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Generalized basic hypergeometric series with unconnected bases

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1. *Introduction.* In a series of recent papers Verma and Upadhyay (7, 8, 9) developed the theory of basic hypergeometric series with two bases q and $q^{\frac{1}{2}}$. These investigations were made in an attempt to discover a summation formula for a bilateral basic hypergeometric series ${}_2\Psi_2$ analogous to that for a ${}_2H_2$ (cf. Bailey (2, 3)) and in finding relations between certain q-infinite products. In one of their papers they mentioned that it did not seem possible to develop the corresponding general theory for basic series with two unconnected bases q and q_1. A recent paper by Andrews (1) indicates that transformations between basic hypergeometric series with two unconnected bases can be very interesting and useful in the study of 'mock' theta functions and their extensions. Besides this interest, such a theory also enables one to extend the entire existing transformation theory of the generalized basic hypergeometric series.

Prof. Agarwal recently suggested that it may be possible to find out general transformations between basic hypergeometric series with unconnected bases q and q_1, by considering a Slater-type of basic contour integral (5, 6) and choosing the parameters in a suitable manner so as to include only a definite number of poles, at a time, in a particular strip.

In this paper this idea has been developed to obtain the most general transformations between series with two bases q and q_1 ($|q| < 1$, $|q_1| < 1$). In fact, one can easily extend the method of this paper, to obtain transformations for basic hypergeometric series involving any finite number of independent bases.

2. *Definitions and notation.* Let

$$[\alpha^a; n] = (1-\alpha^a)(1-\alpha^{a+1})\dots(1-\alpha^{a+n-1}), \quad [\alpha^a; 0] = 1 \quad \text{and} \quad |\alpha| < 1.$$

The generalized 'bibasic' hypergeometric is then defined as

$$ {}_{A+B}\Phi_{C+D}\left[\begin{matrix} q^{(a)}: q_1^{(b)}; \\ q^{(c)}: q_1^{(d)}; \end{matrix} x \right] \equiv {}_{A+B}\Phi_{C+D}\left[\begin{matrix} (a): (b); \\ (c): (d); \end{matrix} x \right] $$
$$ = \sum_{n=0}^{\infty} \frac{[q^{(a)}; n][q_1^{(b)}; n]}{[q^{(c)}; n][q_1^{(d)}; n]} x^n q^{\frac{1}{2}\lambda n(n+1)} q_1^{\frac{1}{2}\lambda_1 n(n+1)}, $$

where $\lambda, \lambda_1 > 0$, $|q| < 1$, $|q_1| < 1$ and for $\lambda = 0 = \lambda_1$, $|x| < 1$.

In the numerator and the denominator the terms before the colon are on the base q and those after it are on the base q_1. As usual (a_N) stands for the sequence of N parameters $a_1, a_2, \dots, a_N$. When $N = A$ it will be dropped out.

Lastly, we denote by

$$\prod^{P}\left[\begin{matrix} x^{(\alpha)}; \\ x^{(\beta)} \end{matrix} x\right] \quad \text{or simply} \quad \prod^{P}\left[\begin{matrix} (\alpha); \\ (\beta) \end{matrix} x\right]$$

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the product $\prod_{u=0}^{P-1}\left[\frac{(1-x^{(\alpha)+u})}{(1-x^{(\beta)+u})}\right]$.

For P infinite it will be dropped from the product symbol. A term of the type $(h)'-h_r$ in the series or in the infinite product stands for the sequence

$$h_1-h_r,\quad h_2-h_r,\quad \ldots,\quad h_{r-1}-h_r,\quad h_{r+1}-h_r,\quad \ldots,\quad h_H-h_r$$

from which the term h_r-h_r is omitted.

The bases q and q_1 are assumed to be such that $|q|<1$, $|q_1|<1$ and $|q|<|q_1|$ for definiteness.

