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A note on the Maxwell's eigenvalues on thin sets

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We analyse the Maxwell's spectrum on thin tubular neighbourhoods of embedded surfaces of $\mathbb{R}^3$. We show that the Maxwell's eigenvalues converge to the Laplacian eigenvalues of the surface as the thin parameter tends to zero. To achieve this, we reformulate the problem in terms of the spectrum of the Hodge Laplacian with relative conditions acting on co-closed differential 1-forms. The result leads to new examples of domains where the Faber–Krahn inequality for Maxwell's eigenvalues fails, examples of domains with any number of arbitrarily small eigenvalues, and underlines the failure of spectral stability under singular perturbations changing the topology of the domain. Additionally, we explicitly produce Maxwell's eigenfunctions on product domains with the product metric, extending previous constructions valid in the Euclidean case.

Keywords: eigenvalue asymptotics; Hodge Laplacian; Maxwell's equations; thin domain limit; product metric

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1. Introduction and statement of the main results

Let Ω be a bounded domain in $\mathbb{R}^3$. The second-order reformulation of the time-harmonic Maxwell's system

$$\begin{cases} \operatorname{curl} E = \mathrm{i}\eta H, & \text{in } \Omega, \\ \operatorname{curl} H = -\mathrm{i}\eta E, & \text{in } \Omega, \\ \nu \times E = 0, & \text{on } \partial\Omega \end{cases}$$

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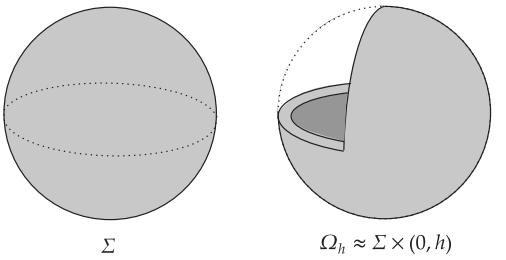


Figure 1. Surface Σ without boundary and domain Ω_h.

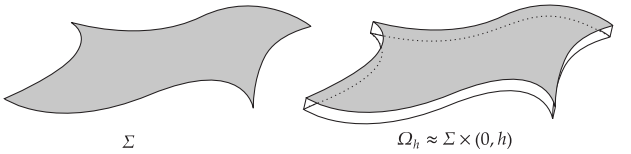


Figure 2. Surface Σ with boundary and domain Ω_h.

is given by

$$\begin{cases} \operatorname{curl}\operatorname{curl} E = \lambda E, & \text{in } \Omega, \\ \nu \times E = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the outer normal to $\partial\Omega$ and $\lambda := \eta^2$ is the eigenvalue. If Ω is sufficiently regular, e.g., if $\partial\Omega$ is Lipschitz, it is well-known that problem (1.1) has $\lambda = 0$ as an eigenvalue of infinite multiplicity and it further admits a sequence of non-negative eigenvalues of finite multiplicity (corresponding to divergence-free eigenfunctions)

$$0 \leq \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_j(\Omega) \leq \cdots \nearrow +\infty,$$

where the eigenvalues are repeated according to their multiplicity.

We are mainly interested in the dependence of $\lambda_j(\Omega)$ upon the perturbation of the domain Ω; in particular, we will assume that Ω is a thin domain, described by tubes of size h around smooth embedded surfaces (with or without boundary). More precisely, if Σ is a smooth, embedded, orientable, compact surface in $\mathbb{R}^3$ (with or without boundary), we define, for all $h > 0$ sufficiently small, the tube Ω_h by

$$\Omega_h := \{x + t\nu(x) : t \in (0,h), x \in \Sigma\}, \tag{1.2}$$

where ν is a choice of a unit normal vector field on Σ, and $\nu(x)$ is the corresponding unit normal vector at $x \in \Sigma$. Note that if Σ has a boundary, Ω_h is just a piecewise smooth, Lipschitz domain. See Figures 1 and 2.

If the boundary is just Lipschitz, problem (1.1), and in particular, the boundary condition $\nu \times E|_{\partial\Omega_h} = 0$, has to be interpreted in a suitable weak sense, see [11]. We now state our main result.

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THEOREM 1.1. *Let Σ be a smooth, compact, embedded, orientable surface in $\mathbb{R}^3$, and let Ω_h be the tube of size h around Σ defined by (1.2). Let $\{\lambda_j(\Omega_h)\}_{j=1}^{\infty}$ be the sequence of Maxwell's eigenvalues. Then, for all $j \in \mathbb{N}$:*

(i) if $\partial\Sigma = \emptyset$, $\lim_{h\to 0^+} \lambda_j(\Omega_h) = \mu_j$, where $\{\mu_j\}_{j=1}^{\infty}$ are the Laplacian eigenvalues on Σ;
(ii) if $\partial\Sigma \neq \emptyset$, $\lim_{h\to 0^+} \lambda_j(\Omega_h) = \mu_j^D$, where $\{\mu_j^D\}_{j=1}^{\infty}$ are the Dirichlet Laplacian eigenvalues on Σ.

An immediate consequence of Theorem 1.1 is that we can always find examples of domains Ω with any number of arbitrarily small eigenvalues $\lambda_j(\Omega)$, in the class of domains with prescribed volume $|\Omega|$ (see [3] for a recent related result).

COROLLARY 1.2. *For any $N \in \mathbb{N}$ and $\epsilon > 0$, there exists a domain Ω with $|\Omega| = 1$ and $\lambda_j(\Omega) \leq \epsilon$ for all $j = 1, ..., N$. Moreover, the domain Ω can be chosen to be homeomorphic to a ball.*

Proof. Let us first prove the result in the case of Ω not homeomorphic to a ball. Let Σ be a Cheeger's dumbbell [13, p. 79] with $N-1$ thin passages such that $\mu_1, ..., \mu_N < \frac{\epsilon}{2}$ (in particular, $\mu_1 = 0$). Choose a small enough $h > 0$ guaranteeing that $\lambda_j(\Omega_h) < \epsilon$ for $j = 1, ...N$. Now, for h small, $|\Omega_h| \approx |\Sigma|h$ and we can possibly reduce h in such a way that $|\Omega_h| < 1$. Let $\Omega := \frac{\Omega_h}{|\Omega_h|^{1/3}}$, so that $|\Omega| = 1$. Then $\lambda_j(\Omega) = |\Omega_h|^{2/3}\lambda_j(\Omega_h) < \epsilon$ for all $j = 1, ..., N$.

To produce a domain homeomorphic to a ball, replace Σ in the above construction with $\Sigma_\delta := \Sigma \setminus B_\delta$, where $B_\delta \subset \Sigma$ is a small geodesic disk with $\delta > 0$, chosen in such a way that the Dirichlet eigenvalues μ_j^D in Σ_δ satisfy $\mu_1^D, ..., \mu_N^D < \frac{2\epsilon}{3}$. This is possible because the Dirichlet spectrum of Σ_δ converges to the spectrum of the Laplacian on Σ as $\delta \to 0^+$, see e.g., [16]. The rest of the construction is as in the previous part of the proof. It is sufficient to note that a thin tube around Σ_δ (or, equivalently, $\Sigma_\delta \times (0, h)$) is homeomorphic to a ball. □

Corollary 1.2 implies that a Faber–Krahn inequality cannot hold for the first Maxwell's eigenvalue, nor for other functions of the eigenvalues like the sum or the product (or other elementary symmetric functions) of the first N eigenvalues, as already highlighted in [30] (see also [41]).

Combining Corollary 1.2 with the examples of convex domains of fixed volume and arbitrarily large first eigenvalue (see e.g., [30, 41], or simply take $(0, \delta) \times (0, \delta) \times (0, 1/\delta^2)$ which has large first eigenvalue when δ is small by Theorem 3.1) we conclude that for any $N \in \mathbb{N}$ and any $\epsilon, M > 0$, there exist domains Ω, ω homeomorphic to a ball with $|\Omega| = |\omega| = 1$, such that $\lambda_j(\Omega) > M$ and $\lambda_j(\omega) < \epsilon$, for all $j = 1, ..., N$.

To prove Theorem 1.1, it is convenient to change perspective and interpret problem (1.1) as an eigenvalue problem for the Hodge Laplacian acting on co-closed differential 1-forms with relative boundary conditions on a Riemannian manifold (M, g) (see also problem (3.1)):

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$$\begin{cases} \Delta u = \lambda u\,, & \text{in } M \\ \delta u = 0\,, & \text{in } M \\ i^* u = 0\,, & \text{on } \partial M, \end{cases} \tag{1.3}$$

where now u is a differential 1-form on M, $\Delta = d\delta + \delta d$ is the Hodge Laplacian associated with the metric g acting on differential forms, d is the exterior derivative, δ is the codifferential associated with the metric g and $i : \partial M \to M$ is the canonical inclusion. We refer to Section 2 for more details. When M is a bounded domain in $\mathbb{R}^3$ and the metric is the Euclidean one, Problem (1.1) restricted to divergence-free fields and Problem (1.3) coincide (under the canonical identification of vector fields and 1-forms).

It is worth recalling the following fact. For small $h > 0$, the domain Ω_h with the Euclidean metric is quasi-isometric to the manifold $M = \Sigma \times (0, h)$ with the product metric $g_p = g_\Sigma \times dt^2$, where g_Σ is the induced metric on Σ from the ambient Euclidean space. Using the product structure of the metric, we are able to explicitly describe all the eigenvalues of Problem (1.3) in $M = \Sigma \times (0, h)$ and the associated eigenfunctions. Concerning the eigenvalues, we have the following (see Theorems 3.1 and 3.4)

THEOREM 1.3. *Let $M = \Sigma \times (0, h)$, $h > 0$, (Σ, g_Σ) be a compact Riemannian surface (without boundary) and $g_p = g_\Sigma + dt^2$ be the product metric on M. Then the spectrum of (1.3) on (M, g_p) is given by the union of the following four families:*

(i) $\mu_k + \eta_j(h)$, $k \geq 2$, $j \geq 1$;
(ii) $\mu_k + d_j(h)$, $k \geq 2$, $j \geq 1$;
(iii) $d_j(h)$, $j \geq 1$, *each repeated* 2γ *times;*
(iv) 0 *with multiplicity* 1.

Here μ_k are the eigenvalues of the Laplacian on (Σ, g_Σ) (with multiplicities), $\eta_j(h), d_j(h)$ are the Neumann and Dirichlet eigenvalues on $(0, h)$, and γ is the genus of the surface.

THEOREM 1.4. *Let $M = \Sigma \times (0, h)$, $h > 0$, (Σ, g_Σ) be a compact Riemannian surface with non-empty boundary $\partial\Sigma$ and $g_p = g_\Sigma + dt^2$ be the product metric on M. Then the spectrum of (1.3) on (M, g_p) is given by the union of the following three sequences:*

(i) $\mu_k^D + \eta_j(h)$, $k, j \geq 1$;
(ii) $\mu_k^N + d_j(h)$, $k \geq 2$, $j \geq 1$;
(iii) $d_j(h)$, $j \geq 1$, *each repeated* $2\gamma + b$ *times.*

Here μ_k^D, μ_k^N are the eigenvalues of the Laplacian on (Σ, g_Σ) with Dirichlet and Neumann boundary conditions, respectively (with multiplicities), $\eta_j(h), d_j(h)$ are the Neumann and Dirichlet eigenvalues on $(0, h)$, γ is the genus of the surface, and $b + 1$ is the number of connected components of $\partial\Sigma$.

