exists $\gamma > 0$ such that $\int \psi\, d\mu \geq \gamma h_\mu(f)$ for all $\mu \in \mathcal{M}^f(X)$, then $\hat{I} = \operatorname{int} I$.*

Remark. The hypothesis of Proposition 2.6 holds if ψ is uniformly positive and $h_{\text{top}}(f) < \infty$; thus, in this case Theorem C applies to all $\alpha \in \operatorname{int} I$.

The measures in the conditional variational principle equation (2.15) need not be supported on K_α unless they are ergodic. Thus, in general, we have the following extended version of (2.14):

$$\sup\{\dim_u \mu \mid \mu \in \mathcal{M}^f_E(K_\alpha)\} \leq \hat{\mathcal{F}}(\alpha) \leq \mathcal{S}(\alpha). \tag{2.16}$$

It is of interest to know when all three quantities in (2.16) are equal. By Theorem B, this is the case when R_α is differentiable at $(q(\alpha), \mathcal{S}(\alpha))$, provided an equilibrium state ν exists with $\int \psi\, d\nu > 0$ and $\int u\, d\nu > 0$; furthermore, in this case the supremum is achieved. Corollary 2.5 gives equality in (2.16) under weaker conditions, but in this case the supremum may not be achieved.

2.4. *Concavity when $u = \psi$.* There are several important examples of multifractal spectra for which $u = \psi$; these are the *non-mixed* spectra, which have been more thoroughly studied. For these spectra, $\mathcal{S}$ takes a simpler form and Theorem A can be strengthened.

With $\psi = u$, we have $R_\alpha(q, t) = P(q\varphi - (\alpha q + t)u)$, and so T_α is given by

$$T_\alpha(q) = \inf\{t \in \mathbb{R} \mid P(q\varphi - (\alpha q + t)u) \leq 0\}.$$

Observe that $T_0 \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$T_0(q) = \inf\{t \in \mathbb{R} \mid P(q\varphi - tu) \leq 0\}, \tag{2.17}$$

and we have $T_\alpha(q) = T_0(q) - \alpha q$. In particular,

$$\mathcal{S}(\alpha) = \inf\{T_0(q) - \alpha q \mid q \in \mathbb{R}\} \tag{2.18}$$

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is the Legendre transform of T_0. Recall that the relationship of T_0 and $\mathcal{S}$ as a Legendre transform pair can be seen geometrically as follows: for every α, the value of $\mathcal{S}(\alpha)$ is the y-intercept of the subtangent line to the graph of T_0 with slope α. (This line is unique by convexity, although it need not intersect the graph at a unique point.)

PROPOSITION 2.7. *Let* $Q := \{q \in \mathbb{R} \mid T_0(q) < \infty\}$. *Then one of the following three cases holds:*

(1) $Q = \mathbb{R}$;
(2) $Q = (q_0, \infty)$ *or* $[q_0, \infty)$ *for some* $q_0 \geq 0$;
(3) $Q = (-\infty, q_0)$ *or* $(-\infty, q_0]$ *for some* $q_0 \leq 0$.

By the convexity of T_0, the following limits exist for every $q \in \operatorname{int} Q$:

$$D^- T_0(q) := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon}(T_0(q) - T_0(q - \epsilon)),$$
$$D^+ T_0(q) := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon}(T_0(q + \epsilon) - T_0(q)).$$

Writing $J(q) := [D^- T_0(q), D^+ T_0(q)]$, we see that $\operatorname{int} J(q) = \emptyset$ if and only if T_0 is differentiable at q. Furthermore, because the sets $\operatorname{int} J(q)$ are disjoint for different values of q (a consequence of convexity), $\operatorname{int} J(q)$ is non-empty for at most countably many values of q. Observe that $\mathcal{S}$ is an affine function of α on each interval $J(q)$, and conversely, $\mathcal{S}$ is strictly concave on every interval that is disjoint from every $\operatorname{int} J(q)$.

For every q, the graph of T_0 admits subtangents at q with every slope in the interval $J(q)$. Denote by $\tilde{I}$ the set of values of α corresponding to subtangents where T_0 is stably finite: $\tilde{I} := \bigcup_{q \in \operatorname{int} Q} J(q)$. Note that if Q is open, then $\tilde{I} = I$. However, when $Q = [q_0, \infty)$ or $Q = (-\infty, q_0]$, this need not be the case.

THEOREM D. *Let* φ, ψ, u *be as in Theorem A, and suppose that* $u = \psi$. *Then* $\mathcal{S}|_{\tilde{I}}$ *is the concave hull of* $\hat{\mathcal{F}}|_{\tilde{I}}$:

$$\mathcal{S}(\alpha) = \inf\{\tilde{\mathcal{S}}(\alpha) \mid \tilde{\mathcal{S}} \geq \hat{\mathcal{F}} \text{ is concave on } \tilde{I}\} \quad \text{for all } \alpha \in \tilde{I}.$$

Remark. When $u \neq \psi$, the spectrum $\mathcal{S}$ is not necessarily concave; examples where concavity fails are given in [**13**, **60**].

For the equivalence between the Legendre transform description of $\mathcal{S}$ and the conditional variational principle, we can often apply Proposition 2.6 to show that Theorem C applies on all of $\operatorname{int} I$.

PROPOSITION 2.8. *Let* X *be a compact metric space, let* $f : X \to X$ *be continuous, and fix* φ, ψ, $u \in C(X)$ *such that* $\psi \equiv u$ *satisfies* ***(P)***. *Suppose that every ergodic measure* μ *with positive entropy has* $\int u \, d\mu > 0$, *and suppose further that* $\dim_u X < \infty$. *Then we have* $\int u \, d\mu \geq h_\mu(f)/\dim_u X$ *for all* $\mu \in \mathcal{M}^f(X)$, *whence Proposition 2.6 and Theorem C give (2.15) for all* $\alpha \in \operatorname{int} I$.

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2.5. *Birkhoff spectrum and coarse spectra.* We temporarily restrict our attention to the case where $\psi = u \equiv 1$. In this case the predicted spectrum takes on a particularly simple form: indeed, we see that

$$T_0(q) = \inf\{t \in \mathbb{R} \mid P(q\varphi - t) \le 0\} = \inf\{t \in \mathbb{R} \mid P(q\varphi) - t \le 0\} = P(q\varphi),$$

and so $\mathcal{S}$ is the Legendre transform of the convex function $q \mapsto P(q\varphi)$.

For $\psi = u \equiv 1$, the level sets are

$$K_\alpha^\varphi = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\} \tag{2.19}$$

and the multifractal spectrum associated to (φ, ψ, u) is the entropy spectrum for Birkhoff averages

$$\mathcal{F}(\alpha) = \mathcal{B}_E(\alpha) = h_{\text{top}}(K_\alpha^\varphi).$$

We will refer to this as the *fine spectrum* to distinguish it from the *coarse spectrum*, which we now define.

Given a neighbourhood $U \subset \mathbb{R}$ of α, consider the approximate level set

$$G_n(U) := \left\{ x \in X \;\middle|\; \frac{1}{n} S_n\varphi(x) \in U \right\}, \tag{2.20}$$

and let

$$\Lambda_n^\delta(U) := \min\{\#E \mid B(E, n, \delta) \supset G_n(U)\}, \tag{2.21}$$

where $B(E, n, \delta) := \bigcup_{x\in E} B(x, n, \delta)$. The coarse multifractal spectra are the functions $\underline{\mathcal{F}}, \overline{\mathcal{F}} \colon \mathbb{R} \to \mathbb{R}$ given by

$$\underline{\mathcal{F}}(\alpha) := \inf_{U\ni\alpha} \lim_{\delta\to 0} \underline{\lim_{n\to\infty}} \frac{1}{n} \log \Lambda_n^\delta(U),$$

$$\overline{\mathcal{F}}(\alpha) := \inf_{U\ni\alpha} \lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \Lambda_n^\delta(U).$$

The following result is a consequence of [**32**, Theorem B] and strengthens Theorem A in this setting.

THEOREM E. *Let $\varphi \in C(X)$. Then for all α,*

$$\mathcal{F}(\alpha) \le \underline{\mathcal{F}}(\alpha) \le \overline{\mathcal{F}}(\alpha) \le \mathcal{S}(\alpha). \tag{2.22}$$

For convenience and concreteness, we state a corollary that collects our main results in the setting where $\psi = u = 1$ and the entropy map is upper semi-continuous (which implies existence of equilibrium states).

COROLLARY 2.9. (Entropy spectrum for Birkhoff averages) *Let X be a compact metric space, $f \colon X \to X$ be continuous, and $\varphi \in C(X)$. Suppose that the entropy map $\mathcal{M}^f(X) \to \mathbb{R}$ is upper semi-continuous. Define $T_0(q) = P(q\varphi)$ and let $J(q) = [D^- T_0(q), D^+ T_0(q)]$. Then the predicted spectrum is $\mathcal{S}(\alpha) = \inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\}$ and the predicted domain $I = \{\alpha \mid \mathcal{S}(\alpha) \ge 0\}$ is given by*

$$I = \Big[\lim_{q\to-\infty} D^- T_0(q), \lim_{q\to\infty} D^+ T_0(q) \Big] = \left\{ \int \varphi \, d\mu \;\middle|\; \mu \in \mathcal{M}^f(X) \right\}. \tag{2.23}$$

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For every $\alpha \notin I$ we have $K_\alpha^\varphi = \emptyset$, while for every $\alpha \in \operatorname{int} I \setminus \bigcup_{q\in\mathbb{R}} \operatorname{int} J(q)$, we have

$$\begin{aligned} h_{\mathrm{top}}(K_\alpha^\varphi) &= \underline{\mathcal{F}}(\alpha) = \overline{\mathcal{F}}(\alpha) \\ &= \inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\} \\ &= \sup\left\{ h_\mu(f) \;\middle|\; \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha \right\} \\ &= \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_E^f(K_\alpha^\varphi)\}. \end{aligned} \tag{2.24}$$

Finally, if there exist compact invariant sets $X_n \subset X$ such that $P_{X_n}(q\varphi) \to P_X(q\varphi)$ for all q and $q \mapsto P_{X_n}(q\varphi)$ is differentiable on $\mathbb{R}$ for all n, then equation (2.24) holds for every $\alpha \in \operatorname{int} I$.

2.6. *Conformal maps and Lyapunov spectra.* The map $f\colon X \to X$ is *conformal* with conformal derivative $f'(x)$ if for every $x \in X$ we have

$$f'(x) = \lim_{\substack{y,z\to x \\ y\neq z}} \frac{d(f(y), f(z))}{d(y, z)}, \tag{2.25}$$

where $f'\colon X \to [0, \infty]$ is continuous. We restrict our attention to maps without critical points or singularities, that is, maps for which $0 < f'(x) < \infty$ for all $x \in X$.

When f is conformal, the Lyapunov exponent of a point x is $\lambda(x) = \lim(1/n)S_n \log f'(x)$, provided the limit exists, and the level sets for Lyapunov exponents are

$$K_\alpha^\lambda = \{x \in X \mid \lambda(x) = \alpha\}.$$

The *entropy spectrum for Lyapunov exponents* $\mathcal{L}_E$ is the multifractal spectrum associated to $\varphi = \log f'$ and $\psi = u = 1$. This is nothing but the Birkhoff spectrum for the potential φ, and so has already been discussed in §2.5.

One may also consider the *dimension spectrum for Lyapunov exponents* $\mathcal{L}_D$, which is obtained by replacing $h_{\mathrm{top}}K_\alpha$ with $\dim_H K_\alpha$. Here this means replacing $u = 1$ with $u = \log f'$. Indeed, when $u = \log f'$ is uniformly positive (that is, when f is uniformly expanding), it was shown in [**15**] that $\dim_u Z = \dim_H Z$ for all $Z \subset X$, so the u-dimension in this case coincides with the Hausdorff dimension.

The relationship between u-dimension and Hausdorff dimension for non-uniformly expanding systems is slightly more delicate. We consider sets all of whose points have positive lower Lyapunov exponent and satisfy the following *tempered contraction* condition for $u = \log f'$:

$$\inf_{0\le k\le n<\infty} \{S_{n-k}u(f^k(x)) + n\epsilon\} > -\infty \quad \text{for every } \epsilon > 0. \tag{2.26}$$

In our examples, we will have $f'(x) \ge 1$ everywhere, which implies that every $x \in X$ satisfies (2.26). For the time being, however, we allow $f'(x) < 1$.

Recall that our results apply to level sets in the set $\hat{X}$ given in (2.9). With $\psi \equiv 1$, we have that $\hat{X} = \{x \in X \mid \underline{\lim}_{n\to\infty}(1/n)S_n u(x) > 0\}$; consider the set $X^\star = \{x \in \hat{X} \mid$ (2.26) holds$\}$. Given $Z \subset X^\star$, we have

$$\dim_H Z = \inf\{t \ge 0 \mid P_Z(-tu) \le 0\}$$

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by [**31**]. In particular, using Proposition 2.2, we see that every such set has $\dim_u Z = \dim_H Z$.

If the Lyapunov exponent of x exists (that is, if $\lim_{n\to\infty}(1/n)S_n u(x)$ exists), then x satisfies (2.26) by [**31**, Proposition 5.5]. Thus, if $Z \subset X$ is such that the Lyapunov exponent exists and is positive for every $x \in Z$, then $\dim_u Z = \dim_H Z$. In particular, this shows that for *any* conformal map (not necessarily uniformly expanding) without critical points or singularities, the mixed multifractal spectrum associated to $\varphi = u = \log f'$ and $\psi = 1$ is equal to the dimension spectrum for Lyapunov exponents, at least for $0 < \alpha < \infty$. (The case $\alpha = 0$ is more subtle.) Thus, for every conformal map without critical points or singularities, we have $\mathcal{L}_D(\alpha) = (1/\alpha)\mathcal{L}_E(\alpha)$ for all $\alpha > 0$.