3. *A general transformation.* We now proceed to deduce the following general transformation

$$\begin{aligned}
&\frac{1}{t}\sum_{u=0}^{r-1}\Pi\begin{bmatrix}(a)+f_1,(b)-f_1;q\\1,(e)+f_1,(f)'-f_1\end{bmatrix}\Pi\begin{bmatrix}(c)+m[f_1]+u_1,(d)-m[f_1]-u_1;q_1\\(g)+m[f_1]+u_1,(h)-m[f_1]-u_1\end{bmatrix}\\
&\quad\times{}_\rho\Phi_\sigma\begin{bmatrix}(e)+f_1,1-(b)+f_1:1-(d)+m[f_1]+u_1,(g)+m[f_1]+u_1;Q_1\\(a)+f_1,1-(f)+f_1:1-(h)+m[f_1]+u_1,(c)+m[f_1]+u_1;q^{F-B}q_1^{H-D}\end{bmatrix}\\
&\quad+\text{idem }(f_1;f_2,f_3,\ldots,f_F)\\
&\quad+\frac{1}{t_1}\Pi\begin{bmatrix}(a)+M[h_1],(b)-M[h_1];q\\(e)+M[h_1],(f)-M[h_1]\end{bmatrix}\Pi\begin{bmatrix}(c)+h_1,(d)-h_1;q_1\\(g)+h_1,(h)'-h_1,1\end{bmatrix}\\
&\quad\times{}_\rho\Phi_\sigma\begin{bmatrix}(e)+M[h_1],1-(b)+M[h_1]:(g)+h_1,1-(d)+h_1;Q_2\\(a)+M[h_1],1-(f)+M[h_1]:(c)+h_1,1-(h)+h_1;q^{F-B}q_1^{H-D}\end{bmatrix}\\
&\quad+\text{idem }(h_1;h_2,h_3,\ldots,h_H)\\
&=\frac{1}{t}\sum_{u=0}^{r-1}\Pi\begin{bmatrix}(a)-e_1,(b)+e_1;q\\1,(e)'-e_1,(f)+e_1\end{bmatrix}\Pi\begin{bmatrix}(c)-m[e_1]-u_1,(d)+m[e_1]+u_1;q_1\\(g)-m[e_1]-u_1,(h)+m[e_1]+u_1\end{bmatrix}\\
&\quad\times{}_\sigma\Phi_\rho\begin{bmatrix}(f)+e_1,1-(a)+e_1:(h)+m[e_1]+u_1,1-(c)+m[e_1]+u_1;Q_3\\(b)+e_1,1-(e)+e_1:(d)+m[e_1]+u_1,1-(g)+m[e_1]+u_1;q^{E-A}q_1^{G-C}\end{bmatrix}\\
&\quad+\text{idem }(e_1;e_2,e_3,\ldots,e_E)\\
&\quad+\frac{1}{t_1}\Pi\begin{bmatrix}(a)-M[g_1],(b)+M[g_1];q\\(e)-M[g_1],(f)+M[g_1]\end{bmatrix}\Pi\begin{bmatrix}(d)+g_1,(c)-g_1;q_1\\1,(h)+g_1,(g)'-g_1\end{bmatrix}\\
&\quad\times{}_\sigma\Phi_\rho\begin{bmatrix}(f)+M[g_1],1-(a)+M[g_1]:(h)+g_1,1-(c)+g_1;Q_4\\(b)+M[g_1],1-(e)+M[g_1]:(d)+g_1,1-(g)+g_1;q^{E-A}q_1^{G-C}\end{bmatrix}\\
&\quad+\text{idem }(g_1;g_2,g_3,\ldots,g_G),
\end{aligned}\tag{3·1}$$

where $$\rho=B+D+E+G,\quad \sigma=A+C+F+H,\quad u_1\equiv 2\pi iu/t,$$

$$Q_\mu=(-1)^{B+D-F-H}\,q^{\Sigma(b)-\Sigma(f)+(F-B)\xi_\mu}q_1^{\Sigma(d)-\Sigma(h)+(H-D)\eta_\mu}$$

with $$\mu=1,2;\quad \xi_1=f_1,\quad \xi_2=M[h_1],\quad \eta_1=m[f_1]+u_1,\quad \eta_2=h_1;$$

$$Q_\nu=(-1)^{A+C-E-G}\,q^{\Sigma(a)-\Sigma(e)+(E-A)\xi_\nu}q_1^{\Sigma(c)-\Sigma(g)+(G-C)\eta_\nu}$$

with $$\nu=3,4;\quad \xi_3=e_1,\quad \xi_4=M[g_1],\quad \eta_3=m[e_1]+u_1,\quad \eta_4=g_1;$$