We note that in both cases, as $h \to 0^+$, all eigenvalues diverge to $+\infty$ except $\mu_k + \eta_1(h)$ $(k \geq 1)$ if $\partial\Sigma = \emptyset$ and $\mu_k^D + \eta_1(h)$ $(k \geq 1)$ if $\partial\Sigma \neq \emptyset$. In fact $\eta_1(h) = 0$ for all

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h. The quasi-isometry between (M, g_p) and (M, g_E), where g_p is the product metric and g_E is the Euclidean metric, finally allows us to conclude that the corresponding eigenvalues are at most at distance Ch from each other, concluding therefore the proof of Theorem 1.1.

Theorems 1.3 and 1.4 should be compared with the case of 'flat' product domains of $\mathbb{R}^3$ of the form $\Omega = \omega \times I$, where $\omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$. For such domains, it is well-known that the eigenvalues $\lambda_j(\Omega)$ belong to three different families (in the following list we keep the notation of [15]):

(i) the TE-modes, $\lambda^{\mathrm{TE}}_{jm}(\Omega) = \lambda_j(-\Delta^{\mathrm{neu}}_\omega) + \lambda_m(-\Delta^{\mathrm{dir}}_I)$, $j \geq 2$, $m \geq 1$;
(ii) the TM-modes, $\lambda^{\mathrm{TM}}_{jm}(\Omega) = \lambda_j(-\Delta^{\mathrm{dir}}_\omega) + \lambda_m(-\Delta^{\mathrm{neu}}_I)$, $j \geq 1$, $m \geq 1$;
(iii) when ω is not simply connected, and $\partial\omega$ has D connected components, the TEM modes: $\lambda^{\mathrm{TEM}}_{dm}(\Omega) = \lambda_m(-\Delta^{\mathrm{dir}}_I)$, $1 \leq d \leq D-1$, $m \geq 1$;

Theorems 1.3 and 1.4 say that the TE-TM-TEM description of the eigenvalues given in [15] continues to hold in the Riemannian setting, that is, when we replace ω with a Riemannian surface Σ, therefore generalizing the construction valid for straight cylinders to possibly curved ones. The only difference is that the Maxwell's eigenvalues will now be described in terms of the eigenvalues of the Laplacian on the surface Σ (with Dirichlet or Neumann boundary conditions on $\partial\Sigma$ when $\partial\Sigma \neq \emptyset$, as in the flat case).

From the description in [15], we easily see that the limiting spectrum of the Maxwell's operator on the flat cylinder $\omega \times I$ as $|I| \to 0^+$ coincides with the Dirichlet spectrum of the Laplacian on ω: this is a particular case of Theorem 1.1.

Another implication of Theorem 1.1 is that there is no spectral stability under singular domain perturbation when a change of topology is involved. More precisely, let $\Sigma_\delta = \omega \setminus B_\delta \subset \mathbb{R}^2$, where ω is a simply connected planar domain, B_δ is a disk of radius δ centred at some $x \in \omega$, $B_\delta \subset \omega$ for all $\delta > 0$ sufficiently small. Let $\Omega_{h,\delta} = \Sigma_\delta \times (0, h)$. Let $h > 0$ be fixed. Then the Maxwell's eigenvalues on $\Omega_{h,\delta}$ are just those given by Theorem 1.4:

(i) $\mu^D_k(\delta) + \frac{\pi^2(j-1)^2}{h^2}$, $k, j \geq 1$, where $\mu^D_k(\delta)$ are the Dirichlet eigenvalues on Σ_δ;
(ii) $\mu^N_k(\delta) + \frac{\pi^2 j^2}{h^2}$, $k \geq 2$, $j \geq 1$, where $\mu^N_k(\delta)$ are the Neumann eigenvalues on Σ_δ;
(iii) $\frac{\pi^2 j^2}{h^2}$, $j \geq 1$.

When $\delta \to 0^+$, $\Omega_{h,\delta}$ converges (in the sense of Hausdorff convergence) to $\Omega_h = \omega \times (0, h)$. The Maxwell's spectrum on Ω_h is given by Theorem 1.4; however, we note that, since ω is simply connected, we do not have the third family of eigenvalues:

(i) $\mu^D_k + \frac{\pi^2(j-1)^2}{h^2}$, $k, j \geq 1$, where μ^D_k are the Dirichlet eigenvalues on $\omega = \Sigma_0$;
(ii) $\mu^N_k + \frac{\pi^2 j^2}{h^2}$, $k \geq 2$, $j \geq 1$, where μ^N_k are the Neumann eigenvalues on $\omega = \Sigma_0$.

The first two families of eigenvalues behave continuously in δ (this follows from the spectral stability of the Dirichlet and Neumann eigenvalues in Euclidean domains upon removal of a small ball). On the contrary, the eigenvalues of the third family

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clearly admit a limit as $\delta \to 0^+$ (they do not depend on δ), but the limits are not Maxwell's eigenvalues in the limit domain.

Associated with the families of eigenvalues in Theorems 1.3 and 1.4, there are families of eigenfunctions that we describe more explicitly in Theorems 3.1 and 3.4. In these theorems, the eigenfunctions are interpreted as eigenfunctions of the Hodge Laplacian (Problem 3.1), that is, they are 1-forms. Finally, we prove that the eigenfunctions in Ω_h and the limit eigenfunctions in Σ converge in a suitable sense as $h \to 0^+$, see Theorem 5.1 and Corollary 5.2.

The analysis of eigenvalue problems for differential operators on thin domains is a classical topic that has experienced a noticeable growth in recent years, see e.g., [4, 5, 7, 8, 26, 29, 31] and references therein. Our analysis was inspired by the well-known result in [42]: the Neumann eigenvalues of a thin tube around a closed embedded hypersurface in $\mathbb{R}^n$ converge to the eigenvalues of the Laplacian on the surface. Since then, the analysis of the behaviour of the spectrum in the thin limit turned out to be useful in the study of many other spectral problems, e.g., the hot spot conjecture [32], the clamped plate equation [12], Navier–Stokes equations [35–37], and quantum waveguides [17–19, 40]. It is worth mentioning the recent paper [33], in which the authors perform an asymptotic analysis of the spectrum of the Dirac operator in tubes around hypersurfaces in $\mathbb{R}^n$. When $n = 3$, these are the domains considered in the present paper. When the tube shrinks to the surface, they are able to identify the effective Schrödinger operator driving the dynamics, and write an asymptotic expansion of the eigenvalues. From the geometric point of view, the analysis of the spectrum of the Hodge Laplacian acting on p-forms on domains with thin parts has been considered by various authors, see e.g., [1–3]. However, also from the geometric point of view, we were not aware of a result in the spirit of [42] for p-forms. This is the main motivation of the present note, which focuses on $p = 1$ due to the relation of the problem on forms with the Maxwell's problem. We finally remark that in the last ten years, there has been an upsurge of interest in the connection between the spectrum of the Maxwell's operator and the underlying geometry, mainly in the Euclidean setting, see, for instance, [6, 9, 20–22], where, for instance, the role of the topology of the domain in the spectral properties of the Maxwell's system prominently appears.

The present note is organized as follows. Section 2 contains a few geometric preliminaries and the description of the connection between Problems (1.1) and (1.3). In Section 3, we prove Theorems 1.3 and 1.4, namely, we describe the Maxwell's eigenvalues and the eigenfunction of the product manifold $(M, g_p) = (\Sigma \times (0, L), g_p)$. In Section 4, we prove our main Theorem 1.1. In Section 5, we establish a convergence result for eigenfunctions. Finally, we remark that in this note we identify the Maxwell's problem with the eigenvalue problem for the Hodge Laplacian restricted on co-closed 1-forms with relative conditions. For the reader's convenience, in Appendix A, we describe the spectrum of the full Hodge Laplacian with relative boundary conditions in the product manifold (M, g_p).

We conclude this section by underlining that this article is purposely written to be accessible to both mathematical analysts and differential/spectral geometers, and therefore, it may contain details that are usually omitted in a research article.

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2. Geometric preliminaries and the interpretation of problem (1.1) as an eigenvalue problem for the Hodge Laplacian

2.1. Notation and functional spaces

Let us first describe the functional spaces that are involved in the analysis of the curl curl equation in the case of a Lipschitz bounded domain $\Omega \subset \mathbb{R}^3$. The ambient Hilbert space will be $L^2(\Omega)^3$. Let $\nabla H_0^1(\Omega) := \{\nabla u : u \in H_0^1(\Omega)\}$ and $H(\operatorname{div} 0, \Omega) := \{E \in L^2(\Omega)^3 : \operatorname{div} E = 0\}$. We recall that we have the classical Helmholtz decomposition

$$L^2(\Omega)^3 = \nabla H_0^1(\Omega) \oplus H(\operatorname{div} 0, \Omega). \tag{2.1}$$

Let us also define the space

$$H_0(\operatorname{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \operatorname{curl} u \in L^2(\Omega)^3,\ \nu \times u|_{\partial\Omega} = 0\},$$

and similarly

$$H(\operatorname{div}, \Omega) = \{u \in L^2(\Omega)^3 : \operatorname{div} u \in L^2(\Omega)\}.$$

In view of Weber's compactness result (see [44]), the space

$$X_N(\Omega) := H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$$

is compactly embedded in $L^2(\Omega)^3$. Let us now consider the weak formulation of Equation (1.1), which reads

$$\int_\Omega \langle \operatorname{curl} E, \operatorname{curl} H\rangle = \lambda \int_\Omega \langle E, H\rangle \tag{2.2}$$

for all $H \in H_0(\operatorname{curl}, \Omega)$. One sees immediately that, if $\lambda = 0$, any $E = \nabla u$, $u \in H_0^1(\Omega)$ is a solution of (2.2). In fact, the space $\nabla H_0^1(\Omega) := \{\nabla u : u \in H_0^1(\Omega)\}$ is contained in the kernel of the operator curl.

There are now two (equivalent) ways of studying the spectrum of this operator. Either we study it in the Hilbert space $L^2(\Omega)^3$, and then $\lambda = 0$ is a point of essential spectrum of the operator; or we restrict the Hilbert space to $H(\operatorname{div} 0, \Omega) = \{E \in L^2(\Omega)^3 : \operatorname{div} E = 0\}$, which corresponds to restrict the domain of the operator curl curl to the orthogonal of $\nabla H_0^1(\Omega)$. We proceed with this second option.

Thus, in the Hilbert space $H(\operatorname{div} 0, \Omega)$, which is endowed with the usual $L^2(\Omega)^3$-norm, we consider the sesquilinear form

$$Q(E, H) = \int_\Omega \langle \operatorname{curl} E, \operatorname{curl} H\rangle$$

with domain $\operatorname{dom}(Q) = H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div} 0, \Omega)$, which is compactly embedded in $H(\operatorname{div} 0, \Omega)$. By the second representation theorem, there exists a unique positive self-adjoint operator T such that

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$$(Tu, v) = Q(u, v),$$

for all $u \in \mathrm{dom}(T) := \{u \in \mathrm{dom}(Q) : T^{1/2}u \in \mathrm{dom}(Q)\}$, $v \in \mathrm{dom}(Q)$. Moreover, the compact embedding of $\mathrm{dom}(Q)$ into the ambient Hilbert space $H(\mathrm{div}\, 0, \Omega)$ implies that the resolvent T^{-1} is compact as an operator in $H(\mathrm{div}\, 0, \Omega)$. By standard spectral theory, we deduce that the spectrum of T coincides with its discrete spectrum and can be described by a sequence of non-negative, isolated eigenvalues of finite multiplicity

$$0 \leq \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_j(\Omega) \leq \cdots \nearrow +\infty,$$

where the eigenvalues are repeated according to their multiplicity.