COROLLARY 2.10. (Dimension spectrum for Lyapunov exponents) *Let X be a compact metric space, and let $f\colon X \to X$ be continuous and conformal without critical points or singularities, and suppose that $u = \log f'$ satisfies **(P)**. Suppose also that the entropy map $\mathcal{M}^f(X) \to \mathbb{R}$ is upper semi-continuous, and write $\lambda(\mu) = \int u\, d\mu = \int \log f'\, d\mu$ for the Lyapunov exponent of a measure μ.*

Write $T_0(q) := P(q \log f')$, and let $J(q) = [D^- T_0(q), D^+ T_0(q)]$. Then for all $\alpha > 0$, the predicted spectrum is $\mathcal{S}(\alpha) = (1/\alpha)\inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\}$ and the predicted domain $I = \{\alpha \mid \mathcal{S}(\alpha) \geq 0\}$ is given by

$$I = \Big[\lim_{q\to-\infty} D^- T_0(q), \lim_{q\to\infty} D^+ T_0(q)\Big] = \{\lambda(\mu) \mid \mu \in \mathcal{M}^f(X)\}. \tag{2.27}$$

For every $\alpha \notin I$ we have $K_\alpha^\lambda = \emptyset$, while for every $\alpha \in \operatorname{int} I \setminus \bigcup_{q\in\mathbb{R}} \operatorname{int} J(q)$, we have

$$\begin{aligned} \dim_H(K_\alpha^\lambda) &= \frac{1}{\alpha}\inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\} \\ &= \sup\left\{\frac{h_\mu(f)}{\lambda(\mu)} \;\middle|\; \mu \in \mathcal{M}^f(X),\, \lambda(\mu) = \alpha\right\} \\ &= \sup\left\{\frac{h_\mu(f)}{\lambda(\mu)} \;\middle|\; \mu \in \mathcal{M}_E^f(K_\alpha^\lambda)\right\}. \end{aligned} \tag{2.28}$$

Finally, if there exist compact invariant sets $X_n \subset X$ such that $P_{X_n}(q\varphi) \to P_X(q\varphi)$ for all q and $q \mapsto P_{X_n}(q\varphi)$ is differentiable on $\mathbb{R}$ for all n, then (2.28) holds for every $\alpha \in \operatorname{int} I$.

It is worth mentioning that because $u \neq \psi$, the dimension spectrum for Lyapunov exponents is not a Legendre transform and need not be concave. Examples where concavity fails have been given in [**60**].

2.7. *Dimension spectrum for weak Gibbs measures.* Let μ be a Borel probability measure on a separable metric space X. The *pointwise dimension* of μ at $x \in X$ is

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r},$$

provided the limit exists. Denoting the level sets by $K_\alpha^{d_\mu} := \{x \in X \mid d_\mu(x) = \alpha\}$, the *dimension spectrum for pointwise dimensions* of μ is

$$\mathcal{D}_D(\alpha) := \dim_H K_\alpha^{d_\mu}.$$

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This definition does not require a dynamical system in the background. However, we will focus on the case where μ is a weak Gibbs measure for a potential $\phi \in C(X)$ and a conformal map $f: X \to X$, that is, where

$$\begin{aligned} P(\phi) &= \lim_{\delta \to 0} \underline{\lim}_{n \to \infty} \frac{1}{n}(-\log \mu(B(x, n, \delta)) + S_n\phi(x)) \\ &= \lim_{\delta \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n}(-\log \mu(B(x, n, \delta)) + S_n\phi(x)) \end{aligned} \tag{2.29}$$

for every $x \in X$. Note that when μ is ergodic we have (2.29) μ-almost everywhere by the Brin–Katok entropy formula and the Birkhoff ergodic theorem. The key for a weak Gibbs measure is that (2.29) holds everywhere, not just almost everywhere. We allow for non-invariant weak Gibbs measures as long as (2.29) holds.

The dimension spectrum $\mathcal{D}_D$ is obtained in the present framework by taking $\psi = u = \log f'$ and $\varphi = P(\phi) - \phi$. Observe that from equation (2.29), we have† $\underline{\lim}(1/n)S_n\varphi(x) \geq 0$ for all $x \in X$.

We consider non-uniformly expanding conformal maps with $f'(x) \geq 1$ for all $x \in X$; thus $u(x) \geq 0$. It is often the case that even though we may have $\underline{\lim}(1/n)S_n u(x) = 0$ for some $x \in X$, we nevertheless have

$$\underline{\lim}_{n \to \infty} \frac{1}{n} S_n(u + \varphi)(x) > 0 \tag{2.30}$$

for all $x \in X$. (See §4.2.)

COROLLARY 2.11. (Dimension spectrum for pointwise dimensions) *Let X be a compact metric space, and let $f: X \to X$ be continuous and conformal with $1 \leq f'(x) < \infty$ for all $x \in X$. Suppose that the entropy map $\mathcal{M}^f(X) \to \mathbb{R}$ is upper semi-continuous.*

Let μ be a weak Gibbs measure in the sense of (2.29) for $\phi \in C(X)$, and set $\varphi = P(\phi) - \phi$ and $u = \log f'$. Suppose that (2.30) holds for all $x \in X$ and that u satisfies ***(P)****. Furthermore, suppose that* $\dim_u X < \infty$ *and every measure $\nu \in \mathcal{M}^f(X)$ with $\lambda(\nu) := \int u \, d\nu = 0$ has $h_\nu(f) = 0$. Let*

$$T_0(q) := \inf\{t \in \mathbb{R} \mid P(q\varphi - t \log a) \leq 0\},$$

and let $J(q) = [D^- T_0(q), D^+ T_0(q)]$. Then the predicted spectrum is $\mathcal{S}(\alpha) = \inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\}$ and the predicted domain $I = \{\alpha \mid \mathcal{S}(\alpha) \geq 0\}$ is given by

$$I = \Big[\lim_{q \to -\infty} D^- T_0(q), \lim_{q \to \infty} D^+ T_0(q)\Big]. \tag{2.31}$$

For every $\alpha \notin I$ we have $K_\alpha^{d_\mu} = \emptyset$, while for every $\alpha \in \operatorname{int} I \setminus \bigcup_{q \in \mathbb{R}} \operatorname{int} J(q)$, we have

$$\begin{aligned} \dim_H(K_\alpha^{d_\mu}) &= \inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\} \\ &= \sup\left\{\frac{h_\nu(f)}{\lambda(\nu)} \;\middle|\; \nu \in \mathcal{M}^f(X), \frac{\int \varphi \, d\nu}{\lambda(\nu)} = \alpha\right\} \\ &= \sup\left\{\frac{h_\nu(f)}{\lambda(\nu)} \;\middle|\; \nu \in \mathcal{M}_E^f(K_\alpha^{d_\mu})\right\}. \end{aligned} \tag{2.32}$$

† Our choice of φ differs from some other papers in the literature, which use $\varphi = \phi - P(\phi)$. We use the opposite sign so as to maintain consistency between the form of the Legendre transform in equation (2.32) and that already given in equation (2.24)

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Finally, if there exist compact invariant sets $X_n \subset X$ such that $P_{X_n}(q\varphi) \to P_X(q\varphi)$ for all q and $(q, t) \mapsto P_{X_n}(q\varphi - tu)$ is differentiable on $\mathbb{R}^2$ for all n, then (2.32) holds for every $\alpha \in \operatorname{int} I$.

Remark. In fact, Corollary 2.11 holds under slightly more relaxed assumptions on f'; namely, we can have $f'(x) < 1$ provided f has no critical points and equation (2.26) holds for every $x \in K_\alpha^{\varphi,u} \cup K_\alpha^{d_\mu}$ for which $\underline{\lim}(1/n)S_n u(x) > 0$. Similarly, the condition that equation (2.30) holds for all $x \in X$ can be relaxed to the condition that it holds for every $x \in K_\alpha^{\varphi,u} \cup K_\alpha^{d_\mu}$ with $0 \le \alpha < \infty$.

3. *History and survey of known results*

In this section we describe some of the settings in which the multifractal formalism has been established to date. Surveys of the literature can also be found in [**4**, **97**], among others.

3.1. *Results for $\mathcal{D}_D$.*

The dimension spectrum for pointwise dimensions has received the most attention of any multifractal spectrum. It was introduced in [**51**], although [**16**] already considered similar notions in the study of turbulence.

In order to establish the multifractal formalism for the dimension spectrum of a measure μ, some information about the structure of μ is required. In particular, μ is usually assumed to have some sort of Gibbs property for a system f. The easiest case is when f is uniformly expanding and conformal: the latter hypothesis is automatically fulfilled if we study one-dimensional differentiable maps (real or complex).

We first mention the relevant papers for the uniformly expanding case. Markov maps of the interval were studied in [**33**], which dealt with the dimension spectrum for Markov measures, and also in [**105**], which dealt with more general Gibbs measures. In the complex plane, measures of maximal entropy for rational maps with uniformly expanding Julia sets were studied in [**79**]. Gibbs measures corresponding to Hölder continuous potentials on arbitrary conformal repellers were studied in [**99**]. Here the thermodynamic approach yields a family of full measures for the dimension spectrum. Many of these results follow from Corollary 2.11 (see §4.1.2), and also from the results in [**13**], since for uniformly expanding conformal maps it was shown in that paper that $\dim_u = \dim_H$ when $u = \log f'$. Asymptotically additive potentials and C^1 average conformal repellers were studied in [**25**].

Non-Markov interval maps were studied by Hofbauer [**54**], by approximating f with a Markov map and φ, u by locally constant functions. Despite the thermodynamic nature of this approach, the approximation techniques permit stronger results that apply even when the pressure function is non-differentiable.

The dimension spectrum for Gibbs measures has also been studied for maps with both expanding and contracting directions, provided the dynamics on each of those directions are conformal (which is the case if the expanding and contracting distributions are each one-dimensional). The geodesic flow on a surface of constant negative curvature was studied in [**111**], while Axiom A surface diffeomorphisms were studied in [**100**, **114**, **115**], and conformal Axiom A flows were studied in [**96**].

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Prior to this, the dimension spectrum for measures arising as a result of a geometric construction (rather than as invariant measures for a dynamical system) was studied in [**21**, **26**, **34**], where the thermodynamically defined function T_0 is replaced by a function related to the Hentschel–Procaccia dimension spectrum for μ. This line of inquiry has been vigorously continued, see [**36**, **52**, **76**, **77**, **80**, **119**], among many others. In this paper, we focus on measures arising as invariant measures for dynamical systems.

Within this context, there are three primary ways to weaken the hypotheses on the triple (μ, f, φ). In the first place, one can consider potentials φ that are not Hölder continuous, which often requires weakening the Gibbs property for μ. Second, one can replace the uniform hyperbolicity of f with something weaker. Third, one can remove the requirement that f be conformal.

In the symbolic setting, the dimension spectrum for a class of measures called g-measures was studied in [**84**, **85**]. Weak Gibbs measures for continuous circle maps were studied in [**46**]. These examples include cases where the pressure function is not differentiable, and the proofs rely on orbit-gluing techniques.

The easiest class of non-uniformly hyperbolic maps to consider are expanding Markov interval maps with parabolic fixed points, such as the Manneville–Pomeau map $x \mapsto x + x^{1+\beta} \pmod 1$, where $\beta > 0$, which has a parabolic fixed point at 0. On the symbolic models for these systems, partial results for the dimension spectrum of equilibrium states of suitably regular measures were obtained in [**104**]. Further results for the dimension spectrum of the measure of maximal entropy for the original system were given in [**82**], and more general weak Gibbs measures were studied in [**126**]. In the invertible regime, horseshoe maps with parabolic fixed points were considered in [**8**].

All of the results in the previous paragraph make use of inducing schemes and the thermodynamic approach; see §4.2 for a description of similar results that can be obtained using Corollary 2.11 (without using inducing schemes).

For Markov interval maps with a parabolic fixed point, the dimension spectrum for weak Gibbs measures associated to continuous potentials (a broader class of potentials than those above) was studied in [**66**] using what are essentially orbit-gluing techniques. A general result using thermodynamic techniques that adapts Corollary 2.11 using ideas from [**54**] is given in [**32**, Theorem 3.8].

One-dimensional maps with critical points are more difficult to study; in this setting, the presence of the critical point prevents a direct application of Corollary 2.11, and so results beyond those given in this paper are needed. Using inducing schemes and the corresponding thermodynamics, the dimension spectra of various conformal measures have been examined in [**62**, **122**].

Non-conformal maps require new methods; in particular, one requires a non-conformal version of Bowen's equation and a non-additive version of the thermodynamic formalism [**3**, **35**]. Dimension spectra for various non-conformal maps have been studied in [**65**, **74**, **86**].

Finally, we mention that the relationship given in Theorem D between the multifractal spectrum and a globally defined function has been presented already in [**123**], which shows that the Legendre transform of a certain free energy function gives the concave hull of $\mathcal{D}_D(\alpha)$ in a very general setting. The free energy function in question is given not in terms

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of the topological pressure, but in terms of the measure μ being studied, and is related to the Hentschel–Procaccia spectrum.

3.2. *Results for $\mathcal{L}_D$.* The dimension spectrum for Lyapunov exponents was studied for one-dimensional full-branched Markov maps by Bohr and Rand [**18**]. The formalism was extended to Axiom A surface diffeomorphisms in [**100**], to conformal repellers in [**125**], and to conformal Axiom A flows in [**96**], using thermodynamic techniques in each case. See §4.1.2 for applications of Corollary 2.10 in some of these settings; because the maps in question are uniformly expanding, the results in [**13**] also apply here.