and r is a positive integer such that

$$rt_1\leqslant t<(r+1)t_1.\tag{3·2}$$

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Also m_α is the sequence of least integers for which

$$-\frac{\pi}{t_1} < \Im\{m[(f)], M[(h)], -m[(e)], -M[(g)]\} < \frac{\pi}{t_1}, \qquad (3\cdot3\,a\text{–}d)$$

where the numbers $(\alpha)+(2\pi i/t)m_{(\alpha)}$ have been abbreviated to $m[(\alpha)]$ and $(\alpha)+(2\pi i/t_1)m_{(\alpha)}$ to $M[(\alpha)]$.

The parameters in (3·1) are subject to the restrictions

$$\left.\begin{array}{llllll} \text{(I)} & F > B & \text{or} & F = B & \text{and} & \Re[\Sigma(b)-\Sigma(f)] > 0, \\ \text{(II)} & H > D & \text{or} & H = D & \text{and} & \Re[\Sigma(d)-\Sigma(h)] > 0, \\ \text{(III)} & E > A & \text{or} & E = A & \text{and} & \Re[\Sigma(a)-\Sigma(c)] > 0, \\ \text{(IV)} & G > C & \text{or} & G = C & \text{and} & \Re[\Sigma(c)-\Sigma(g)] > 0. \end{array}\right\} \qquad (3\cdot4)$$

To prove (3·1), let us consider the contour integral

$$I_{P,R} = \frac{1}{2\pi i}\int_L \overset{P}{\Pi}(s)\,ds,$$

where the contour L is the rectangle

$$A(-i\pi/t_1), \quad B(2R-i\pi/t_1), \quad C(2R+i\pi/t_1), \quad D(i\pi/t_1),$$

$P < R$ in the s-plane and

$$\overset{P}{\Pi}(s) = \overset{P}{\Pi}\begin{bmatrix}(a)+s,(b)-s;q\\(e)+s,(f)-s\end{bmatrix}\overset{P}{\Pi}\begin{bmatrix}(c)+s,(d)-s;q_1\\(g)+s,(h)-s\end{bmatrix}.$$

Let q and q_1 be supposed real, $0<q<1$, $0<q_1<1$, $q=e^{-t}$, $q_1=e^{-t_1}$, $t>0$, $t_1>0$, $t>t_1$. The restrictions that q and q_1 are real are not as drastic as they might appear to be, for once we prove (3·1) for q and q_1 real, the result for complex values could be deduced by making an appeal to the method of analytic continuation. The restrictions are introduced only to shorten the proof by ensuring that the strip contours considered all lie parallel to the real axis. All other parameters in (3·1) are, in general, complex numbers, subject to the restriction that no two denominator parameters coincide, to avoid double poles.

Now $q^{f_1-s}=e^{-t(f_1-s)}$ has a period $2\pi i/t$ and hence $\{\Pi[f_1-s;q]\}^{-1}$ has poles at all the points $s=n+m[f_1]$ in the s-plane, one set in every strip of width $2\pi/t$. The condition (3·3 (a)) ensures that there is at least one such sequence of poles within $ABCD$. However, by virtue of (3·2) there can only be r such sequences of poles inside the strip $ABCD$. The restriction (3·2) is imposed to make the number of such sequences in a given strip definite.

Next, for a product of the type $\{\Pi[h_1-s;q_1]\}^{-1}$, one has poles at all the points $s=n+M[h_1]$, in the s-plane, one set in every strip of width $2\pi/t_1$. The condition (3·3 (b)) ensures that there is only one sequence of such poles within the contour $ABCD$.