2.2. Hodge Laplacian and geometric functional setting

The weak formulation (2.2) and the functional setting described in Section 2.1, which apparently seem tied to Euclidean three-dimensional domains, can be translated into the context of general compact Riemannian manifolds.

Let (M, g) be a compact, orientable, n-dimensional Riemannian manifold. Let $\Omega^p(M)$ denotes the vector space of smooth differential p-forms on the differentiable manifold M.

By d we denote the exterior derivative: $d : \Omega^p(M) \to \Omega^{p+1}(M)$, which is the ordinary differential of a function for $p = 0$. For example, in $\mathbb{R}^3$ with Cartesian coordinates (x, y, z), for a 0-form $f = f(x, y, z)$ we have $df = \partial_x f dx + \partial_y f dy + \partial_y f dz$. For a 1-form $f = f_1 dx + f_2 dy + f_3 dz$ we have $df = (\partial_x f_2 - \partial_y f_1) dx \wedge dy + (\partial_y f_3 - \partial_z f_2) dy \wedge dz + (\partial_z f_1 - \partial_x f_3) dx \wedge dz$, etc.

The metric g allows us to define a Hodge-star operator $\star : \Omega^p(M) \to \Omega^{n-p}(M)$. More concretely, we first note that the Riemannian metric induces a scalar product on the space of p-forms, which we denote by $\langle \cdot, \cdot \rangle_g$. Then, for any $\omega \in \Omega^p(M)$, the Hodge-star operator $\star$ is defined by the following identity

$$\phi \wedge (\star\omega) = \langle \phi, \omega \rangle_g dv_g\,, \quad \forall \phi \in \Omega^p(M),$$

where dv_g is the volume n-form for the metric g. For example, in $\mathbb{R}^3$ with the Euclidean metric g_E and the canonical basis dx, dy, dz of 1-forms, one has: $\star dx = dy \wedge dz$, $\star dy = dz \wedge dx$, $\star dz = dx \wedge dy$, $\star 1 = dx \wedge dy \wedge dz = dv_{g_E}$, $\star(dx \wedge dy \wedge dz) = \star dv_{g_E} = 1$, etc. In particular, for a 1-form $f = f_1 dx + f_2 dy + f_3 dz$, $\star df = (\partial_y f_3 - \partial_z f_2) dx + (\partial_z f_1 - \partial_x f_3) dy + (\partial_x f_2 - \partial_y f_1) dz$ which can be identified with $\mathrm{curl} f$.

The Hodge $\star$ allows us to define a codifferential $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$:

$$\delta\omega = (-1)^{n(p+1)+1} \star d \star .$$

For example, if $p = 1$, we always have $\delta = - \star d \star$. For a 1-form in $\mathbb{R}^3$, $f = f_1 dx + f_2 dy + f_3 dz$, we have $\delta f = -\partial_x f_1 - \partial_y f_2 - \partial_z f_3 = -\mathrm{div} f$ (for the Euclidean metric g_E). For $n = 3$, $p = 2$, $\delta = \star d \star$. Note that δ depends on the metric g.

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Finally, we can define the Hodge Laplacian $\Delta : \Omega^p(M) \to \Omega^p(M)$ by

$$\Delta\omega = (\delta d + d\delta)\omega.$$

Note that Δ depends on the metric g. We will omit the dependence of δ and Δ on the metric g when it is clear from the context; otherwise, we will write δ_g, Δ_g. Further note that in $\mathbb{R}^n$ we have $\delta f = 0$, for any function (or, equivalently, 0-form) f, hence $-\Delta f = \delta df = -\mathrm{div}\nabla f$ is the usual Laplacian on functions. In $\mathbb{R}^3$, given a 1- form $f = f_1 dx + f_2 dy + f_3 dz$, the Hodge Laplacian acts as

$$\Delta f = \mathrm{curl}\,\mathrm{curl} f - \nabla\,\mathrm{div} f.$$

Hence, the eigenvalue equation in (1.1) corresponds to $\Delta E = \lambda E$ when E is *co-closed*, that is, when $\delta E = 0$ ($-\mathrm{div} E = 0$). Here, with abuse of notation, we have identified the vector field E with its dual 1-form. The boundary condition $\nu \times E = 0$ on $\partial\Omega$ can be translated into $i^*E = 0$ on $\partial\Omega$, where $i : \partial\Omega \to \Omega$ is the canonical inclusion. This condition forces E to be normal to the boundary. In the terminology of differential forms, $i^*E = 0$ on $\partial\Omega$ is called a *relative* condition. In the case of 0-forms, the relative condition corresponds to the Dirichlet boundary condition.

We now consider the functional setting for the Hodge Laplacian acting on forms. Having a scalar product induced by the metric on $\Omega^p(M)$, the definitions of L^2 spaces and Sobolev spaces extend naturally to differential p-forms: the space $L^2\Omega^p(M)$ is defined as the completion of $\Omega^p(M)$ with respect to the L^2-inner product on forms: $\int_M \langle\omega_1, \omega_2\rangle_g dv_g$. The Sobolev spaces $H^m\Omega^p(M)$, $m \in \mathbb{N}$ are defined analogously, through the natural connection ∇ on (M, g) induced by the Riemannian metric (which allows to differentiate forms). It is then possible to define the analogous of the spaces $H_0(\mathrm{curl}, \Omega)$, $H(\mathrm{div}, \Omega)$, $H(\mathrm{div}0, \Omega)$ on (M, g) for differential forms of any degree. More precisely, we have

$$X_N(M, g) := \\ \{\omega \in L^2\Omega^p(M) : d\omega \in L^2\Omega^{p+1}(M), \delta\omega \in L^2\Omega^{p-1}(M), \text{ and } i^*\omega = 0 \text{ on } \partial M\},$$

where $i : \partial M \to M$ is the canonical inclusion (if $\partial M = \emptyset$ the last condition in the definition of X_N is empty). In the case of non-empty boundary, this is the space of differential p-forms in L^2 with differential and codifferential in L^2 and satisfying the *relative* boundary conditions (namely, they are normal to ∂M).

We also recall the fundamental Hodge–Morrey decomposition:

$$L^2\Omega^p(M) = d\Omega_R^{p-1}(M) \oplus \delta\Omega^{p+1}(M) \oplus \mathcal{H}_R(M) \tag{2.3}$$

where $\Omega_R^{p-1}(M) := \{\omega \in \Omega^{p-1}(M) : i^*\omega = 0 \text{ on } \partial M\}$ and $\mathcal{H}_R(M) := \{\omega \in \Omega^p(M) : d\omega = \delta\omega = 0\,, i^*\omega = 0 \text{ on } \partial M\}$. By abuse of notation, the spaces in the decompositions denote the closure of the corresponding spaces of smooth p-forms with respect to the L^2 norm. In the case of a domain of $\mathbb{R}^3$ and $p = 1$, $L^2\Omega^1(\Omega) = L^2(\Omega)^3$ and $d\Omega^0(M) = \nabla H_0^1$ (up to the isomorphism identifying 1-forms with vector fields), and therefore we recover the Helmholtz decomposition (2.1). The analogous Hodge–Morrey decomposition holds for any Sobolev space $H^m\Omega^p(M)$, $m \geq 1$.

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We can now see that problem (2.2) corresponds to

$$\int_M \langle d\omega, d\varphi\rangle_g dv_g = \lambda \int_M \langle \omega, \varphi\rangle_g dv_g \tag{2.4}$$

for all $\varphi \in X_N(M,g)$ such that $\delta\varphi = 0$ in M, in the unknown $\omega \in X_N(M,g)$, $\delta\omega = 0$ and $\lambda \in \mathbb{R}$.

Finally, we recall Gaffney's inequality

$$\|\omega\|^2_{H^1\Omega^p(M)} \leq C_G \left(\|d\omega\|^2_{L^2\Omega^{p+1}(M)} + \|\delta\omega\|^2_{L^2\Omega^{p-1}(M)} + \|\omega\|^2_{L^2\Omega^p(M)} \right) \tag{2.5}$$

which holds for $\omega \in X_N(M,g)$. It is clear now that all the discussion in Section 2.1 applies to this more general setting, since the embedding $H^1\Omega^p(M) \to L^2\Omega^p(M)$ is compact. This means that problem (2.4) is associated with a compact, self-adjoint operator in $L^2\Omega^p(M)$ with nonnegative, discrete spectrum. Gaffney's inequality was originally proved in [24, 25] for manifolds without boundary, and in [23] for manifolds with boundary. For manifolds with boundary, it also holds under very mild smoothness assumptions on ∂M, see [34]. More precisely, for Euclidean domains, a Lipschitz condition on the boundary and a uniform outer ball condition are sufficient to guarantee the validity of Gaffney's inequality.

For more details on Sobolev spaces of p-forms, Hodge–Morrey decomposition, Gaffney's inequality, and for other relevant results of functional analysis, we refer to [43]. For more information on the spectrum of the Hodge Laplacian on p-forms (with different types of boundary conditions), we refer, e.g., to [27, 28, 38, 39].

The geometric framework described above allows for a more general approach to eigenvalue problems of type (1.1) and helps to avoid some technicalities that may arise from the explicit use of coordinates. It gives a geometric meaning to the decomposition of the Maxwell's spectrum in the TE, TM, and TEM modes for domains $\omega \times I$ (where $\omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$) contained in [15]. Moreover, even though it is not the purpose of the present paper, it can be applied in any dimension and any ambient Riemannian manifold.

We refer to [27] for a nice introduction to eigenvalue problems for the Laplacian on p-forms on manifolds with boundary. See also [41] for a detailed exposition on the spectrum of the Laplacian on p-forms on convex Euclidean domains.

3. Hodge Laplacian spectrum on co-closed 1-forms with relative conditions on three-dimensional product manifolds

Throughout this section, we shall denote by M the following product manifold of dimension 3:

$$M = \Sigma \times I,$$

where (Σ, g_Σ) is a compact Riemannian surface, $I = (0,h)$ is an interval, and $h > 0$. The surface Σ is compact, connected, smooth, and it is allowed to have a smooth boundary (with any number of connected components) and a possibly non-trivial topology. By g_Σ we denote a smooth Riemannian metric on Σ. By x we shall denote a point of Σ and by $t \in (0,h)$ the usual coordinate on I.

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We consider the manifold M endowed with the product metric $g_p = g_\Sigma \times dt^2$. With this choice, (M, g_p) is a product Riemannian manifold. Note that at any point $q = (x, t) \in M$, we have the canonical orthogonal decomposition $T_qM = T_x\Sigma \oplus T_tI$.

We consider the following eigenvalue problem

$$\begin{cases} \Delta\omega = \lambda\omega\,, & \text{in } M \\ i^*\omega = 0\,, & \text{on } \partial M \end{cases} \tag{3.1}$$

restricted to the space of *co-closed* 1*-forms*, that is, 1-forms ω verifying $\delta\omega = 0$. To be more precise, we should have written δ_{g_p} and Δ_{g_p} since the co-differential depends on the metric. We shall omit the subscript when its meaning is clear from the context. Here $i : \partial M \to M$ is the canonical inclusion. This problem is the eigenvalue problem for the Hodge Laplacian restricted to co-closed one-forms with *relative* boundary conditions, see [27].