More recently, using similar thermodynamic and approximation techniques to those in [**54**], Hofbauer studied piecewise monotonic interval maps in [**55**]; note that the lack of any finite Markov structure makes this case significantly more difficult than that studied in [**18**].

Markov maps of the interval with parabolic fixed points were studied in [**104**], where results were obtained on part of the domain of the spectrum; complete results in this setting were given in [**82**]. These proofs use inducing schemes and the thermodynamic approach; see §4.2 for some applications of Corollary 2.10 in this non-uniformly hyperbolic context.

Interval maps with parabolic fixed points were studied again in [**49**], and rational maps whose Julia sets contain parabolic points were considered in [**47**, **48**] using the orbit-gluing approach. Results using the thermodynamic approach are obtained in [**32**, Theorem 3.5].

As with dimension spectra, the fully non-uniform case of multimodal maps cannot be treated with the results in this paper. The Lyapunov spectrum for multimodal maps has been studied by Iommi and Todd [**62**] using inducing schemes and thermodynamic results.

For non-conformal maps, life is again more difficult; some results on the *entropy* spectrum for Lyapunov exponents are given in [**7**].

3.3. *Results for $\mathcal{B}_E$.* The entropy spectrum for Birkhoff averages (or something equivalent to it) of arbitrary continuous potentials was studied using orbit-gluing techniques for shifts of finite type in [**38**, **39**]. This was extended to systems with specification in [**41**, **117**], and to the weaker g-almost product property in [**103**]. For systems with specification, the pressure of level sets (which generalizes entropy) was studied in [**120**].

The thermodynamic approach was initially applied to systems with specification and sufficiently regular potential functions in [**116**], and some results on subshifts of finite type are given in [**101**]. In [**13**], it was observed that differentiability of the pressure function $q \mapsto P(q\varphi)$ is the key, leading to a version of Corollary 2.9. A similar result for flows was shown in [**5**]. The thermodynamic approach can be extended to all continuous potentials if differentiability holds on a dense vector subspace of $C(X)$ [**32**].

More recently, a version of Corollary 2.9 for almost additive potential functions was proved in [**45**]; non-additive versions of the Birkhoff spectrum have also been studied in [**1**, **6**], and a broad class of non-additive functions obtained from matrix-value cocycles are studied in [**23**, **43**, **44**].

3.4. *Results for mixed spectra.* One of the simplest mixed spectra to study is the dimension spectrum for Birkhoff averages. For uniformly expanding conformal maps,

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we have $\dim_u = \dim_H$ when $u = \log f'$, and so this spectrum was studied in [**13**], using thermodynamic techniques and assuming that the pressure function $(q, t) \mapsto P(q\varphi - t \log f')$ is differentiable. The corresponding results for flows were proved in [**5**].

For non-uniformly expanding maps with parabolic fixed points, this spectrum was studied using orbit-gluing techniques in [**63**]. Section 4.2 gives results in this setting using the thermodynamic approach and applying Corollary 2.5; see also [**32**, Theorems 3.6 and 3.9]. As before, these results do not apply to multimodal maps, which are considered in [**28**].

Various non-conformal systems and their associated dimension spectra for Birkhoff averages have been studied in [**11**, **67**, **106**, **107**].

More general mixed spectra, such as the entropy spectrum for pointwise dimensions and the dimension spectrum for local entropies, were mentioned in the general framework of [**9**], but no results were proven. The $\dim_u$-spectra studied in this paper were introduced in [**13**], where the formalism was proven under assumptions of differentiability of the pressure function.

3.5. *Irregular set.* For the local asymptotic quantities used in defining a multifractal spectrum, the irregular set of points on which the lower and upper limits disagree is usually invisible to invariant measures. Thus, thermodynamic methods do not suffice, and one needs orbit-gluing techniques to show that the irregular set has full dimension, which is often the case provided it is non-empty.

For mixing shifts of finite type, the irregular set for Birkhoff averages was shown to have full entropy in [**39**]. A more general result for u-dimensions and shifts with specification was shown in [**13**, **15**], and in [**121**] it is shown that the irregular set carries full topological pressure for systems with specification.

For uniformly expanding conformal maps with a Markov structure, the Hausdorff and packing dimensions of various sorts of irregular sets, including more refined sets defined as those points x for which the quantities $\phi_n(x)$ have a specified accumulation set, are studied in the series of papers [**87**, **90**, **93**, **94**]. For systems with specification, analogous results for topological entropy are given in [**27**].

For maps with parabolic fixed points, the irregular set for Lyapunov exponents was studied in [**104**] and more recently in [**47**–**49**]. The latter set of papers contains detailed information about not just the irregular set itself, but the set of points whose sequence of approximate Lyapunov exponents has a specified lower and upper limit.

For multimodal interval maps, the Hausdorff dimension of the irregular set for Birkhoff averages was studied in [**28**], and the irregular set for pointwise dimensions was studied in [**122**].

Another sort of irregular set, comprising points for which the forwards and backwards averages disagree, was studied in [**113**] for hyperbolic surface diffeomorphisms.

3.6. *Coarse spectra.* Coarse multifractal spectra have received rather less attention than their fine counterparts, and what attention they have received has been focused predominantly on the coarse spectrum for pointwise dimensions. This spectrum was studied in various settings in [**37**, **89**, **108**, **109**, **123**]. An analogue of Theorem D has been

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given in [**91**], showing that the Legendre transform of the coarse spectrum for dimensions is the Rényi spectrum. General conditions under which the coarse and fine spectra agree are given in [**32**].

3.7. *Higher-order spectra.* A natural generalization is to allow the local quantities $\phi_n(x)$ to take values in a more general space than $\mathbb{R}$. In its simplest form, replacing $\mathbb{R}$ with $\mathbb{R}^2$, this amounts to considering the spectrum $\mathcal{F}\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $\mathcal{F}(\alpha, \beta) = \dim_u(K_\alpha \cap K_\beta)$. These higher-order spectra exhibit various new phenomena: in particular, the domain of the spectrum need not be convex, and its interior may be disconnected [**14**]. Nevertheless, many of the same techniques can be used: the thermodynamic approach was applied in [**14**] to show that differentiability of the appropriate cross-section of the pressure function still leads to a conditional variational principle (see also [**4**, **32**]). A related notion of 'rotation sets' is also studied; see [**75**], for example.

Using orbit-gluing techniques, higher-order spectra for subshifts of finite type were studied in [**38**, **39**], and also [**87**, **90**, **93**, **94**]. (The latter series of papers also studies more general local quantities.) For systems with specification, multifractal spectra for quantities taking values in $\mathbb{R}^d$ were studied in [**117**], and Banach-space valued quantities in [**41**]. Spectra associated to almost-additive potentials were studied in [**2**].

3.8. *Other multifractal results.* There are a great many papers in the literature addressing other questions of a multifractal nature. We mention just a few of these.

In [**64**, **88**], the level sets K_α are further partitioned into sets $K_{\alpha,\gamma}$, where γ controls the *rate* of convergence $\phi_n(x) \to \alpha$, and the sizes of these sets are examined.

Multiple ergodic averages of the form $(1/n)\sum_{k=1}^{n} \varphi_1(f^k x)\varphi_2(f^{2k}x)\cdots\varphi_\ell(f^{\ell k}x)$, and their corresponding level sets, are considered in [**40**, **42**].

In [**57**–**59**], various multifractal spectra are studied in the context of a topological Markov chain with *infinitely* many symbols; this is related to iterated function systems with countably many branches [**73**].

In [**1**, **68**, **118**], the asymptotic behaviour captured by the level sets K_α is localized: given $\xi \in C(X)$, one studies the level set $K_\xi = \{x \in X \mid \phi_n(x) \to \xi(x)\}$.

In [**50**, **92**], the local quantity $\phi_\infty(x)$ is defined in terms of first return times to a small neighbourhood of x.

Various asymptotic quantities associated to actions of Kleinian and Fuchsian groups are considered in [**71**, **72**, **81**].

A notion of multifractal rigidity is studied in [**4**, **10**, **12**], based on the idea that under certain circumstances, a finite number of multifractal spectra can serve as complete invariants within a particular class of dynamical systems.

4. *Examples and applications*

We describe various classes of systems and multifractal spectra to which the results in §2 can be applied. Many of these systems and spectra have already been studied in the papers surveyed in §3. In cases where previous work has used the thermodynamic approach, we obtain new results in the presence of phase transitions, which occurs for non-Hölder potentials and for non-uniformly hyperbolic maps. In cases where previous work has

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used the orbit-gluing approach, we obtain new results in the form of the strengthened conditional variational principle equation (1.6).

4.1. *Uniformly hyperbolic maps.*

4.1.1. *Entropy spectrum for Birkhoff averages.* Let (X, σ) be a transitive subshift of finite type (similar results hold for Axiom A diffeomorphisms, uniformly expanding repellers, and expansive maps with specification). It follows from [**20**] that every Hölder potential φ has a unique equilibrium state.

The entropy map is upper semi-continuous by expansivity, so uniqueness of the equilibrium state is equivalent to differentiability of the pressure function in every direction. That is, ϕ has a unique equilibrium state if and only if the map $q \mapsto P(\phi + q\psi)$ is differentiable at $q = 0$ for every $\psi \in C(X)$ (see [**69**, **112**]). Thus, P is differentiable on the set of Hölder continuous potentials.

Consider the entropy spectrum for Birkhoff averages of a Hölder potential φ. By Corollary 2.9, we have $I = \{\int \varphi \, d\mu \mid \mu \in \mathcal{M}^f(X)\}$, $K_\alpha^\varphi = \emptyset$ for all $\alpha \notin I$, and the spectrum is given by (2.24) for all $\alpha \in \operatorname{int} I$. In particular, for every $\alpha \in \operatorname{int} I$, there is a full measure ν_α, which is obtained as the equilibrium state for $q\varphi$, where $q = q(\alpha)$ is such that $(d/dq)P(q\varphi) = \alpha$. This example is also given in [**13**].

For general continuous potentials (not necessarily Hölder), it was shown in [**117**] that we have

$$\begin{aligned} h_{\text{top}}(K_\alpha^\varphi) &= \sup\left\{ h_\mu(f) \;\middle|\; \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha \right\} \\ &= \inf\{P(q\varphi) - q\alpha \mid q \in \mathbb{R}\} \end{aligned} \tag{4.1}$$

whenever $K_\alpha^\varphi \neq \emptyset$. This establishes part, but not all, of (2.24).

To establish the final equality in (2.24), we need to know more about the thermodynamics of $q\varphi$. The function $T_0 \colon q \mapsto P(q\varphi)$ need not be differentiable [**53**, **56**, **102**]. For example, Pesin and Zhang show in [**102**] that if f is a uniformly piecewise expanding full-branched Markov map on [0, 1] and

$$\varphi(x) = \begin{cases} -(1 - \log x)^{-\beta}, & x \in (0, 1], \\ 0, & x = 0, \end{cases} \tag{4.2}$$

for some $0 < \beta \leq 1$, then there exists $q_0 > 0$ at which T_0 is not differentiable. Thus the results in [**13**] do not apply, while Corollary 2.9 shows that the formalism holds for all $\alpha \in \operatorname{int} I \setminus (D^- T_0(q_0), D^+ T_0(q_0))$; for all such α we have (2.24).

To extend the formalism to the interval $(D^- T_0(q_0), D^+ T_0(q_0))$, on which the predicted spectrum is an affine function of α, we need to use the last part of Corollary 2.9, thus we need to find a family of compact f-invariant sets X_n such that $P_{X_n}(q\varphi) \to P_X(q\varphi)$ for all q.

In Pesin and Zhang's example, the map f is topologically (semi-)conjugate† to the full shift Σ_d^+. The potential φ is Hölder on every compact subset of Σ_d^+ that does not contain the point $000\ldots$. Let X_n be the set of $x \in [0, 1]$ whose coding does not have n

† It is conjugate on a total probability set, and hence by the variational principle, the pressure function for (X, f) is entirely determined by the pressure function for (Σ_d^+, σ).

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consecutive zeros. Then $q \mapsto P_{X_n}(q\varphi)$ is differentiable for all q. It remains only to show that $P_{X_n}(q\varphi) \to P_X(q\varphi)$.

PROPOSITION 4.1. *Let X be a transitive subshift of finite type, and $\mathcal{O} \subset X$ a periodic orbit. For $\epsilon > 0$, let $X_\epsilon := \{x \in X \mid d(\sigma^n(x), \mathcal{O}) \geq \epsilon \text{ for all } n\}$. Then the X_ϵ are compact σ-invariant sets such that for every $\phi \in C(X)$, we have $\lim_{\epsilon\to 0} P_{X_\epsilon}(\phi) = P_X(\phi)$.*

By Proposition 4.1 and Corollary 2.9 we have the following result, which does not follow either from [**13**] or [**117**].

THEOREM 4.2. (Entropy spectrum for Birkhoff averages of a non-Hölder potential) *Let (X, σ) be a transitive subshift of finite type, and let $\varphi \in C(X)$ be locally Hölder continuous everywhere except a single periodic orbit. Then in addition to (4.1), the level sets $K_\alpha^\varphi = \{x \mid (1/n)S_n\varphi(x) \to \alpha\}$ satisfy $h_{\text{top}} K_\alpha^\varphi = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_E^f(K_\alpha^\varphi)\}$ for every $\alpha \in \text{int}\{\int \varphi \, d\mu \mid \mu \in \mathcal{M}^f(X)\}$.*

4.1.2. *Dimension spectra for conformal repellers.* Let M be a smooth manifold and $f\colon M \to M$ a $C^{1+\epsilon}$ map. Let $X \subset M$ be a transitive conformal repeller for f. That is, there exists an open set $U \subset M$ such that $X = \bigcap_{n\geq 1} f^{-n}(U)$, and an ϵ-Hölder function $a\colon U \to (1, \infty)$ such that $Df(x)/a(x)$ is an isometry for every $x \in U$.

