Blocks of decimal digits

K. ROBIN McLEAN

Recently a friend kindly made me a birthday card whose background consisted of rows and rows of digits, some 3000 in all. There appeared to be no discernible pattern in the digits. Perhaps they had been taken from a table of random numbers. They were certainly not the opening digits of the decimal parts of π or $\sqrt{2}$, although they might, so far as I knew, have been consecutive digits of either number in some section remote from the decimal point.

On thinking about this, I realised that they must be the opening digits of the decimal part of the square root of some whole number. Indeed, they must be the opening digits of the decimal parts of the square roots of infinitely many positive integers. It is remarkably easy to prove this and the argument is simple enough to be appreciated and understood in school classrooms at GCSE level.

To illustrate the method, we work with a block of 2 decimal places rather than 3000. Let us suppose that we wish to find a positive integer N for which the decimal expansion of N consists of some integer, n, followed by the decimal .47…. Then

$$0 < n + 0.47 < \sqrt{N} < n + 0.48.$$

So

$$(n + 0.47)^2 < N < (n + 0.48)^2. \tag{1}$$

Provided that the left- and right-hand sides of this inequality differ by more than 1, we can always find an integer, N, in the gap between them. Thus we seek an integer n such that

$$(n + 0.48)^2 - (n + 0.47)^2 > 1.$$

Using the difference of two squares, we get $0.01\,(2n + 0.95) > 1$. This inequality is certainly satisfied when $0.02n \geqslant 1$, that is, when $n \geqslant 50$. Choosing $n = 50$ and referring back to (1), the integer N must lie between $50.47^2 = 2547.2209$ and $50.48^2 = 2548.2304$. This forces N to be 2548 and, sure enough, $\sqrt{2548} = 50.47\ldots$.

To generate infinitely many values of $\sqrt{N}$, we could take in succession $n = 51, 52, 53, \ldots$ and use the inequality (1) to find an appropriate integer N for each n. Alternatively, we could use the succession $n = 50$, $n = 500$, $n = 5000$, $n = 50000$, … . This leads to

$$\sqrt{2548} = 50.47\ldots$$

$$\sqrt{250480} = 500.47\ldots$$

$$\sqrt{25004800} = 5000.47\ldots$$

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$$\sqrt{2500048000} = 50000.47\ldots$$

$$\ldots \quad \ldots \quad \ldots$$

and it is clear that the obvious pattern can be extended indefinitely.

More advanced methods allow us to prove a theorem that generalises this result to any number of decimal places and to other functions such as x^k (with $0 < k < 1$) and $\ln x$.

Theorem 1: Let $d_1, d_2, \ldots, d_r$ be any finite sequence of decimal digits and let $f(x)$ be any differentiable function of a real variable x such that

(a) $f(x) \to \infty$ as $x \to \infty$,

(b) $f'(x) > 0$ for all $x > 0$, and

(c) $f'(x) \to 0$ as $x \to \infty$.

Then there are infinitely many positive integers N such that the decimal part of $f(N)$ begins with the digits $d_1, d_2, \ldots, d_r$.

Proof: Let $d = 10^{-1}d_1 + 10^{-2}d_2 + \ldots + 10^{-r}d_r$. It is sufficient to show that, given any $M > 0$, there are positive integers, n and N such that $M < N$ and

$$n + d \leqslant f(N) < n + d + 10^{-r}. \tag{2}$$

Since $f'(x) \to 0$ as $x \to \infty$, there exists N_1 such that $M < N_1$ and

$$0 < f'(x) < 10^{-r} \text{ for all } x > N_1.$$

Let n be the least integer that exceeds $f(N_1)$. As $f(x) \to \infty$ as $x \to \infty$, there exists N_2 such that $n + 1 < f(N_2)$. So $f(N_1) < n < n + 1 < f(N_2)$.

