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Hartshorne's question on cofiniteness of complexes

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Abstract. Let d be a positive integer, and let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R. We answer Hartshorne's question on cofiniteness of complexes posed in Hartshorne (1970, *Invent. Math.* 9, 145–164) in the cases $\dim R = d$ or $\dim R/\mathfrak{a} = d-1$ or $\operatorname{ara}(\mathfrak{a}) = d-1$, show that if $d \leqslant 2$, then a complex $X \in \mathrm{D}_{\sqsubset}(R)$ is $\mathfrak{a}$-cofinite if and only if each homology module $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite; if R is regular local, $\mathfrak{a}$ is perfect and $d \leqslant 2$, then $X \in \mathrm{D}(R)$ is $\mathfrak{a}$-cofinite if and only if every $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite; if $d \geqslant 3$, then $X \in \mathrm{D}_{\sqsubset}(R)$ is $\mathfrak{a}$-cofinite and $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j \leqslant d-2$ and $i \in \mathbb{Z}$ if and only if every $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite.

1 Introduction

Throughout this article, R is a commutative Noetherian ring with nonzero identity, and $\operatorname{Spec} R$ denotes the set of prime ideals of R. For an ideal $\mathfrak{a}$ of R, set

$$\mathrm{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R | \mathfrak{a} \subseteq \mathfrak{p}\}.$$

By an *R-complex* X, we mean a sequence of R-modules

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots,$$

such that $d_n d_{n+1} = 0$ for $n \in \mathbb{Z}$. The category of R-complexes and their morphisms is denoted by $\mathrm{C}(R)$. The derived category of R-complexes is denoted by $\mathrm{D}(R)$. The full subcategories of $\mathrm{D}(R)$ consisting of homology bounded above, bounded below, and bounded R-complexes are denoted by $\mathrm{D}_{\sqsubset}(R)$, $\mathrm{D}_{\sqsupset}(R)$, and $\mathrm{D}_{\square}(R)$, respectively. We also denote by $\mathrm{D}^{\mathrm{f}}(R)$ the full subcategory of $\mathrm{D}(R)$ consisting of R-complexes with finitely generated homology.

In his algebraic geometry seminars of 1961–2, Grothendieck founded the theory of local cohomology and raised a few questions on the finiteness properties of local cohomology modules (see [14, Conjectures 1.1 and 1.2]). For an ideal $\mathfrak{a}$ of R and an R-module M, the *ith local cohomology* of M with respect to $\mathfrak{a}$ is

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_{t>0} \operatorname{Ext}^i_R(R/\mathfrak{a}^t, M).$$

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He specifically conjectured that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_\mathfrak{a}(M))$ is finitely generated for all finitely generated R-modules M and $i \geqslant 0$, which had been answered affirmatively in the same seminar when $(R, \mathfrak{m})$ is local and $\mathfrak{a} = \mathfrak{m}$. In 1970, Hartshorne [15] provided a counterexample to show that this conjecture is not true in general. He then defined an R-module M to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for every $i \geqslant 0$. When R is an $\mathfrak{a}$-adically complete regular ring of finite Krull dimension, he defined an R-complex X to be $\mathfrak{a}$-*cofinite* if $X \simeq \mathrm{RHom}_R(Y, \mathrm{R}\Gamma_\mathfrak{a}(R))$ for some complex $Y \in \mathrm{D}^{\mathrm{f}}(R)$. Hartshorne [15] also proceeded to pose some questions in this respect.

Question 1 *Is the local cohomology module* $\mathrm{H}^i_\mathfrak{a}(M)$ $\mathfrak{a}$*-cofinite for every finitely generated* R*-module* M *and every* $i \geqslant 0$*?*

Question 2 *Is the category* $\mathcal{M}(R, \mathfrak{a})_{cof}$ *of* $\mathfrak{a}$*-cofinite* R*-modules an abelian subcategory of the category of* R*-modules?*

Question 3 *Is it true that an* R*-complex* X *is* $\mathfrak{a}$*-cofinite if and only if the homology module* $\mathrm{H}_i(X)$ *is* $\mathfrak{a}$*-cofinite for every* $i \in \mathbb{Z}$*?*

By providing a counterexample, he showed that the answers to these questions are negative in general, and further established affirmative answers to Questions 1 and 2, and to Question 3 for bounded above complexes when R is an $\mathfrak{a}$-adically complete regular ring with $\dim R < \infty$ and $\mathfrak{a}$ is a principal ideal generated by a nonzero divisor, or when $\mathfrak{a}$ is a prime ideal of a complete regular local ring R with $\dim R/\mathfrak{a} = 1$ (see [15, Propositions 6.2 and 7.6, Corollaries 6.3 and 7.7, and Theorem 7.5]).

If the local cohomology module $\mathrm{H}^i_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite, then the set $\mathrm{Ass}_R \mathrm{H}^i_\mathfrak{a}(M)$ of associated primes and the Bass numbers $\mu^j_R(\mathfrak{p}, \mathrm{H}^i_\mathfrak{a}(M))$ are finite for all $\mathfrak{p} \in \mathrm{Spec} R$ and $i, j \geqslant 0$. This observation reveals the utmost significance of Question 1 as an affirmative answer to this question in any case would readily set fourth affirmative answers to Huneke's questions (see [16]). In the following years, Hartshorne's results on Questions 1 and 2 were systematically extended and polished by commutative algebra practitioners in several stages (see [1–5, 10, 18–20]). Whereas not much attention has been brought to Question 3. The most striking result on Question 3 is the following.

Theorem I [11] *Let* $\mathfrak{a}$ *be a proper ideal of a complete Gorenstein local domain* R *with* $\dim R/\mathfrak{a} = 1$*. Then an* R*-complex* $X \in \mathrm{D}_\square(R)$ *is* $\mathfrak{a}$*-cofinite if and only if* $\mathrm{H}_i(X)$ *is* $\mathfrak{a}$*-cofinite for every* $i \in \mathbb{Z}$*.*

In fact, these problems are inextricably linked. A piece of evidence that inspires and supports this speculation depends on the fact that all these problems are true in one case, but not in another. To shed some light on this revelation, Faridian [12, Section 2.4] defined an R-complex X to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R X \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{RHom}_R(R/\mathfrak{a}, X) \in \mathrm{D}^{\mathrm{f}}(R)$, this definition coincides with that of Hartshorne. He showed that if Question 3 holds true for bounded complexes then Questions 1 and 2 also hold true, and obtained the following relatively complete answer to Question 3.

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Theorem II [10, 12] *Let $\mathfrak{a}$ be a proper ideal of an $\mathfrak{a}$-adically complete ring R. Suppose that either $\mathrm{cd}(\mathfrak{a}, R) \leqslant 1$, or $\dim R \leqslant 2$, or $\dim R/\mathfrak{a} \leqslant 1$. Then an R-complex $X \in \mathrm{D}_{\square}(R)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$.*

Let d be a positive integer. The primary objective of this article is to answer Question 3 in the cases $\dim R/\mathfrak{a} = d-1$, or $\dim R = d$, or $\mathrm{ara}(\mathfrak{a}) = d-1$ based on the extensive use of certain spectral sequences. More precisely, we show the following theorem.

