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On integrals involving classical polynomials

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1. We know ((6), p. 343) that

$$\int_0^\infty\int_0^\infty F(\alpha x+\beta y,\alpha' x+\beta' y)\,dx\,dy=\frac{1}{\kappa}\int_0^\infty\int_0^\infty F(u,v)\,du\,dv, \tag{1·1}$$

and

$$\int_{-\infty}^\infty\int_{-\infty}^\infty F(\alpha x+\beta y,\alpha' x+\beta' y)\,dx\,dy=\frac{1}{\kappa}\int_{-\infty}^\infty\int_{-\infty}^\infty F(u,v)\,du\,dv, \tag{1·2}$$

where α,β,α' and β' are parameters such that

$$\kappa=\begin{vmatrix}\alpha & \alpha'\\ \beta & \beta'\end{vmatrix}\neq 0. \tag{1·3}$$

Consider $$F(\alpha x+\beta y,\alpha' x+\beta' y)=f_1(\alpha x+\beta y)f_2(\alpha' x+\beta' y).$$

Then (1·1) and (1·2) are transformed to

$$\int_0^\infty\int_0^\infty f_1(\alpha x+\beta y)f_2(\alpha' x+\beta' y)\,dx\,dy=\frac{1}{\kappa}\int_0^\infty f_1(u)\,du\int_0^\infty f_2(v)\,dv \tag{1·4}$$

and

$$\int_{-\infty}^\infty\int_{-\infty}^\infty f_1(\alpha x+\beta y)f_2(\alpha' x+\beta' y)\,dx\,dy=\frac{1}{\kappa}\int_{-\infty}^\infty f_1(u)\,du\int_{-\infty}^\infty f_2(v)\,dv \tag{1·5}$$

respectively. From (1·4) and (1·5) we see that the double integrals are transformed to the products of two simple integrals. We shall now employ these results to evaluate certain integrals.

2. Consider

$$f_1(\alpha x+\beta y)=(\alpha x+\beta y)^{\mu-1}\exp\left[-p(\alpha x+\beta y)^{\gamma_1}\right]L_m^s(\alpha x+\beta y,\gamma,p),$$
$$f_2(\alpha' x+\beta' y)=(\alpha' x+\beta' y)^{\sigma-1}\exp\left[-q(\alpha' x+\beta' y)^{\gamma_2}\right]L_n^t(\alpha' x+\beta' y,\gamma_2,q).$$

Then by (1·4), we get

$$\int_0^\infty\int_0^\infty(\alpha x+\beta y)^{\mu-1}(\alpha' x+\beta' y)^{\sigma-1}\exp\left[-p(\alpha x+\beta y)^{\gamma_1}-q(\alpha' x+\beta' y)^{\gamma_2}\right]$$
$$\times L_m^s(\alpha x+\beta y,\gamma,p)\,L_n^t(\alpha' x+\beta' y,\gamma_2,q)\,dx\,dy$$
$$=\frac{1}{\kappa}\int_0^\infty u^{\mu-1}\exp\left(-pu^{\gamma_1}\right)L_m^s(u,\gamma_1,p)\,du\int_0^\infty v^{\sigma-1}\exp\left(-qv^{\gamma_2}\right)L_n^t(v,\gamma_2,q)\,dv,$$

which by using result (18) of ((5), p. 6)

$$=\frac{1}{\kappa}\left\{\frac{(-1)^{m+n}\Gamma(\mu-s)\,\Gamma(\mu/\gamma_1)}{m!\,\Gamma(\mu-s-m)\,\gamma_1\,(p)^{\mu/\gamma_1}}\times\frac{\Gamma(\sigma-k)\,\Gamma(\sigma/\gamma_2)}{n!\,\Gamma(\sigma-t-n)\,\gamma_2(q)^{\sigma/\gamma_2}}\right\}. \tag{2·1}$$

