There is another, and completely different, way to look at the paradox. A Möbius map is a function of the form $f(z) = \dfrac{az + b}{cz + d}$, where $ad - bc \neq 0$, and z is a complex variable. One of the truly basic facts about Möbius maps is that given any distinct z_1, z_2 and z_3, and any distinct w_1, w_2 and w_3, there is a unique Möbius map f such that $f(z_1) = w_1$, $f(z_2) = w_2$ and $f(z_3) = w_3$.

We shall now apply this to Simpson's paradox. Given numbers r_1 and r_2 with $0 < r_1 < r_2$, and any ξ in (r_1, r_2), there is a Möbius map f with $f(\infty) = r_1, f(1) = \xi$ and $f(0) = r_2$. Thus there are numbers a, b, c and d such that $f(z) = \dfrac{az + b}{cz + d}$, $r_1 = f(\infty) = \dfrac{a}{c}$, $r_2 = f(0) = \dfrac{b}{d}$, and $\dfrac{a + b}{c + d} = f(1) = \xi$. This again shows that given r_1 and r_2 we can find data a, b, c and d such that $\dfrac{a}{c} = r_1$, $\dfrac{b}{d} = r_2$, and such that $\dfrac{a + b}{c + d}$ lies in any pre-assigned place in the interval (r_1, r_2).

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102.55 Cable to outer space?

The website 'Brilliant Maths and Science' (brilliant.org) set a problem for the week of 2nd October 2017 about the possibility of a cable moored on the equator going up to outer space and kept in place through the earth's rotation. This would be similar to the way in which stationary satellites stay up, their weight being just right to give them the acceleration needed to continue in a circular orbit at a rate of one revolution per day.

The solution given on the website was to equate the total weight of the cable with the total 'centrifugal force' on it, each total being found by integration; but this ignored the fact that the cable is moored at the ground, so that there could be a tension at the bottom end.

Instead, consider a small part of the cable, between height y and $y + \delta y$ above the earth's surface. Let the cable have mass m kg per metre, the earth's radius be r, the acceleration due to gravity at the earth's surface be g and the earth's rate of rotation be ω radians per second.

Let the tensions at the ends of the small part be T and $T + \delta T$. Then the total downward force on the small piece is close to $\dfrac{m\,\delta y\, gr^2}{(r + y)^2} - \delta T$, and this has to be close to $m\,\delta y\,(r + y)\,\omega^2$ by Newton's law, $(r + y)\,\omega^2$ being the acceleration arising from the circular motion. Letting δy tend to 0, we get the equation

$$\frac{dT}{dy} = \frac{mgr^2}{(r + y)^2} - m(r + y)\,\omega^2, \tag{1}$$

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with solution

$$T = -\frac{mgr^2}{r+y} - m\left(ry + \frac{y^2}{2}\right)\omega^2 + c.$$

To find c we use the condition that the tension is 0 at the top end of the cable; so if the cable length is l,

$$0 = -\frac{mgr^2}{r+l} - m\left(rl + \frac{l^2}{2}\right)\omega^2 + c.$$

This gives us c and makes

$$T = \frac{mgr^2}{r+l} + m\left(rl + \frac{l^2}{2}\right)\omega^2 - \frac{mgr^2}{r+y} - m\left(ry + \frac{y^2}{2}\right)\omega^2. \qquad (2)$$

At this point I was stuck for quite a while, because it appeared that there could be a solution for any value of l, so how could l be found? Eventually I realised that the formula for T could give negative values of T, which would imply that we are dealing with a rod instead of a cable. A rod could of course have positive or negative tension, but in a cable the tension has to be non-negative.