Theorem III *Let d be a positive integer, and let $\mathfrak{a}$ be a proper ideal of R so that either $\dim R/\mathfrak{a} = d-1$, or $\dim R = d$, or $\mathrm{ara}(\mathfrak{a}) = d-1$.*

(1) *If $d \leqslant 2$, then a complex $X \in \mathrm{D}_{\sqsubset}(R)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$ (see Theorem 3.1).*

(2) *Assume that R is regular local and $\mathfrak{a}$ is perfect. If $d \leqslant 2$, then a complex $X \in \mathrm{D}(R)$ is $\mathfrak{a}$-cofinite if and only if each $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite (see Theorem 3.3).*

(3) *If $d \geqslant 3$, then a complex $X \in \mathrm{D}_{\sqsubset}(R)$ is $\mathfrak{a}$-cofinite and $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for all i and $j \leqslant d-2$ if and only if each $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite (see Theorem 3.7).*

2 Preliminaries

This section collects some notions and facts about complexes for use throughout this article. For terminology, we shall follow [6] and [17].

Associated primes and support. Let M be an R-module. The set $\mathrm{Ass}_R M$ of *associated primes* of M is the set of prime ideals $\mathfrak{p}$ of R such that there exists a cyclic submodule N of M with $\mathfrak{p} = \mathrm{Ann}_R N$, the annihilator of N. The *support* of M is the set

$$\mathrm{Supp}_R M := \{\mathfrak{p} \in \mathrm{Spec} R \,|\, M_{\mathfrak{p}} \neq 0\}.$$

For an R-complex X, the *support* of X is defined as

$$\mathrm{Supp}_R X := \{\mathfrak{p} \in \mathrm{Spec} R \,|\, X_{\mathfrak{p}} \ncong 0\}.$$

By flatness of $R_{\mathfrak{p}}$ over R, one has $\mathrm{Supp}_R X = \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}_R \mathrm{H}_i(X)$.

Arithmetic rank and cohomological dimension. The *arithmetic rank* of the ideal $\mathfrak{a}$, denoted by $\mathrm{ara}(\mathfrak{a})$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$, that is,

$$\mathrm{ara}(\mathfrak{a}) := \min\{n \geqslant 0 | \exists\ a_1, \ldots, a_n \in R \text{ with } \mathrm{Rad}(a_1, \ldots, a_n) = \mathrm{Rad}(\mathfrak{a})\}.$$

The *arithmetic rank* of $\mathfrak{a}$ in R with respect to an R-module M, denoted by $\mathrm{ara}_M(\mathfrak{a})$, is defined by the arithmetic rank of the ideal $(\mathfrak{a} + \mathrm{Ann}_R M)/\mathrm{Ann}_R M$ in the ring $R/\mathrm{Ann}_R M$.

For each R-module M, set

$$\mathrm{hd}(\mathfrak{a}, M) = \sup\{n \geqslant 0 | \mathrm{H}^{\mathfrak{a}}_n(M) \neq 0\}, \mathrm{cd}(\mathfrak{a}, M) := \sup\{n \in \mathbb{Z} | \mathrm{H}^n_{\mathfrak{a}}(M) \neq 0\}.$$

The *cohomological dimension* of the ideal $\mathfrak{a}$ is

$$\mathrm{cd}(\mathfrak{a}, R) := \sup\{\mathrm{cd}(\mathfrak{a}, M) | M \text{ is an } R\text{-module}\}.$$

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Local cohomology and homology. Following [8], an R-complex I is called *semi-injective* if $\mathrm{Hom}_R(\alpha, I)$ is a surjective quasi-isomorphism for any injective quasi-isomorphism α in $\mathrm{C}(R)$. An R-complex F is called *semi-flat* if $\beta \otimes_R F$ is an injective quasi-isomorphism for every injective quasi-isomorphism β in $\mathrm{C}(R)$. Let X be an R-complex. A *semi-injective resolution* of X is a quasi-isomorphism $X \xrightarrow{\simeq} I$ where I is semi-injective. A *semi-flat resolution* of X is a quasi-isomorphism $F \xrightarrow{\simeq} X$ where F is semi-flat. By [8, Theorem 5.3.19], every R-complex has a semi-injective resolution. By [8, Theorem 5.2.14 and Corollary 5.4.10], every R-complex has a semi-flat resolution.

Let M be an R-module. The $\mathfrak{a}$-*torsion submodule* of M is

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \,|\, \mathfrak{a}^t m = 0 \text{ for some integer } t\} \cong \varinjlim \mathrm{Hom}_R(R/\mathfrak{a}^t, M).$$

The module M is called $\mathfrak{a}$-*torsion* if $\Gamma_{\mathfrak{a}}(M) = M$, or equivalently, $\mathrm{Supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$. The association $M \mapsto \Gamma_{\mathfrak{a}}(M)$ extends to define a left exact additive functor on the category $\mathrm{C}(R)$, its right derived functor, denoted by $\mathrm{R}\Gamma_{\mathfrak{a}}(-)$, can be computed by $\mathrm{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$, where $X \xrightarrow{\simeq} I$ is a semi-injective resolution of X. For $i \in \mathbb{Z}$, the *ith local cohomology* of X with support in $\mathfrak{a}$ is the R-module

$$\mathrm{H}^i_{\mathfrak{a}}(X) := \mathrm{H}_{-i}(\mathrm{R}\Gamma_{\mathfrak{a}}(X)).$$

The $\mathfrak{a}$-*adic completion* of M is

$$\Lambda^{\mathfrak{a}}(M) := \varprojlim (R/\mathfrak{a}^t \otimes_R M).$$

The map $M \mapsto \Lambda^{\mathfrak{a}}(M)$ extends to an additive functor on the category $\mathrm{C}(R)$. Following [17], this functor admits a left derived functor that is denoted by $\mathrm{L}\Lambda^{\mathfrak{a}}(-)$, and can be computed by $\mathrm{L}\Lambda^{\mathfrak{a}}(X) = \Lambda^{\mathfrak{a}}(F)$, where $F \xrightarrow{\simeq} X$ is a semi-flat resolution of X. For $i \in \mathbb{Z}$, the *ith local homology* of X with respect to $\mathfrak{a}$ is the R-module

$$\mathrm{H}^{\mathfrak{a}}_i(X) := \mathrm{H}_i(\mathrm{L}\Lambda^{\mathfrak{a}}(X)).$$

For an R-complex $X \in \mathrm{D}(R)$, set

$$\inf X = \inf\{n \in \mathbb{Z} \,|\, \mathrm{H}_n(X) \neq 0\} \text{ and } \sup X = \sup\{n \in \mathbb{Z} \,|\, \mathrm{H}_n(X) \neq 0\}.$$

The right derived functor of the homomorphism functor for R-complexes is denoted by $\mathrm{RHom}_R(-,-)$, and $- \otimes^{\mathrm{L}}_R -$ is the left derived functor of the tensor product functor for R-complexes. For R-complexes X, Y and $i \in \mathbb{Z}$, let

$$\mathrm{Ext}^i_R(X, Y) := \mathrm{H}_{-i}(\mathrm{RHom}_R(X, Y)), \quad \mathrm{Tor}^R_i(X, Y) := \mathrm{H}_i(X \otimes^{\mathrm{L}}_R Y).$$

We need the next convergent spectral sequences, which are very useful in this article.