In particular, when $p=q=\gamma_1=\gamma_2=\mu=\sigma=1$, then

$$\int_0^\infty\int_0^\infty\exp\left[-(\alpha x+\beta y)-(\alpha' x+\beta' y)\right]L_m^s(\alpha x+\beta y)\,L_n^t(\alpha' x+\beta' y)\,dx\,dy$$
$$=\frac{(s)_m(t)_n}{\kappa.m!\,n!}\quad\Re(s)>0,\quad\Re(t)>0. \tag{2·2}$$

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Again, consider

$$f_1(\alpha x+\beta y) = (\alpha x+\beta y)^a \exp\left[-p(\alpha x+\beta y)^\gamma\right] L_n^a(\alpha x+\beta y, \gamma, p)\, L_m^b(\alpha x+\beta y, \delta, q),$$

$$f_2(\alpha'+\beta' y) = (\alpha' x+\beta' y)^c \exp\left[-r(\alpha' x+\beta' y)^\mu\right] L_\lambda^c(\alpha' x+\beta' y, \mu, r)\, L_\omega^a(\alpha' x+\beta' y, \sigma, s).$$

Then proceeding as above and using results (14) and (15) of ((5); p. 5), we get

$$\int_0^\infty\int_0^\infty (\alpha x+\beta y)^a (\alpha' x+\beta' y)^c \exp\left\{-p(\alpha x+\beta y)^\gamma - r(\alpha' x+\beta' y)^\mu\right\} L_n^a(\alpha x+\beta y, \gamma, p)$$
$$\times L_m^b(\alpha x+\beta y, \delta, q)\, L_\lambda^c(\alpha' x+\beta' y, \mu, r)\, L_w^d(\alpha'+\beta' y, \sigma, s)\, dx\, dy$$

$$= \frac{1}{\kappa} \frac{(-1)^{m\delta+m+\omega\sigma+\omega} (q\delta)^m (\delta\sigma)^\lambda \Gamma\left(\dfrac{a+m\delta+1}{\gamma}\right) \Gamma\left(\dfrac{c+\omega\sigma+1}{\mu}\right)}{m!\,\omega!\,\gamma(p)\,\dfrac{a+m\delta+1}{\gamma}\,\mu(r)\,\dfrac{c+\omega\sigma+1}{\mu}} \quad \text{for} \quad n = m\delta, \lambda = \omega\sigma$$

$$= 0 \qquad \text{for} \quad n > m\delta \quad \text{or} \quad \lambda > \omega\sigma.$$

Similar results can be derived for generalized Hermite polynomials.

3. Now we derive some results involving derivatives of some functions. Recall (1·4), and let

$$f_1(\alpha x+\beta y) = (\alpha x+\beta y)^{a+1} \exp\left[-(\alpha x+\beta y)\right] \prod_{\epsilon=1}^{\kappa} D^{m_\epsilon} L_{n_\epsilon}^{a_\epsilon}(\alpha x+\beta y)\, DL_n^a(\alpha x+\beta y),$$

$$f_2(\alpha' x+\beta' y) = (\alpha' x+\beta' y)^{b+1} \exp\left[-(\alpha' x+\beta' y)\right] \prod_{\epsilon=1}^{l} D^{p_\epsilon} C_{q_\epsilon}^{\lambda}(\alpha' x+\beta' y)\, DL_q^b(\alpha' x+\beta' y).$$