We now describe the eigenvalues and eigenfunctions of (3.1) in the product manifold M. We start from the case $\partial\Sigma = \emptyset$.

3.1. Eigenvalues and eigenfunctions when $\partial\Sigma = \emptyset$

THEOREM 3.1. *Let (M, g_p) be a product Riemannian manifold, with $M = \Sigma\times(0, h)$, $h > 0$, (Σ, g_Σ) a compact Riemannian surface (without boundary) and g_p the product metric. Let*

$$\begin{aligned} &(\mu_k, w_k)_{k\geq 1} \text{ be the eigencouples of the Laplacian on } (\Sigma, g_\Sigma);\\ &(\eta_j(h), v_j)_{j\geq 1} \text{ be the eigencouples of the Neumann Laplacian on } (0, h);\\ &(d_j(h), u_j)_{j\geq 1} \text{ be the eigencouples of the Dirichlet Laplacian on } (0, h). \end{aligned}$$

Then the spectrum of (3.1) is given by the union of the following four families:

(i) $\mu_k + \eta_j(h)$, $k \geq 2$, $j \geq 1$;
(ii) $\mu_k + d_j(h)$, $k \geq 2$, $j \geq 1$;
(iii) $d_j(h)$, $j \geq 1$, *each repeated* 2γ *times;*
(iv) 0 *with multiplicity* 1.

Here γ is the genus of the surface. The corresponding eigenfunctions are given by

(i) $F_{jk}(x, t) = \delta d(w_k(x)v_j(t)dt)$, $k \geq 2$, $j \geq 1$.
(ii) $F_{jk}(x, t) = \star d(w_k(x)u_j(t)dt)$, $k \geq 2$, $j \geq 1$.
(iii) $F_{jk}(x, t) = H_k(x)u_j(t)$, *where* $\{H_k\}_{k=1}^{2\gamma}$ *is a basis of harmonic* 1*-forms on* Σ.
(iv) $F(x, t) = dt$.

REMARK 3.2. We can recognize the three families of modes described in [15] for cylinders and balls in $\mathbb{R}^3$. In particular, our first family corresponds to the TM modes in [15], the second family corresponds to the TE modes, while, if the topology is not trivial, our third family corresponds to the TEM modes.

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Proof. **First family.** We look for eigenfunctions F of the form

$$F = \delta d(f dt),$$

where f is a smooth function on M. We have the following facts:

- $F = d_\Sigma f_t + (\Delta_\Sigma f)dt$, where d_Σ is the differential on Σ, and f_t indicates the derivative of f with respect to t;
- we have $\delta F = \delta^2 d(f dt) = 0$, hence F is co-closed;
- we have then

$$\Delta F = \delta dF = d_\Sigma(\Delta_\Sigma f_t - f_{ttt}) + (\Delta_\Sigma^2 f - \Delta_\Sigma f_{tt})dt;$$

- the boundary condition $i^* F = 0$ reads $d_\Sigma f_t = 0$.

Here and in what follows, by Δ_Σ we denote the Laplacian on Σ for the metric g_Σ and by δ_Σ we denote the codifferential on Σ for the metric g_Σ. Therefore, we need to solve

$$\begin{cases} d_\Sigma(\Delta_\Sigma f_t - f_{ttt}) + (\Delta_\Sigma^2 f - \Delta_\Sigma f_{tt})dt = \lambda(d_\Sigma f_t + (\Delta_\Sigma f)dt)\,, & \text{in } M \\ d_\Sigma f_t = 0\,, & \text{on } \Sigma \times \{0, h\}. \end{cases} \tag{3.2}$$

According to the separation-of-variables ansatz, we look for solutions of the form $f(x,t) = w(x)v(t)$. The equation preserves the separation of variables, and therefore, we obtain:

$$\begin{cases} v\Delta_\Sigma^2 w - v''\Delta_\Sigma w - \lambda v \Delta_\Sigma w = 0, & \text{in } M \\ v'(0) d_\Sigma w = v'(h) d_\Sigma w = 0\,, & \text{on } \Sigma \times \{0, h\}. \end{cases} \tag{3.3}$$

If $w = \text{const}$ on Σ, we would have $F = 0$. Moreover, we can choose w such that $\int_\Sigma w = 0$, since the corresponding F would not change. Hence, the boundary condition reads $v'(0) = v'(h) = 0$.

Note that if $f(x,t) = w(x)v(t)$ with $\int_\Sigma w = 0$ solves (3.3), then it solves

$$d_\Sigma(\Delta_\Sigma f_t - f_{ttt}) = \lambda d_\Sigma f_t\,, \quad \text{in } M \tag{3.4}$$

thus it solves (3.2).

A standard ansatz to construct a solution is to require that there exist constants c such that

$$\begin{cases} \Delta_\Sigma^2 w + c\Delta_\Sigma w - \lambda \Delta_\Sigma w = 0\,, & \text{in } \Sigma\,, \\ -v''(t) = cv(t)\,, & \text{in } (0,h)\,, \\ v'(0) = v'(h) = 0. \end{cases}$$

The first equation implies that w is an eigenfunction of the Laplacian on Σ with eigenvalue $\mu = \lambda - c > 0$, since we are in the subspace of $H^1(\Sigma)$ of functions w

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with zero mean over Σ. Note that $\mu_1 = 0$, $\mu_2 > 0$. On the other hand, v must be a Neumann eigenfunction on $(0, h)$ with eigenvalue c. We conclude that

$$\lambda = \mu_k + \eta_j(h)\,, \quad k \geq 2, j \geq 1.$$

Second family. Consider now the following functions:

$$F = \star d(f dt)$$

where f is a smooth function in M. One checks that

$$F = \star d_\Sigma f.$$

It is immediate to check that $\delta F = -\delta^2 \star (f dt) = 0$, so F is co-closed. Hence

$$\Delta F = \delta d F = \star d_\Sigma(\Delta_\Sigma f - f_{tt}).$$

Hence (3.1) becomes

$$\begin{cases} \star d_\Sigma(\Delta_\Sigma f - f_{tt}) = \lambda \star d_\Sigma f\,, & \text{in } M\,, \\ \star d_\Sigma f = 0\,, & \text{on } \Sigma \times \{0, h\}. \end{cases} \tag{3.5}$$

which is equivalent to

$$\begin{cases} d_\Sigma(\Delta_\Sigma f - f_{tt}) = \lambda d_\Sigma f\,, & \text{in } M\,, \\ d_\Sigma f = 0\,, & \text{on } \Sigma \times \{0, h\}. \end{cases} \tag{3.6}$$

Again, the separation-of-variables ansatz suggests to look for solutions of the form $f(x,t) = w(x)u(t)$. We obtain

$$\begin{cases} -u'' d_\Sigma w + u\, d_\Sigma \Delta_\Sigma w = \lambda u d_\Sigma w\,, & \text{in } M \\ u(0) d_\Sigma w = u(h) d_\Sigma w = 0. \end{cases}$$

Note that we can take w such that $\int_\Sigma w = 0$. Indeed, adding to f any function $\phi(t)$ which depends only on t, we would have $\star d(f dt + \phi dt) = \star d(f dt) + \star d(\phi dt)$ and $d(\phi(t) dt) = 0$.

The same argument used for the first family shows that w cannot be constant, otherwise $f = u(t)$ and hence $F = 0$. This implies that necessarily $-u''(t) = cu(t)$ for some constant c and $u(0) = u(h) = 0$. This implies that $c = d_j(h)$, a Dirichlet eigenvalue on $(0, h)$, and that w must solve

$$d_\Sigma(\Delta_\Sigma w + d_j(h) - \lambda) = 0,$$

hence

$$\Delta_\Sigma w - (\lambda - d_j(h)) w = c$$

for some constant w. But since w has zero mean, integrating the previous in Σ we get $c = 0$. Therefore, w is an eigenfunction of Δ_Σ with eigenvalue $\lambda - d_j(h) > 0$. We conclude that

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$$\lambda = \mu_k + d_j(h)\,, \quad k \geq 2, j \geq 1.$$

Third family. This family arises when the topology of Σ is not trivial. We set

$$F = u(t)H(x)$$

where H is a harmonic 1-form on Σ. The space of harmonic 1-forms on a compact surface is finite-dimensional and has dimension 2γ, where γ is the genus of the surface. Proceeding as above, we find that u must be some Dirichlet eigenfunction on $(0, h)$ with eigenvalue $d_j(h)$, $j \geq 1$. The resulting eigenvalues are

$$\lambda = d_j(h)$$

each one repeated 2γ times.

Fourth family (eigenfunctions with zero eigenvalue). One eigenfunction is left out from this analysis, which is dt. We have that dt is harmonic and satisfies the relative boundary conditions. We remark that this is the only zero eigenvalue of the Hodge Laplacian on 1-forms (not just restricted to co-closed forms) on $\Sigma \times (0, L)$. This is not surprising since the relative cohomology in degree 1 for $\Sigma \times (0, L)$ has dimension 1.

Completeness of the eigenfunctions. To complete the proof of Theorem 3.1, it remains to establish that the 4 families of eigenfunctions found above span the whole of the Hilbert space L^2. Assume that there exists a co-closed 1-form ω, satisfying the relative boundary conditions, and orthogonal to all the eigenfunctions in the three families and to dt. We will show that ω must be zero.

Note that ω is a 1-form satisfying $\delta\omega = 0$ in M and $i^*\omega = 0$ on ∂M. We start by testing with the eigenfunctions of the first family, which are of the form

$$F = d_\Sigma f_t + (\Delta_\Sigma f)dt$$

where $f = w_k v_j$, $k \geq 2$, $j \geq 1$, with w_k and v_j as in the statement of the Theorem. At any $(x, t) \in M$ we can write $\omega = \omega_\Sigma + \omega_t dt$, where $\langle \omega_\Sigma, dt\rangle_g = 0$ in M and ω_t is a smooth function defined in M. The coefficients of ω_Σ are smooth functions on M. Then, for all $k \geq 2$, $j \geq 1$, we have

$$\begin{aligned}
0 &= \int_0^h \int_\Sigma \langle d_\Sigma f_t + (\Delta_\Sigma f)dt, \omega_\Sigma + \omega_t dt\rangle_g dv_{g_\Sigma} dt \\
&= \int_0^h \int_\Sigma \langle d_\Sigma f_t, \omega_\Sigma\rangle_{g_\Sigma} + \Delta_\Sigma f \omega_t dv_{g_\Sigma} dt \\
&= -\int_0^h \int_\Sigma f_t \delta_\Sigma \omega_\Sigma dv_{g_\Sigma} dt + \mu_k \int_0^h \int_\Sigma w_k v_j \omega_t dv_{g_\Sigma} dt \\
&= \int_0^h \int_\Sigma f_t (\omega_t)_t dv_{g_\Sigma} dt + \mu_k \int_0^h \int_\Sigma w_k v_j \omega_t dv_{g_\Sigma} dt. \qquad (3.7)
\end{aligned}$$

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Here we have used that $0 = \delta\omega = \delta_\Sigma\omega_\Sigma + (\omega_t)_t$, where $(\omega_t)_t := \partial_t\omega_t$ since the metric is in product form. Then

$$\begin{aligned} 0 &= \int_0^h \int_\Sigma f_t(\omega_t)_t dv_{g_\Sigma} dt + \mu_k \int_0^h \int_\Sigma w_k v_j \omega_t dv_{g_\Sigma} dt \\ &= (n_j(h) + \mu_k) \int_0^h \int_\Sigma w_k v_j \omega_t dv_{g_\Sigma} dt. \end{aligned} \tag{3.8}$$

This implies that ω_t must be constant on Σ for any t (that is, ω_t is a function of t only). In particular, again from $\delta_\Sigma\omega_\Sigma = -(\omega_t)_t$ we get, from Stokes theorem

$$0 = \int_\Sigma \delta_\Sigma\omega_\Sigma = (\omega_t)_t|\Sigma|$$

which means that ω_t is constant in M. The orthogonality with dt implies that $\omega_t = 0$.