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Further, the line AD may be indented, if necessary, to ensure that all the poles in the increasing sequences

$$s = n+m[(f)]+u_1 \quad \text{and} \quad s = M[(h)]+n; \quad n = 0, 1, \ldots, P-1; u = 0, 1, 2, \ldots, r-1$$

lie to the right of AD and the sequences of decreasing poles

$$s = -m[(e)]-n-u_1 \quad \text{and} \quad s = -M[(g)]-n; \quad n = 0, 1, \ldots, P-1; u = 0, 1, 2, \ldots, r-1$$

lie to the left of it. Then

$$\frac{1}{2\pi i}\int_L \prod^P(s)\,ds = \frac{1}{2\pi i}\left[\int_{AB}+\int_{BC}+\int_{CD}+\int_{DA}\right]\prod^P(s)\,ds.$$

On the line AB and CD, $s = \sigma \mp i\pi/t_1$, $0 \leqslant \sigma \leqslant 2R$, and since for $s = \sigma + i\tau$

$$1-q^{\sigma+\Re a} \leqslant |1-q^{a+s}| \leqslant 1+q^{\sigma+\Re a},$$

$$q^{\Re a-\sigma}(1-q^{\sigma-\Re a}) \leqslant |1-q^{a-s}| \leqslant q^{\Re a-\sigma}(1+q^{\sigma-\Re a}),$$

we have

$$|\prod^P(\sigma \mp i\tau)| \leqslant q^{PK}q_1^{PK_1}\prod^{P\prime}(\sigma),$$

where

$$K = \sigma(F-B)+\Re[\Sigma(b)-\Sigma(f)],$$

$$K_1 = \sigma(H-D)+\Re[\Sigma(d)-\Sigma(h)],$$

and

$$\prod^{P\prime}(\sigma) = \prod^P\left[\begin{matrix}-q^{\sigma+\Re(a)}, -q^{\sigma-\Re(b)}; q\\ q^{\sigma+\Re(e)}, q^{\sigma-\Re(f)}\end{matrix}\right]\prod^P\left[\begin{matrix}-q_1^{\sigma+\Re(c)}, -q_1^{\sigma-\Re(d)}; q_1\\ q_1^{\sigma+\Re(g)}, q_1^{\sigma-\Re(h)}\end{matrix}\right].$$

It is easy to see that $\prod^{P\prime}(\sigma)$ is a bounded function of σ and tends to one as $\sigma \to \infty$. Moreover, since

$$\int_0^\infty q^{\sigma(F-B)P}q_1^{\sigma(H-D)P}\prod^{P\prime}(\sigma)\,d\sigma$$

converges under the set of conditions (3·4 (I) and (II)), the integrals along the lines AB and CD tend to zero as R and P tend to infinity.

On BC: $s = 2R+ir_1$, $-\pi/t_1 \leqslant r_1 \leqslant \pi/t_1$, and hence it can be easily shown that the integral along BC tends to zero under the set of conditions (3·4 (I) and (II)) as R and P tend to infinity.

Thus, we obtain by the calculus of residues, that

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\pi/t_1}^{i\pi/t_1}\Pi(s)\,ds &= \frac{1}{t}\sum_{n=0}^{r-1}\Pi\left[\begin{matrix}(a)+f_1, (b)-f_1; q\\ 1, (e)+f_1, (f')'-f_1\end{matrix}\right]\Pi\left[\begin{matrix}(c)+m[f_1]+u_1, (d)-m[f_1]-u_1; q_1\\ (g)+m[f_1]+u_1, (h)-m[f_1]-u_1\end{matrix}\right]\\ &\times {}_\rho\Phi_\sigma\left[\begin{matrix}(e)+f_1, 1-(b)+f_1: (g)+m[f_1]+u_1, 1-(d)+m[f_1]+u_1; Q_1\\ (a)+f_1, 1-(f)+f_1: (c)+m[f_1]+u_1, 1-(h)+m[f_1]+u_1; q^{F-B}q_1^{H-D}\end{matrix}\right]\\ &+\text{idem}\,(f_1; f_2, f_3, \ldots, f_F)\\ &+\frac{1}{t_1}\Pi\left[\begin{matrix}(a)+M[h_1], (b)-M[h_1]; q\\ (e)+M[h_1], (f)-M[h_1]\end{matrix}\right]\Pi\left[\begin{matrix}(c)+h_1, (d)-h_1; q_1\\ (g)+h_1, (h)'-h_1, 1\end{matrix}\right]\\ &\times {}_\rho\Phi_\sigma\left[\begin{matrix}(e)+M[h_1], 1-(b)+M[h_1]: (g)+h_1, 1-(d)+h_1; Q_2\\ (a)+M[h_1], 1-(f)+M[h_1]: (c)+h_1, 1-(h)+h_1\end{matrix}\right]\\ &+\text{idem}\,(h_1; h_2, h_3, \ldots, h_H). \qquad (3\cdot5)\end{aligned}$$