Then

$$\int_0^\infty\int_0^\infty (\alpha x+\beta y)^{a+1} (\alpha'+\beta' y)^{b+1} \exp\left[-(\alpha x+\beta y)-(\alpha' x+\beta' y)\right] \prod_{\epsilon=1}^{\kappa} D^{m_\epsilon} L_{n_\epsilon}^{a_\epsilon}(\alpha x+\beta y)$$
$$\times DL_n^a(\alpha x+\beta y) \prod_{\epsilon=1}^{l} D^{\beta_\epsilon} C_{q_\epsilon}^{\lambda}(\alpha' x+\beta' y)\, DL_q^b(\alpha' x+\beta' y)\, dx\, dy$$

$$= \frac{1}{\kappa} \int_0^\infty u^{a+1} e^{-u} \prod_{\epsilon=1}^{\kappa} D^{m_\epsilon} L_{r_\epsilon}^{a_\epsilon}(u)\, DL_n^a(u)\, du \int_0^\infty v^{l+1} e^{-\nu} \prod_{\epsilon=1}^{l} D^{p_\epsilon} C_{q_\epsilon}^{\lambda}(v)\, DL_q^b(v)\, dv,$$

which with the help of (2·4) and (2·6) of ((3), p. 173)

$$= \frac{1}{\kappa} (-1)^{n_1+n_2+\ldots+n_k} \Gamma(1+a+n)\, \Gamma(1+b+p)$$
$$\times \left[\frac{\Gamma(\frac{1}{2})\,\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}\right]^l \prod_{\epsilon=1}^{\kappa} \left[\frac{(n_\epsilon-m_\epsilon+1)_{m_\epsilon}}{n_\epsilon!}\right] \prod_{\epsilon=1}^{l} \left[\frac{(2\lambda)_{p_\epsilon}(q_\epsilon-p_\epsilon+1)_{q_\epsilon}}{p_\epsilon!\,(\lambda+p_\epsilon)}\right], \quad (3\cdot1)$$

where $\sum_{\epsilon=1}^{\kappa}(n_\epsilon-m_\epsilon) = n-1; \sum_{\epsilon=1}^{l}(q_\epsilon-p_\epsilon) = q-1; a > -2, b > -2$ and κ is the same as (1·3). Similarly we can prove by (2·5) and (2·7) of ((3); p. 173)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-(\alpha x+\beta y)^2 - (\alpha' x+\beta' y)^2\right] \prod_{\epsilon=1}^{\kappa} D^{m_\epsilon} H_{n_\epsilon}(\alpha x+\beta y) \prod_{\epsilon=1}^{l}$$
$$\times D^{p_\epsilon} L_{q_\epsilon}^{\gamma_\epsilon}(\alpha' x+\beta' y)\, DH_n(\alpha x+\beta y).\, DH_q(\alpha' x+\beta' y)\, dx\, dy \quad (3\cdot2)$$

$$= \frac{4\pi}{\kappa} n!\, q!\, (-1)^{n_1+n_2+\ldots+n_k} \prod_{\epsilon=1}^{\kappa} \left[2^{n_\epsilon}(n_\epsilon-m_\epsilon+1)_{m_\epsilon}\right] \prod_{\epsilon=1}^{l} \left[\frac{(q_\epsilon-p_\epsilon+1)_{p_\epsilon}}{q_\epsilon!}\right],$$

where $\sum_{\epsilon=1}^{\kappa}(n_\epsilon-m_\epsilon) = n-1; \sum_{\epsilon=1}^{l}(q_\epsilon-p_\epsilon) = q-1.$

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4. *Bateman's, Rice's, Legendre's and Sister Celine's polynomials.* We follow the definitions of Bateman's polynomial $Z_n(x)$, Rice's polynomial, $H_n(a,b,x)$, Legendre polynomial $P_n(x)$, Sister Celine's polynomial $f_n(x)$ and

$$f_n\left[\begin{matrix}a_1,\ldots,a_p;x\\ b_1,\ldots,b_q\end{matrix}\right]=\frac{1}{\sqrt{\pi}}\int_0^\infty y^{-\frac{1}{2}}e^{-y}f_n\left[\begin{matrix}a_1,\ldots,a_p;xy\\ \frac{1}{2},b_1,\ldots,bq\end{matrix}\right]dy \tag{4·1}$$