Now, let us consider the second family of eigenfunctions:

$$F = \star d_\Sigma f$$

where $f = w_k u_j$, $k \geq 2$, $j \geq 1$, with w_k and u_j as in the statement of the Theorem. We therefore have, for any $k \geq 2$, $j \geq 1$:

$$0 = \int_0^h \int_\Sigma \langle \star d_\Sigma f, \omega \rangle_g dv_{g_\Sigma} dt = \int_0^h u_j \left(\int_\Sigma \langle \star d_\Sigma w_k, \omega_\Sigma \rangle_{g_\Sigma} dv_{g_\Sigma} \right) dt.$$

This means that a.e.

$$\int_\Sigma \langle \star d_\Sigma w_k, \omega_\Sigma \rangle_{g_\Sigma} dv_{g_\Sigma} = 0$$

which implies

$$\int_\Sigma w_k \star d_\Sigma \omega_\Sigma = 0$$

for all k. This means that $\star d_\Sigma\omega_\Sigma$ is constant on $\Sigma \times \{t\}$ for all t, namely, $\star d_\Sigma\omega_\Sigma = z(t)$. By Stokes theorem

$$0 = \int_\Sigma d_\Sigma\omega_\Sigma = z(t)|\Sigma|$$

hence $z(t) = 0$ and for any t, ω_Σ is closed on Σ.

Now we consider the third family of eigenfunctions $H_k(x)u_j(t)$, $j \geq 1$, where u_j are the Dirichlet eigenfunctions on $(0, h)$ and $\{H_k\}_{k=1}^{2g}$, is a basis of harmonic 1-forms in Σ. Proceeding as above, we find that ω_Σ is co-exact for any t, which means that $\omega_\Sigma = - \star d_\Sigma \star \phi\, dv_{g_\Sigma}$ for some function ϕ. Since we have proved that ω is closed, we conclude that $\Delta_\Sigma\phi = 0$ for all t, hence ϕ is constant and $\omega_\Sigma = 0$. Hence $\omega = cdt$ for some constant c. Since ω must be orthogonal to dt (the last eigenfunction, associated with the eigenvalue 0), necessarily $c = 0$. □

We observe that all the eigenvalues diverge to $+\infty$ as $h \to 0^+$, except for the family $\mu_k + \eta_1(h) = \mu_k$, $k \geq 2$, and the zero eigenvalue (which is μ_1). We restate this result in the following corollary.

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COROLLARY 3.3. *Let $\lambda_j(h, g_p)$ denote the eigenvalues of the product manifold (M, g_p) where $M = \Sigma \times (0, h)$. Then*

$$\lim_{L \to 0} \lambda_j(h, g_p) = \mu_j,$$

where μ_j are the Laplacian eigenvalues of (Σ, g_Σ).

3.2. Eigenvalues and eigenfunctions when $\partial\Sigma \neq \emptyset$

We consider now the case when Σ has a boundary. The arguments remain essentially the same, up to some minor changes that we now underline.

The first difference comes from the ansatz for the families of eigenfunctions. By separation of variables, the eigenfunctions must be in the form $f(x,t) = w(x)u(t)$, where the component $w(x)$ (which was an eigenfunction of the Laplacian on Σ in Theorem 3.1) must now satisfy some boundary conditions on $\partial\Sigma$. The second difference that one can note with respect to the proof of the completeness of eigenfunctions in Theorem 3.1 is that, when using the Stokes theorem on Σ, we have boundary terms; however, orthogonality with respect to harmonic 1-forms associated with boundary components of $\partial\Sigma$ allows to perform the same proof above, with almost no change. In fact, in the case where $\partial\Sigma \neq \emptyset$, there are, in general, additional harmonic 1-forms related to the connected components of $\partial\Sigma$.

THEOREM 3.4. *Let (M, g_p) be a product Riemannian manifold, with $M = \Sigma \times (0, h)$, $h > 0$, (Σ, g_Σ) a compact Riemannian surface with non-empty boundary, and g_p the product metric. Let*

$(\mu_k^D, w_k^D)_{k \geq 1}$ *be the eigencouples of the Dirichlet Laplacian on* (Σ, g_Σ);
$(\mu_k^N, w_k^N)_{k \geq 1}$ *be the eigencouples of the Neumann Laplacian on* (Σ, g_Σ);
$(\eta_j(h), v_j)_{j \geq 1}$ *be the eigencouples of the Neumann Laplacian on* $(0, h)$;
$(d_j(h), u_j)_{j \geq 1}$ *be the eigencouples of the Dirichlet Laplacian on* $(0, h)$.

Then the spectrum of (3.1) is given by the union of the following three sequences:

(i) $\mu_k^D + \eta_j(h)$, $k, j \geq 1$;
(ii) $\mu_k^N + d_j(h)$, $k \geq 2$, $j \geq 1$;
(iii) $d_j(h)$, $j \geq 1$, *each repeated* $2\gamma + b$ *times.*

Here γ is the genus of the surface and $b+1$ is the number of connected components of $\partial\Sigma$. The corresponding eigenfunctions are given by

(i) $F_{jk}(x,t) = \delta d(w_k^D(x)v_j(t)dt)$, $k \geq 1$, $j \geq 1$.
(ii) $F_{jk}(x,t) = \star d(w_k^N(x)u_j(t)dt)$, $k \geq 2$, $j \geq 1$.
(iii) $F_{jk}(x,t) = H_k(x)u_j(t)$, *where* $\{H_k\}_{k=1}^{2\gamma+b}$ *is a basis of harmonic* 1*-forms on* Σ *with relative conditions.*

Proof. During this proof, we will use the same notation for the families of eigenfunctions, as introduced in the proof of Theorem 3.1.

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First family. For the first family $F = d_\Sigma f_t + (\Delta_\Sigma f)dt$, the additional boundary condition reads $\Delta_\Sigma f = 0$ and $d_{\partial\Sigma} f_t = 0$ on $\partial\Sigma \times (0,h)$, which, with the ansatz $f(x,t) = w(x)v(t)$ becomes

$$\Delta_\Sigma w|_{\partial\Sigma} v(t) = 0 \quad \text{and} \quad d_{\partial\Sigma} w|_{\partial\Sigma} v'(t) = 0\,, \quad t \in (0,h).$$

Recall that the equations to be solved are

$$\begin{cases} \Delta_\Sigma^2 w + (c-\lambda)\Delta_\Sigma w = 0\,, & \text{in } \Sigma \\ -v''(t) = cv(t)\,, & \text{in } (0,h) \\ \Delta_\Sigma w|_{\partial\Sigma} v(t) = 0\,, & \text{on } \partial\Sigma \times (0,h)\,, \\ d_{\partial\Sigma} w|_{\partial\Sigma} v'(t) = 0\,, & \text{on } \partial\Sigma \times (0,h) \\ d_\Sigma w v'(0) = d_\Sigma g v'(h) = 0. \end{cases}$$

Note that $w \equiv$ const on Σ is not admissible, otherwise $F = 0$; the last condition then reads $v'(0) = v'(h) = 0$. Therefore, $c = \eta_j(h)$ is a Neumann eigenvalue on $(0,h)$. Setting $\phi = \Delta_\Sigma w$, we have that ϕ is an eigenfunction of the Dirichlet Laplacian on Σ with eigenvalue $\mu^D = \lambda - \eta_j(h)$, $j \geq 1$. Now, for $j = 1$ we have $\eta_1(h) = 0$ and $v(t) =$ const. Also, $\Delta_\Sigma(\phi + \mu^D w) = 0$, which means that $w = \phi + har$ is a linear combination of a Dirichlet eigenfunction ϕ and a harmonic function har. However, we may choose w to be a Dirichlet eigenfunction, because for $j = 0$ we have $F = \Delta_\Sigma(\phi + har)dt = (\Delta_\Sigma \phi)dt$. For $\eta_j(h) \neq \eta_1(h) = 0$, the boundary condition imposes that w is constant on any connected component of the boundary. In this case, we have that

$$\begin{cases} \Delta_\Sigma(\phi + \mu^D w) = 0\,, & \text{in } \Sigma\,, \\ \phi + \mu^D w = \text{const}\,, & \text{on each connect. comp. of } \partial\Sigma. \end{cases} \tag{3.9}$$

If Σ has just one connected component of the boundary, then the only solution is $\phi + \mu^D w \equiv$ const; we may then choose w to be an eigenfunction of the Dirichlet Laplacian on Σ with eigenvalue μ^D, since adding a constant does not change F. If $\partial\Sigma$ has $b+1$ connected components, there exists b independent solutions of (3.9), which contribute to the spectrum. Altogether, we have identified a family of eigenvalues in the form

$$\lambda = \mu_k^D + \eta_j(h)\,, \quad k, j \geq 1$$

and if we have $b+1$ connected components of $\partial\Omega$, we have b copies of $\eta_2(h), \eta_3(h), \cdots$, which are the positive Neumann eigenvalues on $(0,h)$, or, equivalently, we have b copies of $d_1(h), d_2(h), ...$ which are the Dirichlet eigenvalues on $(0,h)$. We will list these eigenvalues corresponding to non-trivial topology of the boundary $\partial\Sigma$ in the third family here below.

Second family. For the second family, the ansatz is $F = \star d_\Sigma f$. We recall that for Σ without boundary, the boundary condition at $\Sigma \times \{0,h\}$ reads $\star d_\Sigma f = 0$, which is equivalent to $d_\Sigma f = 0$.

Considering now the $\partial\Sigma \neq \emptyset$ case and setting $f(x,t) = w(x)u(t)$, the condition $d_\Sigma f = 0$ amounts to $u(0)d_\Sigma w = u(h)d_\Sigma w = 0$. On the lateral boundary $\partial\Sigma \times (0,h)$,

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the boundary condition implies that $u(t) \star d_\Sigma w$ must be normal, forcing $\partial_{\nu_\Sigma} w = 0$ on $\partial\Sigma$. We end up with

$$\begin{cases} u''(t) d_\Sigma w - u(t) d_\Sigma \Delta_\Sigma w = -\lambda u(t) d_\Sigma w\,, & \text{in } M \\ u(0) d_\Sigma w = u(h) d_\Sigma w = 0\,, & \\ \partial_{\nu_\Sigma} w = 0\,, & \text{on } \partial\Sigma. \end{cases}$$

Following the proof of Theorem 3.1, we conclude that all the eigenvalues are given by

$$\lambda = \mu_k^N + d_j(h)\,, \quad j = 1, ..., k = 2, ...$$

with μ_k^N and $d_j(h)$ as in the statement of the Theorem.

