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Simultaneous linearization of germs of commuting holomorphic diffeomorphisms

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Abstract. Let $\alpha_1, \ldots, \alpha_n$ be irrational numbers which are linearly independent over the rationals, and $f_1, \ldots, f_n$ commuting germs of holomorphic diffeomorphisms in $\mathbb{C}$ such that $f_k(0) = 0$, $f_k'(0) = e^{2\pi i\alpha_k}$, $k = 1, \ldots, n$. Moser showed that $f_1, \ldots, f_n$ are simultaneously linearizable (i.e. conjugate by a germ of holomorphic diffeomorphism h to the rigid rotations $R_{\alpha_k}(z) = e^{2\pi i\alpha_k}z$) if the vector of rotation numbers $(\alpha_1, \ldots, \alpha_n)$ satisfies a Diophantine condition. Adapting Yoccoz's renormalization to the setting of commuting germs, we show that simultaneous linearization holds in the presence of a weaker Brjuno-type condition $\mathcal{B}(\alpha_1, \ldots, \alpha_n) < +\infty$, where $\mathcal{B}(\alpha_1, \ldots, \alpha_n)$ is a multivariable analogue of the Brjuno function. If there are no periodic orbits for the action of the germs $f_1, \ldots, f_n$ in a neighbourhood of the origin, then a weaker arithmetic condition $\mathcal{B}'(\alpha_1, \ldots, \alpha_n) < +\infty$ analogous to Perez-Marco's condition for linearization in the absence of periodic orbits is shown to suffice for linearizability. Normalizing the germs to be univalent on the unit disc, in both cases the Siegel discs are shown to contain discs of radii $Ce^{-2\pi\mathcal{B}}$, $Ce^{-2\pi\mathcal{B}'}$ for some universal constant C.

1. *Introduction*

A germ $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbb{R} - \mathbb{Q}/\mathbb{Z}$ of a holomorphic diffeomorphism fixing the origin in $\mathbb{C}$ is said to be linearizable if it is analytically conjugate to the rigid rotation $R_\alpha(z) = e^{2\pi i\alpha}z$. The number α is called the rotation number of f, and the maximal domain of linearization is called the Siegel disc of f. Siegel [**Sie42**] showed that if α satisfies a Diophantine condition

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^\tau}$$

for all integers p, q for some $C, \tau > 0$, then f is linearizable. Brjuno [**Brj71**] showed that linearizability holds under the more general Brjuno condition, defined equivalently in [**SMY97**] in terms of the Gauss map as follows.

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Let $(\alpha^{(i)})_{i\geq 0}$ denote the fractions arising from the continued fraction algorithm (given by iterating the Gauss map $\alpha \mapsto 1/\alpha \pmod 1$) applied to α, then the Brjuno condition is

$$\mathcal{B}(\alpha) := \sum_{n=0}^{\infty} \alpha^{(0)} \cdots \alpha^{(n-1)} \log\left(\frac{1}{\alpha^{(n)}}\right) < +\infty.$$

Yoccoz [**Yoc95**] proved the optimality of the Brjuno condition: if $\mathcal{B}(\alpha) = +\infty$, then there exists a non-linearizable germ f with rotation number α. Perez-Marco [**PM93**] showed that the weaker arithmetic condition

$$\mathcal{B}'(\alpha) := \sum_{n=0}^{\infty} \alpha^{(0)} \cdots \alpha^{(n-1)} \log\log\left(\frac{e}{\alpha^{(n)}}\right) < +\infty$$

is sufficient for linearizability in the absence of periodic orbits accumulating the fixed point, and is optimal in this case: if $\mathcal{B}'(\alpha) = +\infty$, then there exists a non-linearizable germ f with rotation number α with no periodic orbits in a neighbourhood of the origin.

It is well known that N commuting germs $f_1, \ldots, f_N$ are either simultaneously linearizable or non-linearizable. Moser [**Mos90**] has shown that if the rotation numbers $\alpha_1, \ldots, \alpha_N$ satisfy a Diophantine condition

$$\max_{1\leq k\leq N} |q\alpha_k - p_k| \geq \frac{C}{q^\tau}$$

for some $C > 0$, $\tau > 0$, then the germs $f_1, \ldots, f_N$ are simultaneously linearizable. More precisely, Moser proves that commuting analytic circle diffeomorphisms whose rotation numbers satisfy the above condition and are sufficiently close to the corresponding rotations are simultaneously analytically linearizable; by the dictionary between circle maps and germs established by Perez-Marco in [**PM97**], this local linearization result implies the corresponding global linearization result for germs. We remark that a Diophantine condition is indeed necessary; Perez-Marco has constructed in [**PM95**] uncountable families of commuting non-linearizable germs.

Generalizing Moser's result, we show that multidimensional analogues of Brjuno's and Perez-Marco's conditions, given by arithmetic functions $\mathcal{B}(\alpha_1, \ldots, \alpha_N)$, $\mathcal{B}'(\alpha_1, \ldots, \alpha_N)$ (defined in the next section), suffice for linearizability in the presence and absence of periodic orbits.