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Similarly, considering a rectangular strip to the left of the imaginary axis and proceeding exactly as above, we get

$$\frac{1}{2\pi i}\int_{-i\pi/t_1}^{i\pi/t_1}\Pi(s)\,ds=\frac{1}{t}\sum_{u=0}^{r-1}\Pi\begin{bmatrix}(b)+e_1,(a)-e_1;q\\1,(f)+e_1,(e)'-e_1\end{bmatrix}\Pi\begin{bmatrix}(d)+m[e_1]+u_1,(c)-m[e_1]-u_1;q_1\\(h)+m[e_1]+u_1,(g)-m[e_1]-u_1\end{bmatrix}$$

$$\times{}_\sigma\Phi_\rho\begin{bmatrix}(f)+e_1,1-(a)+e_1:(h)+m[e_1]+u_1,1-(c)+m[e_1]+u_1;Q_3\\(b)+e_1,1-(e)+e_1:(d)+m[e_1]+u_1,1-(g)+m[e_1]+u_1;q^{E-A}q_1^{G-C}\end{bmatrix}$$

$$+\text{idem}\,(e_1;e_2,e_3,\ldots,e_E)$$

$$+\frac{1}{t_1}\Pi\begin{bmatrix}(b)+M[g_1],(a)-M[g_1];q\\(f)+M[g_1],(e)-M[g_1]\end{bmatrix}\Pi\begin{bmatrix}(d)+g_1,(c)-g_1;q_1\\(h)+g_1,(g)'-g_1,1\end{bmatrix}$$

$$\times{}_\sigma\Phi_\rho\begin{bmatrix}(f)+M[g_1],1-(a)+M[g_1]:(h)+g_1,1-(c)+g_1;Q_4\\(b)+M[g_1],1-(e)+M[g_1]:(d)+g_1,1-(g)+g_1;q^{E-A}q_1^{G-C}\end{bmatrix}$$

$$+\text{idem}\,(g_1;g_2,g_3,\ldots,g_G),\tag{3·6}$$

under the set of conditions (3·4(III) and (IV)).

Equating the right-hand sides of (3·5), (3·6), we get the required transformation (3·1).

In this transformation if we take $q=q_1$ and $C=D=G=H=0$, we get the general transformation due to Slater (6), which, in turn, incorporates the general transformation of basic hypergeometric series due to Sears (4).

It may further be noted that proceeding exactly as above one could even derive general transformations of basic hypergeometric series with more than two bases. Transformations of such 'polybasic' series would naturally be more complicated and hence are not mentioned here.

4. *A general transformation between products of two series.* The transformation (3·1) readily gives a transformation between the products of two generalized basic hypergeometric series, in which one of the series is on the base q and the other is on the base q_1. In fact, we prove that

$$\frac{1}{t}\sum_{u=0}^{r-1}\Pi\begin{bmatrix}(a)+f_1,(b)-f_1;q\\1,(e)+f_1,(f)'-f_1\end{bmatrix}\Pi\begin{bmatrix}(d)-m[f_1]-u_1,(c)+m[f_1]+u_1;q_1\\(h)-m[f_1]-u_1,(g)+m[f_1]+u_1\end{bmatrix}$$

$$\times{}_\delta\Phi_\mu^{(q)}\begin{bmatrix}(e)+f_1,1-(b)+f_1;\xi/x\\(a)+f_1,1-(f)+f_1;q^{F-B}\end{bmatrix}$$

$$\times{}_\nu\Phi_\varepsilon^{(q_1)}\begin{bmatrix}(g)+m[f_1]+u_1,1-(d)+m[f_1]+u_1;x\xi_1\\(c)+m[f_1]+u_1,1-(h)+m[f_1]+u_1;q_1^{H-D}\end{bmatrix}+\text{idem}\,(f_1;f_2,\ldots,f_F)$$

$$+\frac{1}{t_1}\Pi\begin{bmatrix}(a)+M[h_1],(b)-M[h_1];q\\(e)+M[h_1],(f)-M[h_1]\end{bmatrix}\Pi\begin{bmatrix}(c)+h_1,(d)-h_1;q_1\\(g)+h_1,(h)'-h_1,1\end{bmatrix}$$