Third family. Finally, for genus $\gamma \geq 1$, we have a third family given by $u_j(t) H_k(x)$, where $\{H_k(x)\}_{k=1}^{2\gamma}$ is such that $\{H_k\}_{k=1}^{2\gamma} \cup \{d_\Sigma \psi_k\}_{k=1}^{b}$ is a basis of the harmonic 1-forms on Σ with relative boundary conditions. Here ψ_k are defined by $\Delta\psi_k = 0$ in Σ, $\psi_k = \text{const} \neq 0$ on $\partial\Sigma_k$ and $\psi_k = 0$ on $\partial\Sigma_i$, $i \neq k$, for $k = 1, ..., b$, where Σ_i, $i = 1, ..., b+1$ are the connected components on $\partial\Sigma$ (these are the functions found in the analysis of the first family). In fact, the first relative cohomology of a surface Σ of genus γ and $b+1$ boundary components is isomorphic to $\mathbb{Z}^{2\gamma+b}$.

Thus, we get the additional family of eigenvalues

$$\lambda = d_j(h)\,, \quad j \geq 1,$$

each repeated 2γ times. We also include in this family the eigenfunctions defined through the functions ψ_k found in the analysis of the first family.

The completeness of eigenfunctions is proved exactly as in Theorem 3.1. □

We observe that all the eigenvalues diverge to $+\infty$ as $h \to 0^+$, except for the family $\mu_k^D + \eta_1(h) = \mu_k^D$, $k \geq 2$. We state these results explicitly in the following corollary.

COROLLARY 3.5. *Let $\lambda_j(h, g_p)$ denote the eigenvalues of the product manifold (M, g_p) where $M = \Sigma \times (0, L)$. Then*

$$\lim_{L \to 0} \lambda_j(h, g_p) = \mu_j^D,$$

where μ_j^D are the Dirichlet Laplacian eigenvalues of (Σ, g_Σ).

4. Convergence of eigenvalues on tubes around embedded surfaces: proof of Theorem 1.1

We will now proceed to prove Theorem 1.1. The proof strategy will be the following: we relate the eigenvalues of the tube Ω_h with those of the product manifold (M, g_p), where

$$M = \Sigma \times (0, h)$$

and g_p is the product metric; then we show that for $h \to 0^+$, the two sequences of eigenvalues become arbitrarily close and conclude using Theorems 3.1 and 3.4.

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Throughout this section, Σ is a smooth, compact, embedded orientable surface in $\mathbb{R}^3$, and g_Σ is the Riemannian metric on Σ induced by the ambient Euclidean space. Let Ω_h be defined by (1.2), namely, $\Omega_h := \{x + t\nu : t \in (0,h)\,, x \in \Sigma\}$ where ν is a choice of the unit normal to Σ vector field to σ. Since Σ is embedded and smooth, there exists $h_0 > 0$ such that, for all $h \in (0, h_0)$, the parallel surface to Σ at distance h is smooth, and moreover Ω_h is diffeomorphic to $M := \Sigma \times (0,h)$ through $\phi : \Sigma \times (0,h) \to \Omega_h$. We may then use Fermi coordinates $(x,t) \in M = \Sigma \times (0,h)$. Moreover, the domain Ω_h with the Euclidean metric g_E is isometric to $(M, \phi^* g_E)$, where $\phi^* g_E$ is the pull-back of the Euclidean metric through ϕ. To abbreviate, we write

$$g_F := \phi^* g_E$$

for the pull-back of the Euclidean metric on M. One also sees that

$$g_F = g_p\,, \quad \text{on } \Sigma \times \{0\},$$

where $g_p = g_\Sigma + dt^2$ is the product metric on $M = \Sigma \times (0,h)$. Since M is compact, we deduce that, for all $h > 0$ sufficiently small,

$$\|g_F - g_p\|_{C^2(M)} < Ch\,, \quad \|g_F^{-1} - g_p^{-1}\|_{C^2(M)} < Ch, \tag{4.1}$$

uniformly in M. See e.g., [10, § 2]. From now on, by C we denote a positive constant which does not depend on h and which may be re-defined line by line.

By $\lambda_j(h, g_p)$ we denote the eigenvalues of (3.1) (restricted to co-closed forms) on (M, g_p) and by $\lambda_j(h, g_F)$ the eigenvalues of (3.1) (restricted to co-closed forms) on (M, g_F). From the discussion above, we have that

$$\lambda_j(\Omega_h) = \lambda_j(h, g_F)$$

where $\lambda_j(\Omega_h)$ are the eigenvalues of (1.1).

To prove Theorem 1.1, we compare the Rayleigh quotients defining $\lambda_j(h, g_F)$ and $\lambda_j(h, g_p)$. We have

$$\lambda_j(h, g_F) = \inf_{\substack{U \subset V_F \\ \dim U = j}} \max_{0 \neq u \in U} \frac{\int_M |du|^2_{g_F} dv_{g_F}}{\int_M |u|^2_{g_F} dv_{g_F}}.$$

where $V_F = \{u : \delta_{g_F} u = 0 \text{ in } M, i^* u = 0 \text{ on } \partial M\}$, that is, the subspace of 1-forms that are co-closed and normal to ∂M. Analogously, we have that

$$\lambda_j(h, g_p) = \inf_{\substack{U \subset V_p \\ \dim U = j}} \max_{0 \neq u \in U} \frac{\int_M |du|^2_{g_p} dv_{g_p}}{\int_M |u|^2_{g_p} dv_{g_p}}, \tag{4.2}$$

where $V_p = \{u : \delta_{g_p} u = 0 \text{ in } M, i^* u = 0 \text{ on } \partial M\}$. The two spaces V_F and V_p are not the same since the codifferential δ depends on the metric.

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However, given u such that $\delta_{g_F} u \neq 0$, but $\delta_{g_p} u = 0$, we can replace it by $u + dv$ where v satisfies $\delta_{g_F} dv = -\delta_{g_F} u$, $v = 0$ on ∂M. This is just

$$\begin{cases} \Delta_F v = -\delta_{g_F} u\,, & \text{in } M\,, \\ v = 0\,, & \text{on } \partial M. \end{cases} \tag{4.3}$$

That is, we have projected u on the subspace of co-closed 1-forms for the metric g_F. But now

$$d(u + dv) = du + d^2 v = du.$$

Summarizing, for any $u \in V_p$, there exists a unique $\widetilde{u} \in V_F$, $\tilde{u} = u + dv$, with $\delta_{g_F} \widetilde{u} = 0$, $du = d\tilde{u}$, and $\tilde{u}$ is tangential for g_F. This last fact follows since for a solution v of (4.3), dv is tangential, and since a 1-form u is tangential for g_F if and only if it is tangential for g_p. This is due to the structure of the metrics g_F, g_p, which, in local coordinates, assume the form of a block diagonal matrix splitting the components along $T\Sigma$ and $(0, h)$. Hence, we can write

$$\lambda_j(h, g_F) = \inf_{\substack{U \subset V_p \\ \dim U = j}} \max_{0 \neq u \in U} \frac{\int_M |du|^2_{g_F} dv_{g_F}}{\int_M |u + dv|^2_{g_F} dv_{g_F}}. \tag{4.4}$$

Now, we have

$$\int_M |dv|^2_{g_F} = -\int_M \langle u, dv \rangle_{g_F} = -\int_M \langle u, dv \rangle_{g_F} + \int_M \langle u, dv \rangle_{g_p},$$

where the first identity follows by multiplying (4.3) by v and integrating by parts. We have added the last summand, which equals zero since $\delta_{g_p} u = 0$. Hence

$$\int_M |dv|^2_{g_F} \leq \left| \int_M \langle u, dv \rangle_{g_F} - \int_M \langle u, dv \rangle_{g_p} \right| \tag{4.5}$$

From (4.1) and standard manipulations of the right-hand side of (4.5) we deduce that there exists a constant C not depending on u, v such that

$$\int_M |dv|^2_{g_F} \leq Ch \int_M |u|^2_{g_F}. \tag{4.6}$$

Again, using (4.1) and (4.6), we deduce the existence of a constant $C > 0$ not depending on u such that, for all $u \in V_p$,

$$(1 - Ch) \frac{\int_M |du|^2_{g_p} dv_{g_p}}{\int_M |u|^2_{g_p} dv_{g_p}} \leq \frac{\int_M |du|^2_{g_F} dv_{g_F}}{\int_M |u + dv|^2_{g_F} dv_{g_F}} \leq (1 + Ch) \frac{\int_M |du|^2_{g_p} dv_{g_p}}{\int_M |u|^2_{g_p} dv_{g_p}}. \tag{4.7}$$

The result now follows from (4.2) and (4.4): for all j, we have

$$(1 - Ch)\lambda_j(h, g_p) \leq \lambda_j(h, g_F) \leq (1 + Ch)\lambda_j(h, g_p). \tag{4.8}$$

This concludes the proof of Theorem 1.1 since $\lambda_j(h, g_F) = \lambda_j(\Omega)$.

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5. Convergence of eigenfunctions

In this section, we establish a convergence result for eigenfunctions. Throughout this section

$$M = \Sigma \times (0, h).$$

We start by observing that, if we consider the eigenvalues $\Lambda_j(h, g_p)$ of the (full) Hodge Laplacian with relative boundary conditions on (M, g_p), namely,

$$\begin{cases} \Delta\omega = \Lambda\omega\,, & \text{in } M\,, \\ i^*\omega = i^*\delta\omega = 0\,, & \text{on } \partial M, \end{cases} \tag{5.1}$$

then for all $j \in \mathbb{N}$, $\lim_{h\to 0^+} \Lambda_j(h, g_p) = \mu_j$, where μ_j, μ_j^D are the eigenvalues of the Laplacian on Σ (if $\partial M = \emptyset$). If $\partial\Sigma \neq \emptyset$, the same statement holds with μ_j^D as the limit eigenvalue. Moreover, for any fixed $j \in \mathbb{N}$, there exists $h_j > 0$ such that $\Lambda_j(h, g_p) = \mu_j$ for all $h < h_j$, and a basis of the corresponding eigenspace is given by $\{w_i(x)dt\}_{i=1}^{m_j}$, where w_i are the eigenfunctions of the Laplacian on Σ (with Dirichlet conditions if $\partial\Sigma \neq \emptyset$) associated with μ_j. We refer to Appendix A for more details on the full Hodge Laplacian spectrum on product manifolds.

Moreover, $\delta w_j(x)dt = 0$, hence the only eigenfunctions associated with Hodge Laplacian eigenvalues that admit a finite limit are co-closed. It is not difficult to see that these eigenfunctions correspond (up to scalar multiples) to the eigenfunctions $\delta d(w_j(x)dt)$ of the first family in Theorems 3.1 and 3.4, which in turn are exactly those providing the co-closed eigenvalues that have a finite limit. Now note that $\delta d(w_j dt) = \Delta(w_j dt) = \mu_j w_j dt$. All these statements still hold when $\partial\Omega \neq \emptyset$, up to replacing μ_j with μ_j^D.

Let us now consider problem (5.1) on (M, g_F). The Hodge–Morrey decomposition (2.3) implies that the spectrum of (5.1) on (M, g_F) is given by the union of the co-closed spectrum (i.e., the spectrum of (5.1) restricted to co-closed 1-forms), and the spectrum of (5.1) restricted to the space of gradients of functions in $H_0^1(M)$. But, as $h \to 0^+$, all the eigenvalues of (5.1) associated with eigenfunctions that are gradients of functions in $H_0^1(M)$ diverge to $+\infty$. Hence, if we denote by $\Lambda_j(h, g_F)$ the spectrum of (5.1) for (M, g_F), we have that for any fixed $j \in \mathbb{N}$ there exists $h_0 > 0$ such that $\Lambda_i(h, g_G) = \lambda_i(h, g_F)$ for all $i = 1, ..., j$. Recall that by $\lambda_j(h, g_F)$ and $\lambda_j(h, g_p)$ we have denoted the spectrum of (5.1) restricted to co-closed forms.