THEOREM 1.1. *Let $f_1, \ldots, f_N$ be N commuting germs with rationally independent irrational rotation numbers $\alpha_1, \ldots, \alpha_N$ univalent on the unit disc $\mathbb{D}$. If $\mathcal{B}(\alpha_1, \ldots, \alpha_N) < +\infty$; then $f_1, \ldots, f_N$ are simultaneously linearizable, and their common Siegel disc contains the disc of radius $Ce^{-2\pi\mathcal{B}(\alpha_1,\ldots,\alpha_N)}$ around the origin for some universal constant C.*

In the Appendix A, it is shown that the above arithmetic condition is implied by Moser's Diophantine condition. By a periodic orbit for the action of a collection of commuting maps $f_1, \ldots, f_N$ univalent on $\mathbb{D}$ we mean a finite set $\mathcal{O} \subset \mathbb{D}$ invariant under the maps f_j, i.e. $f_j(\mathcal{O}) = \mathcal{O}$, $j = 1, \ldots, N$.

THEOREM 1.2. *Let $f_1, \ldots, f_N$ be N commuting germs with rationally independent irrational rotation numbers $\alpha_1, \ldots, \alpha_N$, univalent on the unit disc $\mathbb{D}$ with no periodic*

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orbits in $\mathbb{D}$. *If* $\mathcal{B}'(\alpha_1, \ldots, \alpha_N) < +\infty$, *then* $f_1, \ldots, f_N$ *are simultaneously linearizable, and their common Siegel disc contains the disc of radius* $Ce^{-2\pi\mathcal{B}'(\alpha_1,\ldots,\alpha_N)}$ *around the origin for some universal constant* C.

2. *Brjuno-type functions*

We define Gauss maps for N-tuples of fractions as follows.

Let $\alpha_1, \ldots, \alpha_N \in (0, 1)$ be irrationals linearly independent over $\mathbb{Q}$, and $w \in \{1, \ldots, N\}$. We define irrationals $(\tilde{\alpha_1}, \ldots, \tilde{\alpha_N}) = G(\alpha_1, \ldots, \alpha_N, w)$, $0 < \tilde{\alpha_i} < 1/2$ as follows.

Let $\hat{\alpha_i} = \alpha_i$ for $i \neq w$ and $\hat{\alpha_i} = 1$ for $i = w$.

Let $a_i = a_i(\alpha_1, \ldots, \alpha_N, w)$ be integers such that

$$|a_i\alpha_w - \hat{\alpha}_i| = \min_{k\in\mathbb{Z}} |k\alpha_w - \hat{\alpha}_i| < \tfrac{1}{2}\alpha_w.$$

Let

$$\tilde{\alpha}_i := \frac{|a_i\alpha_w - \hat{\alpha}_i|}{\alpha_w} \in (0, 1/2), \quad \epsilon_i = \operatorname{sgn}(a_i\alpha_w - \hat{\alpha}_i).$$

Given an infinite word $\overline{w} = (w(n))_{n\in\mathbb{N}} \in \mathcal{C} := \{1, \ldots, N\}^{\mathbb{N}}$, let $\alpha_i^{(0)} = \alpha_i$ and $(\alpha_1^{(n+1)}, \ldots, \alpha_N^{(n+1)}) = G(\alpha_1^{(n)}, \ldots, \alpha_N^{(n)}, w(n))$ for $n \geq 0$. We define

$$\mathcal{B}(\alpha_1, \ldots, \alpha_n, \overline{w}) := \sum_{n=0}^{\infty} \alpha_{w(0)}^{(0)} \cdots \alpha_{w(n-1)}^{(n-1)} \log\left(\frac{1}{\alpha_{w(n)}^{(n)}}\right),$$

$$\mathcal{B}'(\alpha_1, \ldots, \alpha_n, \overline{w}) := \sum_{n=0}^{\infty} \alpha_{w(0)}^{(0)} \cdots \alpha_{w(n-1)}^{(n-1)} \log\log\left(\frac{1}{\alpha_{w(n)}^{(n)}}\right)$$

and

$$\mathcal{B}(\alpha_1, \ldots, \alpha_n) := \inf_{\overline{w}\in\mathcal{C}} \mathcal{B}(\alpha_1, \ldots, \alpha_n, \overline{w}),$$

$$\mathcal{B}'(\alpha_1, \ldots, \alpha_n) := \inf_{\overline{w}\in\mathcal{C}} \mathcal{B}'(\alpha_1, \ldots, \alpha_n, \overline{w}).$$

3. *Renormalization of commuting germs*

In this section, we generalize Yoccoz's renormalization for germs [**Yoc95**] to the setting of N commuting germs. Throughout, $C > 0$ will denote various universal constants, $\mathbb{D}_r$ the disc $\{|z| < r\}$, $\mathbb{H}_y$ the half-plane $\{\operatorname{Im} z > y\}$ and $S(a, y)$ the vertical strip $\{|\operatorname{Re} z| \leq a, \operatorname{Im} z \geq y\}$.

Let

$$\mathcal{S}_{\alpha,r} := \{f \mid f \text{ is univalent on } \mathbb{D}_r,\ f(0) = 0,\ f'(0) = e^{2\pi i\alpha}\}$$

and

$$\hat{\mathcal{S}}_{\alpha,y} := \left\{F \mid F \text{ is univalent on } \mathbb{H}_y, \lim_{\operatorname{Im} z\to+\infty} (F(z) - (z+\alpha)) = 0,\ F(z+1) = F(z) + 1\right\}$$

be the corresponding space of lifts to the half-plane $\mathbb{H}_y$ via the universal covering $E : \mathbb{H} \to \mathbb{D}^*$, $E(z) = e^{2\pi iz}$, where $y = (1/2\pi)\log(1/r)$. These spaces are compact (with respect to

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the topology of uniform convergence on compacta), and satisfy

$$|F(z) - (z + \alpha)| \leq Ce^{-2\pi \operatorname{Im} z},$$
$$|F'(z) - 1| \leq Ce^{-2\pi \operatorname{Im} z}$$

for all $F \in \hat{\mathcal{S}}_{\alpha,y}$, $\operatorname{Im} z \geq C + y$. We call α the rotation number of F. By the Koebe one-quarter theorem, the inverse F^{-1} is well defined and univalent in $\mathbb{H}_{C+y}$ for a universal constant C, and satisfies the same estimates as F for C large enough.