Lemma 2.1 *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R with $\mathfrak{b} \subseteq \mathfrak{a}$. For an R-complex $X \in \mathrm{D}_{\square}(R)$, there exists a third quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Ext}^{-p}_R(R/\mathfrak{a}, \mathrm{H}^{-q}_{\mathfrak{b}}(X)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{-p-q}_R(R/\mathfrak{a}, X).$$

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In particular, one has two spectral sequences

$$E^2_{p,q} = \mathrm{Ext}^{-p}_R(R/\mathfrak{a}, \mathrm{H}_q(X)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{-p-q}_R(R/\mathfrak{a}, X),$$

$$E^2_{p,q} = \mathrm{Ext}^{-p}_R(R/\mathfrak{a}, \mathrm{H}^{-q}_{\mathfrak{a}}(X)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{-p-q}_R(R/\mathfrak{a}, X).$$

Proof This follows by the isomorphisms $\mathrm{R}\Gamma_{\mathfrak{b}}(\mathrm{RHom}_R(R/\mathfrak{a}, X)) \simeq \mathrm{RHom}_R(R/\mathfrak{a}, \mathrm{R}\Gamma_{\mathfrak{b}}(X)) \simeq \mathrm{RHom}_R(R/\mathfrak{a}, X)$. ∎

Lemma 2.2 *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R with $\mathfrak{b} \subseteq \mathfrak{a}$. For an R-complex $X \in \mathrm{D}_{\sqsupset}(R)$, there exists a first quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^R_p(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{b}}_q(X)) \underset{p}{\Longrightarrow} \mathrm{Tor}^R_{p+q}(R/\mathfrak{a}, X).$$

In particular, one has two spectral sequences:

$$E^2_{p,q} = \mathrm{Tor}^R_p(R/\mathfrak{a}, \mathrm{H}_q(X)) \underset{p}{\Longrightarrow} \mathrm{Tor}^R_{p+q}(R/\mathfrak{a}, X),$$

$$E^2_{p,q} = \mathrm{Tor}^R_p(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_q(X)) \underset{p}{\Longrightarrow} \mathrm{Tor}^R_{p+q}(R/\mathfrak{a}, X).$$

Proof It follows by the isomorphisms $\mathrm{L}\Lambda^{\mathfrak{b}}(R/\mathfrak{a} \otimes^{\mathrm{L}}_R X) \simeq R/\mathfrak{a} \otimes^{\mathrm{L}}_R \mathrm{L}\Lambda^{\mathfrak{b}}(X) \simeq R/\mathfrak{a} \otimes^{\mathrm{L}}_R X$. ∎

Recall that a *Serre subcategory* $\mathcal{S}$ of the category of R-modules is a class such that for any short exact sequence $0 \to L \to M \to N \to 0$, M is in $\mathcal{S}$ if and only if L and N are in $\mathcal{S}$. The next lemma is frequently used through the article.

Lemma 2.3 *Let $\mathcal{S}$ be a Serre subcategory of R-modules, and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R with $\mathfrak{b} \subseteq \mathfrak{a}$, and let $X \in \mathrm{D}_{\sqsubset}(R)$ and $s \geqslant 0, t \geqslant -\sup X$ such that:*

(1) $\mathrm{Ext}^{s+t}_R(R/\mathfrak{a}, X)$ *is in* $\mathcal{S}$,
(2) $\mathrm{Ext}^{s+1+i}_R(R/\mathfrak{a}, \mathrm{H}^{t-i}_{\mathfrak{b}}(X))$ *is in* $\mathcal{S}$ *for all* $1 \leqslant i \leqslant t + \sup X$,
(3) $\mathrm{Ext}^{s-1-i}_R(R/\mathfrak{a}, \mathrm{H}^{t+i}_{\mathfrak{b}}(X))$ *is in* $\mathcal{S}$ *for all* $1 \leqslant i \leqslant s-1$.

Then the R-module $\mathrm{Ext}^s_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{b}}(X))$ belongs to $\mathcal{S}$.

Proof Consider the first spectral sequence in Lemma 2.1,

$$E^2_{p,q} = \mathrm{Ext}^{-p}_R(R/\mathfrak{a}, \mathrm{H}^{-q}_{\mathfrak{b}}(X)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{-p-q}_R(R/\mathfrak{a}, X).$$

For $s = 0$, there is a finite filtration

$$0 = U^{-t-\sup X-1} \subseteq U^{-t-\sup X} \subseteq \cdots \subseteq U^0 = \mathrm{Ext}^t_R(R/\mathfrak{a}, X),$$

such that $U^p/U^{p-1} \cong E^\infty_{p,-t-p}$ for $p + \sup X \geqslant -t$. Let $r \geqslant 2$. Consider the differential

$$0 = E^r_{r,-t-r+1} \xrightarrow{d^r_{r,-t-r+1}} E^r_{0,-t} \xrightarrow{d^r_{0,-t}} E^r_{-r,-t+r-1}.$$

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We obtain the following short exact sequence:

$$0 \to E_{0,-t}^{r+1} \to E_{0,-t}^{r} \to \mathrm{im}d_{0,-t}^{r} \to 0.$$

Since $-t+r-1 \geqslant -t+1$ and $E_{-r,-t+r-1}^{r}$ is a subquotient of $E_{-r,-t+r-1}^{2}$, it follows that $\mathrm{im}d_{0,-t}^{r} \in \mathcal{S}$ for $r \geqslant 2$. Since $\sup X$ is finite, $E_{0,-t}^{r} \cong E_{0,-t}^{\infty} \cong U^{0}/U^{-1} \in \mathcal{S}$ for $r \gg 0$. By using the above sequence inductively, one has $\mathrm{Hom}_{R}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{b}}^{t}(X)) \cong E_{0,-t}^{2} \in \mathcal{S}$. For $s = 1$, there exists a finite filtration

$$0 = U^{-t-\sup X-2} \subseteq U^{-t-\sup X-1} \subseteq \cdots \subseteq U^{0} = \mathrm{Ext}_{R}^{t+1}(R/\mathfrak{a}, X),$$

so that $U^{p}/U^{p-1} \cong E_{p,-t-1-p}^{\infty}$ for $p + \sup X \geqslant -t-1$. Let $r \geqslant 2$. Consider the differential

$$0 = E_{-1+r,-t-r+1}^{r} \xrightarrow{d_{-1+r,-t-r+1}^{r}} E_{-1,-t}^{r} \xrightarrow{d_{-1,-t}^{r}} E_{-1-r,-t+r-1}^{r}.$$