as given by (2). Recall (1·4) and consider

$$f_1(\alpha x+\beta y)=(\alpha x+\beta y)^{-\frac{1}{2}}\exp\left[-(\alpha x+\beta y)\right]f_n\left[\begin{matrix}a_1,\ldots,a_p;(\alpha x+\beta y)(\alpha' x+\beta' y)\\ \frac{1}{2},b_1,\ldots,b_q\end{matrix}\right],$$

$$f_2(\alpha' x+\beta' y)=(\alpha' x+\beta' y)^{-\frac{1}{2}}\exp\left[(\alpha' x+\beta' y)\right]f_m\left[\begin{matrix}a_1,\ldots,a_p;(\alpha x+\beta y)(\alpha' x+\beta' y)\\ \frac{1}{2},b_1,\ldots,b_q\end{matrix}\right].$$

Then

$$\begin{aligned}&\int_0^\infty\int_0^\infty(\alpha x+\beta y)^{-\frac{1}{2}}(\alpha' x+\beta' y)^{-\frac{1}{2}}\exp\left[-(\alpha x+\beta y)-(\alpha' x+\beta' y)\right]\\ &\quad\times f_n\left[\begin{matrix}a_1,\ldots,a_p;(\alpha x+\beta y)(\alpha' x+\beta' y)\\ \frac{1}{2},b_1,\ldots,b_q\end{matrix}\right]f_m\left[\begin{matrix}[a_1,\ldots,a_p;(\alpha x+\beta y)(\alpha' x+\beta' y)]\\ \frac{1}{2},b_1,\ldots,b_q\end{matrix}\right]dx\,dy\\ &=\frac{\pi}{\kappa}f_n\left[\begin{matrix}a_1,\ldots,a_p;u\\ b_1,\ldots,b_q\end{matrix}\right]f_m\left[\begin{matrix}a_1,\ldots,a_p;v\\ b_1,\ldots,b_q\end{matrix}\right].\end{aligned} \tag{4·2}$$

Particular cases.

(*a*) When $p=1, q=1, a_1=\frac{1}{2}, b_1=1$; Then (4·2) becomes

$$\begin{aligned}&\int_0^\infty\int_0^\infty(\alpha x+\beta y)^{-\frac{1}{2}}(\alpha' x+\beta' y)^{-\frac{1}{2}}\exp\left[-(\alpha x+\beta y)-(\alpha' x+\beta' y)\right]\\ &\quad\times f_n[-;1;(\alpha x+\beta y)(\alpha' x+\beta' y)]\\ &\quad\times f_m[-;1;(\alpha x+\beta y)(\alpha' x+\beta' y)]\,dx\,dy=\pi/\kappa\,Z_n(u)\,Z_m(v).\end{aligned} \tag{4·3}$$

(*b*) When $p=1, q=0, a_1=\frac{1}{2}$, then (4·2) becomes

$$\begin{aligned}&\int_0^\infty\int_0^\infty(\alpha x+\beta y)^{-\frac{1}{2}}(\alpha' x+\beta' y)^{-\frac{1}{2}}\exp\left[-(\alpha x+\beta y)-(\alpha' x+\beta' y)\right]\\ &\quad\times f_n[-;-;(\alpha x+\beta y)(\alpha' x+\beta' y)]f_m[-;-;(\alpha x+\beta y)(\alpha' x+\beta' y)]\,dx\,dy\\ &=(\pi/\kappa)\,P_n(1-2u)\,P_m(1-2v).\end{aligned} \tag{4·4}$$

(*c*) When $p=2, q=1, a_1=\frac{1}{2}, a_2=a, b_1=b$, then (4·2) becomes

$$\begin{aligned}&\int_0^\infty\int_0^\infty(\alpha x+\beta y)^{-\frac{1}{2}}(\alpha' x+\beta y)^{-\frac{1}{2}}\exp\left[-(\alpha x+\beta y)-(\alpha' x+\beta' y)\right]\\ &\quad\times f_n[a;b;(\alpha x+\beta y)(a' x+\beta' y)]f_m[a;b;(ax+\beta y)(\alpha' x+\beta' y)]\,dx\,dy\\ &=(\pi/\kappa)\,H_n(a,b,u)\,H_m(a,b,v).\end{aligned} \tag{4·5}$$

Similarly numerous other results can be derived.