3.1. *Uniformizing maps.* We recall the uniformizing maps defined by Yoccoz in order to perform the renormalization. For $\alpha \in \mathbb{R}$, let $t(\alpha) > 0$ be such that $|F(z) - (z + \alpha)| \leq \alpha/4$ for $\operatorname{Im} z \geq t(\alpha)$, $F \in \hat{\mathcal{S}}_{\alpha,0}$ (by the above estimates, we can take $t(\alpha) = (1/2\pi)\log(1/\alpha) + C$ for a universal constant C).

Let l be the line segment $[it(\alpha), +i\infty]$. Then l, $F(l)$ and the segment $s = [it(\alpha), F(it(\alpha))]$ bound a domain $\mathcal{U}$ in $\mathbb{H}$. The proximity of F to the translation by α shows that any point z in the domain $\mathcal{V} = \{\operatorname{Im} z \geq \frac{1}{2}|\operatorname{Re} z| + t(\alpha) + 1\}$ has an iterate $F^n(z)$ contained in $\mathcal{U}$.

Pasting the boundary segments l and $F(l)$ of $\mathcal{U}$ by F, the quotient $\pi : \overline{\mathcal{U}} \to \mathcal{S} := \overline{\mathcal{U}}/(l \sim F(l))$ gives a Riemann surface $\mathcal{S}$ which is biholomorphic to the punctured disc $\mathbb{D}^*$. Let $k : \mathcal{S} \to \mathbb{D}^*$ be a uniformization and $K : \overline{\mathcal{U}} \to \mathbb{H}$ a lift of k satisfying $F \circ K = k \circ \pi$. The lift K satisfies $K(F(z)) = K(z) + 1$ for points z in the left boundary segment l of $\mathcal{U}$. We can therefore extend K univalently to $\mathcal{V}$ by defining $K(z) := K(F^j(z)) - j$, where $z \in \mathcal{V}$, $F^j(z) \in \mathcal{U}$. The extended map K satisfies $K(F(z)) = K(z) + 1$ whenever z, $F(z) \in \mathcal{V}$, and is called a *uniformizing map* for F [**Yoc95**, Proposition, p. 28].

For $C > 0$ large enough, the domain $\mathcal{U}'$ bounded by $l' = [i(t(\alpha) + C), +i\infty]$, $F(l')$ and $s' = [i(t(\alpha) + C), F(i(t(\alpha) + C))]$ is contained in $\mathcal{V}$, and Yoccoz shows that the following estimate holds [**Yoc95**, Proposition 1, p. 31].

PROPOSITION 3.1. *For $z \in S(2, t(\alpha) + C) \subset \mathcal{V}$,*

$$\left| K'(z) - \frac{1}{\alpha} \right| \leq \frac{1}{3} \frac{1}{(\operatorname{Im} z - t(\alpha))^2}.$$

Composing K with a translation, we may assume that $K(s')$ is contained in the closure of the lower half-plane with $K(s') \cap \mathbb{R} \neq \emptyset$. The above estimate then yields (for C large enough) the following on integration.

PROPOSITION 3.2. *For $z \in S(2, t(\alpha) + C)$,*

$$\left| K(z) - \frac{1}{\alpha}(z - i(t(\alpha) + C)) \right| \leq \frac{2}{\alpha}.$$

We may assume C is large enough so that, for $|j| \leq a(\alpha) = [10/\alpha] + 1$, the maps F^j are univalent on $\mathbb{H}_{t(\alpha)+C}$ mapping into $\mathbb{H}_{t(\alpha)+C-1}$. For $n \leq a(\alpha)$, let

$$\mathcal{A}(F, n) = \bigcup_{j=-n}^{n} F^j(\mathcal{U}').$$

Taking C large enough, $\mathcal{A}(F, a(\alpha))$ contains the strip $S(8, t(\alpha) + C + 1)$. The map K extends to $\mathcal{A}(F, n)$, putting $K(F^j(z)) = K(z) + j$, and, by the above proposition, the

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image $K(l')$ of the vertical line l' is close enough to the positive imaginary axis so that

$$K(\mathcal{A}(F, a(\alpha))) = \bigcup_{j=-a(\alpha)}^{j=a(\alpha)} T^j(K(\mathcal{U}'))$$

covers the strip $S(8/\alpha, 0)$, where $T(z) = z + 1$ is the translation by one.

3.2. *Simultaneous renormalization.* Given N commuting maps $F_j \in \hat{\mathcal{S}}_{\alpha_j}$, $j = 1, \ldots, N$ with irrational rationally independent rotation numbers α_j, $j = 1, \ldots, N$ and $w \in \{1, \ldots, N\}$, we define a renormalization operator $\mathcal{R}$ giving N commuting maps $(\tilde{F}_1, \ldots, \tilde{F}_N) = \mathcal{R}(F_1, \ldots, F_N, w)$ with rotation numbers $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N) = G(\alpha_1, \ldots, \alpha_N, w)$ as follows.

