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A NOTE ON SUBADDITIVITY OF VALUE AT RISKS (VARS): A NEW CONNECTION TO COMONOTONICITY

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Abstract

In this paper, we provide a new property of value at risk (VaR), which is a standard risk measure that is widely used in quantitative financial risk management. We show that the subadditivity of VaR for given loss random variables holds for any confidence level if and only if those are comonotonic. This result also gives a new equivalent condition for the comonotonicity of random vectors.

Keywords: Value at risk (VaR); risk measures; quantitative risk management; comonotonicity

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1. Introduction

Value at risk (VaR) is a standard risk measure used to capture tail risks in quantitative financial risk management. Mathematically, VaR is defined as the left continuous version of the generalized inverse function F_X^{-1} of the distribution function $F_X(x) = \mathbb{P}(X \le x)$ of a random variable $X\colon \Omega \to \mathbb{R}$, which represents an uncertain amount of financial loss, that is,

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R};\, F_X(x) \ge \alpha\} \equiv F_X^{-1}(\alpha),$$

where $\alpha \in (0, 1)$ represents the confidence level of a risk manager and $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space. VaR is widely used in financial practice because it is tractable and easy to understand. However, VaR lacks subadditivity, which is one of the important properties for a risk measure to appropriately reflect the effect of risk diversification. Here, a risk measure ρ is called subadditive if and only if

$$\rho(X + Y) \le \rho(X) + \rho(Y) \tag{1}$$

for any random variables X, Y defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Note that we can easily find an example where the subadditivity of VaR is violated. For instance, if we assume that $p, q \in (0, 1)$ satisfy $(1 - p)(1 - q) \le \alpha < 1 - \max\{p, q\}$ for a given $\alpha \in (0, 1)$, we see that

$$\mathrm{VaR}_\alpha(X + Y) > \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$$

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for a random variable X (resp. Y) whose distribution is given by the Bernoulli distribution with parameter p (resp. q).

In this paper, we consider the subadditivity of VaRs from another perspective. Specifically, we investigate a pair of random variables (X, Y) satisfying the subadditivity (1) with $\rho = \mathrm{VaR}_\alpha$ for 'any confidence level $\alpha \in (0, 1)$'. More generally, we say that VaR is subadditive for a random vector $(X_1, \dots, X_n)$ if and only if the inequality

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right) \le \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i), \quad \alpha \in (0, 1) \tag{2}$$

holds. Then we show that such a random vector $(X_1, \dots, X_n)$ is limited to the case where $(X_1, \dots, X_n)$ is comonotonic. This result also gives a new equivalent condition for the comonotonicity of random vectors.

2. Main results

To begin, we introduce the definition of the comonotonicity of random vectors.

Definition 1. Let $n \in \mathbb{N}$.

(i) A set $A \subset \mathbb{R}^n$ is called comonotonic if and only if, for any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in A$ with $x_i < y_i$ for some $i = 1, \dots, n$, it follows that $x_j \le y_j$ for each $j = 1, \dots, n$.

(ii) A random vector $(X_1, \dots, X_n)$ is called comonotonic if and only if the support of the distribution of $(X_1, \dots, X_n)$ is a comonotonic set.

The following theorem is our main result.

Theorem 1. *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and let $X_1, \dots, X_n$ be integrable random variables. Then the following statements are equivalent.*

(i) *For any $\alpha \in (0, 1)$, it holds that $\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$.*

(ii) *$(X_1, \dots, X_n)$ is comonotonic.*

Remark 1. *(Equivalent conditions for comonotonicity.)* The following properties are well known as equivalent conditions for the comonotonicity of a random vector $(X_1, \dots, X_n)$ (see [5], [6], [9]).

(E1) The dependency structure of $(X_1, \dots, X_n)$ is represented by the following copula:

$$M(u_1, \dots, u_n) = \min\{u_1, \dots, u_n\}, \quad u_1, \dots, u_n \in [0, 1].$$

(E2) There exists a uniformly distributed random variable U such that $(X_1, \dots, X_n)$ and $(F_{X_1}^{-1}(U), \dots, F_{X_n}^{-1}(U))$ have the same distribution.

(E3) There exist a random variable Z and non-decreasing functions $f_1, \dots, f_n$ such that $(X_1, \dots, X_n)$ and $(f_1(Z), \dots, f_n(Z))$ have the same distribution.

If $X_1, \dots, X_n$ are integrable, the comonotonicity of $(X_1, \dots, X_n)$ is equivalent to the following condition [1, 2, 10].

