CONVOLUTION HEMIGROUPS OF BOUNDED VARIATION ON A LIE PROJECTIVE GROUP

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1. *Introduction*

In a previous article [**11**] we studied the central limit theorem for infinitesimal triangular arrays of probability measures on a Lie group. Since the work of Berg [**2**] and more recently of Bendikov [**1**] similar studies appeared to be urgent for the infinite-dimensional torus group and beyond that for the class of Lie projective groups which among others contains all compact groups. Consequently we started extending the existing theory within the enlarged framework of Lie projective groups. In the present contribution continuous convolution hemigroups $(\mu(s,t))$ of probability measures on a Lie projective group G are investigated with respect to generation, representation and occurrence as limits of non-commutative infinitesimal triangular arrays.

The layout of our exposition is as follows. In Section 2 some facts on Lie projective groups G with Lie algebra $\mathscr{L}(G)$, Lie system $\mathfrak{H}$, projective basis $(X_i)_{i\in I}$ and corresponding projective weak coordinate system $(x_i)_{i\in I}$ are collected to make the reader familiar with the setting. Section 3 contains the basic methodical result yielding the intended generalisations. It is shown in Proposition 3.3 that a hemigroup $(\mu(s,t))$ in the set $\mathfrak{M}^1(G)$ of probability measures on G corresponds to a triplet (a,B,η) in the set $\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ of characteristics of bounded variation if and only if for each $H\in\mathfrak{H}$ the projection $(p_H(\mu(s,t)))$ in $\mathfrak{M}^1(G/H)$ corresponds to a certain triplet (a^H,B^H,η^H) in the set $\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$. Here the correspondence between hemigroups and triplets is achieved via weak backward evolution equations with respect to $(X_i)_{i\in I}$ and $(x_i)_{i\in I}$ (see Definition 3.2). Applying the reduction method based on this result, the convergence of infinitesimal arrays $\{\mu_{n\ell}:(n,\ell)\in\mathbb{N}^2\}$ in $\mathfrak{M}^1(G)$ towards hemigroups $(\mu(s,t))$ of continuous weak bounded variation is proved in Section 4. In fact, sufficient conditions involving the triplet (a,B,η) are given in order that the sequence $(\mu_n(s,t))_{n\geqslant 1}$ of 'scaled' convolution products $\mu_{n,k_n(s)+1}*\ldots*\mu_{n,k_n(t)}$ converges weakly to $(\mu(s,t))$. Moreover, necessary and sufficient conditions are given for the limiting hemigroup to be a diffusion hemigroup. Section 5 is devoted to the existence and uniqueness results leading to the one-to-one correspondence between the set $\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ and the set of hemigroups of continuous weak bounded variation on G. In Section 6 we illustrate the technique of Lie projectivity of the infinite-dimensional torus group and of the p-adic solenoidal group. Moreover, we sketch a little-known example of a hemigroup appearing in atomic physics and derive from it some thoughts towards a perturbation theory for hemigroups.

For the subsequent analysis the repetition of basic notation seems to be in order.

Let $\mathbb{S}:=\{(s,t)\in\mathbb{R}^2:0\leqslant s\leqslant t\}$. For an ordered vector space V we put $V_+:=\{x\in V:x$ is positive$\}$. For a non-void set I let δ_{ij}, $i,j\in I$, be the Kronecker delta. If H is a subset

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of I then $\mathsf{C}H$ denotes the complement of H in I and 1_H denotes the indicator function of H. By $\mathbb{M}_I$ we denote the space of $I\times I$ real matrices. By $\mathbb{M}_I^+$ we denote the subset of symmetric positive semidefinite matrices $B=(b_{ij})_{i,j\in I}$ in $\mathbb{M}_I$ in the sense that for every finite subset J of I the submatrix $(b_{ij})_{i,j\in J}$ is symmetric and positive semidefinite. A function $B:\mathbb{R}_+\longrightarrow\mathbb{M}_I$ is said to be *increasing* if $B(t)-B(s)\in\mathbb{M}_I^+$ for all $(s,t)\in\mathbb{S}$.

Let E be a locally compact space. By $\mathscr{C}^b(E)$ we denote the space of real-valued bounded continuous functions on E furnished with the supremum norm $\|\cdot\|_0$. $\mathscr{C}^0(E)$ and $\mathscr{K}(E)$ are the subspaces of functions vanishing at infinity and the functions with compact support, respectively. $\mathfrak{M}_+(E)$ denotes the space of positive Radon measures on E, $\mathfrak{M}_+^b(E)$ denotes the subset of bounded positive measures on E equipped with the weak topology $\mathscr{T}_w$, and $\mathfrak{M}^1(E)$ denotes the subset of probability measures on E. The Dirac measure in $x\in E$ will be denoted by ε_x.

Let G be a locally compact group with identity e. Let $\mathfrak{U}(e)$ be the system of all measurable neighbourhoods of e in G. Let $\mathscr{C}_e(G)$ be the subspace of $\mathscr{C}^b(G)$ containing functions vanishing in some neighbourhood of the identity. For a function $f:G\longrightarrow\mathbb{R}$ the function $f^*:G\longrightarrow\mathbb{R}$ is defined by $f^*(x):=f(x^{-1})$. For every $\mu\in\mathfrak{M}^1(G)$ the probability operator T_μ of μ is the contraction on $\mathscr{C}^0(G)$ defined by $T_\mu f(x):=\int f(xy)\,\mu(dy)$. For closed normal subgroups H,M of G with $M\subseteq H$ we write p_H^M for the canonical mapping $xM\longrightarrow xH$. In particular $p_H:=p_H^{\{e\}}$.