Let K be a uniformizing map for $F = F_w$ with associated domains $\mathcal{U}, \mathcal{U}', \mathcal{V}$ as defined above.

Let $\hat{F}_i = F_i$ for $i \neq w$, and $\hat{F}_i = T$ for $i = w$, where $T(z) = z + 1$ is the translation by one.

We may assume the universal constant $C > 2$ is chosen large enough so that the maps $F_w^{\epsilon_i a_i} \circ \hat{F}_i^{-\epsilon_i}$ are univalent on the strip $S(2, t(\alpha) + C)$, and map it into $\mathcal{V}$. On the domain $\mathcal{X} := K(S(2, t(\alpha) + C))$, we define the maps

$$\tilde{F}_i := K \circ (F_w^{\epsilon_i a_i} \circ \hat{F}_i^{-\epsilon_i}) \circ K^{-1}.$$

Since K conjugates F_w to the translation T, the maps $\tilde{F}_i$ commute with T and therefore extend univalently to the union of translates $\mathcal{Y} := \bigcup_{n\in\mathbb{Z}} T^n(\mathcal{X})$ by setting $\tilde{F}_i(T^n(z)) := \tilde{F}_i(z) + n$ for $z \in \mathcal{X}$. Since the image $K(s')$ of the segment s' lies in the lower half-plane, $\mathcal{Y}$ contains the upper half-plane, so we obtain N commuting maps $\tilde{F}_j \in \hat{\mathcal{S}}_{\tilde{\alpha}_j,0}$, $j = 1, \ldots, N$.

LEMMA 3.3. *Given $G \in \hat{\mathcal{S}}_{\tilde{\beta},\tilde{y}}$ commuting with $\tilde{F}_1, \ldots, \tilde{F}_N$, where $|\tilde{\beta}| < 1/2$, $\tilde{y} \geq 0$, the conjugate $G^* = K^{-1} \circ G \circ K$ defines an element of $\hat{\mathcal{S}}_{\beta,y}$ commuting with $F_1, \ldots, F_N$, where $\beta = \alpha_w\tilde{\beta}$ and $y = \alpha_w\tilde{y} + t(\alpha_w) + (1 + \alpha_w)C + 2$.*

Proof. It follows from Proposition 3.2 that the image of the strip $S(2/\alpha_w, y)$ under K is contained in the strip $S(4/\alpha_w, \tilde{y} + C)$. For C large enough, the maps G, $\tilde{F}_j^{\epsilon} \circ T^k$, $G \circ \tilde{F}_j^{\epsilon} \circ T^k$, $\epsilon = \pm 1$, $j = 1, \ldots, N$, $|k| \leq [1/\alpha_w] + 1$ map $S(4/\alpha_w, \tilde{y} + C)$ into $S(8/\alpha_w, \tilde{y}) \subset K(\mathcal{A}(F_w, a(\alpha_w)))$. So the conjugates of these maps by K^{-1} are univalent in $S(2/\alpha_w, y)$; the conjugates include G^*, $F_1, \ldots, F_N$, T and commute because $G \circ \tilde{F}_j^{\epsilon} \circ T^k = \tilde{F}_j^{\epsilon} \circ T^k \circ G$. Since G^* commutes with T, it extends to a univalent map on the union of translates of $S(2/\alpha_w, y)$ under T which contains $\mathbb{H}_y$. From Proposition 3.2, $K(z) = z/\alpha_w + o(z)$, $K^{-1}(z) = \alpha_w z + o(z)$ as $\operatorname{Im} z \to +\infty$ (with $\operatorname{Re} z$ bounded), from which it follows that

$$\lim_{\operatorname{Im} z\to+\infty} G^*(z) - z = \alpha_w\tilde{\beta},$$

so $G^* \in \hat{\mathcal{S}}_{\beta,y}$. □

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3.3. *Iterating the renormalization.* Given maps $F_1, \dots, F_N$ as above and an infinite word $\overline{w} = (w(n))_{n\geq 0} \in \{1, \dots, N\}^{\mathbb{N}}$, we define

$$(F_1^{(0)}, \dots, F_N^{(0)}) := (F_1, \dots, F_N),$$
$$(F_1^{(n+1)}, \dots, F_N^{(n+1)}) := \mathcal{R}(F_1^{(n)}, \dots, F_N^{(n)}, w(n)), \quad n \geq 0.$$

We denote by $\tilde{F}_n$ the map $F_{w(n)}^{(n)}$ with respect to which we are renormalizing at the nth stage and by $\tilde{\alpha}_n$ its rotation number $\alpha_{w(n)}^{(n)}$. Let K_n be the uniformizing map of $\tilde{F}_n$.

For $0 \leq m \leq n$, we define finite families of commuting maps $\mathcal{F}_{m,n}$ univalent on $\mathbb{H}_{y_{m,n}}$ as follows.

Let $y_{n,n} := t(\tilde{\alpha}_n)$. Let $\mathcal{F}_{n,n}$ be the collection of maps $\{\tilde{F}_n^j : 0 \leq j \leq [1/\tilde{\alpha}_n]\}$ univalent on $\mathbb{H}_{y_{m,n}}$. Let $\epsilon_{n,n} := \tilde{\alpha}_n$.