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(E4) Each random vector $(\tilde{X}_1, \dots, \tilde{X}_n) \in \mathcal{R}(F_{X_1}, \dots, F_{X_n})$ satisfies

$$\sum_{i=1}^{n} \tilde{X}_i \leq_{\mathrm{cx}} \sum_{i=1}^{n} X_i,$$

where $\mathcal{R}(F_{X_1}, \dots, F_{X_n})$ is the Fréchet class with marginal distributions $F_{X_1}, \dots, F_{X_n}$; that is,

$$\mathcal{R}(F_{X_1}, \dots, F_{X_n}) = \{(\tilde{X}_1, \dots, \tilde{X}_n);\, F_{\tilde{X}_i} = F_{X_i},\ i = 1, \dots, n\},$$

and we write $X \leq_{\mathrm{cx}} Y$ for random variables X and Y if $\mathbb{E}[X] = \mathbb{E}[Y]$ and

$$\mathbb{E}[\max\{X - c, 0\}] \leq \mathbb{E}[\max\{Y - c, 0\}], \quad c \in \mathbb{R}.$$

Furthermore, for the square-integrable case, the Pareto optimality for the convex order $\leq_{\mathrm{cx}}$ and the optimal coupling condition under the Wasserstein distance (when $n = 2$) have been separately studied as equivalent conditions for the comonotonicity. See [3] and [4] for details.

Our result implies that (2) is also equivalent to the comonotonicity of $(X_1, \dots, X_n)$. Moreover, we see that (2) implies

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i), \quad \alpha \in (0, 1), \tag{3}$$

due to Theorem 1 and the comonotonic subadditivity of VaRs, and hence (2) and (3) are equivalent.

Remark 2. *(Elliptic distributions.)* Elliptic distributions are known to be consistent with the subadditivity of VaRs. That is, if $(X_1, \dots, X_n)$ is elliptically distributed, the subadditivity of VaRs,

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right) \leq \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i), \tag{4}$$

is guaranteed; thus we can use the VaR as if it is a coherent risk measure when $(X_1, \dots, X_n)$ follows an elliptic distribution, including normal distributions and t-distributions. However, we must keep in mind that (4) does not always hold for 'all' $\alpha \in (0, 1)$.

Indeed, Theorem 1 in [7] implies that if $(X_1, \dots, X_n)$ is elliptically distributed and square-integrable, (4) holds for each $\alpha \in [1/2, 1)$. Similarly to the proof of Theorem 1 in [7], if $\alpha \in (0, 1/2]$, we can verify the opposite inequality; that is, we see the 'superadditivity' of the VaR:

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right) \geq \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i), \quad \alpha \in (0, 1/2].$$

Thus, if we further assume (2), we obtain

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i), \quad \alpha \in (0, 1/2].$$

This implies that

$$\sigma_i \sigma_j (1 - \rho_{ij}) = 0, \quad i, j = 1, \dots, n, \tag{5}$$

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where σ_i is the standard deviation of X_i and ρ_{ij} is the correlation of X_i and X_j. In other words, for any $i, j = 1, \dots, n$, it holds that 'if both X_i and X_j are non-deterministic, $\rho_{ij} = 1$'. Then we can verify that $(X_1, \dots, X_n)$ is comonotonic. Indeed, if given $i = 1, \dots, n$ such that $\sigma_i > 0$ and letting $Z = (X_i - \mathbb{E}[X_i])/\sigma_i$ and $f_j(z) = \mathbb{E}[X_j] + \sigma_j z$ for each $j = 1, \dots, n$, we see that $(X_1, \dots, X_n)$ has the same distribution as $(f_1(Z), \dots, f_n(Z))$.

Proof of Theorem 1. It suffices to show that (i) implies (ii). Put $S = X_1 + \cdots + X_n$. Because $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, Lemma A.32 in [8] implies that there is a uniformly distributed random variable U such that $S = F_S^{-1}(U)$ a.s. Now we define

$$\hat{X}_i = F_{X_i}^{-1}(U), \quad i = 1, \dots, n.$$

Then Lemma A.23 in [8] implies that the distribution of $\hat{X}_i$ is the same as that of X_i for each $i = 1, \dots, n$. Thus we get

$$\mathbb{E}[Z] = 0, \tag{6}$$

where $Z = \hat{S} - S$ and $\hat{S} = \hat{X}_1 + \cdots + \hat{X}_n$.

Meanwhile, from condition (i), we have that

$$F_S^{-1}(\alpha) \le \sum_{i=1}^{n} F_{X_i}^{-1}(\alpha), \quad \alpha \in (0, 1).$$

Substituting $\alpha = U$, we get

$$S = F_S^{-1}(U) \le \hat{S} \quad \text{a.s.},$$

so $Z \ge 0$ a.s. Combining this with (6), we see that Z must be equal to zero a.s.; that is,

$$\hat{S} = S \quad \text{a.s.} \tag{7}$$

Because $(\hat{X}_1, \dots, \hat{X}_1)$ is comonotonic, condition (E4) holds for $(\hat{X}_1, \dots, \hat{X}_n)$ and thus

$$\sum_{i=1}^{n} \tilde{X}_i \le_{\mathrm{cx}} \hat{S}, \quad (\tilde{X}_1, \dots, \tilde{X}_n) \in \mathcal{R}(F_{X_1}, \dots, F_{X_n}).$$

Combining this with (7) and using (E4) again, we conclude that $(X_1, \dots, X_n)$ is comonotonic. □

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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