We are now ready to prove the following

THEOREM 5.1. *Let μ_j be an eigenvalue of the Laplacian on Σ (with Dirichlet conditions if $\partial\Sigma \neq \emptyset$) and let w be an associated L^2-normalized eigenfunction. Then there exists $h_0 > 0$ and an eigenfunction u of problem (5.1) on (M, g_F) associated with $\Lambda_j(h, g_F)$ satisfying*

$$\left\| h^{-1/2} w dt - u \right\|_{L^2\Omega^1(M)} \leq Ch$$

for all $h < h_0$, where $C > 0$ is a constant depending only on μ_j and Σ.

In view of the previous discussion, from Theorem 5.1, we deduce the following corollary.

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COROLLARY 5.2. *Let μ_j be an eigenvalue of the Laplacian on Σ (with Dirichlet conditions if $\partial\Sigma \neq \emptyset$) and let w be an associated L^2-normalized eigenfunction. Then there exists $h_0 > 0$ and an eigenfunction u of problem (3.1) associated with $\lambda_j(h, g_F)$ satisfying*

$$\left\| h^{-1/2} w dt - u \right\|_{L^2\Omega^1(M)} \leq Ch$$

for all $h < h_0$, where $C > 0$ is a constant depending only on μ_j and Σ.

This last corollary, translated in terms of solutions of Maxwell's problem (1.1), says that, for all $h > 0$ sufficiently small, given an eigenfunction w on the limit surface Σ associated with a limit eigenvalue μ_j (or μ_j^D if Σ has a boundary), there exists an eigenfunction u of (1.1) associated with $\lambda_j(\Omega_h)$ which is close in $L^2(\Omega)^3$ to the constant extension of w in the normal direction to Σ.

Proof of Theorem 5.1. Let $\{w_i\}_{i=1}^{m_j}$ is an orthonormal basis of the eigenspace associated with μ_j. First, note that there exists $h_0 > 0$, such that, for any $h < h_0$ an orthonormal basis of the eigenspace associated with $\Lambda_j(h, g_p)$ is given by $\{h^{-1/2} w_i dt\}_{i=1}^{m_j}$. Let $v = h^{-1/2} \sum_{i=1}^{m_j} a_i w_i$ with $\sum_{i=1}^{m_j} a_i^2 = 1$. Then v is an L^2-normalized eigenfunction of (M, g_p) associated with $\Lambda_j(h, g_p)$. Possibly choosing a smaller h_0, we may further assume that $\Lambda_{j-1}(h, g_p) < \Lambda_j(h, g_p) = \Lambda_{j+1}(h, g_p) = \cdots = \Lambda_{j+m_j-1}(h, g_p) < \Lambda_{j+m_j}(h, g_p)$ and $\Lambda_{j-1}(h, g_F) < \Lambda_j(h, g_F) = \Lambda_{j+1}(h, g_F) = \cdots = \Lambda_{j+m_j-1}(h, g_F) < \Lambda_{j+m_j}(h, g_F)$.

From (4.8) and from the fact that, for h_0 sufficiently small $\lambda_j(h, g_p) = \Lambda_j(h, g_p)$ and $\lambda_j(h, g_F) = \Lambda_j(h, g_F)$, we have that $|\Lambda_j(h, g_F) - \Lambda_j(h, g_p)| < Ch$, and the multiplicities of $\Lambda_j(h, g_F)$ and $\Lambda_j(h, g_p)$ coincide. Since the eigenfunctions in the product metric are explicit (see Section 3), we can assume $v \in H^2\Omega^1(M)$ for the metric g_p (and also for g_F), uniformly in h, as the eigenfunctions of the Laplacian in Σ (with Dirichlet conditions in the case Σ has a boundary) are smooth. Let u be an eigenfunction associated with $\Lambda_j(h, g_F)$ such that $\int_M \langle u_i^j, (u - v) \rangle_{g_F} dv_{g_F} = 0$ for all $i = 1, ..., m_j$, where u_i^j is a L^2-orthonormal basis of the eigenspace Θ_h associated with $\Lambda_j(h, g_F)$ (for the metric g_F). Note that if h_0 is sufficiently small, then $u \neq 0$.

Now, consider the following problem

$$(\Delta_{g_F} - \Lambda_j(h, g_F))(u - v) = (\Delta_{g_F} - \Delta_{g_p})v - (\Lambda_j(h, g_F) - \Lambda_j(h, g_p))v \tag{5.2}$$

in (M, g_F). On the boundary, we have that $i^*(u - v) = 0$ (the outer unit normal is the same for both the metrics). Moreover, $i^*\delta_{g_F}(u - v) = i^*(\delta_{g_p} v - \delta_{g_F} v) = O(h)$ by (4.1); indeed, $\delta_{g_F} u = \delta_{g_p} v = 0$, since, upon choosing h small enough, u and v must be eigenfunctions of the Hodge operator restricted to co-closed 1-forms. Then $u - v$ belongs to the orthogonal of the kernel of $\Delta_{g_F} - \Lambda_j(h, g_F)$, it is normal to the boundary and its divergence is small ($O(h)$) at the boundary. Then, if $F_h = (\Delta_{g_F} - \Delta_{g_p})v - (\lambda_j(h, g_F) - \lambda_j(h, g_p))v$, $q_h = i^*(\delta_{g_p} v - \delta_{g_F} v)$, the form $w = u - v$ solves the following inhomogeneous problem for the Hodge Laplacian with relative boundary conditions:

$$\begin{cases} (\Delta_{g_F} - \lambda_j(h))w = F_h\,, & \text{in } M\,, \\ i^* w = 0, \;\; i^*\delta_{g_F} w = q_h, & \text{on } \partial M. \end{cases}$$

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Since $w \in \Theta_h^\perp$ for each fixed $h > 0$, the Fredholm alternative implies that this problem is solvable provided that $F_h \in \Theta_h^\perp$, which holds true in view of the identity (5.2).

Interpreting this problem in weak form, we obtain the $L^2\Omega^1(M)$ (for the metric g_F) a priori: estimate

$$\operatorname{dist}(\sigma(\Delta_{g_F}) \setminus \{\Lambda_j(\Omega_h)\}, \Lambda_j(\Omega_h))\|w\|^2_{L^2\Omega^1(M)} \leq C(\|F_h\|^2_{L^2\Omega^1(M)} + \|q_h\|^2_{L^2(\partial M)}),$$

where $C > 0$ does not depend on h. Now note that

$$\|F_h\|_{L^2\Omega^1(M)} \leq C\big(\|g_p^{-1} - g_F^{-1}\|_{C^2(M)})\|v\|_{H^2\Omega^1(M)} + |\Lambda_j(\Omega_h) - \Lambda_j(h))|\|v\|_{L^2\Omega^1(M)}\big)$$

and

$$\|q_h\|_{L^2(\partial\Omega_h)} \leq C\,\|g_p^{-1} - g_F^{-1}\|_{C^2(M)}\,\|v\|_{H^2(\Omega_h)}$$

by the Trace inequality and the definition of q_h. Note that the constant $C > 0$ needs to be possibly redefined, but it still does not depend on h. This concludes the proof in view of (4.1) and of the convergence of the eigenvalues established in Section 4. □

Remark 5.3. Concerning stronger notions of convergence, it is natural to expect that the resolvent operator of the Maxwell's system converges to that of the (Dirichlet) Laplacian on Σ. We believe that this is the case. However, the proof of this fact is beyond the scope of the paper, which is devoted to the analysis of the eigenvalues and the consequences for the spectral geometry of this operator. Although the convergence in the case of the product metric could be achieved with standard arguments, the convergence in the Euclidean setting is more delicate. Indeed, the natural extension operator, taking a function w on Σ to a 1-form wdt in M, does not produce, in general, co-closed forms; and even more dramatically, the projection of wdt on the space of co-closed forms may depend on the small parameter h.

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References

1. C. Anné and B. Colbois. Spectre du laplacien agissant sur les p-formes différentielles et écrasement d'anses. *Math. Ann. (3)* **303** (1995), 545–573.
2. C. Anné and O. Post. Wildly perturbed manifolds: norm resolvent and spectral convergence. *J. Spectr. Theory (1)* **11** (2021), 229–279.
3. C. Anné and J. Takahashi. Small eigenvalues of the Hodge-Laplacian with sectional curvature bounded below. *Ann. Global Anal. Geom. (1)* **68** (2025),15.
4. J. M. Arrieta, J. C. Nakasato and M. C. Pereira. The p-Laplacian equation in thin domains: the unfolding approach. *J. Differ. Equ.* **274** (2021), 1–34.
5. J. M. Arrieta and M. Villanueva-Pesqueira. Elliptic and parabolic problems in thin domains with doubly weak oscillatory boundary. *Commun. Pure Appl. Anal. (14)* **19** (2020), 1891–1914.
6. S. Bögli, F. Ferraresso, M. Marletta and C. Tretter. Spectral analysis and domain truncation for Maxwell's equations. *J. Math. Pures Appl.* ***(9)*****170** (2023), 96–135.
7. D. Borisov and P. Freitas. Asymptotics of Dirichlet eigenvalues and eigenfunctions of the Laplacian on thin domains in $\mathbb{R}^d$. *J. Funct. Anal. (3)* **258** (2010), 893–912.
8. B. Brandolini, F. Chiacchio and J. J. Langford. Eigenvalue estimates for p-Laplace problems on domains expressed in Fermi coordinates. *J. Math. Anal. Appl. (2)* **540** (2024), 128616.
9. P. Briet, M. Cassier, T. Ourmières-Bonafos and M. Zaccaron. Geometric spectral properties of electromagnetic waveguides, arXiv. (2025).
10. J. Brisson. Tubular excision and Steklov eigenvalues. *J. Geom. Anal. (5)* **32** (2022), 166, 24.
11. A. Buffa, M. Costabel and D. Sheen. On traces for $h(curl, \omega)$ in lipschitz domains. *J. Math. Anal. Appl. (2)* **276** (2002), 845–867.
12. D. Buoso and F. Ferraresso. Spectral convergence analysis for the Reissner-Mindlin system in any dimension. *J. Differ. Equ.* **422** (2025), 386–425.
13. I Chavel. *Eigenvalues in Riemannian Geometry, Volume 115 of Pure and Applied Mathematics*. (Academic Press, Inc, Orlando, FL, 1984). Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
14. B. Colbois. The spectrum of Laplacian acting on differential p-forms. In *SéMinaire de théOrie Spectrale et géométrie. Année 1995-1996*, pp. 25–37 (Univ. de Grenoble I, Institut Fourier, St. Martin d'Hères, 1996).
15. M. Costabel and M. Dauge. Maxwell eigenmodes in product domains. *In Maxwell's equations—nalysis and numerics, volume 24 of Radon Ser. Comput. Appl. Math.*, pp. 171–198 (De Gruyter, Berlin, 2019).
16. G. Courtois. Spectrum of manifolds with holes. *J. Funct. Anal. (1)* **134** (1995), 194–221.
17. P. Exner and O. Post. Convergence of spectra of graph-like thin manifolds. *J. Geom. Phys. (1)* **54** (2005), 77–115.
18. P. Exner and O. Post. Convergence of resonances on thin branched quantum waveguides. *J. Math. Phys. (9)* **48** (2007), 43.
19. P. Exner and O. Post. First order operators on shrinking graph-like spaces. *J. Phys. A (1)* **58** (2025), 305202, 13.
20. F. Ferraresso and M. Marletta. Essential spectrum for dissipative Maxwell equations in domains with cylindrical ends. *J. Math. Anal. Appl. (1)* **536** (2024), 128174, 21.
21. N. D. Filonov. The Maxwell operator in a cylinder with coefficients that do not depend on the transverse variables. *Algebra i Analiz. (1)* **32** (2020), 187–207.
22. N. D. Filonov. Maxwell operator in a cylinder. *Appl. Anal. (12)* **103** (2024), 2141–2174.
23. K. O. Friedrichs. Differential forms on Riemannian manifolds. *Commun. Pure Appl. Math.* **8** (1955), 551–590.
24. M. P. Gaffney. The harmonic operator for exterior differential forms. *Proc. Natl. Acad. Sci. USA.* **37** (1951), 48–50.
25. M. P. Gaffney. Hilbert space methods in the theory of harmonic integrals. *Trans. Am. Math. Soc.* **78** (1955), 426–444.