For $0 < m \leq n$, assume $y_{m,n}$ and $\mathcal{F}_{m,n}$ have been defined and satisfy the following induction hypotheses.

(1) The maps in $\mathcal{F}_{m,n}$ are univalent in $\mathbb{H}_{y_{m,n}}$.

(2) $F_1^{(m)}, \dots, F_N^{(m)}, T$ commute with the elements of $\mathcal{F}_{m,n}$.

(3) The rotation numbers of the maps in $\mathcal{F}_{m,n}$ are contained in $[0, 1)$ and are $\epsilon_{m,n}$-dense in $[0, 1]$ (every point in $[0, 1]$ is at distance less than $\epsilon_{m,n}$ from these rotation numbers).

Let

$$y_{m-1,n} := \tilde{\alpha}_{m-1} y_{m,n} + t(\tilde{\alpha}_{m-1}) + (1 + \tilde{\alpha}_{m-1})C + 2.$$

By Lemma 3.3, the conjugates $\tilde{\mathcal{F}}_{m-1,n} := \{G^* = K_{m-1}^{-1} \circ G \circ K_{m-1} : G \in \mathcal{F}_{m,n}\}$ are univalent on $\mathbb{H}_{y_{m-1,n}}$, commute with $F_1^{(m-1)}, \dots, F_N^{(m-1)}, T$ and, taking C large enough, we may assume (since they have rotation numbers in $[0, 1)$) that they map this half-plane into $\mathbb{H}_{y_{m-1,n}-2}$. For $0 \leq j \leq [1/\tilde{\alpha}_{m-1}]$, the maps $\tilde{F}_{m-1}^j$ are univalent on $\mathbb{H}_{\tilde{\alpha}_{m-1}+C} \supset \mathbb{H}_{y_{m-1,n}-2}$; for each $G^* \in \tilde{\mathcal{F}}_{m-1,n}$, we can choose $0 \leq j(G^*) \leq [1/\tilde{\alpha}_{m-1}]$ such that the rotation numbers of the maps $\{\tilde{F}_{m-1}^j \circ G^* : G^* \in \tilde{\mathcal{F}}_{m-1,n}, 0 \leq j \leq j(G^*)\}$ are contained in $[0, 1)$, are $\tilde{\alpha}_{m-1}\epsilon_{m,n}$-dense in $[0, 1]$, and these maps are univalent on $\mathbb{H}_{y_{m-1,n}}$.

Let $\mathcal{F}_{m-1,n} = \{\tilde{F}_{m-1}^j \circ G^* : G^* \in \tilde{\mathcal{F}}_{m-1,n}, 0 \leq j \leq j(G^*)\}$ be this collection of maps, and $\epsilon_{m-1,n} = \tilde{\alpha}_{m-1}\epsilon_{m,n}$. Then the induction hypotheses (1), (2) and (3) above are satisfied for $y_{m-1,n}$, $\mathcal{F}_{m-1,n}$ and $\epsilon_{m-1,n}$. By descending induction on m, we get the following proposition.

PROPOSITION 3.4. *For any word $(w(n))_{n\geq 0} \in \{1, \dots, N\}^{\mathbb{N}}$, there are families of maps $\mathcal{F}_{0,n}$ univalent on $\mathbb{H}_{y_{0,n}}$ commuting with $F_1, \dots, F_N, T$, where*

$$y_{0,n} \leq \sum_{j=0}^{n} \alpha_{w(0)}^{(0)} \cdots \alpha_{w(j-1)}^{(j-1)} t(\alpha_{w(j)}^{(j)}) + C'$$

for a universal constant C'. Moreover, the rotation numbers of the maps $F \in \mathcal{F}_{0,n}$ are $\epsilon_{0,n}$-dense in $[0, 1]$, where $\epsilon_{0,n} = \alpha_{w(0)}^{(0)} \cdots \alpha_{w(n)}^{(n)}$.

Proof. Assume that $y_{m+1,n}$ satisfies

$$y_{m+1,n} \leq \sum_{j=m+1}^{n} \alpha_{w(m+1)}^{(m+1)} \cdots \alpha_{w(j-1)}^{(j-1)} t(\alpha_{w(j)}^{(j)}) + 2(C+1)\left(1 + \sum_{j=m+1}^{n-2} \alpha_{w(j)}^{(j)} \cdots \alpha_{w(n-2)}^{(n-2)}\right)$$

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(for an $m+1 \le n$). Then the definition of $y_{m,n}$ gives

$$\begin{aligned} y_{m,n} &\le \sum_{j=m}^{n} \alpha_{w(m)}^{(m)} \cdots \alpha_{w(j-1)}^{(j-1)} t(\alpha_{w(j)}^{(j)}) + \alpha_{w(m)}^{(m)} 2(C+1)\left(1 + \sum_{j=m+1}^{n-2} \alpha_{w(j)}^{(j)} \cdots \alpha_{w(n-2)}^{(n-2)}\right) \\ &\quad + (1 + \alpha_{w(m)}^{(m)})C + 2 \\ &\le \sum_{j=m}^{n} \alpha_{w(m)}^{(m)} \cdots \alpha_{w(j-1)}^{(j-1)} t(\alpha_{w(j)}^{(j)}) + 2(C+1)\left(1 + \sum_{j=m}^{n-2} \alpha_{w(j)}^{(j)} \cdots \alpha_{w(n-2)}^{(n-2)}\right) \\ &\le \sum_{j=m}^{n} \alpha_{w(m)}^{(m)} \cdots \alpha_{w(j-1)}^{(j-1)} t(\alpha_{w(j)}^{(j)}) + C', \end{aligned}$$

where $C' = 4(C+1)$, using $\alpha_{w(j)}^{(j)} < 1/2$ for all j. By induction, we get the required inequality for $y_{0,n}$. □