As $-t+r-1 \geqslant -t+1$, it follows that $\mathrm{im}d_{-1,-t}^{r} \in \mathcal{S}$ for $r \geqslant 2$. Since $E_{-1,-t}^{r} \cong E_{-1,-t}^{\infty} \cong U^{-1}/U^{-2} \in \mathcal{S}$ for $r \gg 0$, the short exact sequence

$$0 \to E_{-1,-t}^{r+1} \to E_{-1,-t}^{r} \to \mathrm{im}d_{-1,-t}^{r} \to 0$$

implies that $\mathrm{Ext}_{R}^{1}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{b}}^{t}(X)) \cong E_{-1,-t}^{2} \in \mathcal{S}$. For $s \geqslant 2$, consider the filtration

$$0 = U^{-s-t-\sup X-1} \subseteq U^{-s-t-\sup X} \subseteq \cdots \subseteq U^{0} = \mathrm{Ext}_{R}^{s+t}(R/\mathfrak{a}, X),$$

where $U^{p}/U^{p-1} \cong E_{p,-s-t-p}^{\infty}$ for $-s-t \leqslant p + \sup X$. Let $r \geqslant 2$. Consider the differential

$$E_{-s+r,-t-r+1}^{r} \xrightarrow{d_{-s+r,-t-r+1}^{r}} E_{-s,-t}^{r} \xrightarrow{d_{-s,-t}^{r}} E_{-s-r,-t+r-1}^{r}.$$

As $E_{-s-r,-t+r-1}^{r} = 0$ for $r \geqslant t + \sup X + 2$ and $E_{-s+r,-t-r+1}^{r} = 0$ for $r \geqslant s+1$, it follows from the conditions that $\mathrm{im}d_{-s+r,-t-r+1}^{r}$ and $\mathrm{im}d_{-s,-t}^{r}$ are in $\mathcal{S}$ for $r \geqslant 2$. Let $r \geqslant s+1$. Then $E_{-s+r,-t-r+1}^{r} = 0$. So we have a short exact sequence

$$0 \to E_{-s,-t}^{r+1} \to E_{-s,-t}^{r} \to \mathrm{im}d_{-s,-t}^{r} \to 0.$$

Since $E_{-s,-t}^{r} \cong E_{-s,-t}^{\infty} \cong U^{-s}/U^{-s-1} \in \mathcal{S}$ for $r \gg 0$, it follows from the above sequence that $E_{-s,-t}^{s+1} \in \mathcal{S}$. Thus the following exact sequence

$$0 \to \mathrm{im}d_{0,-t-s+1}^{s} \to \mathrm{ker}d_{-s,-t}^{s} \to E_{-s,-t}^{s+1} \to 0$$

implies that $\mathrm{ker}d_{-s,-t}^{s} \in \mathcal{S}$, and the next exact sequence

$$0 \to \mathrm{ker}d_{-s,-t}^{s} \to E_{-s,-t}^{s} \to \mathrm{im}d_{-s,-t}^{s} \to 0$$

yields that $E_{-s,-t}^{s} \in \mathcal{S}$. By repeating this process, we obtain that $\mathrm{Ext}_{R}^{s}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{b}}^{t}(X)) \cong E_{-s,-t}^{2} \in \mathcal{S}$. The proof is complete. ∎

Using the first quadrant spectral in Lemma 2.2, by similar arguments, one can prove the following dual result.

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Lemma 2.4 *Let $\mathcal{S}$ be a Serre subcategory of R-modules, and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R with $\mathfrak{b} \subseteq \mathfrak{a}$, and let $X \in \mathrm{D}_{\sqsupset}(R)$ and $s \geqslant 0, t \geqslant \inf X$ such that:*

(1) $\mathrm{Tor}^R_{s+t}(R/\mathfrak{a}, X)$ *is in* $\mathcal{S}$,
(2) $\mathrm{Tor}^R_{s+1+i}(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{b}}_{t-i}(X))$ *is in* $\mathcal{S}$ *for all* $1 \leqslant i \leqslant t - \inf X$,
(3) $\mathrm{Tor}^R_{s-1-i}(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{b}}_{t+i}(X))$ *is in* $\mathcal{S}$ *for all* $1 \leqslant i \leqslant s - 1$.

Then the R-module $\mathrm{Tor}^R_s(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{b}}_t(X))$ belongs to $\mathcal{S}$.

3 Cofiniteness of complexes

Let d be a positive integer, and let $\mathfrak{a}$ be an ideal of R such that either $\dim R = d$ or $\dim R/\mathfrak{a} = d - 1$ or $\mathrm{ara}(\mathfrak{a}) = d - 1$. By providing criterions for determining cofiniteness, we answer Hartshorne's Question 3 in $d \leqslant 2$ and $d \geqslant 3$, respectively, in this section.

We begin with the first main theorem of this section, which answers Hartshorne's Question 3 for complexes in $\mathrm{D}_{\sqsubset}(R)$ under some conditions.

Theorem 3.1 *Let $\mathfrak{a}$ be a proper ideal of R and $X \in \mathrm{D}_{\sqsubset}(R)$. If $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$, then X is $\mathfrak{a}$-cofinite. The converse holds when either of the following holds:*

(1) $\dim R/\mathfrak{a} \leqslant 1$.
(2) $\mathrm{ara}(\mathfrak{a}) \leqslant 1$.
(3) $\dim R \leqslant 2$.
(4) $\dim_R \mathrm{H}_i(X) \leqslant 1$ *for every* $i \in \mathbb{Z}$.
(5) $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$.
(6) $X \in \mathrm{D}_{\square}(R)$ *and* $\mathrm{hd}(\mathfrak{a}, \mathrm{H}_i(X)) \leqslant 1$ *for* $i \in \mathbb{Z}$.
(7) $\mathrm{cd}(\mathfrak{a}, R) \leqslant 1$ *and* $X \in \mathrm{D}_{\square}(R)$.

Proof The first statement follows from [10, Theorem 3.5] or [12, Lemma 2.4.4].

If $\dim R/\mathfrak{a} \leqslant 1$, then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j = 0, 1$ by [19, Theorem 2.3]. If $\mathrm{ara}(\mathfrak{a}) \leqslant 1$, then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j = 0, 1$ by [12, Lemma 2.2.13]. If $\dim R \leqslant 2$, then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j = 0, 1$ by [20, Corollary 2.4]. If $\dim_R \mathrm{H}_i(X) \leqslant 1$, then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j = 0, 1$ by [3, Proposition 2.6]. If $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$, then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j = 0, 1$. Assume that either of the conditions (1)–(5) holds. Set $\mathfrak{b} = 0$ and $s = 0, 1$ in Lemma 2.3. We use induction on $t \geqslant -\sup X$ to prove that $\mathrm{H}^t(X) = \mathrm{H}_{-t}(X)$ is $\mathfrak{a}$-cofinite. The case $t = -\sup X$ is obvious by Lemma 2.3 and the preceding results. Suppose $t > -\sup X$. Then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i > -t$ by the induction, so $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_{-t}(X))$ is finitely generated for $j = 0, 1$ by Lemma 2.3. Thus, $\mathrm{H}_{-t}(X)$ is $\mathfrak{a}$-cofinite for $t \in \mathbb{Z}$ by the preceding results.