As before, every Hölder continuous potential ϕ on X has a unique equilibrium state, which is a Gibbs measure. Because f is expansive, the entropy map is upper semi-continuous, so every continuous potential has an equilibrium state and the pressure function P is differentiable on the space of Hölder continuous potentials. This thermodynamic information is all we need to apply the results from §2.

For example, the dimension spectrum for Lyapunov exponents can be computed via Corollary 2.10, which shows that with $I = \{\lambda(\mu) \mid \mu \in \mathcal{M}^f(X)\}$, we have $K_\alpha^\lambda = \emptyset$ whenever $\alpha \notin I$, while the spectrum satisfies (2.28) for all $\alpha \in \text{int } I$. This was originally shown in [**125**].

For other potentials $\varphi \in C(X)$ not equal to $\log a$, the dimension spectrum for Birkhoff averages of φ is a mixed spectrum of the sort studied in [**13**]. Let $I = \{\int \varphi \, d\mu \mid \mu \in \mathcal{M}^f(X)\}$ and define $T_\alpha(q)$ by $P(q(\varphi - \alpha) - T_\alpha(q)\log a) = 0$. It follows from [**13**] or Theorem B that if P is differentiable on $\text{span}\{\varphi, \log a\}$, then

$$\begin{aligned} \dim_H(K_\alpha^\varphi) &= \inf\{T_\alpha(q) \mid q \in \mathbb{R}\} \\ &= \sup\left\{\frac{h_\mu(f)}{\lambda(\mu)} \,\middle|\, \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha\right\} \\ &= \sup\left\{\frac{h_\mu(f)}{\lambda(\mu)} \,\middle|\, \mu \in \mathcal{M}_E^f(K_\alpha^\varphi)\right\} \end{aligned} \tag{4.3}$$

for every $\alpha \in \text{int } I$. If φ is not Hölder continuous, then the pressure function is not necessarily differentiable on the span of $\{\varphi, \log a\}$; thus, the following result is new.

THEOREM 4.3. (Dimension spectrum for Birkhoff averages of a non-Hölder potential) *Let $f\colon M \to M$ be $C^{1+\epsilon}$ and let X be a conformal repeller for f. Let $\varphi \in C(X)$ be Hölder continuous everywhere except for a single periodic orbit. Then with I as above, we have $K_\alpha^\varphi = \emptyset$ for all $\alpha \notin I$, while $\dim_H(K_\alpha^\varphi)$ is given by (4.3) for every $\alpha \in \text{int } I$.*

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Proof. By Proposition 4.1, there exist compact invariant sets $X_n \subset X$ on which $P_{X_n}\colon (q, t) \mapsto P_{X_n}(q\varphi - t \log a)$ is differentiable, and furthermore $P_{X_n}(\phi) \to P_X(\phi)$ for all $\phi \in C(X)$, so Corollary 2.5 shows that (4.3) holds for all $\alpha \in \mathbb{R}$. □

Returning to the case where $\phi \in C(X)$ is Hölder continuous, we recall that its unique equilibrium state μ is a Gibbs measure, and so Corollary 2.11 recovers one of the main results of [**99**], namely that for all $\alpha \in \operatorname{int} I$, the dimension spectrum for pointwise dimensions of μ is given by (2.32).

4.2. *Maps with parabolic fixed points.* Let (X, f) be a non-uniformly expanding $C^{1+\epsilon}$ repeller on the real line or the circle. That is, let $M = \mathbb{R}$ or S^1, let $X \subset M$ be a compact set, $U \supset X$ an open set, and $f\colon U \to M$ a $C^{1+\epsilon}$ map such that $X = \bigcap_{n\geq 1} f^{-n}(U)$. Assume that $|f'(x)| \geq 1$ for all $x \in X$, with equality at exactly one point p, which is fixed by f. Finally, assume that near the fixed point we have $|f'(x)| \geq 1 + cd(x, p)^{\epsilon}$ for some $c > 0$.

PROPOSITION 4.4. *In this setting, the function $u = \log |f'|$ satisfies **(P)**, and so $\dim_u$ is a well-defined Carathéodory dimension characteristic.*

Remark. Proposition 4.4 gives the first example of a setting in which u-dimension is a Carathéodory dimension characteristic despite the fact that u is not uniformly positive. Although we consider here the case when f is $C^{1+\epsilon}$ and has a specific form near p, one can ask about C^1 maps without this extra assumption; see the remark after Lemma 5.7.

Example 4.5. Define $f\colon [0, 1] \to [0, 1]$ by

$$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{for } 0 \leq x \leq \dfrac{1}{2}, \\ \dfrac{2x-1}{x} & \text{for } \dfrac{1}{2} < x \leq 1. \end{cases}$$

Identifying the endpoints of the interval, we can think of f as a smooth map of the unit circle with a single parabolic fixed point. We have $f'(x) = (1-x)^{-2}$ for $x \in [0, 1/2]$ and $f'(x) = x^{-2}$ for $x \in (1/2, 1]$; thus, f satisfies the conditions above.

As in Proposition 4.1, there exist compact f-invariant sets $X_n \subset X \setminus \{p\}$ such that $\lim_{n\to\infty} P_{X_n}(\phi) = P_X(\phi)$ for all $\phi \in C(X)$. The sets X_n can be chosen to be locally maximal and transitive, so in particular, since f is uniformly expanding on each X_n, the pressure function P_{X_n} is differentiable on $\operatorname{span}\{\phi, u\}$ for all Hölder ϕ, even through P_X may not be.

4.2.1. *Dimension spectrum for Lyapunov exponents.* It is shown in [**49**] that for $I = \{\lambda(\mu) \mid \mu \in \mathcal{M}^f(X)\}$ and $\alpha \in \operatorname{int} I$, we have

$$\begin{aligned} \dim_H(K_\alpha^\lambda) &= \frac{1}{\alpha} \inf\{P(q \log f') - q\alpha \mid q \in \mathbb{R}\} \\ &= \sup\left\{ \frac{h_\mu(f)}{\lambda(\mu)} \;\middle|\; \mu \in \mathcal{M}^f(X), \lambda(\mu) = \alpha \right\}. \end{aligned} \tag{4.4}$$

In light of the above discussion, Corollary 2.10 extends this as follows.

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THEOREM 4.6. (Dimension spectrum for Lyapunov exponents of a non-uniformly expanding map) *Let (X, f) be as above. Then in addition to (4.4), the level sets K_α^λ satisfy*

$$\dim_H(K_\alpha^\lambda) = \sup\left\{ \frac{h_\mu(f)}{\lambda(\mu)} \;\middle|\; \mu \in \mathcal{M}_E^f(K_\alpha^\lambda) \right\}.$$

Remark. Although Theorem 4.6 is stronger than the corresponding results in [**49**], we observe that our thermodynamic techniques do not suffice to prove the results in [**49**] regarding the irregular set or points with zero lower Lyapunov exponent.

4.2.2. *Dimension spectrum for Birkhoff averages.* Moving to the dimension spectrum for Birkhoff averages on a non-uniformly expanding conformal repeller, it is shown in [**63**] that for $\varphi \in C(X)$, $I = \{\int \varphi \, d\mu \mid \mu \in \mathcal{M}^f(X)\}$, and $\alpha \in I \setminus \{\varphi(p)\}$, we have

$$\begin{aligned} \dim_H(K_\alpha^\varphi) &= \inf\{T_\alpha(q) \mid q \in \mathbb{R}\} \\ &= \sup\left\{ \frac{h_\mu(f)}{\lambda(\mu)} \;\middle|\; \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha \right\}. \end{aligned} \tag{4.5}$$

Under the stronger assumption that φ is Hölder continuous, we can use Theorem A and Corollary 2.5 to extend this on $\operatorname{int} I \setminus \{\varphi(p)\}$.

LEMMA 4.7. *Let f be a conformal expanding map with a parabolic fixed point p satisfying the conditions above, and let $\varphi \in C(X)$. Then for every $\alpha \neq \varphi(p)$, we have $K_\alpha \subset \hat{X}$, whence $\hat{K}_\alpha = K_\alpha$.*

This implies the following new result.

THEOREM 4.8. (Dimension spectrum for Birkhoff averages of a non-uniformly expanding map) *Let (X, f) be as above and let $\varphi \in C(X)$ be Hölder continuous. Then in addition to equation (4.5), the level sets $K_\alpha^\varphi = \{x \mid (1/n)S_n\varphi(x) \to \alpha\}$ satisfy*

$$\dim_H(K_\alpha^\varphi) = \sup\left\{ \frac{h_\mu(f)}{\lambda(\mu)} \;\middle|\; \mu \in \mathcal{M}_E^f(K_\alpha^\varphi) \right\}$$

for every $\alpha \in \operatorname{int}\{\int \varphi \, d\mu \mid \mu \in \mathcal{M}^f(X)\} \setminus \{\varphi(p)\}$.

Remark. As in the previous section, we observe that our thermodynamic techniques do not yield the stronger results in [**63**] on the irregular set, the precise shape of the spectrum, and the value of $\dim_H K_{\varphi(p)}$.

4.2.3. *Dimension spectrum for pointwise dimensions.* Finally, we consider the dimension spectrum for pointwise dimensions. Let μ be a weak Gibbs measure for some potential $\phi \in C(X)$. (For the existence of such measures, see [**66**, **70**].) Writing $\varphi = P(\phi) - \phi$, it is shown in [**66**] that for $I = \{(\int \varphi \, d\mu)/\lambda(\mu) \mid \mu \in \mathcal{M}^f(X), \lambda(\mu) > 0\}$ and $\alpha \in \operatorname{int} I$, we have

$$\begin{aligned} \dim_H(K_\alpha^{d_\mu}) &= \inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\} \\ &= \sup\left\{ \frac{h_\nu(f)}{\lambda(\nu)} \;\middle|\; \nu \in \mathcal{M}^f(X), \frac{\int \varphi \, d\nu}{\lambda(\nu)} = \alpha \right\}. \end{aligned} \tag{4.6}$$

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If we assume, in addition, that ϕ is Hölder continuous and $P(\phi) > \phi(p)$, then we can obtain this and more from Corollary 2.11.

LEMMA 4.9. *With X, f, ϕ, and μ as above, we have (2.30) for all $x \in X$.*

Corollary 2.11 and Lemma 4.9, give the following result.

THEOREM 4.10. (Dimension spectrum for pointwise dimensions of a non-uniformly expanding map) *Let X, f, ϕ, and μ be as above, and suppose that* $\dim_u X < \infty$. *Then in addition to (4.6), the level sets $K_\alpha^{d_\mu} = \{x \mid d_\mu(x) = \alpha\}$ satisfy*

$$\dim_H(K_\alpha^{d_\mu}) = \sup\left\{ \frac{h_\nu(f)}{\lambda(\nu)} \;\middle|\; \nu \in \mathcal{M}_E^f(K_\alpha^{d_\mu}) \right\}$$

for every $\alpha \in \operatorname{int} I$.

Proof. Write $u = \log a$, then any measure $\nu \in \mathcal{M}_E^f(X)$ with $\int u \, d\nu = 0$ must have $\nu(\{p\}) = 1$, whence $h_\nu(f) = 0$. This verifies the hypotheses of Corollary 2.11. □

4.3. *Other non-uniformly hyperbolic maps.*

4.3.1. *Maps with critical points: continuous potentials.* Let I be a compact interval and $f : I \to I$ a topologically exact smooth map with finitely many non-flat critical points. Assume that all periodic points of f are repelling, and that every critical point c has the property that $\lim_{n\to\infty} |(f^n)'(c)| = \infty$.

It was shown by Blokh that any continuous topologically mixing interval map has the specification property [**17**, **24**]. Thus, the orbit-gluing results in [**103**] can be applied to any continuous φ (note that the results in [**117**] do not apply because f is not expansive). For the level sets $K_\alpha^\varphi = \{x \mid (1/n)S_n\varphi(x) \to \alpha\}$, this yields

$$\begin{aligned} h_{\text{top}}(K_\alpha^\varphi) &= \sup\left\{ h_\mu(f) \;\middle|\; \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha \right\} \\ &= \inf\{P(q\varphi) - q\alpha \mid q \in \mathbb{R}\} \end{aligned} \tag{4.7}$$

for all $\alpha \in \operatorname{int}\{\int \varphi \, d\mu \mid \mu \in \mathcal{M}^f(X)\}$. Once again, we can extend (4.7) given appropriate thermodynamic information. It was shown in [**78**] that every Hölder continuous potential has a unique equilibrium state, and so by upper semi-continuity of the entropy map [**22**, Lemma 2.3], the pressure function is differentiable on the space of Hölder potentials. In particular, Corollary 2.9 gives the following result.

THEOREM 4.11. (Entropy spectrum for Birkhoff averages of a map with critical points) *Fix $f \in \mathcal{H}$ and let $\varphi \in C([0, 1])$ be Hölder continuous. Then for every $\alpha \in \operatorname{int} I$, in addition to (4.7), the level set K_α^φ satisfies*

$$h_{\text{top}}(K_\alpha^\varphi) = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_E^f(K_\alpha^\varphi)\}.$$

4.3.2. *Maps with critical points: discontinuous potentials.* For maps with critical points, the geometric potential $\varphi(x) = -\log|f'(x)|$ is no longer continuous or even

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bounded, and so the results presented here do not apply directly to the multifractal analysis of the spectra associated to this potential or to the various dimension spectra associated to f. Nevertheless, the thermodynamics of the potentials $t\varphi$ have recently been studied in [**23**, **61**, **98**], and it is reasonable to expect that these thermodynamic results can be used to obtain multifractal information. This has been done in [**62**, **122**] using inducing schemes; a more abstract deduction of (weaker) multifractal results using methods along the lines of this paper has been given in [**29**].