4. *Linearization of commuting germs*

LEMMA 4.1. *Let $\mathcal{F} = (f_t)_{t\in A}$ be a commuting family of maps univalent on a disc $\mathbb{D}_r$ such that $f_t(0) = 0$, $f_t'(0) = e^{2\pi i t}$, the set of rotation numbers A contains an irrational and is dense in* [0, 1]. *Then the maps f_t are simultaneously linearizable with a Siegel disc containing $\mathbb{D}_r$.*

Proof. The family $\mathcal{F}$, being a normalized family of univalent functions on a disc, is normal. Fix a t_0 in A which is irrational. For any sequence $t_n \in A$ converging to $t \in [0, 1]$, any normal limit f of the sequence (f_{t_n}) must have rotation number t and commute with f_{t_0}, but t_0, being irrational, the power series of f is uniquely determined by these two conditions. So the sequence (f_{t_n}) has a unique normal limit and is normally convergent to a map, which we denote by f_t. Since A is dense, we get a commuting family of maps $\{f_t : t \in [0, 1]\}$ univalent on $\mathbb{D}_r$; as above, $t_n \to t$ implies $f_{t_n}(z) \to f_t(z)$ for t_n, $t \in [0, 1]$, so $t \mapsto f_t(z)$ is continuous for fixed z. Let

$$h(z) := \int_0^1 e^{-2\pi i t} f_t(z)\, dt, \quad z \in \mathbb{D}_r.$$

Then h is holomorphic, $h(0) = 0$ and $h'(0) = 1$ (Riemann sums for the integral converge normally to h). For $t \in \mathbb{R}$, we let $f_t := f_{\bar{t}}$, where $\bar{t} \in [0, 1]$ is the fractional part of t. Then the maps f_t satisfy $f_t \circ f_s = f_{t+s}$ for all real t, s, which gives $h(f_t(z)) = e^{2\pi i t} h(z)$, so h linearizes the maps f_t. □

Proof of Theorem 1.1. Let $F_1, \ldots, F_N$ be lifts of $f_1, \ldots, f_N$ with rotation numbers $0 < \alpha_1, \ldots, \alpha_N < 1$. Let $\overline{w} = (w(n))$ be a word such that $\mathcal{B}(\alpha_1, \ldots, \alpha_N, \overline{w}) \le \mathcal{B}(\alpha_1, \ldots, \alpha_N) + 1$. Since $t(\alpha_{w(n)}^{(n)}) = (1/2\pi)\log(1/\alpha) + C$ and $0 < \alpha_{w(n)}^{(n)} < 1/2$ for $n \ge 1$,

$$\sum_{n=0}^{\infty} \alpha_{w(0)}^{(0)} \cdots \alpha_{w(n-1)}^{(n-1)} t(\alpha_{w(n)}^{(n)}) \le \mathcal{B}(\alpha_1, \ldots, \alpha_N, \overline{w}) + C''$$

for a universal constant C''. Let $\mathcal{F}_{0,n}$ be the families of maps given by Proposition 3.4, and

$$\mathcal{F} = \{f : E \circ F = f \circ E \text{ for some } F \in \mathcal{F}_{0,n}, n \ge 0\}.$$

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Then, by the estimate in Proposition 3.4, the family $\mathcal{F}$ satisfies the hypotheses of Lemma 4.1 on a disc $\mathbb{D}_r$, where $r = C'''e^{-2\pi\mathcal{B}(\alpha_1,\ldots,\alpha_N)}$ for some universal constant C'''; these maps commute with $f_1, \ldots, f_N$, hence $f_1, \ldots, f_N$ are linearizable with a common Siegel disc containing $\mathbb{D}_r$. □

Proof of Theorem 1.2. For a map $F \in \hat{\mathcal{S}}_{\alpha,0}$ without fixed points, as Perez-Marco shows in [**PM93**], we can take $t(\alpha) = (1/2\pi)\log\log(e/\alpha)$ as the height above which F is close to translation by α. We note that the set of fixed points in $\mathbb{H}_C$ (where $C > 0$) of any F_j is finite modulo the translation T, and hence, if non-empty, gives rise to a periodic orbit (modulo T) for $F_1, \ldots, F_N, T$. Since $F_1^{(n+1)}, \ldots, F_N^{(n+1)}, T$ are conjugates by K_n of the group generated by $F_1^{(n)}, \ldots, F_N^{(n)}, T$, a periodic orbit of $F_1^{(n+1)}, \ldots, F_N^{(n+1)}, T$ corresponds to a periodic orbit for $F_1^{(n)}, \ldots, F_N^{(n)}, T$. It follows that if $F_1, \ldots, F_N, T$ have no periodic orbits, then neither do $F_1^{(n)}, \ldots, F_N^{(n)}, T$. In particular, $F_{w(n)}^{(n)}$ has no fixed points modulo T, and so we can take $t(\alpha_{w(n)}^{(n)}) = (1/2\pi)\log\log(e/\alpha_{w(n)}^{(n)})$ for all n. The proof then follows the same lines as that of Theorem 1.1 above. □

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A. *Appendix*

We prove that Moser's Diophantine condition implies the Brjuno-type condition.