Assume that (6) holds true. By [12, Corollary 2.2.17], $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{Tor}^R_j(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for $j = 0, 1$. As X is $\mathfrak{a}$-cofinite, it follows from [12, Lemma 2.4.2] that $\mathrm{Tor}^R_i(R/\mathfrak{a}, X)$ is finitely generated for all i. Set $\mathfrak{b} = 0$ and $s = 0, 1$ in Lemma 2.4. By induction on $i \geqslant \inf X$, one has $\mathrm{Tor}^R_j(R/\mathfrak{a}, \mathrm{H}_i(X))$ is

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finitely generated for $j = 0, 1$. Thus, $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i \in \mathbb{Z}$. If $\mathrm{cd}(\mathfrak{a}, R) \leqslant 1$, then $\mathrm{hd}(\mathfrak{a}, \mathrm{H}_i(X)) \leqslant 1$ by [13, Corollary 3.2], it follows by (6) that $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i \in \mathbb{Z}$. ∎

The next lemma gives a criterion of $\mathfrak{a}$-cofiniteness for complexes in $\mathrm{D}_{\sqsupset}(R)$.

Lemma 3.2 *Let $\mathfrak{a}$ be a proper ideal of R with $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$ (e.g., R is a hereditary ring). For an R-complex $X \in \mathrm{D}_{\sqsupset}(R)$, the following statements are equivalent:*

(1) *X is $\mathfrak{a}$-cofinite;*
(2) *$\mathrm{Supp}_R X \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Tor}_j^R(R/\mathfrak{a}, X)$ is finitely generated for all j;*
(3) *$\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$.*

Proof As $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$, one can choose a projective resolution $0 \to P_1 \to P_0 \to R/\mathfrak{a} \to 0$. Set $P : 0 \to P_1 \to P_0 \to 0$. Then, $\mathrm{H}_0(P) = R/\mathfrak{a}$ and $\mathrm{H}_1(P) = 0$.

(1) ⇒ (2) By [21, Theorem 10.85], for $i \in \mathbb{Z}$, one has a Künneth short exact sequence

$$0 \to \mathrm{Ext}_R^1(\mathrm{H}_0(P), \mathrm{H}_{i+1}(X)) \to \mathrm{H}_i(\mathrm{Hom}_R(P, X)) \to \mathrm{Hom}_R(\mathrm{H}_0(P), \mathrm{H}_i(X)) \to 0.$$

Since $\mathrm{H}_n(\mathrm{Hom}_R(P, X))$ is finitely generated for all n, it follows from the above sequence that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ and $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_i(X))$ are finitely generated. Thus, $\mathrm{Tor}_j^R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated for all i, j by [12, Corollary 2.2.17] as $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$. For $j \geqslant \inf X$, consider the second spectral sequence in Lemma 2.2,

$$E^2_{p,q} = \mathrm{Tor}_p^R(R/\mathfrak{a}, \mathrm{H}_q(X)) \underset{p}{\Longrightarrow} \mathrm{Tor}_{p+q}^R(R/\mathfrak{a}, X).$$

There exists a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^{j-\inf X} = \mathrm{Tor}_j^R(R/\mathfrak{a}, X),$$

such that $U^p/U^{p-1} \cong E^\infty_{p,j-p}$ for $p \leqslant j - \inf X$. As $E^\infty_{p,j-p}$ is a subquotient of $E^2_{p,j-p}$, $E^\infty_{p,j-p}$ is finitely generated for $0 \leqslant p \leqslant j$. A successive use of the short exact sequence

$$0 \to U^{p-1} \to U^p \to U^p/U^{p-1} \to 0$$

implies that $\mathrm{Tor}_j^R(R/\mathfrak{a}, X)$ is finitely generated.

(2) ⇒ (3) As $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$, it follows by [12, Corollary 2.2.17] that $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $R/\mathfrak{a} \otimes_R \mathrm{H}_i(X)$ and $\mathrm{Tor}_1^R(R/\mathfrak{a}, \mathrm{H}_i(X))$ are finitely generated. Set $\mathfrak{b} = 0$ and $s = 0, 1$ in Lemma 2.4. We use induction on $t \geqslant \inf X$ to prove that $\mathrm{H}_t(X)$ is $\mathfrak{a}$-cofinite. The case $t = \inf X$ is obvious by Lemma 2.4. Suppose that $t > \inf X$. Then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i < t$ by the induction, so $R/\mathfrak{a} \otimes_R \mathrm{H}_t(X)$ and $\mathrm{Tor}_1^R(R/\mathfrak{a}, \mathrm{H}_t(X))$ are finitely generated by Lemma 2.4. Hence $\mathrm{H}_t(X)$ is $\mathfrak{a}$-cofinite for $t \in \mathbb{Z}$.

(3) ⇒ (1) By [21, Theorem 10.85], for $n \in \mathbb{Z}$, one has a Künneth short exact sequence

$$0 \to \mathrm{Ext}_R^1(\mathrm{H}_0(P), \mathrm{H}_{n+1}(X)) \to \mathrm{H}_n(\mathrm{Hom}_R(P, X)) \to \mathrm{Hom}_R(\mathrm{H}_0(P), \mathrm{H}_n(X)) \to 0.$$

As $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_i(X)), \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ are finitely generated for all i, it follows from the above sequence that $\mathrm{Ext}_R^n(R/\mathfrak{a}, X)$ is finitely generated, and hence X is $\mathfrak{a}$-cofinite. ∎

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A local ring is *Cohen–Macaulay* if some system of parameters is a regular sequence. A nonzero finitely generated R-module M is *perfect* if $\mathrm{pd}_R M = \min\{i \mid \mathrm{Ext}^i_R(M,R) \neq 0\}$. An ideal $\mathfrak{a}$ is called *perfect* if $R/\mathfrak{a}$ is a perfect R-module. A semi-injective R-complex I is *minimal* if and only if the zero complex is the only acyclic subcomplex of I. By [8, Appendix B.26], every complex has a minimal semi-injective resolution.

The following theorem answers Hartshorne's Question 3 completely.

Theorem 3.3 *Let $\mathfrak{a}$ be a proper ideal of R. Suppose that either of the next conditions holds:*

(1) $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$.
(2) $\mathfrak{a}$ *is a principal ideal generated by a nonzero divisor.*
(3) $(R,\mathfrak{m})$ *is local Cohen–Macaulay with* $\dim R \leqslant 2$ *and* $\mathfrak{a}$ *is perfect so that* $\mathrm{pd}_R R/\mathfrak{a} < \infty$.
(4) $(R,\mathfrak{m})$ *is local Cohen–Macaulay and* $\mathfrak{a}$ *is perfect so that* $\dim R/\mathfrak{a} \leqslant 1$ *and* $\mathrm{pd}_R R/\mathfrak{a} < \infty$.

Then an R-complex $X \in \mathrm{D}(R)$ is $\mathfrak{a}$-cofinite if and only if every $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite.