$$\times{}_\mu\Phi_\delta^{(q)}\begin{bmatrix}(f)-M[h_1],1-(a)-M[h_1];\alpha x\\(b)-M[h_1],1-(e)-M[h_1],q^{F-B}\end{bmatrix}{}_\nu\Phi_\varepsilon^{(q_1)}\begin{bmatrix}(g)+h_1,1-(d)+h_1;\alpha_1 x\\(c)+h_1,1-(h)+h_1;q_1^{H-D}\end{bmatrix}$$

$$+\text{idem}\,(h_1;h_2,\ldots,h_H)$$

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$$= \frac{1}{t}\sum_{u=0}^{r-1} \Pi\begin{bmatrix}(a)-e_1,(b)+e_1;q\\1,(e)'-e_1,(f)+e_1\end{bmatrix}\Pi\begin{bmatrix}(c)-m[e_1]-u_1,(d)+m[e_1]+u_1;q_1\\(g)-m[e_1]-u_1,(h)+m[e_1]+u_1\end{bmatrix}$$
$$\times {}_\mu\Phi_\delta^{(q)}\begin{bmatrix}(f)+e_1,1-(a)+e_1;\beta x\\(b)+e_1,1-(e)+e_1;q^{E-A}\end{bmatrix}$$
$$\times {}_\nu\Phi_\epsilon^{(q_1)}\begin{bmatrix}1-(d)-m[e_1]-u_1,(g)-m[e_1]-u_1;\beta_1 x\\1-(h)-m[e_1]-u_1,(c)-m[e_1]-u_1;q_1^{G-C}\end{bmatrix}+\text{idem}\,(e_1;e_2,\ldots,e_E)$$
$$+\frac{1}{t_1}\Pi\begin{bmatrix}(a)-M[g_1],(b)+M[g_1];q\\(e)-M[g_1],(f)+M[g_1]\end{bmatrix}\Pi\begin{bmatrix}(d)+g_1,(c)-g_1;q_1\\(h)+g_1,(g)'-g_1,1\end{bmatrix}$$
$$\times {}_\mu\Phi_\delta^{(q)}\begin{bmatrix}(f)+M[g_1],1-(a)+M[g_1];\gamma x\\(b)+M[g_1],1-(e)+M[g_1];q^{E-A}\end{bmatrix}{}_\epsilon\Phi_\nu^{(q_1)}\begin{bmatrix}(h)+g_1,1-(c)+g_1;\gamma_1/x\\(d)+g_1,1-(g)+g_1;q_1^{G-C}\end{bmatrix}$$
$$+\text{idem}\,(g_1;g_2,\ldots,g_G), \tag{4·1}$$

where
$$B+E=\delta,\quad C+H=\epsilon,\quad A+F=\mu,\quad D+G=\nu;$$
$$\xi=(-1)^{B-F}\quad q^{\Sigma(b)-\Sigma(f)+(F-B)f_1},$$
$$\xi_1=(-1)^{D-H}\quad q_1^{\Sigma(d)-\Sigma(h)+(H-D)[m[f_1]+u_1]},$$
$$\alpha=(-1)^{A-E}\quad q^{\Sigma(a)-\Sigma(e)+(A-E)M[h_1]},$$
$$\alpha_1=(-1)^{H-D}\quad q_1^{\Sigma(d)-\Sigma(h)+(H-D)h_1},$$
$$\beta=(-1)^{A-E}\quad q^{\Sigma(a)-\Sigma(e)+[E-A]e_1},$$
$$\beta_1=(-1)^{H-D}\quad q_1^{\Sigma(d)-\Sigma(h)+(D-H)(m[e_1]+u_1)},$$
$$\gamma=(-1)^{A-E}\quad q^{\Sigma(a)-\Sigma(e)+(E-A)M[g_1]},$$
$$\gamma_1=(-1)^{G-C}\quad q_1^{\Sigma(c)-\Sigma(g)+(G-C)g_1}$$

provided the following conditions of convergence are satisfied:

(I) $F>B$ or $F=B$ and $|\xi/x|<1$;
(II) $H>D$ or $H=D$ and $|\xi_1 x|<1$ and $|x\beta_1|<1$, $|x\alpha_1|<1$;
(III) $E>A$ or $E=A$ and $|\beta x|<1$, $|x\alpha|<1$ and $|x\gamma|<1$;
(IV) $G>C$ or $G=C$ and $|\gamma_1/x|<1$.