Let $N \geq 2$ be an integer and $\tau > 0$. We will write $a \ll b$ for quantities satisfying $a \leq kb$, where k is a constant depending only on N, τ, and $a \asymp b$ for $a \ll b$, $b \ll a$. For $C > 0$, let $DC_N(C, \tau)$ be the set of N-tuples $(\alpha_1, \ldots, \alpha_N)$ such that $\alpha_1, \ldots, \alpha_N$ are rationally independent irrationals in the interval $(0, 1)$ satisfying

$$\max_{1\leq j\leq N} |q\alpha_j - p_j| \geq \frac{C}{|q|^\tau}$$

for all integers $p_1, \ldots, p_N, q$ with $q \neq 0$. Using Khintchine's transference principle, it can be shown (see [**Sch80**, Ch. IV]) that any $(\alpha_1, \ldots, \alpha_N) \in DC_N(C, \tau)$ satisfies

$$|(\alpha_1, \ldots, \alpha_N, 1) \cdot (p_1, \ldots, p_N, q)| \geq \frac{C'}{\|(p_1, \ldots, p_N, q)\|^{\tau'}}$$

for all $(p_1, \ldots, p_N, q) \in \mathbb{Z}^{N+1} - \{0\}$, where $\|\cdot\|$ denotes the sup norm on $\mathbb{R}^{N+1}$, $C' \gg C$ and $\tau' = N\tau + N - 1$. Conversely, if an inequality

$$|(\alpha_1, \ldots, \alpha_N, 1) \cdot (p_1, \ldots, p_N, q)| \geq \frac{C'}{\|(p_1, \ldots, p_N, q)\|^{\tau'}}$$

holds for all $(p_1, \ldots, p_N, q) \in \mathbb{Z}^{N+1} - \{0\}$, then $(\alpha_1, \ldots, \alpha_N) \in DC_N(C, \tau)$, where $C \gg C'$ and $\tau = (\tau' + 1 - N)/N$.

Letting $(\alpha_1, \ldots, \alpha_N) \mapsto (\tilde{\alpha_1}, \ldots, \tilde{\alpha_N}) = G(\alpha_1, \ldots, \alpha_N, w)$, denote the Gauss maps defined in §2 (where $w \in \{1, \ldots, N\}$); we have the following lemma.

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LEMMA A.1.

$$(\alpha_1, \ldots, \alpha_N) \in DC_N(C, \tau) \Rightarrow (\tilde{\alpha_1}, \ldots, \tilde{\alpha_N}) \in DC_N(C', \tau),$$

where $C' \gg C\alpha_w^{\tau'-1}$, $\tau' = N\tau + N - 1$.

Proof. Let $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_N, 1)$, $\overrightarrow{\beta} = (\tilde{\alpha_1}, \ldots, \tilde{\alpha_N}, 1) \in \mathbb{R}^{N+1}$. We can write

$$\overrightarrow{\beta} = \frac{1}{\alpha_w} A \overrightarrow{\alpha},$$

where $A \in GL(N+1, \mathbb{Z})$ is a matrix whose largest entry is bounded by $1/\alpha_w$, hence $\|A^T\| \ll 1/\alpha_w$ (where here $\|\cdot\|$ denotes the operator norm). Let $\tau' = N\tau + N - 1$. For $\overrightarrow{k} = (p_1, \ldots, p_N, q)$, we have

$$\begin{aligned} |\overrightarrow{\beta} \cdot \overrightarrow{k}| &= \frac{1}{\alpha_w} |\overrightarrow{\alpha} \cdot A^T \overrightarrow{k}| \\ &\gg \frac{C}{\alpha_w} \frac{1}{\|A^T \overrightarrow{k}\|^{\tau'}} \\ &\geq \frac{C}{\alpha_w} \frac{1}{\|A^T\|^{\tau'}} \frac{1}{\|\overrightarrow{k}\|^{\tau'}} \\ &\gg \frac{C\alpha_w{}^{\tau'-1}}{\|\overrightarrow{k}\|^{\tau'}}. \end{aligned}$$

The conclusion of the lemma follows from the remarks above. □

LEMMA A.2. *For* $(\alpha_1, \ldots, \alpha_N) \in DC_N(C, \tau)$, *given* $w \in \{1, \ldots, N\}$ *and* $(\tilde{\alpha_1}, \ldots, \tilde{\alpha_N}) = G(\alpha_1, \ldots, \alpha_N, w)$, *there exists* $1 \leq j = j(w) \leq N$ *such that* $\tilde{\alpha_j} \gg C\alpha_w^{\tau}$.