Proof Assume that (1) holds true. For any n, set $X_{>n} : \ldots \to X_{n+1} \to \mathrm{Im}d_{n+1} \to 0$ and $X_{\leqslant n} : 0 \to X_n/\mathrm{Im}d_{n+1} \to X_{n-1} \to \ldots$. We have a short exact sequence of R-complexes

$$(*) \qquad 0 \to X_{>n} \to X \to X_{\leqslant n} \to 0$$

with $X_{>n} \in \mathrm{D}_{\sqsupset}(R)$ and $X_{\leqslant n} \in \mathrm{D}_{\sqsubset}(R)$. If $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i \in \mathbb{Z}$, then $X_{\leqslant n}$ is $\mathfrak{a}$-cofinite by Theorem 3.1 and $X_{>n}$ is $\mathfrak{a}$-cofinite by Lemma 3.2. Thus the long exact sequence

$$\cdots \to \mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X_{>n})) \to \mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X)) \to \mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X_{\leqslant n})) \to \cdots$$

implies that $\mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X))$ is finitely generated for all i, so X is $\mathfrak{a}$-cofinite. Conversely, let X be $\mathfrak{a}$-cofinite and $P \xrightarrow{\simeq} R/\mathfrak{a}$ a projective resolution with $P : 0 \to P_1 \to P_0 \to 0$. As $\mathrm{H}_1(P) = 0$, [21, Theorem 10.85] yields a short exact sequence

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_0(P), \mathrm{H}_{i+1}(X)) \to \mathrm{H}_i(\mathrm{Hom}_R(P,X)) \to \mathrm{Hom}_R(\mathrm{H}_0(P), \mathrm{H}_i(X)) \to 0$$

for $i \in \mathbb{Z}$. Since $\mathrm{H}_n(\mathrm{Hom}_R(P,X))$ is finitely generated for all n, it follows from the above sequence that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ are finitely generated. Hence $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i \in \mathbb{Z}$ as $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X)) = 0$ for $j \geqslant 2$.

If the condition (2) holds true, then $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$, as claimed.

Assume that $\mathrm{pd}_R R/\mathfrak{a} < \infty$. Let $\mathrm{H}_i(X)$ be $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$. Apply the functor $\mathrm{RHom}_R(R/\mathfrak{a}, -)$ to the sequence $(*)$ to get an exact triangle

$$\mathrm{RHom}_R(R/\mathfrak{a}, X_{>n}) \to \mathrm{RHom}_R(R/\mathfrak{a}, X) \to \mathrm{RHom}_R(R/\mathfrak{a}, X_{\leqslant n}) \rightsquigarrow .$$

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For $i \in \mathbb{Z}$, choose $n = i + \mathrm{pd}_R R/\mathfrak{a}$. From the long exact cohomology sequence, we have

$$\begin{aligned}\mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X_{>n})) \to \mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X)) \to \mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X_{\leqslant n}))\\ \to \mathrm{H}_{i-1}(\mathrm{RHom}_R(R/\mathfrak{a}, X_{>n})).\end{aligned}$$

As $\inf X_{>n} > i + \mathrm{pd}_R R/\mathfrak{a}$, $\inf \mathrm{RHom}_R(R/\mathfrak{a}, X_{>n}) \geqslant -\mathrm{pd}_R R/\mathfrak{a} + \inf X_{>n} > i$ by [8, Theorem 8.1.8], so $\mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X_{>n})) = 0 = \mathrm{H}_{i-1}(\mathrm{RHom}_R(R/\mathfrak{a}, X_{>n}))$. If either $\dim R \leqslant 2$ or $\dim R/\mathfrak{a} \leqslant 1$, then $X_{\leqslant n}$ is $\mathfrak{a}$-cofinite by Theorem 3.1, it implies that

$$\mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X)) \cong \mathrm{H}_i(\mathrm{RHom}_R(R/\mathfrak{a}, X_{\leqslant n}))$$

is finitely generated. Thus X is $\mathfrak{a}$-cofinite.

Assume that (3) holds true. Let X be $\mathfrak{a}$-cofinite and $\dim R \leqslant 2$. Then $\mathrm{pd}_R R/\mathfrak{a} \leqslant \dim R \leqslant 2$. If $\mathrm{pd}_R R/\mathfrak{a} \leqslant 1$, then $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i \in \mathbb{Z}$ by (1). Assume that $\mathrm{pd}_R R/\mathfrak{a} = 2$. In this case, $\dim R/\mathfrak{a} = 0$ by the Auslander–Buchsbaum formula as $R/\mathfrak{a}$ is Cohen–Macaualy by [7, Theorem 2.1.5]. So there exists a finite filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = R/\mathfrak{a}$$

such that $N_j/N_{j-1} \cong R/\mathfrak{m}$ for $1 \leqslant j \leqslant r$. Let $X \xrightarrow{\simeq} I$ be a minimal semi-injective resolution. For $i \in \mathbb{Z}$, the exact sequence $0 \to R/\mathfrak{m} \to N_2 \to R/\mathfrak{m} \to 0$ yields a short exact sequence

$$0 \to \mathrm{Hom}_R(R/\mathfrak{m}, I_i) \to \mathrm{Hom}_R(N_2, I_i) \to \mathrm{Hom}_R(R/\mathfrak{m}, I_i) \to 0.$$

As $\mathrm{Hom}_R(R/\mathfrak{m}, I)$ has zero differentials by [8, Lemma 8.2.16] and $I \simeq X$ is $\mathfrak{a}$-cofinite, it follows by [9, Corollary 1] that $\mathrm{Hom}_R(R/\mathfrak{m}, I_i) \cong \mathrm{H}_i(\mathrm{Hom}_R(R/\mathfrak{m}, I))$ is finitely generated, so is $\mathrm{Hom}_R(N_2, I_i)$ by the above sequence. Continuing this process, we see that $\mathrm{Hom}_R(R/\mathfrak{a}, I_i)$ is finitely generated. Hence I_i is Artinian $\mathfrak{a}$-cofinite by [18, Proposition 4.1] for $i \in \mathbb{Z}$. Note that the subcategory of Artinian $\mathfrak{a}$-cofinite R-modules is Serre, one has $\mathrm{H}_i(X) \cong \mathrm{H}_i(I)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$, as desired.

Assume that (4) holds true. Let X be $\mathfrak{a}$-cofinite. We may assume that $\dim R/\mathfrak{a} = 1$. By [7, Theorem 2.1.5], $R/\mathfrak{a}$ is Cohen–Macaualy, it follows by [7, Theorems 2.1.2(a) and 2.1.5(a)] that $\mathrm{Ass}_R R/\mathfrak{a} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$, where $\mathfrak{p}_j \in \mathrm{V}(\mathfrak{a})$ with $\dim R/\mathfrak{p}_j = 1$ for $j = 1, \ldots, s$. By Prime Avoidance Theorem, there exists $x \in \mathfrak{m} \backslash (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$ that is $R/\mathfrak{a}$-regular and $\dim R/(\mathfrak{a} + xR) = 0$. As $R/(\mathfrak{a} + xR) \otimes^{\mathrm{L}}_R R/xR \in \mathrm{D}^{\mathrm{f}}_{\square}(R)$, $\mathrm{Supp}_R(R/(\mathfrak{a} + xR) \otimes^{\mathrm{L}}_R R/xR) = \mathrm{Supp}_R(R/(\mathfrak{a} + xR)) \cap \mathrm{Supp}_R R/xR \subseteq \mathrm{V}(\mathfrak{a})$, it follows by [12, Lemma 2.4.2] that

$$\mathrm{RHom}_R(R/(\mathfrak{a} + xR), \mathrm{RHom}_R(R/xR, X)) \simeq \mathrm{RHom}_R(R/(\mathfrak{a} + xR) \otimes^{\mathrm{L}}_R R/xR, X) \in \mathrm{D}^{\mathrm{f}}(R).$$

Consider the projective resolution $Q : 0 \to R \xrightarrow{x} R \to 0$ of R/xR. As $\mathrm{H}_1(Q) = 0$, [21, Theorem 10.85] yields a short exact sequence