The mean value theorem and condition (b) imply that f is strictly increasing, so $N_1 < N_2$. By the intermediate value theorem, there exists N_3 such that $N_1 < N_3 < N_2$ and $f(N_3) = n + d$. Similarly there exists N_4 such that $N_3 < N_4 < N_2$ and $f(N_4) = n + d + 10^{-r}$. The mean value theorem shows that

$$f(N_4) - f(N_3) = (N_4 - N_3)f'(\xi) \tag{3}$$

for some ξ such that $N_3 < \xi < N_4$. But $N_1 < N_3 < \xi$, so $0 < f'(\xi) < 10^{-r}$. From (3),

$$N_4 - N_3 = \frac{f(N_4) - f(N_3)}{f'(\xi)} = \frac{10^{-r}}{f'(\xi)} > 1.$$

So there is an integer N such that $N_3 < N < N_4$. As f is strictly increasing, we get $f(N_3) < f(N) < f(N_4)$, giving (2) as desired.

A statement equivalent to Theorem 1 is that, if $f(x)$ satisfies the conditions of the theorem, then the decimal parts of $f(n)$ for $n \in \mathbb{N}$ are dense in the unit interval [0, 1]. This means that each subinterval of [0, 1] contains at least one such decimal part.

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The following lemma is useful when considering the decimal parts of periodic functions.

Lemma: Let $I = \{x \in \mathbb{R} : 0 \leqslant a < x < b\}$ be any open interval of positive real numbers and let α be any positive irrational number. Then there are infinitely many pairs of natural numbers m, n such that $m - n\alpha$ lies in I.

Proof: The decimal parts of α, 2α, 3α, … are all distinct, for otherwise α would be rational. So, given ε in the range $0 < \varepsilon < 1$, there are natural numbers n_1, n_2 such that $n_1 > n_2$ and the decimal parts, f_1 and f_2, of $n_1\alpha$ and $n_2\alpha$ respectively are within ε of each other. As $\alpha > 0$, we may take

$$n_1\alpha = m_1 + f_1 \tag{4}$$

$$\text{and} \qquad n_2\alpha = m_2 + f_2, \tag{5}$$

where $m_1, m_2 \in \mathbb{N}$. Now $n_1 > n_2$ and $\alpha > 0$, so $m_1 \geqslant m_2$. We claim that there are natural numbers p and q such that

$$0 < p - q\alpha < \varepsilon. \tag{6}$$

From (4) and (5), $(m_1 - m_2) - (n_1 - n_2)\alpha = f_2 - f_1$. If $f_2 > f_1$, (6) follows immediately by taking $p = m_1 - m_2$ and $q = n_1 - n_2$. Otherwise, $f_1 > f_2$ and

$$0 < (n_1 - n_2)\alpha - (m_1 - m_2) < \varepsilon.$$

Hence there is a positive integer k such that

$$1 - \varepsilon < k\{(n_1 - n_2)\alpha - (m_1 - m_2)\} < 1.$$

This yields

$$0 < 1 + k(m_1 - m_2) - k(n_1 - n_2)\alpha < \varepsilon,$$

Taking $p = 1 + k(m_1 - m_2)$ and $q = k(n_1 - n_2)$ gives (6) as desired.

To complete the proof, take any positive integer N (as large as we wish), choose $\varepsilon = \dfrac{b - a}{N}$ and note that (6) ensures that at least N sufficiently large multiples of $p - q\alpha$ lie in I.

To illustrate the use of the lemma, we apply it to values of periodic functions in the next theorem.

Theorem 2: Let $d_1, d_2, \ldots, d_r$ be any finite sequence of decimal digits. Then there are infinitely many positive integers N such that the decimal part of $\sin N$ begins with the digits $d_1, d_2, \ldots, d_r$. The same result holds when $\sin N$ is replaced in turn by each of $\cos N$, $\tan N$, $\operatorname{cosec} N$, $\sec N$ and $\cot N$.

Proof: Let $d = 10^{-1}d_1 + 10^{-2}d_2 + \ldots + 10^{-r}d_r$. As $\sin x$ is continuous and monotonic over the range $0 \leqslant x \leqslant \frac{1}{2}\pi$, increasing from 0 to 1, there are real numbers a and b in this range such that $\sin a = d$ and $\sin b = d + 10^{-r}$. Now π is irrational, so, by our lemma, there are

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infinitely many pairs of natural numbers N and n such that $N - 2n\pi$ lies between a and b. Hence $\sin N = \sin(N - 2n\pi)$ lies between d and $d + 10^{-r}$ as desired. The proofs for each of the other trigonometrical ratios are similar.

