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VARIATION ON A THEOREM BY CARATHÉODORY

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Abstract. Carathéodory's theorem on small witnesses for convex hulls of sets is shown to have a natural analogue for finitely supported measures. Contrast is drawn with the much larger witnesses required for multisets, as shown by Bárány and Perles.

§1. *Finitely supported measures.*

Definition 1. A measure μ on $\mathbb{R}^d$ has property $(*)$ if for every closed halfspace h containing the origin, $\mu(h) \geq 1$.

For measures ν and μ, say that ν is a submeasure of μ (written $\nu \leq \mu$) if $\nu(S) \leq \mu(S)$ for every measurable set S.

THEOREM 2. *If μ is a finitely supported measure on $\mathbb{R}^d$ having property $(*)$, then it has a submeasure ν with property $(*)$ such that $\nu(\mathbb{R}^d) \leq d + 1$.*

The bound is best possible: consider the uniform measure on $d + 1$ points in convex position.

Proof. Let μ be supported on the set $S \subseteq \mathbb{R}^d$ of cardinality s. For v, w row vectors in $\mathbb{R}^d$, write $v \in H(w)$ if $v \cdot w^\dagger \geq 0$. Say that u, v are equivalent if for all $w \in S$, $u \in H(w)$ if and only if $v \in H(w)$. Let T denote the collection of equivalence classes and t its cardinality.

Form the $(s + t) \times s$ matrix A as follows. The columns of A are indexed by S, the first s rows are likewise indexed by S, and the remaining rows are indexed by T. For $\sigma \in S$, let $A_{\sigma,\sigma} = -1$; for $\sigma \neq \sigma' \in S$, let $A_{\sigma,\sigma'} = 0$. For $\sigma \in S$ and $\tau \in T$, let $A_{\tau,\sigma} = 1$ if $\tau \subseteq H(\sigma)$ and $A_{\tau,\sigma} = 0$ otherwise.

Let $\mathbb{1}_S$ be the row vector indexed by S, all of whose entries equal 1. Similarly let $\mathbb{1}_T$ be the row vector indexed by T, all of whose entries equal 1. Letting μ' denote any measure on S, consider μ' also as a row vector indexed by S; and let $(-\mu', \mathbb{1}_T)$ denote the row vector of length $s + t$ whose first s entries are $-\mu'(\sigma)$, for σ ranging over S, and whose remaining entries equal 1.

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Let $\mathrm{Val}(\mu')$ denote the value of the following linear program (which we call the primal μ'-LP):

$$\text{minimize} \quad \mathbb{1}_S \cdot x^\dagger \tag{1.1}$$

$$\text{subject to} \quad (-\mu', \mathbb{1}_T)^\dagger \le A \cdot x^\dagger \tag{1.2}$$

$$0 \le x^\dagger. \tag{1.3}$$

By hypothesis the LP is feasible for $\mu' = \mu$. The theorem is equivalent to showing that $\mathrm{Val}(\mu) \le d + 1$.

Clearly, if $\mu' \le \mu$ and the μ'-LP is feasible, then $\mathrm{Val}(\mu') \ge \mathrm{Val}(\mu)$. Let $M(\mu)$ denote the set of measures μ' on S for which the LP is feasible and $\mu' \le \mu$. $M(\mu)$ is compact (and convex); Val is continuous on $M(\mu)$. So Val is maximized at some measure $\nu \in M(\mu)$.

Form the dual of the ν-LP by writing the dual vector as the concatenation (y, z) where y is a row vector of length s indexed by S, and z is a row vector of length t indexed by T. The dual program is as follows:

$$\text{maximize} \quad (y, z) \cdot (-\nu, \mathbb{1}_T)^\dagger \tag{1.4}$$

$$\text{subject to} \quad (y, z) \cdot A \le \mathbb{1}_S \tag{1.5}$$

$$(y, z) \ge 0. \tag{1.6}$$

Let $\bar{x}$ be an optimal solution to the primal ν-LP, and let $(\bar{y}, \bar{z})$ be an optimal solution to the dual ν-LP. By complementary slackness, $(\nu(\sigma) - \bar{x}(\sigma)) \cdot \bar{y}(\sigma) = 0$ for all σ. (Otherwise we could, by reducing $\nu(\sigma)$, increase the dual without affecting the primal.)

Actually, however, more is true:

$$\nu(\sigma) \cdot \bar{y}(\sigma) = 0 \quad \text{for all } \sigma.$$

For otherwise we could, by reducing $\nu(\sigma)$, obtain an LP of higher value, contradicting the extremality of ν.

This implies, first, that $\mathrm{support}(\nu)$ and $\mathrm{support}(\bar{y})$ are disjoint; and second, that the expression $\mathrm{Val}(\nu) = (\bar{y}, \bar{z}) \cdot (-\nu, \mathbb{1}_T)^\dagger$ simplifies to $\mathrm{Val}(\nu) = \bar{z} \cdot \mathbb{1}_T^\dagger$.