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_0(Q), \mathrm{H}_{i+1}(X)) \to \mathrm{H}_i(\mathrm{Hom}_R(Q, X)) \to \mathrm{Hom}_R(\mathrm{H}_0(Q), \mathrm{H}_i(X)) \to 0$$

for any i. Since each $\mathrm{H}_n(\mathrm{Hom}_R(Q, X))$ is Artinian $\mathfrak{a} + xR$-cofinite by the preceding proof, so is $\mathrm{Hom}_R(R/xR, \mathrm{H}_i(X))$, it follows by [9, Proposition 2]

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that $\mathrm{Hom}_R(R/xR, \mathrm{H}_i(X))$ is an $(\mathfrak{a}+xR)/xR$-cofinite R/xR-module. Thus, by [21, Corollary 10.65],

$$\mathrm{Ext}^j_{R/xR}(R/(\mathfrak{a}+xR), \mathrm{Hom}_R(R/xR, \mathrm{H}_i(X))) \cong \mathrm{Ext}^j_R(R/(\mathfrak{a}+xR), \mathrm{H}_i(X))$$

is finitely generated for $j \geqslant 0$. Consider the short exact sequence

$$0 \longrightarrow R/\mathfrak{a} \xrightarrow{x} R/\mathfrak{a} \longrightarrow R/(\mathfrak{a}+xR) \longrightarrow 0,$$

which induces the following exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(R/(\mathfrak{a}+xR), \mathrm{H}_i(X)) \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X)) \xrightarrow{x} \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X)) \\ &\to \mathrm{Ext}^1_R(R/(\mathfrak{a}+xR), \mathrm{H}_i(X)) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X)) \xrightarrow{x} \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X)) \\ &\to \mathrm{Ext}^2_R(R/(\mathfrak{a}+xR), \mathrm{H}_i(X)) \to \cdots. \end{aligned}$$

Then $(0 :_{\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))} x)$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))/x\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ and $(0 :_{\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X))} x)$, $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X))/x\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ are finitely generated, so $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ are $\mathfrak{a}$-cofinite by [18, Corollary 3.4]. Hence [20, Corollary 2.12] implies that $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for $i \in \mathbb{Z}$. ∎

A Noetherian local ring $(R, \mathfrak{m})$ is *regular* if it has a system of parameters generating $\mathfrak{m}$.

Corollary 3.4 *Let $(R, \mathfrak{m})$ be a regular local ring, and let $\mathfrak{a}$ be a proper ideal of R such that either $\dim R \leqslant 2$ and $\mathfrak{a}$ is perfect, or $\mathfrak{a}$ is perfect and $\dim R/\mathfrak{a} \leqslant 1$, or $\mathrm{ara}(\mathfrak{a}) \leqslant 1$, then an R-complex X is $\mathfrak{a}$-cofinite if and only if $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{Z}$.*

Proof As R is regular, R is Cohen–Macaulay and $\mathrm{pd}_R R/\mathfrak{a} < \infty$. If $\dim R \leqslant 2$ and $\mathfrak{a}$ is perfect, then the condition (3) in Theorem 3.3 holds true. If $\mathfrak{a}$ is perfect and $\dim R/\mathfrak{a} \leqslant 1$, then the condition (4) in Theorem 3.3 holds true. If $\mathrm{ara}(\mathfrak{a}) \leqslant 1$, then $\sqrt{\mathfrak{a}} = \sqrt{(x)}$ for some nonzero divisor $x \in R$, so $\mathfrak{a}' := (x)$ satisfies the condition (2) in Theorem 3.3. As $\mathrm{V}(\mathfrak{a}) = \mathrm{V}(\mathfrak{a}')$, X is $\mathfrak{a}$-cofinite if and only if X is $\mathfrak{a}'$-cofinite by [12, Lemma 2.4.2], $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite if and only if $\mathrm{H}_i(X)$ is $\mathfrak{a}'$-cofinite by [12, Corollary 2.2.17], as desired. ∎

Next, we give an answer of Hartshorne's Question 3 when $d \geqslant 3$. To do this, we first generalize the Melkersson's result [19, Theorem 2.3] and Bahmanpour, Naghipour, Sedghi's result [4, Theorem 3.5].

Lemma 3.5 *Let $\mathfrak{a}$ be a proper ideal of R with $\dim R/\mathfrak{a} = d \geqslant 0$. Then an $\mathfrak{a}$-torsion R-module M is $\mathfrak{a}$-cofinite if and only if $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for $i \leqslant d$.*

Proof "Only if" part is trivial.

"If" part. If $d \leqslant 1$, then the assertion holds by [19, Theorem 2.3]. Now let $d \geqslant 2$, and suppose that the result holds for $d-1$. Let $a = (a_1, \ldots, a_t)$. For $s \geqslant 1$, there exists an exact sequence of R-modules

$$0 \to (0 :_M \mathfrak{a}) \to (0 :_M \mathfrak{a}^s) \xrightarrow{f} a_1(0 :_M \mathfrak{a}^s) \oplus \cdots \oplus a_t(0 :_M \mathfrak{a}^s),$$

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where $f(x) = (a_1x, \dots, a_tx)$. As $a_i(0 :_M \mathfrak{a}^s)$ is a submodule of $(0 :_M \mathfrak{a}^{s-1})$ for each i, $(0 :_M \mathfrak{a}^s)$ is finitely generated by induction on s. If $\mathfrak{a}$ is nilpotent, say $\mathfrak{a}^n = 0$ for some integer n, then $M = (0 :_M \mathfrak{a}^n)$ is finitely generated, and so M is $\mathfrak{a}$-cofinite. Now suppose that $\mathfrak{a}$ is not nilpotent. We can choose $n > 0$ such that $(0 :_R \mathfrak{a}^n) = \Gamma_{\mathfrak{a}}(R)$. Set $\bar{R} = R/\Gamma_{\mathfrak{a}}(R)$, $\bar{M} = M/(0 :_M \mathfrak{a}^n)$ and $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\bar{R}$. Since $\Gamma_{\bar{\mathfrak{a}}}(\bar{R}) = 0$, $\bar{\mathfrak{a}}$ contains a nonzero divisor of $\bar{R}$, and so $\dim R/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(R)) = \dim \bar{R}/\bar{\mathfrak{a}} \leqslant d - 1$. The assumption on M and [9, Proposition 1] imply that $\operatorname{Ext}^i_R(R/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(R)), M)$ is finitely generated for $i \leqslant d$. Note that $(0 :_M \mathfrak{a}^n)$ is finitely generated, it follows from the exact sequence $0 \to (0 :_M \mathfrak{a}^n) \to M \to \bar{M} \to 0$ that $\operatorname{Ext}^i_R(R/(\mathfrak{a} + \Gamma_{\mathfrak{a}}(R)), \bar{M})$ is finitely generated for $i \leqslant d$. By induction, $\bar{M}$ is $\mathfrak{a} + \Gamma_{\mathfrak{a}}(R)$-cofinite since $\operatorname{Supp}_R \bar{M} \subseteq \mathrm{V}(\mathfrak{a} + \Gamma_{\mathfrak{a}}(R))$, and hence $\bar{M}$ is $\mathfrak{a}$-cofinite by [9, Proposition 2]. Note that $(0 :_M \mathfrak{a}^n)$ is $\mathfrak{a}$-cofinite, it follows by the above sequence that M is $\mathfrak{a}$-cofinite. ∎