Whilst the decimal parts of periodic functions are easily handled by Theorem 2, those of exponential functions take us to the limits of current knowledge and we can only indicate the intricacy of the situation. Let $I = \{x \in \mathbb{R} : 1 \leqslant a < x < b\}$ be any open interval of real numbers greater than 1. Let us call a given $x \in \mathbb{R}$ *good* if the decimal parts of $x^n (x \in \mathbb{N})$ are dense in the unit interval [0, 1] and let us say that x is *bad* if it is not good. It is known that

(a) almost all $x \in I$ are good, that is to say that the bad ones can be confined in a set of measure zero (see [1]). However,

(b) there are uncountably many bad numbers in I (see [2] and, for further information, [3]).

Despite their ubiquity, it is extremely difficult to find any examples of good numbers and, as far as I am aware, it is not known whether e is good. By contrast, finding bad numbers is quite easy. A celebrated example is the golden ratio $\alpha = \frac{1}{2}(1 + \sqrt{5})$. If $\beta = \frac{1}{2}(1 - \sqrt{5})$, then α and β are roots of the quadratic equation $x^2 - x - 1 = 0$. It is easy to prove by induction that, for all $n \in \mathbb{N}$, $\alpha^n + \beta^n$ is an integer. Now $\beta^n \to 0$ as $n \to \infty$, so, for large n, α^n is close to an integer, so its decimal part begins with either a 0 or a 9, and it is easy to verify that there is no $n \in \mathbb{N}$ for which the decimal part of α^n begins with one of the digits 1, 3, 4, 5 or 7. Readers are invited to find their own examples of bad numbers.

The referee kindly pointed out a related phenomenon that involves blocks of digits. This phenomenon lay at the heart of Problem 100.J in the November 2016 issue of the *Gazette* in which readers were asked to find a natural number n for which 2^n begins with the digits 2016. There is nothing special about the 2 or the 2016 here, for we have the following more general result.

Theorem 3: Let $d_r, d_{r-1}, \ldots, d_1, d_0$ be any given finite sequence of digits in base 10 and let $a\,(> 1)$ be any real number that is not a rational power of 10. Then there are infinitely many natural numbers N such that a^N begins with our given sequence of digits.

Proof: Let $d = d_0 + 10d_1 + \ldots + 10^r d_r$. We wish to show that there are infinitely many pairs of natural numbers n and N such that

$$10^n d < a^N < 10^n (d + 1). \tag{7}$$

Writing logarithms to base a simply as log, (7) is equivalent to

$$n \log 10 + \log d < N < n \log 10 + \log (d + 1). \tag{8}$$

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Let $\alpha = \log 10 = \dfrac{1}{\log_{10} a}$, which is irrational. From (8),

$$\log d < N - n\alpha < \log(d + 1).$$

Applying the lemma above completes the proof.

References

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K. ROBIN McLEAN

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL

e-mail: *krmclean@liv.ac.uk*

The answers to the *Nemo* page from March on logic and inference were:

1.	Robert M. Pirsig	Zen and the art of motorcycle maintenance An inquiry into values	
2.	George Eliot	Daniel Deronda	Chapter 41
3.	Lewis Carroll	Sylvie and Bruno	
4.	Aldous Huxley	Point Counter Point	Chapter 4
5.	Jack London	Mutiny of the Elsinore	Chapter 48
6.	Henry James	The Princess Casamassima	Chapter 12

Congratulations to Henry Ricardo on tracking all of these down. It is time to turn our minds to thoughts about ratio and proportion. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 30th September 2018.

1. The feelings of husband and wife varied, in some measure, inversely with their positions.

2. George's mind, as he paced the pier, was divided between the beauties of Nature and the forthcoming crisis in his affairs in the ratio of one-eighth to the former and seven-eighths to the latter.

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