Observe that the origin lies in the convex hull of $\mathrm{support}(\nu)$. (This is because ν has property $(*)$. More concretely, use the fact that the primal ν-LP is feasible: let x be a feasible vector. The set of inequalities corresponding to S establishes that $x \le \nu$; the set of inequalities corresponding to T establishes that for every vector $\xi \in \mathbb{R}^d$ there is a $\sigma \in \mathrm{support}(x)$ such that $\sigma \in H(\xi)$ and $x(\sigma) > 0$.)

Therefore, the origin lies in the convex hull of $S' = \{\sigma \mid \bar{y}(\sigma) = 0\}$.

By Carathéodory's theorem there is a subset $S'' \subseteq S'$ of cardinality at most $d + 1$ having the origin in its convex hull [4, 5]. The dual ν-LP implies that for each $\sigma \in S''$, $\sum_{\tau : \tau \subseteq H(\sigma)} \bar{z}(\tau) \le 1$. So

$$\mathrm{Val}(\nu) = \bar{z} \cdot \mathbb{1}_T^\dagger = \sum_{\tau \in T} \bar{z}(\tau) \le \sum_{\sigma \in S''} \sum_{\tau : \tau \subseteq H(\sigma)} \bar{z}(\tau) \le d + 1. \qquad \square$$

§2. *The floating body of μ.* There is another very natural way in which to state Theorem 2. Let κ be a positive real and define the *floating body of μ*

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(with parameter κ) by $\text{core}_\kappa^{\mathbb{R}}\,\mu = \{p \in \mathbb{R}^d$: if h is a closed affine halfspace containing p, then $\mu(h) \geq \kappa\}$. (If $\mu(\mathbb{R}^d) > \kappa$, which includes all cases of interest, then equivalently write: $\text{core}_\kappa^{\mathbb{R}}\,\mu =$ the intersection of all closed affine halfspaces h satisfying $\mu(h) > \mu(\mathbb{R}^d) - \kappa$.) The floating body was considered as early as 1822 [7] and subsequently many times, including [1, 3, 8, 11, 12]. The alternative statement of Theorem 2 is: if μ is finitely supported, then for every $p \in \text{core}_\kappa^{\mathbb{R}}\,\mu$ there exists a $\nu \leq \mu$ such that $p \in \text{core}_\kappa^{\mathbb{R}}\,\nu$ and $\nu(\mathbb{R}^d) \leq (d+1)\kappa$.

§3. *Multisets.* Let S be a finite multiset in $\mathbb{R}^d$ of cardinality s. For positive integer $\kappa < s$, let $\text{core}_\kappa^{\mathbb{N}}\,S =$ the intersection of all closed affine halfspaces h having $|S \cap h| > s - \kappa$. (Save only superficial changes to unify treatment with the non-integer case, this definition is from [2]; see also [6, 10].)

Bárány and Perles [2] showed that for every $d \geq 1$ and $\kappa \geq 1$ there is a finite integer F such that for every S and p, if $p \in \text{core}_\kappa^{\mathbb{N}}\,S$ then there exists a submultiset T of S of cardinality at most F for which $p \in \text{core}_\kappa^{\mathbb{N}}\,T$. The smallest integer with this property, $F(d, \kappa)$, is called the Carathéodory number.

In fact, the case $\kappa = 1$ is a transversal (or covering) problem. Since the family of sets in this problem has small Vapnik–Chervonenkis dimension, the associated linear program cannot have large integrality gap [9, Corollary 10.2.7], so a weak form of Carathéodory's theorem (i.e. with a bound larger than $d+1$) is implied by Theorem 2. However, for $\kappa > 1$ this reasoning does not apply, and then finiteness of $F(d, \kappa)$ is not implied by Theorem 2. Bárány and Perles showed that $F(d, \kappa) \leq d^\kappa$ (except for $(d, \kappa) = (2, 2)$ and $(2, 1)$).

As we know from above, the lower bound $\kappa(d+1)$ is tight for the non-integer case for all d and κ. In two dimensions this bound is tight even for the integer case: Bárány and Perles showed that $F(2, \kappa) = 3\kappa$. Strikingly, however, their results imply that in most other cases the integrality gap is indeed large. The gap starts already with $\kappa = 2$: $F(d, 2) = \max\{2(d+1), 1 + d + \lfloor d^2/4 \rfloor\}$. The difference increases for larger κ: (a) for any fixed κ, for d sufficiently large, $F(d, \kappa) \geq d^\kappa/(e^{\kappa-1}\kappa!)$; (b) for $\kappa - 1 > d \geq 5$, $F(d, \kappa) \geq \kappa - 1 + d + \frac{1}{2}\binom{\kappa-1}{\lfloor (d-1)/2 \rfloor}$.

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