4.3.3. *Other non-uniformly expanding examples.* The existence and uniqueness of equilibrium states for a broad class of non-uniformly expanding maps in higher dimensions was studied by Oliveira and Viana [**83**] and by Varandas and Viana [**124**]. The multifractal properties of these systems do not appear to have been studied, nor does the question of whether or not these systems (which may have contracting regions) satisfy specification or any of its variants.

We briefly describe the systems studied in [**124**]. Let M be a compact connected manifold and let $f\colon M \to M$ be a local homeomorphism. Let $L(x)$ be a bounded function such that near x we have $d(f(y), f(z)) \geq L(x)^{-1} d(y, z)$. Assume every point has finitely many preimages and write $\deg_x(f) = \#f^{-1}(x)$. If level sets for the degree are closed, then is it shown in [**124**] that up to considering some iterate f^N of f, we can assume that $\deg_x(f) \geq e^{h_{\text{top}}(f)}$ for all x.

Finally, suppose there are $\sigma > 1$ and an open region $\mathcal{A} \subset M$ such that:

(H1) $L(x) \leq \sigma^{-1}$ for all $x \in M \setminus \mathcal{A}$, and $\sup_{x\in\mathcal{A}} L(x)$ is close to 1 (see [**124**] for precise conditions);

(H2) there exists a covering $\mathcal{P}$ of M by domains of injectivity for f such that $\mathcal{A}$ can be covered by $r < e^{h_{\text{top}}(f)}$ elements of $\mathcal{P}$.

Thus, f is uniformly expanding outside of $\mathcal{A}$, and does not display too much contraction inside $\mathcal{A}$; furthermore, since there are at least $e^{h_{\text{top}}(f)}$ preimages of any given point x, and only r of these can lie in covering of $\mathcal{A}$ by elements of $\mathcal{P}$, every point has at least one preimage in the expanding region. (See [**124**] for examples of specific systems satisfying these conditions.) Now suppose $\varphi \in C(M)$ is Hölder continuous and satisfies

$$\sup \varphi - \inf \varphi < h_{\text{top}}(f) - \log r. \tag{4.8}$$

Then it is proved in [**124**] that there exists a unique equilibrium state for φ. By upper semi-continuity of the entropy map, this implies differentiability of the pressure function at φ. The bounded range condition (4.8) holds for $q\varphi$ for all q in a neighbourhood of $[-1, 1]$. Thus, there exists $q_0 > 1$ such that $T_0\colon q \mapsto P(q\varphi)$ is differentiable on $(-q_0, q_0)$ and we can apply Corollary 2.9 to give the following new result.

THEOREM 4.12. *Let M, f, and φ be as above. Then we have*

$$\begin{aligned} h_{\text{top}}(K_\alpha^\varphi) = \underline{\mathcal{F}}(\alpha) = \overline{\mathcal{F}}(\alpha) &= \inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\} \\ &= \sup\left\{ h_\mu(f) \;\middle|\; \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha \right\} \\ &= \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_E^f(K_\alpha^\varphi)\} \end{aligned}$$

for every $\alpha \in (D^+T_0(-q_0), D^-T_0(q_0))$.

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5. *Proofs*

5.1. *Proof of Proposition 2.1.* It suffices to show that $\int u\,d\mu \geq 0$ whenever $\mu \in \mathcal{M}^f(X)$ is ergodic. Suppose $\mu \in \mathcal{M}_E^f(X)$ is such that $\int u\,d\mu < 0$, and let x be generic for μ, so $(1/n)S_n u(x) \to \int u\,d\mu < 0$. Let $\epsilon > 0$ be such that $S_n u(x) < -n\epsilon$ for all sufficiently large n, and let $\delta > 0$ be such that $|u(z) - u(y)| < \epsilon$ whenever $d(y, z) < \delta$. Then if $y \in X$ and $n \in \mathbb{N}$ are such that n is large and $x \in B(y, n, \delta)$, we have $u(f^k(y)) \leq u(f^k(x)) + \epsilon$ for all $0 \leq k < n$, whence $S_n u(y) \leq S_n u(x) + n\epsilon \leq 0$, contradicting property **(P)**.

5.2. *Proof of Proposition 2.2.* We will use the following lemma a number of times.

LEMMA 5.1. *Given $f : X \to X$, $\eta, \phi \in C(X)$, and $Z \subset X$, suppose that there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha \leq \underline{\lim}_{n\to\infty} \frac{1}{n} S_n\phi(x) \leq \overline{\lim}_{n\to\infty} \frac{1}{n} S_n\phi(x) \leq \beta$$

for every $x \in Z$. Then

$$P_Z(\eta) + \alpha t \leq P_Z(\eta + t\phi) \leq P_Z(\eta) + \beta t \tag{5.1}$$

for all $t > 0$.

Proof. This follows from the proof of Proposition 5.3 in [**31**]. □

Now fix $Z \subset X$ as in the statement of Proposition 2.2 and observe that

$$m_P(Z, 0, -tu, N, \delta) = \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} e^{S_{n_i}(-tu)} = m_u(Z, t, N, \delta)$$

for all $t \in \mathbb{R}$, $N \in \mathbb{N}$, and $\delta > 0$. Thus, $m_P(Z, 0, -tu, \delta) = m_u(Z, t, \delta)$.

Let $t^* = \inf\{t \in \mathbb{R} \mid P_Z(-tu) \leq 0\}$. Given $t < t^*$, we have $P_Z(-tu, \delta) > 0$ for all sufficiently small $\delta > 0$, and hence

$$m_u(Z, t, \delta) = m_P(Z, 0, -tu, \delta) = +\infty,$$

which implies that $\dim_u(Z, \delta) \geq t$. Since $t < t^*$ was arbitrary, this shows that $\dim_u Z \geq t^*$.

Now consider the sets $Z_m = \{x \in Z \mid \underline{\lim}_{n\to\infty}(1/n)S_n u(x) \geq 1/m\}$. By the assumption that $\underline{\lim}(1/n)S_n u(x) > 0$ for all $x \in Z$, we have $\bigcup_{m\geq 1} Z_m = Z$, and hence by (2.4), to show that $\dim_u Z \leq t^*$ it suffices to show that $\dim_u Z_m \leq t^*$ for all m.

To this end, fix $m \in \mathbb{N}$ and $t > t^*$. By Lemma 5.1, we have

$$P_{Z_m}(-tu) \leq P_{Z_m}(-t^*u) - \frac{t - t^*}{m} \leq P_Z(-t^*u) - \frac{t - t^*}{m} < 0,$$

and, hence, for all sufficiently small δ we have

$$m_u(Z_n, t, \delta) = m_P(Z, 0, -tu, \delta) = 0,$$

whence $\dim_u Z_m \leq t$. Since $t > t^*$ was arbitrary, this implies that $\dim_u Z_m \leq t^*$, and we are done.

Remark. Observe that if $Z = \emptyset$, then $P_Z(-tu) = -\infty$ for all $t \in \mathbb{R}$, which agrees with the convention that $\dim_u Z = -\infty$. If $Z \neq \emptyset$, then there exists $x \in Z$ with $\underline{\lim}(1/n)S_n u(x) = \alpha > 0$, whence for all $t < 0$ we have $P_Z(-tu) \geq P_{\{x\}}(-tu) \geq |t|\alpha > 0$, and so it suffices to consider $t \geq 0$ in Bowen's equation.

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5.3. *Proof of Proposition 2.3.* Let G_μ be the set of generic points for μ, so $(1/n)S_n u(x) \to \int u\,d\mu$ for every $x \in G_\mu$, and $h_{\text{top}}(G_\mu) = h_\mu(f)$ by [**19**, Theorem 3]. Using Lemma 5.1, we obtain

$$P_{G_\mu}(-tu) = P_{G_\mu}(0) - t\int u\,d\mu = h_{\text{top}}(G_\mu) - t\int u\,d\mu = h_\mu(f) - t\int u\,d\mu.$$

This is equal to 0 if and only if $t = h_\mu(f)/\int u\,d\mu$, so it follows from the Birkhoff ergodic theorem and Proposition 2.2 that $\dim_u \mu \le \dim_u G_\mu = h_\mu(f)/\int u\,d\mu$.

For the other inequality, we observe that if $Z \subset X$ is such that $\mu(Z) = 1$, then writing $Z' = \{x \in Z \mid (1/n)S_n u(x) \to \int u\,d\mu\}$, we have $\mu(Z') = 1$, whence $h_{\text{top}}(Z') \ge h_\mu(f)$, and so using Lemma 5.1 again, we have

$$P_{Z'}(-tu) = h_{\text{top}}(Z') - t\int u\,d\mu \ge h_\mu(f) - t\int u\,d\mu,$$

whence by Proposition 2.2 we have $\dim_u Z \ge \dim_u Z' \ge h_\mu(f)/\int u\,d\mu$.

5.4. *Proof of Proposition 2.4.*

LEMMA 5.2. *If $\phi \in C(X)$ is such that $P(\phi) \le 0$, then $P(\lambda\phi) \le 0$ for all $\lambda \ge 1$.*

Proof. For every $\mu \in \mathcal{M}^f(X)$, the fact that $P(\phi) \le 0$ implies that $h_\mu(f) + \int \phi\,d\mu \le 0$. In particular, $\int \phi\,d\mu \le 0$, hence for all $\lambda \ge 1$,

$$h_\mu(f) + \int \lambda\phi\,d\mu = \left(h_\mu(f) + \int \phi\,d\mu\right) + (\lambda - 1)\int \phi\,d\mu \le 0 + 0 = 0.$$

It follows that $P(\lambda\phi) = \sup_{\mu\in\mathcal{M}^f(X)}(h_\mu(f) + \int \lambda\phi\,d\mu) \le 0$. □

Now suppose $\mathcal{S}(\alpha) < 0$. Then there exist $q, t \in \mathbb{R}$ with $t < 0$ such that $P(q(\varphi - \alpha\psi) - tu) \le 0$. By Lemma 5.2, we have $R_\alpha(\lambda q, \lambda t) \le 0$ for every $\lambda \ge 1$, and so $T_\alpha(\lambda q) \le \lambda t$. In particular, since $t < 0$, this implies that $\mathcal{S}(\alpha) = \inf T_\alpha = -\infty$, and so $\alpha \notin I$.

5.5. *Proof of Theorem A.* Fix $\alpha \in \mathbb{R}$ and $t < \hat{\mathcal{F}}(\alpha)$. (If $\hat{\mathcal{F}}(\alpha) = -\infty$, then the desired inequality is automatic.) We claim that

$$R_\alpha(q, t) = P(q(\varphi - \alpha\psi) - tu) > 0, \tag{5.2}$$

for every q, and hence $T_\alpha(q) \ge t$. This will in turn imply that

$$\mathcal{S}(\alpha) = \inf\{T_\alpha(q) \mid q \in \mathbb{R}\} \ge t,$$

and since $t < \hat{\mathcal{F}}(\alpha)$ was arbitrary, this suffices. Thus, it remains only to prove (5.2).

LEMMA 5.3. *If $S_n\varphi(x)/S_n\psi(x) \to \alpha$, then $(1/n)S_n(\varphi - \alpha\psi) \to 0$.*

Proof. The proof follows as $|(1/n)S_n(\varphi - \alpha\psi)| = |S_n\varphi(x)/S_n\psi(x) - \alpha| \cdot |(1/n)S_n\psi(x)| \le |S_n\varphi(x)/S_n\psi(x) - \alpha| \sup \psi$. □

Using Bowen's equation (Proposition 2.2), the fact that $t < \hat{\mathcal{F}}(\alpha) = \dim_u \hat{K}_\alpha$ and $\underline{\lim}_{n\to\infty}(1/n)S_n u(x) > 0$ for all $x \in \hat{K}_\alpha$ implies that $P_{\hat{K}_\alpha}(-tu) > 0$. Furthermore, for every $q \in \mathbb{R}$, Lemmas 5.1 and 5.3 imply that $P_{\hat{K}_\alpha}(q(\varphi - \alpha\psi) - tu) = P_{\hat{K}_\alpha}(-tu)$, which establishes (5.2).

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5.6. *Proof of Theorem B.* The key tool is Ruelle's formula for the derivative of pressure.