The formula (1) above for $\frac{dT}{dy}$ shows that $\frac{dT}{dy}$ is itself a decreasing function of y; at $y = 0$, $\frac{dT}{dy} = mg - mr\omega^2$, that is the weight per metre minus the force needed to maintain that metre length in a circular path; this is clearly positive, else nothing could lie on the equator without being tied. Thus to begin with, T is an increasing function of y. That changes once $\frac{dT}{dy} = 0$, which occurs at the height needed for a stationary satellite. Beyond that T decreases, becoming 0 at the top end of the cable. This shows that our condition of T always being non-negative is fulfilled if, and only if, $T \geqslant 0$ when $y = 0$.

So now we need

$$\frac{mgr^2}{r+l} + m\left(rl + \frac{l^2}{2}\right)\omega^2 - \frac{mgr^2}{r} \geqslant 0.$$

Multiplying by $2(r+l)$ and dividing by m gives

$$2gr^2 + \left(2rl + l^2\right)(r+l)\omega^2 - 2gr(r+l) \geqslant 0,$$

which can be rearranged to

$$l\left(\omega^2 l^2 + 3r\omega^2 l + 2r\left(r\omega^2 - g\right)\right) \geqslant 0.$$

We can divide by l as it is known to be positive, and then have the quadratic inequality

$$\omega^2 l^2 + 3r\omega^2 l + 2r\left(r\omega^2 - g\right) \geqslant 0.$$

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The term without l is negative, as we have seen above that $g > r\omega^2$. The term in l, $3r\omega^2 l$, has a positive coefficient. Thus the graph of

$$z = \omega^2 l^2 + 3r\omega^2 l + 2r\left(r\omega^2 - g\right)$$

against l meets the z-axis below the origin and has positive gradient there, so the graph continues to rise. It crosses the l-axis at a positive value of l and then continues to increase for ever. Thus the solution to the inequality is of the form $l \geqslant l_0$ where l_0 is the positive root of the equation

$$\omega^2 l^2 + 3r\omega^2 l + 2r\left(r\omega^2 - g\right) = 0,$$

which we can solve by the usual formula to give

$$l_0 = \frac{-3r\omega^2 + \sqrt{9r^2\omega^4 - 8r\omega^2\left(r\omega^2 - g\right)}}{2\omega^2} = \frac{\sqrt{r\left(r\omega^2 + 8g\right)} - 3r\omega}{2\omega}.$$

This is the minimum length of cable.

With $\omega = \dfrac{2\pi}{24 \times 3600} = 7.272 \times 10^{-5}$ radians per second, $r = 6371000$ m and $g = 9.81$ m/sec^2 we get $l_0 = 144\,216\,000$ m or 144 000 km, to 3 significant figures. This is close to the solution given in Brilliant, using slightly different values for r and g, as that solution implicitly assumed zero tension at the bottom of the cable.

Could any known material sustain the tension involved? The maximum tension occurs where $\dfrac{dT}{dy} = \dfrac{mgr^2}{(r+y)^2} - m(r+y)\omega^2 = 0$, giving $(r+y)^3 = \dfrac{gr^2}{\omega^2}$, so that $y = \sqrt[3]{\dfrac{gr^2}{\omega^2}} - r$, which comes to 3.59×10^7 m or 35 900 km to 3 significant figures; this is the height of a geostationary orbit. With this value of y, the tension formula (2) gives a tension of $1.13 \times 10^{16} m$ Newtons, where m is, as above, the mass per unit length, in kg per metre. A Wikipedia site gives tensile strengths of various materials in MPa, meaning millions of Newtons per square metre. The strongest steel it mentions (called 2800 Maraging steel) has a breaking strength of 2690 MPa and density $8g$ /cm^3.

If a cable of this material has cross-section area A cm^2, then the mass per metre length is $m = 100A \times 0.008 = 0.8A$ kg, and the tension it can sustain before breaking is $2690 \times 10^6 \times A \times 10^{-4} = 269\,000A$ Newtons, while the maximum tension needed is $1.13 \times 10^{16} \times 0.8A = 9.04 \times 10^{15} A$, which is 3.36×10^{10} times the tension the steel can take before breaking. The cable to outer space seems not to be a practical proposition yet!

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