https://doi.org/10.1017/prm.2026.10141 Published online by Cambridge University Press

26. D. Grieser. Thin tubes in mathematical physics, global analysis and spectral geometry. *In Analysis on graphs and its applications, volume 77 of Proc. Sympos. Pure Math.*, Vol. 77, pp. 565–593 (Amer. Math. Soc, Providence, RI, 2008).
27. P. Guerini and A. Savo. The Hodge Laplacian on manifolds with boundary. *In Séminaire de Théorie Spectrale et Géométrie. Vol. 21. Année 2002–2003, volume 21 of Sémin. Théor. Spectr. Géom.* pp. 125–146 (Univ. Grenoble I, Saint-Martin-d'Héres, 2003).
28. P. Guerini and A. Savo. Eigenvalue and gap estimates for the Laplacian acting on p-forms. *Trans. Amer. Math. Soc. (1)* **356** (2004), 319–344.
29. D. Krejčiřík. Spectrum of the Laplacian in a narrow curved strip with combined Dirichlet and Neumann boundary conditions. *ESAIM Control Optim. Calc. Var. (13)* **15** (2009), 555–568.
30. D. Krejčiřík, P. D. Lamberti and M. Zaccaron. A note on the failure of the Faber–Krahn inequality for the vector Laplacian. *ESAIM: COCV*. **31** (2025), 21.
31. D. Krejčiřík, N. Raymond, J. Royer and P. Siegl. Reduction of dimension as a consequence of norm-resolvent convergence and applications. *Mathematika. (2)* **64** (2018), 406–429.
32. D. Krejčiřík and M. Tušek. Location of hot spots in thin curved strips. *J. Differ. Equ. (6)* **266** (2019), 2953–2977.
33. V. Lotoreichik and T. Ourmières-Bonafos. Spectral asymptotics of the Dirac operator in a thin shell. *J. Math. Anal. Appl. (2)* **546** (2025), 129190, 42.
34. M. Mitrea. Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains. *Forum Math. (4)* **13** (2001), 531–567.
35. T. -H. Miura. Navier-Stokes equations in a curved thin domain, Part III: Thin-film limit. *Adv. Diff. Eq. (9–10)* **25** (2020), 457–626.
36. T. -H. Miura. Navier-Stokes equations in a curved thin domain, Part II: Global existence of a strong solution. *J. Math. Fluid Mech. (1)* **23** (2021), 7, 60.
37. T. -H. Miura. Navier-Stokes equations in a curved thin domain, Part I: Uniform estimates for the Stokes operator. *J. Math. Sci. Univ. Tokyo. (2)* **29** (2022), 149–256.
38. S. Raulot and A. Savo. A Reilly formula and eigenvalue estimates for differential forms. *J. Geom. Anal. (3)* **21** (2011), 620–640.
39. S. Raulot and A. Savo. On the spectrum of the Dirichlet-to-Neumann operator acting on forms of a Euclidean domain. *J. Geom. Phys.* **77** (2014), 1–12.
40. J. Rubinstein and M. Schatzman. Variational problems on multiply connected thin strips. I. Basic estimates and convergence of the Laplacian spectrum. *Arch. Ration. Mech. Anal. (4)* **160** (2001), 271–308.
41. A. Savo. Hodge-Laplace eigenvalues of convex bodies. *Trans. Amer. Math. Soc. (4)* **363** (2011), 1789–1804.
42. M. Schatzman. On the eigenvalues of the Laplace operator on a thin set with Neumann boundary conditions. *Appl. Anal. (3–4)* **61** (1996), 293–306.
43. G. Schwarz. *Hodge decomposition. A method for solving boundary value problems, volume 1607 of Lect*, (Springer Verlag, Berlin, 1995).
44. C. Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci. (1)* **2** (1980), 12–25.

Appendix A. Full Hodge Laplacian spectrum with relative conditions on product manifolds

We have seen that the Maxwell's eigenvalues coincide with the eigenvalues of the Hodge Laplacian with relative conditions restricted to the subspace of co-closed differential forms. On a compact Riemannian manifold (M, g), the eigenvalue problem for the Hodge Laplacian with relative conditions acting on p forms reads

$$\begin{cases} \Delta\omega = \lambda\omega\,, & \text{in } M \\ i^*\omega = i^*\delta\omega = 0\,, & \text{on } \partial M. \end{cases} \tag{A.1}$$

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When (M, g) is an Euclidean domain Ω, identifying vectors and 1-form, problem (A.1) reads

$$\begin{cases} \mathrm{curl}\,\mathrm{curl} E - \nabla \mathrm{div} E = \lambda E\,, & \text{in } \Omega\,, \\ \nu \times E = \mathrm{div} E = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{A.2}$$

It is well-known that if $p + q = r$, and if α is a p-form and β is a q-form, then the Laplacian acting on the r-form $\alpha \wedge \beta$ on a product manifold $M \times N$ splits as follows (Künneth formula):

$$\Delta(\alpha \wedge \beta) = \alpha \wedge \Delta\beta + \alpha \wedge \Delta\beta.$$

In the case of closed manifolds M, N, the Hodge Laplacian spectrum of r forms is then given by summing p-eigenvalues of M and q-eigenvalues of N, for any choices of p, q such that $p + q = r$. In the case of 1 forms, the spectrum is then given by sums of Laplacian eigenvalues of M and Hodge Laplacian eigenvalues on 1-forms on N, and sums of Laplacian eigenvalues of N and Hodge Laplacian eigenvalues on 1-forms on M.

We compute now the spectrum of (A.1) for product Riemannian manifolds and 1-forms, and in particular, for $M = \Sigma \times (0, h)$ and $g = g_p$, where (Σ, g_Σ) is a Riemannian surface and $g_p = g_\Sigma + dt^2$ is the product metric. Assume first that Σ has no boundary. Consider the family

$$u_j(t)\omega_k(x)\,, \quad j, k \geq 1,$$

where u_j are the Dirichlet eigenfunctions of $(0, h)$ with eigenvalues $d_j(h)$ and ω_k are the eigenfunctions of the Hodge Laplacian on 1-forms on Σ with eigenvalues μ_k^1. One checks that these are eigenfunctions of the Hodge Laplacian on $\Sigma \times (0, h)$ with relative boundary conditions. The corresponding eigenvalues are given by $d_j(h)+\mu_k^1$, $j, k \geq 1$. Consider now the family

$$w_k(x)v_j(t)dt\,, \quad j, k \geq 1$$

where v_j are the Neumann eigenfunctions of $(0, h)$ with eigenvalues $\eta_j(h)$ and w_k are the eigenfunctions of the Laplacian (on functions) on Σ with eigenvalues μ_k. One checks that these are eigenfunctions of the Hodge Laplacian on $\Sigma \times (0, h)$ with relative boundary conditions. The corresponding eigenvalues are given by $\eta_j(h)+\mu_k$, $j, k \geq 1$. These two families exhaust the entire spectrum. This can be done as in the proofs of Theorems 3.1 and 3.4 (see also [14]).

If Σ has a boundary, it is possible to argue in a similar way. Namely, one just needs to choose eigenfunctions in the form $u_j(t)\omega_k(x)$ with u_j Dirichlet eigenfunctions on $(0, h)$ and ω_k 1-forms with relative conditions on Σ, or in the form $w_k(x)v_j(t)dt$ with $w_k(x)$ Dirichlet eigenfunctions on Σ and $v_j(t)$ Neumann eigenfunctions on $(0, h)$.

Note that the Hodge Laplacian spectrum restricted to co-closed 1-forms (Maxwell's spectrum) is a subset of the Hodge Laplacian spectrum. However, it is not immediate to recognize the explicit form of the co-closed eigenfunctions as

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in Theorems 3.1 and 3.4. Nevertheless, if we denote by $\Lambda_j(h, g_p)$ the eigenvalues of the (full) Hodge Laplacian on (M, g_p) with relative conditions, then, for all $j \in \mathbb{N}$,

$$\lim_{h\to 0^+} \Lambda_j(h, g_p) = \mu_j,$$

where μ_j are the eigenvalues of the Laplacian on (Σ, g_Σ) (with Dirichlet conditions if $\partial\Sigma \neq \emptyset$). More precisely, for each fixed j, there exists h_j sufficiently small such that $\Lambda_j(h, g_p) = \mu_j$ for all $h < h_j$ and the eigenfunctions associated with $\Lambda_j(h, g_p)$ are given by $w_j(x)dt$, with $w_j(x)$ the eigenfunctions of the Laplacian on Σ associated with μ_j (with Dirichlet conditions if $\partial\Sigma \neq \emptyset$). This analysis is actually sufficient to prove Corollaries 3.3 and 3.5, and then Theorem 1.1. However, our method of proof also provides an explicit description of the eigenfunctions on the product manifold, see Theorems 3.1 and 3.4.

Finally, we mention that there is a dual set of boundary conditions for the eigenvalue problem for the Hodge Laplacian. More precisely, we can consider the eigenvalue problem for the Hodge Laplacian with *absolute* boundary conditions:

$$\begin{cases} \Delta\omega = \lambda\omega\,, & \text{in } M\,, \\ i^*\iota_\nu\omega = i^*\iota_\nu d\omega = 0\,, & \text{on } \partial M\,, \end{cases} \tag{A.3}$$

where ι denotes the interior multiplication of differential forms. For 0-forms (functions), this simply reduces to the Neumann boundary conditions. By the Hodge $\star$ isomorphism, which exchanges the two boundary conditions, we have that

$$\lambda^R_{j,p}(M, g) = \lambda^A_{j,n-p}(M, g)$$

where in the above formula, $\lambda^R_{j,p}(M, g)$ and $\lambda^A_{j,p}(M, g)$ denote the eigenvalues of the Hodge Laplacian acting on p-forms on the n-dimensional Riemannian manifold (M, g) with relative and absolute boundary conditions, respectively. In particular, due to the equality $\lambda^R_{j,1} = \lambda^A_{j,2}$ when $n = 3$, we conclude that our analysis of the Hodge Laplacian eigenvalues with relative boundary conditions for 1-forms also yields the same results for 2-forms with absolute boundary conditions. A nice introduction to the Hodge Laplacian on domains with boundary can be found, e.g., in [27].

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