PROPOSITION 5.4. *Fix $\eta, \phi \in C(X)$. If the function $q \mapsto P(\eta + q\phi)$ is differentiable at $q = 0$, and if in addition ν is an equilibrium state for η, then*

$$\frac{d}{dq} P(\eta + q\phi)|_{q=0} = \int \phi \, d\nu. \tag{5.3}$$

Proof. Write $g(q) = P(\eta + q\phi)$. Then for all $q \in \mathbb{R}$, we have

$$\begin{aligned} g(q) &= \sup_{\mu} \left\{ h(\mu) + \int \eta \, d\mu + \int q\phi \, d\mu \right\} \\ &\geq h(\nu) + \int \eta \, d\nu + q \int \phi \, d\nu = P(\eta) + q \int \phi \, d\nu, \end{aligned}$$

whence $g(q) - g(0) = g(q) - P(\eta) \geq q \int \phi \, d\nu$. In particular, for $q > 0$, we get $(1/q)(g(q) - g(0)) \geq \int \phi \, d\nu$, and hence $g'(q) \geq \int \phi \, d\nu$ (recall that differentiability of g was one of the hypotheses), while for $q < 0$, we have $(1/q)(g(q) - g(0)) \leq \int \phi \, d\nu$, and hence $g'(q) \leq \int \phi \, d\nu$, which establishes equality. □

In the setting of Theorem B, we observe that $R_\alpha(q, T_\alpha(q)) = 0$ for all q, and so

$$\begin{aligned} 0 &= \frac{d}{dq} P(q(\varphi - \alpha\psi) - T_\alpha(q)u) \\ &= \frac{d}{dq} P(q(\varphi - \alpha\psi) - \mathcal{S}(\alpha)u) + \frac{d}{dq} T_\alpha(q) \frac{d}{dt} P(q(\varphi - \alpha\psi) - tu)|_{t=\mathcal{S}(\alpha)}. \end{aligned}$$

The fact that q realizes the infimum in (2.12) implies that $(d/dq)T_\alpha(q) = 0$, and hence by Proposition 5.4, we have $\int (\varphi - \alpha\psi) \, d\nu_\alpha = 0$. This yields $\alpha = \int \varphi \, d\nu_\alpha / \int \psi \, d\nu_\alpha$ (recall we assume that $\int \psi \, d\nu_\alpha > 0$), and ergodicity of ν_α implies that $\nu_\alpha(K_\alpha) = 1$. Moreover, $\int u \, d\nu_\alpha$ implies that $\nu_\alpha(\hat{K}_\alpha) = 1$, so $\hat{\mathcal{F}}(\alpha) \geq \dim_u \nu_\alpha$.

Because ν_α is an equilibrium state for $q(\varphi - \alpha\psi) - \mathcal{S}(\alpha)u$, we have

$$0 = P(q(\varphi - \alpha\psi) - \mathcal{S}(\alpha)u) = h(\nu_\alpha) - \mathcal{S}(\alpha) \int u \, d\nu_\alpha,$$

and so

$$\mathcal{S}(\alpha) = \frac{h(\nu_\alpha)}{\int u \, d\nu_\alpha} = \dim_u \nu_\alpha \leq \mathcal{F}(\alpha).$$

5.7. *Proof of Proposition 2.6.* First we show that $\hat{I} \subset \operatorname{int} I$ without any hypotheses regarding $\int \psi \, d\mu$ or $h_\mu(f)$. To this end, fix $\alpha \in \hat{I}$, so $0 \in \operatorname{int} \mathcal{K}(\alpha)$. Thus, there exist $\eta > 0$ and $\mu^\pm \in \mathcal{M}^f(X)$ such that $\int (\varphi - \alpha\psi) \, d\mu^\pm = \pm\eta$. Choosing $\epsilon > 0$ sufficiently small, we have $\int (\varphi - \alpha'\psi) \, d\mu^+ > 0$ and $\int (\varphi - \alpha'\psi) \, d\mu^- < 0$ for all $\alpha' \in (\alpha - \epsilon, \alpha + \epsilon)$. It follows that $0 \in \operatorname{int} \mathcal{K}(\alpha')$ for each such α', so $\hat{I}$ is open.

Furthermore, given $\alpha \in \hat{I}$ and $\eta > 0$, $\mu^\pm \in \mathcal{M}^f(X)$ as before, we have

$$R_\alpha(q, 0) = P(q(\varphi - \alpha\psi)) \geq \max\left(h_{\mu^\pm}(f) + \int q(\varphi - \alpha\psi) \, d\mu^\pm \right) \geq |q|\eta > 0$$

for all $q \neq 0$, whence $T_\alpha(q) > 0$. It follows that $\mathcal{S}(\alpha) \geq 0$, and so $\alpha \in I$. Now since $\hat{I}$ is open and contained in I, we have $\hat{I} \subset \operatorname{int} I$.

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It remains to show that $\operatorname{int} I \subset \hat{I}$. To this end, let $\gamma > 0$ be such that $\int \psi \, d\mu \geq \gamma h_\mu(f)$ for all $\mu \in \mathcal{M}_E^f(X)$, and fix $\alpha \notin \hat{I}$. Then $0 \notin \operatorname{int} \mathcal{K}(\alpha)$, and since $\mathcal{K}(\alpha)$ is an interval, we either have $\mathcal{K}(\alpha) \subset [0, \infty)$ or $\mathcal{K}(\alpha) \subset (-\infty, 0]$.

First consider the case $\mathcal{K}(\alpha) \subset [0, \infty)$. Then $\int (\varphi - \alpha\psi) \, d\mu \geq 0$ for all $\mu \in \mathcal{M}_E^f(X)$, and so for every $\epsilon > 0$ and $q > 0$, we have

$$\begin{aligned} R_{\alpha+\epsilon}(q, t) &= P(q(\varphi - (\alpha + \epsilon)\psi) - tu) \\ &= \sup_{\mu \in \mathcal{M}_E^f(X)} \left(h_\mu(f) + q \int (\varphi - \alpha\psi) \, d\mu - q\epsilon \int \psi \, d\mu - t \int u \, d\mu \right) \\ &\leq \sup_{\mu \in \mathcal{M}_E^f(X)} (h_\mu(f)(1 - q\epsilon\gamma) + |t| \sup u). \end{aligned}$$

For sufficiently large q and for $t < 0$ with $|t|$ sufficiently small, this gives $R_{\alpha+\epsilon}(q, t) \leq 0$, whence $\mathcal{S}(\alpha + \epsilon) \leq T_{\alpha+\epsilon}(q) \leq t < 0$, and so $\alpha + \epsilon \notin I$. Since $\epsilon > 0$ was arbitrary, we have $\alpha \notin \operatorname{int} I$. The proof in the case $\mathcal{K}(\alpha) \subset (-\infty, 0]$ is similar, and we see that $\operatorname{int} I = \hat{I}$.

5.8. *Proof of Theorem C.* Let $\mathcal{M}_\alpha^f(X)$ be as in (1.3), and write

$$\tilde{\mathcal{S}}(\alpha) := \sup \left\{ \frac{h_\mu(f)}{\int u \, d\mu} \middle| \mu \in \mathcal{M}_\alpha^f(X) \right\}.$$

We begin by showing that $\mathcal{S}(\alpha) \geq \tilde{\mathcal{S}}(\alpha)$ for every α: to this end, fix $t < \tilde{\mathcal{S}}(\alpha)$. Then there exists $\mu \in \mathcal{M}_\alpha^f(X)$ such that $h_\mu(f) > t \int u \, d\mu$, and therefore for every $q \in \mathbb{R}$ we have

$$P(q(\varphi - \alpha\psi) - tu) \geq h(\mu) + q \int (\varphi - \alpha\psi) \, d\mu - t \int u \, d\mu > 0,$$

so $T_\alpha(q) > t$. This implies that $\mathcal{S}(\alpha) \geq t$, and since $t < \tilde{\mathcal{S}}(\alpha)$ was arbitrary, one direction is done.

It remains only to show that $\tilde{\mathcal{S}}(\alpha) \geq \mathcal{S}(\alpha)$ when $\alpha \in \hat{I}$. Fix $\alpha \in \hat{I}$, and for each $q, t \in \mathbb{R}$, write $\phi_{q,t} := q(\varphi - \alpha\psi) - tu$. Consider the collections of measures

$$\mathcal{M}^+ := \left\{ \mu \in \mathcal{M}_\alpha^f(X) \middle| \int (\varphi - \alpha\psi) \, d\mu \geq 0 \right\},$$
$$\mathcal{M}^- := \left\{ \mu \in \mathcal{M}_\alpha^f(X) \middle| \int (\varphi - \alpha\psi) \, d\mu \leq 0 \right\},$$

and define two functions $g^\pm \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g^\pm(q, t) := \sup \left\{ h_\mu(f) + \int \phi_{q,t} \, d\mu \middle| \mu \in \mathcal{M}^\pm \right\}.$$

Observe that $P(\phi_{q,t}) = \max(g^+(q, t), g^-(q, t))$, and so $\mathbb{R}^2 = E^+ \cup E^-$, where

$$\begin{aligned} E^\pm &:= \{(q, t) \in \mathbb{R}^2 \mid P(\phi_{q,t}) = g^\pm(q, t)\} \\ &= \{(q, t) \in \mathbb{R}^2 \mid g^\pm(q, t) \geq g^\mp(q, t)\}, \end{aligned} \tag{5.4}$$

It follows from the definitions that $g^-(q, t)$ is non-increasing in q and $g^+(q, t)$ is non-decreasing in q. Using the second characterization in (5.4), this implies that given any

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$(q, t) \in E^-$, we have $(q', t) \in E^-$ and $P(\phi_{q',t}) \geq P(\phi_{q,t})$ for all $q' < q$. Similarly, given any $(q, t) \in E^+$, we have $(q', t) \in E^+$ and $P(\phi_{q',t}) \leq P(\phi_{q,t})$ for all $q' > q$.

Now fix $t < \mathcal{S}(\alpha)$ and let $E^{\pm}(t) := \{q \in \mathbb{R} \mid (q, t) \in E^{\pm}\}$. Because $0 \in \operatorname{int} \mathcal{K}(\alpha)$ (which follows from $\alpha \in \hat{I}$), there exists $\mu \in \mathcal{M}^f(X)$ with $\int(\varphi - \alpha\psi)\, d\mu > 0$, whence $P(\phi_{q,t}) > P(\phi_{0,t})$ for sufficiently large values of q; for such q we have $q \in E^+(t)$, and so $E^+(t) \neq \emptyset$. A similar argument shows that $E^-(t) \neq \emptyset$.

Because both $E^-(t)$ and $E^+(t)$ are non-empty, there exists $q_0 \in \overline{E^-(t)} \cap \overline{E^+(t)}$. If $q_0 \in E^-(t)$, then we have $q_0 + \gamma \in E^+(t)$ for all $\gamma > 0$, while if $q_0 \in E^+(t)$, then $q_0 - \gamma \in E^-(t)$ for all $\gamma > 0$. Without loss of generality, we assume the former case; the proof in the latter case is similar.

Now we have $q^- := q_0 \in E^-(t)$ such that $q^+ \in E^+(t)$ for all $q^+ > q^-$. Let $\delta := \inf\{P(\phi_{q,t}) \mid |q - q^-| \leq 1\}$, and observe that $\delta > 0$. Choose $q^+ > q^-$ so that $q^+ - q^- < (\delta/2)\|\varphi - \alpha\psi\|^{-1}$. Then there exist $\mu^{\pm} \in \mathcal{M}^{\pm}$ such that

$$h_{\mu^{\pm}}(f) + \int \phi_{q^{\pm}}\, d\mu^{\pm} \geq \frac{\delta}{2}.$$

Choose $s \in [0, 1]$ such that $\mu := s\mu^- + (1 - s)\mu^+$ satisfies $\int(\varphi - \alpha\psi)\, d\mu = 0$. Then we have

$$\begin{aligned} h_\mu(f) - t\int u\, d\mu &= h_\mu(f) + \int \phi_{q^-,t}\, d\mu \\ &= s h_{\mu^-} + (1 - s) h_{\mu^+} + s\int \phi_{q^-,t}\, d\mu^- + (1 - s)\int \phi_{q^-,t}\, d\mu^+ \\ &\geq \frac{\delta}{2} + (1 - s)\int (\phi_{q^-,t} - \phi_{q^+,t})\, d\mu^+ \\ &\geq \frac{\delta}{2} - (q^+ - q^-)\|\varphi - \alpha\psi\| > 0, \end{aligned}$$

where the final inequality follows from the choice of q^+. Let $q = q(\alpha)$ be such that $T_\alpha(q) = \mathcal{S}(\alpha)$, such a q exists because $0 \in \operatorname{int} \mathcal{K}(\alpha)$, and observe that

$$\begin{aligned} h_\mu(f) - \mathcal{S}(\alpha)\int u\, d\mu &= h_\mu(f) + \int \phi_{q,\mathcal{S}(\alpha)}\, d\mu \\ &\leq P(\phi_{q,\mathcal{S}(\alpha)}) = 0 < h_\mu(f) - t\int u\, d\mu, \end{aligned}$$

whence $\int u\, d\mu > 0$, so $\mu \in \mathcal{M}^f_\alpha(X)$. Finally, the inequality $h_\mu(f) - t\int u\, d\mu > 0$ implies that $t < h_\mu(f)/\int u\, d\mu$, so $\tilde{\mathcal{S}}(\alpha) \geq \mathcal{S}(\alpha)$ since $t < \mathcal{S}(\alpha)$ was arbitrary.

5.9. *Proof of Proposition 2.7.* The proposition is a direct consequence of the following lemma, which applies even when $u \neq \psi$.

LEMMA 5.5. *For every $\lambda \geq 1$ and $\alpha, q \in \mathbb{R}$, we have $T_\alpha(\lambda q) \leq \lambda T_\alpha(q)$. In particular, if $q \in \mathbb{R}$ is such that $T_\alpha(q) < \infty$, then $T_\alpha(\lambda q) < \infty$ for all $\lambda \geq 1$.*

Proof. We have $T_\alpha(q) < \infty$ if and only if there exists $t < \infty$ such that $P(q(\varphi - \alpha\psi) - tu) \leq 0$. It follows from Lemma 5.2 that in this case $P(\lambda q(\varphi - \alpha\psi) - \lambda t u) \leq 0$ for all $\lambda \geq 1$, whence $T_\alpha(\lambda q) \leq \lambda T_\alpha(q) < \infty$. □

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5.10. *Proof of Theorem D.* To show that $\mathcal{S}$ is the concave hull of $\hat{\mathcal{F}}$ on $\hat{I}$, it suffices to show that

$$T_0(q) = \sup\{\hat{\mathcal{F}}(\alpha) + q\alpha \mid \alpha \in \mathbb{R}\}$$

for every $q \in \operatorname{int} Q$, since then $\mathcal{S}$ is the double Legendre transform of $\hat{\mathcal{F}}$ on $\hat{I}$, which is the concave hull. Thus, we prove the following two inequalities for every $q \in \operatorname{int} Q$:

$$T_0(q) \geq \hat{\mathcal{F}}(\alpha) + q\alpha \quad \text{for all } \alpha \in \mathbb{R}, \tag{5.5}$$

$$T_0(q) \leq \sup\{\hat{\mathcal{F}}(\alpha) + q\alpha \mid \alpha \in \mathbb{R}\}. \tag{5.6}$$

It follows from Theorem A that

$$\hat{\mathcal{F}}(\alpha) \leq \mathcal{S}(\alpha) \leq T_0(q) - q\alpha$$

for every q, α, which shows (5.5).