5. *Polynomials related to generalized Laguerre polynomials.* From ((4), p. 219) we have

$$A_n^{(\alpha+\beta+1)}(t,\epsilon,p)=\sum_{\kappa=0}^{n}A_{n-k}^{\alpha}(x,\epsilon,p)\,A_\kappa^{\beta}(y,\epsilon,p), \tag{5·1}$$

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where $t^\epsilon = x^\epsilon + y^\epsilon$. Now using (1·4), we have

$$\sum_{\kappa=0}^{n} \int_0^\infty \int_0^\infty (\alpha x+\beta y)^{\frac{1}{2}\sigma} (\alpha' x+\beta' y)^{\frac{1}{2}\sigma} \exp\left[p(\alpha x+\beta y)^\epsilon + p(\alpha' x+\beta' y)^\epsilon\right] \times A_{n-k}^{\lambda}(\alpha x+\beta y, \epsilon, p)\, A_\kappa^{\mu}(\alpha' x+\beta' y, \epsilon, p)\, dx\, dy = \frac{1}{\kappa} \int_0^\infty \int_0^\infty u^{\frac{1}{2}\sigma} v^{\frac{1}{2}\sigma} \exp\left[pu^\epsilon + pv^\epsilon\right] A_n^{\lambda+\mu+1}(u+v, \epsilon, p)\, du\, dv.$$

Again since ((1), p. 177)

$$\int_0^\infty \int_0^\infty f(x,y)\, x^\alpha y^\beta\, dx\, dy = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \int_0^\infty z^{\alpha+\beta+1} f(z)\, dz.$$

Therefore with the help of result (18) of ((5); p. 6), we obtain

$$\sum_{\kappa=0}^{n} \int_0^\infty \int_0^\infty (\alpha x+\beta y)^{\frac{1}{2}\sigma} (\alpha' x+\beta' y)^{\frac{1}{2}\sigma} \exp\left[p(\alpha x+\beta y)^\epsilon + p(\alpha' x+\beta' y)^\epsilon\right] \times A_{n-k}^{\lambda}(\alpha x+\beta y, \epsilon, p)\, A_\kappa^{\mu}(\alpha' x+\beta' y, \epsilon, p)\, dx\, dy = \frac{\Gamma(\frac{1}{2}\sigma+1)\,\Gamma(\frac{1}{2}\sigma+1)}{\kappa\,.\,\Gamma(\sigma+2)} \frac{(-1)^n \Gamma(\sigma+1+\mu+2+n)\,\Gamma(\sigma/\epsilon)}{m\,.\,\Gamma(\sigma+\lambda+\mu+2)\,\epsilon(-p)^{(\sigma/\epsilon)}}. \tag{5·2}$$

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REFERENCES

(1) EDWARDS, J. *A treatise on integral calculus*, vol. 2 (New York, 1922).
(2) RAINVILLE, E. D. *Special functions* (New York, 1962).
(3) SINGH, R. P. Some integrals involving the products of orthogonal polynomials and their derivatives, *Proc. Nat. Acad. Sci. India, Part A*, **34** (1964).
(4) SINGH, R. P. Some polynomials related to generalized Laguerre polynomials. *Proc. Nat. Acad. Sci. India, Part A*, **34** (1964).
(5) SINGH, R. P. AND SRIVASTAVA, K. N. A note on generalization of Laguerre and Humbert polynomials. *Ricerca (Napoli)* (2), **14** (1963); settembre-diecembre, 11–21, errata, ibid (2), **15** (1964), maggio-angusto, 63.
(6) WILLIAMSON, B. *An elementary treatise on integral calculus* (1955).

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