Proof. Let $1/\alpha_w = q + \beta$, where $q \geq 1$ is an integer and $|\beta| = \tilde{\alpha_w} < 1/2$. We note that $k_1 \leq q/(q+\beta) \leq k_2$ for universal constants $0 < k_1 < k_2 < 1$. If $\tilde{\alpha_w}\alpha_w = |1 - q\alpha_w| \geq C/|q|^{\tau}$, then $\tilde{\alpha_w} \geq C/|q|^{\tau}$, and we put $j(w) = w$. Otherwise we must have $|q\alpha_k - a_k| \geq C/|q|^{\tau}$ for some $k \neq w$, so

$$\begin{aligned} \tilde{\alpha_k} = \frac{|a_k\alpha_w - \alpha_k|}{\alpha_w} = |a_k - \alpha_k(q+\beta)| &\geq |q\alpha_k - a_k| - \alpha_k|\beta| \\ &\geq \frac{C}{|q|^{\tau}} - \alpha_k|\beta| \\ &\geq C'|q+\beta|^{\tau} - \alpha_k|\beta|, \end{aligned}$$

where $C' \gg C$. If $\alpha_k|\beta| \leq \frac{1}{2}C'/|q+\beta|^{\tau}$, then $\tilde{\alpha_k} \geq \frac{1}{2}C'\alpha_w^{\tau}$, and we put $j(w) = k$; otherwise we have

$$\tilde{\alpha_w} = |\beta| > \frac{1}{\alpha_k}\frac{1}{2}C'\alpha_w^{\tau} > \frac{1}{2}C'\alpha_w^{\tau}$$

and we put $j(w) = w$. □

PROPOSITION A.3. *If* $(\alpha_1, \ldots, \alpha_N) \in DC_N(C, \tau)$ *for some* $C, \tau > 0$, *then* $\mathcal{B}(\alpha_1, \ldots, \alpha_N) < +\infty$.

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Proof. We choose a $w \in \{1, \ldots, N\}$ and define $(\alpha_1^{(n)}, \ldots, \alpha_N^{(n)})_{n\geq 0}$, $(w(n))_{n\geq 0}$, $(C_n)_{n\geq 0}$ inductively by $(\alpha_1^{(0)}, \ldots, \alpha_N^{(0)}) := (\alpha_1, \ldots, \alpha_N)$, $w(0) = w$, $C_0 = C$, and, for $n \geq 0$, $(\alpha_1^{(n+1)}, \ldots, \alpha_N^{(n+1)}) := G(\alpha_1^{(n)}, \ldots, \alpha_N^{(n)}, w(n))$; by Lemma A.1, $(\alpha_1^{(n+1)}, \ldots, \alpha_N^{(n+1)}) \in DC_N(C_{n+1}, \tau)$, where $C_{n+1} := kC_n\alpha_{w(n)}^{(n)(\tau'-1)}$ for some constant $k > 0$ and $\tau' = N\tau + N - 1$. We let $w(n+1) := j(w(n))$ be given by the previous lemma applied to $(\alpha_1^{(n)}, \ldots, \alpha_N^{(n)})$ and $w(n)$, so that (for k taken small enough) $\alpha_{w(n+1)}^{(n+1)} \geq kC_n\alpha_{w(n)}^{(n)\tau}$.

An easy induction gives

$$C_{n+1} = k^{n+1}C(\alpha_{w(0)}^{(0)} \cdots \alpha_{w(n)}^{(n)})^{\tau'-1}.$$

Putting $\tau'' = \max(\tau, \tau' - 1)$, it follows that

$$\frac{1}{\alpha_{w(n+1)}^{(n+1)}} \leq \frac{1}{k^{n+1}C} \frac{1}{(\alpha_{w(0)}^{(0)} \cdots \alpha_{w(n)}^{(n)})^{\tau''}},$$

so

$$\begin{aligned}\sum_{n=0}^{\infty} \alpha_{w(0)}^{(0)} \cdots \alpha_{w(n)}^{(n)} \log \frac{1}{\alpha_{w(n+1)}^{(n+1)}} &\leq \sum_{n=0}^{\infty} (\alpha_{w(0)}^{(0)} \cdots \alpha_{w(n)}^{(n)}) \\ &\quad \times \left((n+1)\log\frac{1}{k} + \log\frac{1}{C} + \tau'' \log \frac{1}{(\alpha_{w(0)}^{(0)} \cdots \alpha_{w(n)}^{(n)})}\right) \\ &< +\infty,\end{aligned}$$

since $\alpha_{w(k)}^{(k)} < 1/2$ for $k \geq 1$. □

References

[Brj71] A. D. Brjuno. Analytical form of differential equations. *Trans. Moscow Math. Soc.* **25** (1971), 131–288.

[Mos90] J. Moser. On commuting circle mappings and simultaneous Diophantine approximations. *Math. Z.* **205** (1990), 105–121.

[PM93] R. Perez-Marco. Sur les dynamiques holomorphes non-linearisables et une conjecture de V.I. Arnold. *Ann. Sci. Éc. Norm. Supér.* **26** (1993), 565–644.

[PM95] R. Perez-Marco. Uncountable number of symmetries for non-linearisable holomorphic dynamics. *Invent. Math.* **119** (1995), 67–127.

[PM97] R. Perez-Marco. Fixed points and circle maps. *Acta Math.* **179**(2) (1997), 243–294.

[Sch80] W. M. Schmidt. *Diophantine Approximation (Lecture Notes in Mathematics, 785)*. Springer, New York, 1980.

[Sie42] C. L. Siegel. Iteration of analytic functions. *Ann. of Math.* (2) **43** (1942), 807–812.

[SMY97] P. Moussa, S. Marmi and J. C. Yoccoz. The Brjuno functions and their regularity properties. *Comm. Math. Phys.* **186** (1997), 265–293.

[Yoc95] J. C. Yoccoz. Théorème de Siegel, nombres de Bruno et polynômes quadratiques. *Astérisque* **231** (1995), 3–88.

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