The symbols $\Phi^{(q)}$ and $\Phi^{(q_1)}$ obviously denote the bases on which the series are taken.

The truth of (4·1) can easily be verified by equating the coefficients of x^m on both sides of it. The resulting identity so obtained comes out to be (3·1) with (c), (d), (g), (h) replaced by $(c)+m$, $(d)-m$, $(g)+m$, $(h)-m$, respectively. This extends a result due to Sears (4) on products of basic series.

We have considered here only one transformation of products of two series, one with base q and other with base q_1. However, one could easily extend these to products of any number of series with base q or q_1 or both by following a method used earlier by one of the present authors (7).

5. *Special types of 'bibasic' generalized basic hypergeometric series*: We shall call the generalized 'bibasic' hypergeometric series

$${}_{A+B}\Phi_{A+B}\begin{bmatrix}(a):(b)\,x\\(c_A):(d_B);q^\lambda q_1^{\lambda_1}\end{bmatrix} \tag{5·1}$$

to be 'q-poised' if $\quad a_r+c_r=a_{r+1}+c_{r+1}\quad$ for $\quad r=1,2,\ldots,A-1.$ (5·2)

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Similarly, we shall call (5·1) a 'q_1-poised' series if

$$b_r + d_r = b_{r+1} + d_{r+1} \quad \text{for} \quad r = 1, 2, \ldots, B-1. \tag{5·3}$$

However, if the relations (5·2) and (5·3) hold simultaneously we shall call the series (5·1) a 'well-poised' series. It may be noted that $q^{a_1+c_1}$ and $q_1^{b_1+d_1}$ may not necessarily be equal.

If (5·2) holds for $r = 1, 2, 3, \ldots, A-2$, then the series (5·1) shall be called a 'q-nearly poised' series, whereas, if any one of the equations in (5·3) breaks down, then the series (5·1) will be called a 'q_1-nearly poised' series. However, if (5·1) is both 'q-nearly-poised' and 'q_1-nearly poised' then we shall simply call it a 'nearly-poised' series.

Lastly, if

$$\Sigma(c) = \Sigma(a) + 1, \tag{5·4}$$

then (5·1) will be said to be 'q-Saalschützian' and if

$$\Sigma(d) = \Sigma(b) + 1, \tag{5·5}$$

then (5·1) will be said to be a 'q_1-Saalschützian' series. But if (5·4) and (5·5) hold simultaneously then we shall call (5·1) simply a 'Saalschützian' series.

6. *A well-poised integral and a well-poised transformation.* Let us consider the contour integral

$$\mathrm{I} = \frac{1}{2\pi i}\int_{-i\pi/t_1}^{i\pi/t_1} \Pi\begin{bmatrix} 1-(b)-s, 1-(b)+\alpha+s; q \\ (e)+s, (e)-\alpha-s \end{bmatrix} \Pi\begin{bmatrix} 1-(d)-s, 1-(d)+\alpha+s; q_1 \\ (g)+s, (g)-\alpha-s \end{bmatrix} q^s\, ds. \tag{6·1}$$

This can be evaluated easily as in (3·1) to obtain the following well-poised transformation:

$$\begin{aligned}
&\frac{1}{t} q^{-e_1}[1-q^{2e_1-\alpha}] \sum_{u=0}^{r-1} \Pi\begin{bmatrix} 1-(b)+\alpha-e_1, 1-(b)+e_1; q \\ (e)+e_1-\alpha, (e)'-e_1, 1 \end{bmatrix} \\
&\quad \times \Pi\begin{bmatrix} 1-(d)+\alpha-m[e_1]-u_1, 1-(d)+m[e_1]+u_1; q_1 \\ (g)+m[e_1]+u_1-\alpha, (g)-m[e_1]-u_1 \end{bmatrix} \\
&\quad \times {}_R\Phi_R\left[\begin{matrix} 1+e_1-\frac{1}{2}\alpha, -[1+e_1-\frac{1}{2}\alpha]^*, (b)+e_1-\alpha, \\ \qquad (e)+e_1-\alpha: (d)-\alpha+m[e_1]+u_1, (g)-\alpha+m[e_1]+u_1; Q_1 \\ e_1-\frac{1}{2}\alpha, -[e_1-\frac{1}{2}\alpha]^*, 1-(b)+e_1, 1-(e)+e_1: \\ \qquad 1-(d)+m[e_1]+u_1, 1-(g)+m[e_1]+u_1; q^{E-B}q_1^{G-D} \end{matrix}\right] \\
&\quad + \text{idem}\,(e_1; e_2, \ldots, e_E) \\
&\quad + \frac{1}{t_1} q^{-M[g_1]}[1-q^{2M[g_1]-\alpha}] \Pi\begin{bmatrix} 1-(b)-M[g_1]+\alpha, 1-(b)+M[g_1]; q \\ (e)+M[g_1]-\alpha, (e)-M[g_1] \end{bmatrix} \\
&\quad \times \Pi\begin{bmatrix} 1-(d)-g_1+\alpha, 1-(d)+g_1; q_1 \\ (g)+g_1-\alpha, (g)'-g_1, 1 \end{bmatrix} \\
&\quad \times {}_R\Phi_R\left[\begin{matrix} 1+M[g_1]-\frac{1}{2}\alpha, -[1+M[g_1]-\frac{1}{2}\alpha]^*, (b)+M[g_1]-\alpha, \\ \qquad (e)+M[g_1]-\alpha: (d)-\alpha+g_1, (g)+g_1-\alpha; Q_2 \\ M[g_1]-\frac{1}{2}\alpha, -[M[g_1]-\frac{1}{2}\alpha]^*, 1-(b)+M[g_1], \\ \qquad 1-(e)+M[g_1]: 1-(d)+g_1, 1-(g)+g_1; q^{E-B}q_1^{G-D} \end{matrix}\right] \\
&\quad + \text{idem}\,(g_1; g_2, \ldots, g_G) = 0,
\end{aligned} \tag{6·2}$$

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where $$R = B+D+E+G+2,$$

$$Q_\mu = (-1)^{B+D-E-G} q^{B+B\alpha-\Sigma(b)-\Sigma(e)-1+(E-B)\xi_\mu} q^{D+D\alpha-\Sigma(g)-\Sigma(d)+(G-D)\eta_\mu},$$

$$\mu = 1, 2; \quad \xi_1 = e_1, \quad \xi_2 = M[g_1], \quad \eta_1 = u_1 + M[e_1], \quad \eta_2 = g_1$$

provided

(I) $E > B$ or $E = B$ and $|Q_1| < 1$,

(II) $G > D$ or $G = D$ and $|Q_2| < 1$.

The parameter -$[\alpha]$* in the series denotes a term of the type $[-q^\alpha]_n$ in the series definition of the 'bibasic' hypergeometric series.

For $D = O = G$, (6·1) reduces to an integral equivalent to one due to Slater (5), from which all the four well-poised transformations of Sears (4) can be deduced.

In (6·1) if we take q_1^s instead of q^s, the integral can still be evaluated to get a result similar to (6·2).

It may be remarked that (6·2) only contains one 'free' parameter α but we could have introduced another 'free' parameter β also by considering β instead of α in the 'q_1-product' in the integrand of (6·1). Suitable variants of (6·1) yield 'well-poised' transformations similar to those for series with one base (see Slater (5)).

In case we do not take the terms q^s or q_1^s in the integrand of (6·1) we obtain an integral representation for a combination of 'bibasic' well-poised hypergeometric series similar to those occurring in (6·2).

7. *Conclusion.* It may be noted that if a 'free' parameter on the base q is required in the transformation (3·1), we may write $A+1$ for A and $B+1$ for B and set $a_1 = 1-x$, $b_1 = x$, to get the desired transformation. Similarly, to deduce a transformation with 'free' parameters in the infinite products on the base q_1, one can replace C and D by $C+1$ and $D+1$ respectively and set $c_1 = 1-y$, $d_1 = y$ (cf. Slater (6)).

Starting from the above types of transformations one could also derive more general product theorems similar to (4·1).

We propose to study in greater details these and allied transformations in a subsequent communication.

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