Turning our attention to (5.6), we see that if $\nu \in \mathcal{M}_E^f(X)$ is such that $h_\nu(f) + q\int \varphi\, d\nu - t\int u\, d\nu > 0$ and $\int u\, d\nu = 0$, then for all $t' > t$, we have

$$\begin{aligned} P(q\varphi - t'u) &\geq h_\nu(f) + q\int \varphi\, d\nu - t'\int u\, d\nu \\ &= h_\nu(f) + q\int \varphi\, d\nu - t\int u\, d\nu > 0 \end{aligned}$$

whence $T_0(q) = +\infty$. Thus, given $q \in \operatorname{int} Q$, $t < T_0(q)$, and potentials of the form $q\varphi - tu$, the supremum in the variational principle may be taken over measures with $\int u\, d\nu > 0$.

For such a q and t, let $\nu \in \mathcal{M}_E^f(X)$ be such that $h(\nu) + q\int \varphi\, d\nu - t\int u\, d\nu > 0$ and $\int u\, d\nu > 0$, and let α be such that $\nu(K_\alpha) = 1$. Then we have

$$(\dim_u \nu + q\alpha - t)u(\nu) > 0,$$

which immediately yields $t < \mathcal{F}(\alpha) - q\alpha$. Since $t < T_0(q)$ was arbitrary, this proves (5.6).

5.11. *Proof of Proposition 2.8.* Let $\mu \in \mathcal{M}_E^f(X)$. By the hypothesis, we have $h_\mu(f) = 0$ whenever $\int u\, d\mu = 0$, and so it remains only to consider measures with $\int u\, d\mu > 0$. For such a measure (2.6) and Proposition 2.3 give $\dim_u X \geq \dim_u \mu = h_\mu(f)/\int u\, d\mu$, which suffices.

5.12. *Proof of Corollary 2.9.* We begin by showing (2.23). The first equality is straightforward; recall that $\alpha \in I$ if and only if $T_\alpha(q) \geq 0$ for all q, which in this case is equivalent to the inequality $P(q\varphi) \geq q\alpha$ for all q. It follows from properties of convex functions that this is true if and only if

$$\alpha \in \Big[\lim_{q\to-\infty} D^- T_0(q), \lim_{q\to\infty} D^+ T_0(q)\Big] = \overline{\bigcup_{q\in\mathbb{R}} J(q)}.$$

For the second equality in (2.23), we first observe that if $\mu \in \mathcal{M}^f(X)$ is such that $\int \varphi\, d\mu = \alpha$, then we have $P(q\varphi) \geq h_\mu(f) + \int q\varphi\, d\mu \geq q\alpha$ for all $q \in \mathbb{R}$. The other inclusion goes as follows. Given $q \in \mathbb{R}$, let $\alpha^+ = D^+ T_0(q)$. For $\gamma > 0$, let μ_γ be an equilibrium state for $(q+\gamma)\varphi$, and let $\mu \in \mathcal{M}^f(X)$ be a weak* limit of μ_γ as $\gamma \to 0$. By upper

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semi-continuity, μ is an equilibrium state for $q\varphi$ and, moreover, $\int \varphi\, d\mu = \lim_{\gamma\to 0} \int \varphi\, d\mu_\gamma = \alpha^+$ since $\int \varphi\, d\mu_\gamma \in J(q+\gamma)$ for all γ. Because $\alpha^+ \in \partial J(q)$, there exists an ergodic component ν^+ of μ which is also an equilibrium state for $q\varphi$ and which has $\int \varphi\, d\nu^+ = \alpha^+$.

A similar argument produces an ergodic measure ν^- which is an equilibrium state for $q\varphi$ and which has $\int \varphi\, d\nu^- = \alpha^-$. Taking convex combinations of $\nu^\pm$ shows that $\{\int \varphi\, d\mu \mid \mu \in \mathcal{M}^f(X)\} \supset J(q)$ for every $q \in \mathbb{R}$, which completes the characterization of I. (Note that the set of values taken by the integral is closed since it is a continuous image of a compact set.)

The completeness result $K_\alpha = \emptyset$ for all $\alpha \notin I$ follows from Theorem A. When α is such that $J(q) = \{\alpha\}$ for some $q \in \mathbb{R}$, the first two lines of equation (2.24) follow from Theorems B and E. When $\alpha \in \partial J(q)$ for some q, they follow from Theorem E and the argument above that produces an ergodic equilibrium state ν for $q\varphi$ with $\int \varphi\, d\nu = \alpha$.

The third line of equation (2.24) follows from Theorem C, and the fourth follows since we proved the first two lines by constructing a full measure. Finally, the last statement of Corollary 2.9 follows from Corollary 2.5.

5.13. *Proof of Corollary 2.10.* Once we establish the form of $\mathcal{S}$ and I, the proof here is virtually identical to the proof of Corollary 2.9. Given $\alpha > 0$ and $t, q \in \mathbb{R}$, we see that the following are equivalent:

$$t < T_\alpha(q) \Leftrightarrow P(qu - q\alpha - tu) > 0 \quad \Leftrightarrow \quad P((q-t)u) - q\alpha > 0$$
$$\Leftrightarrow P(q'u) - q'\alpha > t\alpha \quad \text{where } q' = q - t \quad \Leftrightarrow \quad t < \frac{1}{\alpha}(T_0(q') - q'\alpha).$$

Thus, since $t < \mathcal{S}(\alpha)$ if and only if $t < T_\alpha(q)$ for every q, which is equivalent to $t < (1/\alpha)(T_0(q') - q'\alpha)$ for every q', we see that $\mathcal{S}$ has the form claimed. It follows immediately that for $\alpha > 0$, we have $\mathcal{S}(\alpha) \geq 0$ if and only if $\inf\{T_0(q) - q\alpha \mid q \in \mathbb{R}\} \geq 0$, and then (2.27) follows from (2.23), with the possible exception of $\alpha = 0$.

For $\alpha = 0$, we have $T_\alpha(q) = T_0(q)$ and so $\mathcal{S}(\alpha) = \inf_q T_\alpha(q) = \inf_q T_0(q)$. Thus, $0 \in I$ if and only if $T_0(q) \geq 0$ for all $q \in \mathbb{R}$. This is the same criterion as in Corollary 2.9, which completes the proof of (2.27).

Applying the results from [**31**] as in the discussion, the rest of Corollary 2.10 follows as Corollary 2.9 did.

5.14. *Proof of Corollary 2.11.* Let $\tilde{X} = \{x \in X \mid (2.30) \text{ holds}\}$, and observe that under the assumptions of Corollary 2.11, we have $\tilde{X} = X$ and $X^\star = \hat{X}$. In fact, we prove the corollary under the slightly weaker assumptions that writing $K_\alpha = K_\alpha^{d_\mu} \cup K_\alpha^{\varphi,u}$, we have $K_\alpha \subset \tilde{X}$ and $K_\alpha \cap \hat{X} \subset X^\star$ for every $0 \leq \alpha < \infty$.

PROPOSITION 5.6. *Let X be a compact metric space and $f\colon X \to X$ be conformal (without critical points or singularities). Suppose $\mu \in \mathcal{M}(X)$ and $\phi \in C(X)$ satisfy (2.29), and let $\psi = u = \log a$, $\varphi = \phi - P(\phi)$. Suppose u satisfies **(P)**. Then for every $0 \leq \alpha < \infty$, we have*

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$$\tilde{X} \cap K_\alpha^{\varphi,u} = \tilde{X} \cap K_\alpha^{d_\mu} \subset \hat{X}, \tag{5.7}$$

$$\dim_u(K_\alpha^{\varphi,u} \cap X^\star) = \dim_H(K_\alpha^{d_\mu} \cap X^\star). \tag{5.8}$$

Proof. Fix $x \in \tilde{X}$ and observe that by [**31**, Lemma 6.1], for every $\epsilon > 0$ there exist $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta = \eta(x, \epsilon) > 0$ such that for every $n \in \mathbb{N}$ and $0 < \delta < \delta_0$,

$$B\big(x, \eta\delta e^{-(S_n u(x)+n\epsilon)}\big) \subset B(x, n, \delta) \subset B\big(x, \delta e^{-(S_n u(x)-n\epsilon)}\big). \tag{5.9}$$

Furthermore, by the weak Gibbs property (2.29), for every $\gamma > 0$ we may assume by taking $\delta > 0$ sufficiently small that

$$|S_n\varphi(x) + \log \mu(B(x, n, \delta))| \le n\gamma \tag{5.10}$$

for all large n. Define $r_1, r_2 \colon \mathbb{N} \to (0, \delta)$ by

$$r_1(n) = \delta e^{-(S_n u(x)-n\epsilon)}, \quad r_2(n) = \eta\delta e^{-(S_n u(x)+n\epsilon)},$$

so that (5.9) becomes

$$B(x, r_2(n)) \subset B(x, n, \delta) \subset B(x, r_1(n)). \tag{5.11}$$

Observe that because u is bounded away from $\pm\infty$, there exists $\xi > 0$ such that $r_i(n+1)/r_i(n) \in [\xi^{-1}, \xi]$ for all n. In particular, $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$ can be computed by considering balls with radius $r_i(n)$.

It follows from equations (5.11) and (5.10) that

$$\log \mu(B(x, r_2(n))) \le \log \mu(B(x, n, \delta)) \le -(S_n\varphi(x) - n\gamma),$$

whence

$$\frac{\log \mu(B(x, r_2(n)))}{\log r_2(n)} \ge \frac{-(S_n\varphi(x) - n\gamma)}{-(S_n u(x) + n\epsilon) + \log(\delta\eta)}. \tag{5.12}$$

If $\underline{\lim}(1/n)S_n u(x) = 0$, then there exists $n_k \to \infty$ such that $(1/n_k)S_{n_k}u(x) \to 0$. Since $x \in \tilde{X}$, this implies that $\underline{\lim}(1/n_k)S_{n_k}\varphi(x) > 0$; in particular, we can choose γ small enough that $S_{n_k}\varphi(x) \ge 2n_k\gamma$ for all k, whence (5.12) gives

$$\frac{\log \mu(B(x, r_2(n_k)))}{\log r_2(n_k)} \ge \frac{n_k\gamma}{S_{n_k}u(x) + n_k\epsilon - \log(\delta\eta)}.$$

Taking the limit as $k \to \infty$ gives $\overline{d}_\mu(x) \ge \gamma/\epsilon$, and since $\epsilon > 0$ was arbitrary, this shows that $d_\mu(x) \ne \alpha$ for all $0 \le \alpha < \infty$, whence $K_\alpha^{d_\mu} \cap \tilde{X} \subset \hat{X}$.

Similarly, given $x \in \tilde{X} \setminus \hat{X}$, we have n_k such that $S_{n_k}\varphi(x)/S_{n_k}u(x) \to \infty$, whence $S_n\varphi(x)/S_n u(x) \not\to \alpha$ for any $0 \le \alpha < \infty$; thus, $K_\alpha^{\varphi,u} \cap \tilde{X} \subset \hat{X}$.

Now to establish (5.7), we can restrict our attention to the case where $x \in \hat{X}$, that is, $\underline{\lim}(1/n)S_n u(x) > 0$.

Given $x \in \hat{X}$, fix $\xi > 0$ and suppose ϵ, γ are small enough such that $\xi((1/n)S_n u(x)) > \max\{\epsilon, \gamma\}$ for all large n. Then (5.12) gives

$$\frac{\log \mu(B(x, r_2(n)))}{\log r_2(n)} \ge \frac{S_n\varphi(x) - \xi(S_n u(x)))}{(S_n u(x))(1+\xi) - \log(\delta\eta)},$$

and since ξ can be arbitrarily small, we have $\underline{d}_\mu(x) \ge \underline{\lim}(S_n\varphi(x)/S_n u(x))$.

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Similarly, it follows from equations 5.11 and 5.10 that

$$\log \mu(B(x, r_1(n))) \geq \log \mu(B(x, n, \delta)) \geq -(S_n\varphi(x) + n\gamma),$$

whence with ξ as before, we have

$$\frac{\log \mu(B(x, r_1(n)))}{\log r_1(n)} \leq \frac{-(S_n\varphi(x) + n\gamma)}{-(S_nu(x) - n\epsilon) + \log \delta} \leq \frac{S_n\varphi(x) + \xi(S_nu(x))}{(S_nu(x))(1-\xi) - \log \delta},$$

and so $\overline{d}_\mu(x) \leq \overline{\lim}(S_n\varphi(x)/S_nu(x))$. This finishes the proof of (5.7). As pointed out in §2.6, we have $\dim_u Z = \dim_H Z$ for all $Z \subset X^\star$, and (5.8) follows. □

With Proposition 5.6 in hand, the proof of Corollary 2.11 follows exactly the same lines as the proof of Corollary 2.9, using the results from §§2.2–2.4. The only difference is that we need the hypotheses that $\dim_u X < \infty$ and $\int u\, d\mu > 0$ for every $\mu \in \mathcal{M}^f(X)$ with $h_\mu(f) > 0$ in order to apply Proposition 2.8 and show that Theorem C applies on all of int I. For the form of I, it suffices to observe that the following statements are equivalent:

(1) $\alpha \in \overline{\bigcup_{q\in\mathbb{R}} J(q)}$;
(2) $T_0(q) \geq q\alpha$ for all $q \in \mathbb{R}$;
(3) $P(q\varphi - t \log f') > 0$ for every $q \in \mathbb{R}$ and $t < q\alpha$;
(4) $P(q\varphi - (q\alpha + t') \log f') > 0$ for every $q \in \mathbb{R}$ and $t' < 0$;
(5) $\mathcal{S}(\alpha) \geq 0$.