Lemma 3.6 *If* $\dim R = d \geqslant 1$*, then an* $\mathfrak{a}$*-torsion* R*-module* M *is* $\mathfrak{a}$*-cofinite if and only if* $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is finitely generated for* $i \leqslant d - 1$*.*

Proof For any ideal $\mathfrak{b}$ with $\dim R/\mathfrak{b} \leqslant d - 1$, it follows by Lemma 3.5 that an $\mathfrak{b}$-torsion R-module N is $\mathfrak{b}$-cofinite if and only if $\operatorname{Ext}^i_R(R/\mathfrak{b}, N)$ is finitely generated for $i \leqslant d - 1$. Hence the statement follows from [20, Theorem 2.3]. ∎

Theorem 3.7 *Let* $d \geqslant 3$ *be an integer, and let* $\mathfrak{a}$ *be an ideal of* R *such that either* $\dim R = d$*, or* $\dim R/\mathfrak{a} = d - 1$*, or* $\operatorname{ara}(\mathfrak{a}) = d - 1$*. Then an* R*-complex* $X \in \mathrm{D}_{\sqsubset}(R)$ *is* $\mathfrak{a}$*-cofinite and* $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ *is finitely generated for all* $j \leqslant d - 2$ *and* $i \in \mathbb{Z}$ *if and only if each* $\mathrm{H}_i(X)$ *is* $\mathfrak{a}$*-cofinite.*

Proof "If" part follows from [12, Lemma 2.4.4].

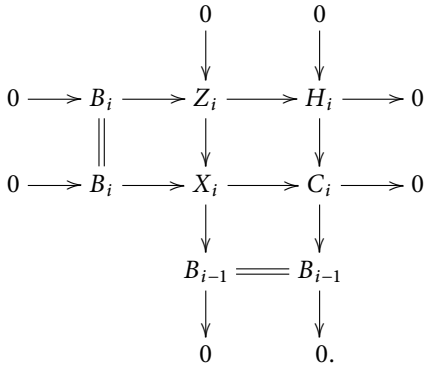
"Only if" part. Using induction on $i \leqslant \sup X$ to show that $\operatorname{Ext}^{d-1}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated. As $\mathrm{H}_i(X) = \mathrm{H}^{-i}(X)$, set $\mathfrak{b} = \{0\}$, $s = d - 1$ and $t = -i$ in Lemma 2.3. If $i = \sup X$, then $\operatorname{Ext}^{d-1}_R(R/\mathfrak{a}, \mathrm{H}_{\sup X}(X))$ is finitely generated by Lemma 2.3. Suppose that $i \leqslant \sup X - 1$ and that the result has been proved for larger values of i. Then $\mathrm{H}_{i+k}(X)$ is $\mathfrak{a}$-cofinite for $k \geqslant 1$ by the induction. As $\operatorname{Ext}^{d+k}_R(R/\mathfrak{a}, \mathrm{H}_{i+k}(X))$ is finitely generated for $1 \leqslant k \leqslant -i + \sup X$ and $\operatorname{Ext}^{d-2-k}_R(R/\mathfrak{a}, \mathrm{H}_{i-k}(X))$ is finitely generated for $1 \leqslant k \leqslant d - 2$, it follows by Lemma 2.3 that $\operatorname{Ext}^{d-1}_R(R/\mathfrak{a}, \mathrm{H}_i(X))$ is finitely generated. Hence, Lemmas 3.5 and 3.6 and [12, Corollary 2.2.13] imply that $\mathrm{H}_i(X)$ is $\mathfrak{a}$-cofinite, as desired. ∎

Next, we give a necessary and sufficient condition under which the homology modules of a complex are $\mathfrak{a}$-cofinite, which is a more general version of [20, Theorem 2.8].

Proposition 3.8 *Let* $d \geqslant 3$ *be a positive integer, and let* $\mathfrak{a}$ *be a proper ideal of* R *such that either* $\dim R = d$*, or* $\dim R/\mathfrak{a} = d - 1$*, or* $\operatorname{ara}(\mathfrak{a}) = d - 1$*, and let* X *be a complex of* $\mathfrak{a}$*-cofinite* R*-modules. Then* $\mathrm{H}_i(X)$ *are* $\mathfrak{a}$*-cofinite for all* $i \in \mathbb{Z}$ *if and only if* $\operatorname{Ext}^j_R(R/\mathfrak{a}, \operatorname{coker} d_i)$ *is finitely generated for all* $j \leqslant d - 2$ *and* $i \in \mathbb{Z}$*, where* $d_i : X_i \to X_{i-1}$*.*

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Proof Set $B_i = \mathrm{im}d_{i+1}$, $Z_i = \ker d_i$, $C_i = \mathrm{coker}d_{i+1}$ and $H_i = \mathrm{H}_i(X)$. We have the following commutative diagram:



For any i, if $\mathrm{Ext}^j_R(R/\mathfrak{a}, C_i)$ is finitely generated for $j \leqslant d-2$, then $\mathrm{Ext}^j_R(R/\mathfrak{a}, B_i)$ is finitely generated for $j \leqslant d-1$ by the second row. Hence, B_i is $\mathfrak{a}$-cofinite by Lemmas 3.5 and 3.6 and [12, Corollary 2.2.13]. As each X_i is $\mathfrak{a}$-cofinite, the second column implies that Z_i is $\mathfrak{a}$-cofinite, and then H_i is $\mathfrak{a}$-cofinite by the first row. Conversely, let each H_i be $\mathfrak{a}$-cofinite. For $j \geqslant 0$, the second row and the third column induce the long exact sequences:

(†)
$$\cdots \to \mathrm{Ext}^j_R(R/\mathfrak{a}, X_i) \to \mathrm{Ext}^j_R(R/\mathfrak{a}, C_i) \to \mathrm{Ext}^{j+1}_R(R/\mathfrak{a}, B_i) \to \mathrm{Ext}^{j+1}_R(R/\mathfrak{a}, X_i) \to \cdots,$$

(‡)
$$\cdots \to \mathrm{Ext}^j_R(R/\mathfrak{a}, H_i) \to \mathrm{Ext}^j_R(R/\mathfrak{a}, C_i) \to \mathrm{Ext}^j_R(R/\mathfrak{a}, B_{i-1}) \to \mathrm{Ext}^{j+1}_R(R/\mathfrak{a}, H_i) \to \cdots.$$

As each $\mathrm{Hom}_R(R/\mathfrak{a}, B_{i-1})$ is finitely generated, so is $\mathrm{Hom}_R(R/\mathfrak{a}, C_i)$ by (‡), and then each $\mathrm{Ext}^1_R(R/\mathfrak{a}, B_i)$ is finitely generated by (†) and each $\mathrm{Ext}^1_R(R/\mathfrak{a}, C_i)$ is finitely generated by (‡). Continuing this process, we obtain the desired statement. ■

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Ethical approval We certify that this article is original and has not been published and will not be submitted elsewhere for publication.

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