5.15. *Proofs of results in examples.*

5.15.1. *Proof of Proposition 4.1.* By passing to an iterate of σ and recoding, we can assume that X is a two-step subshift of finite type on the symbols $0, 1, \ldots, d$, that $\bigcup_n X_n = X \setminus \{0^\infty\}$, and that 01 and 10 are legal words for X. Let $E(k) = \{0, 1, \ldots, d\}^k$, and for every $w \in E(k)$, let $[w] = \{x \in X_A \mid x_i = w_i \text{ for all } 1 \leq i \leq k\}$. Let $E_X(k) = \{w \in E(k) \mid [w] \neq \emptyset\}$, and let $E_n(k)$ be the set of words in $E_X(k)$ which do not contain more than n consecutive appearances of the symbol 0. We estimate P_{X_n}, where $X_n = \{x \in X_A \mid [x_1 \ldots x_k] \in E_n(k) \text{ for all } k\}$.

Define $\phi_k : E_X(k) \to \mathbb{R}$ by $\phi_k(w) = \sup\{S_n\phi(x) \mid x \in [w]\}$. Consider

$$\Lambda_k(X, \phi) = \sum_{w\in E_A(k)} e^{\phi_k(w)}, \quad \Lambda_k(X_n, \phi) = \sum_{w\in E_n(k)} e^{\phi_k(w)},$$

so that $P_X(\phi) = \lim_{k\to\infty}(1/k) \log \Lambda_k(X, \phi)$, and similarly for X_n. Now define $\pi : E_X(k) \to E_n(k)$ by replacing every nth 0 with a 1 in each string of more than n consecutive zeros. We estimate the cardinalities $|\pi^{-1}(w)|$.

Fix $w \in E_n(k)$ and let $J = \{j \in \{1, \ldots, k\} \mid w_i = 0 \text{ for all } j - n \leq i < j\}$. Then we have $|J| \leq k/n$, and furthermore, if $\pi(w') = w$, then $w_i = w'_i$ for all $i \notin J$, while for each $j \in J$ we have $w'_j\{0, 1\}$. Thus, $|\pi^{-1}(w)| \leq 2^{k/n}$.

Furthermore, if $\pi(w') = w$, then w' and w disagree in at most k/n indices, which are separated by at least n. Let $m_1, \ldots, m_p \geq n$ be such that

$$\{m_1 + m_2 + \cdots + m_j \mid 1 \leq j \leq p\} = \{i \in \{1, 2, \ldots, k\} \mid w'_i \neq w_i\},$$

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and observe that $p \le k/n$. Writing $V_\ell(\phi) = \sup\{\phi(x) - \phi(y) \mid x_i = y_i \text{ for all } 1 \le i \le \ell\}$, we have

$$|\phi_k(w') - \phi_k(w)| \le \sum_{i=1}^{p} \sum_{\ell=1}^{m_i} V_\ell(\phi) \le \frac{k}{n} \sum_{\ell=1}^{n} V_\ell(\phi),$$

where the last inequality follows from the fact that $V_\ell(\phi) \le V_{\ell'}(\phi)$ for all $\ell' \ge \ell$. Now we can compare the partition sums, and hence the pressures:

$$\begin{aligned}\Lambda_k(X, \phi) &= \sum_{w \in E_n(k)} \sum_{w' \in \pi^{-1}(w)} e^{\phi_k(w')} \\ &\le 2^{k/n} e^{(k/n)\sum_{\ell=1}^{n} V_\ell(\phi)} \Lambda_k(X_n, \phi), \\ \frac{1}{k} \log \Lambda_k(X, \phi) &\le \frac{1}{n}\left(\log 2 + \sum_{\ell=1}^{n} V_\ell(\phi)\right) + \frac{1}{k} \log \Lambda_k(X_n, \phi).P_{X_n}(\phi).\end{aligned}$$

Take the limit as $k \to \infty$ and observe that since ϕ is uniformly continuous, we have $\lim_{\ell\to\infty} V_\ell(\phi) = 0$, which implies that $\lim_{n\to\infty}(1/n)\sum_{\ell=1}^{n} V_\ell(\phi) = 0$.

5.15.2. *Proof of Proposition 4.4.* We start with the following lemma, which is the only place where the assumption on the form of f near p is used. If f is C^1 and the result of this lemma is shown to hold, then the result of the proposition holds for f.

LEMMA 5.7. *For all $\gamma > 0$ there is a sequence of points $x_n \in B(p, n, \gamma) \cap X$ such that $S_n u(x_n) \to \infty$.*

Remark. If we remove the requirement that $x_n \in X$ and only require that $x_n \in B(p, n, \gamma)$, then Lemma 5.7 can be proved for any non-uniformly expanding C^1 map f without any assumptions on its behaviour near p. However, this would require changing the definition of u-dimension to allow covers by Bowen balls whose centres are in $U \setminus X$, which was overlooked in the application of this result in [**32**].

Proof. In fact, we prove something stronger: for all sufficiently small $\gamma > 0$ and any sequence of points $x_n \in B(p, n, \gamma) \setminus B(p, n+1, \gamma)$, we have $S_n u(x_n) \to \infty$.

Let $\gamma > 0$ be small enough that $|f'(x)| \ge 1 + cd(x, p)^\epsilon$ for all $x \in B(p, \gamma)$ and $\log(1 + r) \ge r/2$ for all $0 < r < \gamma^\epsilon$. Then $u(x) = \log |f'(x)| \ge (c/2)d(x, p)^\epsilon$ in $B(p, \gamma)$ and, in particular,

$$S_n u(x) \ge \frac{c}{2} \sum_{k=0}^{n-1} d(f^k x, p)^\epsilon. \tag{5.13}$$

Because f' is ϵ-Hölder there is $K \in \mathbb{R}$ such that $|f'(x)| \le K d(x, p)^\epsilon$, which implies that

$$d(f(x), p) = \left|\int_p^x f'(y)\, dy\right| \le \int_0^{d(x,p)} (1 + Kt^\epsilon)\, dt \le d(x, p) + \frac{K}{1+\epsilon} d(x, p)^{1+\epsilon}.$$

Using the fact that for $r, L > 0$ we have

$$(r + Lr^{1+\epsilon})^{-\epsilon} = r^{-\epsilon}(1 + Lr^\epsilon)^{-\epsilon} \ge r^{-\epsilon}(1 - \epsilon Lr^\epsilon) = r^{-\epsilon} - \epsilon L,$$

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this yields

$$d(f(x),\, p)^{-\epsilon} \geq d(x,\, p)^{-\epsilon} - \frac{\epsilon}{1+\epsilon} K.$$

We see that if $x_n \in B(p, n\gamma) \setminus B(p, n+1, \gamma)$, then $f^k(x_n)$ has the property that $d(f^{n-k+1}(f^k(x_n)),\, p) \geq \gamma$, and so

$$d(f^k(x_n),\, p)^{-\epsilon} \leq \gamma^{-\epsilon} + \frac{\epsilon}{1+\epsilon} K(n-k+1).$$

Together with (5.13), this shows that $S_n u(x_n) \to \infty$ as $n \to \infty$. □

To prove Proposition 4.4, we must show that **(P)** holds for each $\delta > 0$. Using the fact that f has a generating Markov partition, pass to a refinement $\mathcal{P}$ with the property that $\mathrm{diam}\mathcal{P} < \delta$. Let $\pi : \Sigma_A \to X$ be the coding map for the Markov partition $\mathcal{P}$, and suppose without loss of generality that $\pi(000\cdots) = p$.

Let L_N denote the collection of N-cylinders in Σ_A, and given an N-cylinder $[w]$, let $\phi_N(w) := \sup_{x\in[w]} S_N u(x)$. Observe that to establish **(P)**, it suffices to show that $\lim_{N\to\infty} \inf_{[w]\in L_N} \phi_N(w) = \infty$, since then we may take as our collection E_N the set $\{(x_w, N) \mid [w] \in L_N\}$, where x_w is such that $\phi_N(w) = S_N u(x_w)$.

Let $\gamma := \inf\{u(x) \mid x_0 \neq 0\}$, and observe that $\gamma > 0$. Let $C_N := \phi_N(0^N)$, and observe that by the arguments in the first half of the proof, we have $C_N \to \infty$. Now given $[w] \in L_N$, one of two things happens; either the word w contains a string of at least $\sqrt{N}$ consecutive zeros, in which case $\phi_N(w) \geq C_{\sqrt{N}}$, or w contains at least $\sqrt{N}$ symbols that are not 0, in which case $\phi_N(w) \geq \sqrt{N}\gamma$. It follows that $\inf_{[w]\in L_N} \phi_N(w) \geq \min(\sqrt{N}\gamma, C_{\sqrt{N}})$, which suffices.

5.15.3. *Proof of Lemma 4.7.* Suppose $x \in X \setminus \hat{X}$, so $\underline{\lim}_{n\to\infty}(1/n)S_n u(x) = 0$. Given $\delta > 0$, let $\epsilon(\delta) > 0$ be such that $u(x) \geq \epsilon(\delta)$ for all x with $B(x,\, p) \geq \delta$.

By the hypothesis on x, there exists n_k such that $(1/n_k)S_{n_k} u(x) \to 0$. Let $\delta_k \to 0$ be such that writing $\epsilon_k = \epsilon(\delta_k)$, we have $S_{n_k} u(x) \leq n_k \epsilon_k \delta_k$. Let $q_n(x, \delta) = \#\{k \in \{0, 1, \ldots, n-1\} \mid d(f^k(x),\, p) \geq \delta$, and observe that $S_n u(x) \geq q_n(x, \delta)\epsilon(\delta)$. In particular, we have

$$q_{n_k}(x, \delta_k)\epsilon_k \leq S_{n_k} u(x) \leq n_k \epsilon_k \delta_k,$$

whence $q_{n_k}(x, \delta_k) \leq \delta_k n_k$. It follows immediately that $\lim_{k\to\infty}(1/n_k)S_{n_k}\varphi(x) = \varphi(p)$, and so either $x \in K_{\varphi(p)}$ or $(1/n)S_n\varphi(x)$ does not converge to a limit, in which case $x \notin K_\alpha$ for any α.

5.15.4. *Proof of Lemma 4.9.* Fix $x \in X$ and suppose there is a subsequence n_k is such that $(1/n_k)S_{n_k} u(x) \to 0$. As in the proof of Lemma 4.7, we have $(1/n_k)S_{n_k}\phi(x) \to \phi(p) < P(\phi)$, and so writing $\varphi = P(\phi) - \phi$, we see that $(1/n_k)S_{n_k}\varphi(x) \to P(\phi) - \phi(p) < 0$. It follows that x satisfies (2.30).

6. *Background material on Carathéodory dimension characteristics*

We show that u-dimension is a Carathéodory dimension characteristic even if u is not uniformly positive, so long as condition **(P)** is satisfied. Following [**97**, Chapter 4], recall that a C-structure $\mathcal{C} = \{\mathcal{S}, \mathcal{F}, \xi, \eta, \psi\}$ on a set X is specified by an arbitrary index set $\mathcal{S}$,

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a collection of subsets of X indexed by $\mathcal{S}$ and denoted $\mathcal{F} = \{U_s \mid s \in \mathcal{S}\}$, and functions $\eta, \psi, \xi : \mathcal{S} \to \mathbb{R}^+$ such that:

A1 $\emptyset \in \mathcal{F}$; if $U_s = \emptyset$ then $\eta(s) = \psi(s) = 0$; if $U_s \neq \emptyset$ then $\eta(s) > 0$ and $\psi(s) > 0$;

A2 for any $\delta > 0$ there exists $\epsilon > 0$ such that $\eta(s) \leq \delta$ for any $s \in \mathcal{S}$ with $\psi(s) \leq \epsilon$;

A3 for any $\epsilon > 0$ there exists a finite or countable subcollection $\mathcal{G} \subset \mathcal{S}$ that covers X (that is, $\bigcup_{s\in\mathcal{G}} U_s \supset X$) and has $\psi(\mathcal{G}) := \sup\{\psi(s) \mid s \in \mathcal{G}\} \leq \epsilon$.

A C-structure defines a Carathéodory dimension characteristic as described in [**97**]; §2.1 gives the definition in the specific case of u-dimension.

PROPOSITION 6.1. *If $u \in C(X)$ satisfies **(P)**, then the u-dimension defined in §2.1 is a Carathéodory dimension characteristic.*

Proof. It suffices to show that the candidate C-structure implicit in §2.1 satisfies A1, A2, and A3. Fix $\delta > 0$ and let the index set $\mathcal{S}$ be the union of the index sets from the covers E_N in **(P)**. Let $\mathcal{F}$ be the corresponding Bowen balls $B(x, n, \delta)$, with $\eta(x, n) = e^{-S_n u(x)}$, $\xi(x, n) = 1$, and $\psi(x, n) = 1/n$. Including $\emptyset$ in $\mathcal{F}$, this satisfies A1, and A3 follows since each E_N is a cover of X on which $\psi(x, n) = 1/n \leq 1/N$. Finally, A2 follows from the requirement that $\lim_{N\to\infty} \inf_{(x,n)\in E_N} S_n u(x) = \infty$. □

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