Finally, if G is a Lie group, let $\mathscr{L}(G)$ be the Lie algebra of G and $\exp_G:\mathscr{L}(G)\longrightarrow G$ be the exponential mapping. By $\mathscr{D}(G)$ we denote the space of infinitely differentiable real-valued functions with compact support on G.

2. *Preliminaries on Lie projective groups*

A locally compact group G is called *Lie projective* if there is a system $\mathfrak{H}$ of compact normal subgroups of G such that G/H is a Lie group for all $H\in\mathfrak{H}$ and such that $\bigcap_{H\in\mathfrak{H}}H=\{e\}$. If in addition $\mathfrak{H}$ is filtering downwards, we call $\mathfrak{H}$ a *Lie system* for G. We may and do always suppose $\mathfrak{H}$ to be filtering downwards, since for every Lie projective locally compact group G the system $\mathfrak{H}(G)$ of all compact normal subgroups admitting a Lie factor group is a Lie system for G (see Montgomery and Zippin [**14**]).

Let $\mathfrak{H}$ be a Lie system for a Lie projective group G. Then G is the projective limit $\varprojlim_{H\in\mathfrak{H}}(G/H)$ defined as the closed subgroup

$$\left\{(x_H)_{H\in\mathfrak{H}}\in\prod_{H\in\mathfrak{H}}(G/H):p_H^M(x_M)=x_H \text{ for all } M,H\in\mathfrak{H},\ M\subseteq H\right\}$$

of the product group $\prod_{H\in\mathfrak{H}}(G/H)$. Now the Lie algebra $\mathscr{L}(G)$ of G is by definition the projective limit $\varprojlim_{H\in\mathfrak{H}}\mathscr{L}(G/H)$ of Lie algebras of the ordinary Lie groups G/H with respect to the differentials of the canonical mappings $p_H^M:G/M\longrightarrow G/H$ for $M,H\in\mathfrak{H}$, $M\subseteq H$. That is,

$$\mathscr{L}(G):=\left\{(X^H)_{H\in\mathfrak{H}}\in\prod_{H\in\mathfrak{H}}\mathscr{L}(G/H):dp_H^M(X^H)=X^M \text{ for all } M,H\in\mathfrak{H},\ M\subseteq H\right\}.$$

The canonical mappings $dp_H:\mathscr{L}(G)\longrightarrow\mathscr{L}(G/H)$ are continuous open epimorphisms with $dp_H=dp_H^M\circ dp_M$ for $M,H\in\mathfrak{H}$, $M\subseteq H$. Moreover $\mathscr{L}(G)$ is independent of the choice of the Lie system $\mathfrak{H}$.

Due to Born [**4**, Theorem 2.2] one can always find a Lie system $\mathfrak{H}$ for G admitting a *projective basis* for $\mathscr{L}(G)$ in the sense of the following definition.

DEFINITION 2.1. A family $(X_i)_{i\in I}$ in $\mathscr{L}(G)\backslash\{0\}$ is called a *projective basis* for $\mathscr{L}(G)$ with respect to the Lie system $\mathfrak{H}$ of G, if for every $H\in\mathfrak{H}$ there is a finite subset

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$I_H \subseteq I$ such that $(dp_H(X_i))_{i\in I_H}$ is a basis for the vector space $\mathscr{L}(G/H)$ and $dp_H(X_i) = 0$ for $i \notin I_H$.

For the rest of the paper we fix a Lie system $\mathfrak{H}$ of G such that $\mathscr{L}(G)$ admits a projective basis $(X_i)_{i\in I}$ with respect to $\mathfrak{H}$. According to Born [**4**, Proposition 2.3], for every $(r_i)_{i\in I} \in \mathbb{R}^I$ there exists exactly one vector $X \in \mathscr{L}(G)$ with

$$\lim_{J\in\mathfrak{F}(I)} \sum_{i\in J} r_i X_i = X =: \sum_{i\in I} r_i X_i.$$

In fact,

$$\sum_{i\in I} r_i X_i = \left(\sum_{i\in I_H} r_i \, dp_H(X_i) \right)_{H\in\mathfrak{H}}.$$

Moreover, by Born [**4**, Theorem 2.4], the mapping $(r_i)_{i\in I} \longmapsto \sum_{i\in I} r_i X_i$ is a topological vector space isomorphism of $\mathbb{R}^I$ onto $\mathscr{L}(G)$; hence the cardinality of I depends on G only.

The space $\mathscr{D}(G)$ of test functions is by definition

$$\mathscr{D}(G) := \bigcup_{H\in\mathfrak{H}} \{f^H \circ p_H : f^H \in \mathscr{D}(G/H)\}$$

(see Bruhat [**6**]). The spaces $\mathscr{C}_k(G)$ and $\tilde{\mathscr{C}}_k(G)$ of k times left-, respectively right-differentiable functions has been introduced in Born [**4**, 4.1]. For all $X \in \mathscr{L}(G)$, $H \in \mathfrak{H}$ and $f^H \in \mathscr{C}_1(G/H)$ we have $f^H \circ p_H \in \mathscr{C}_1(G)$ and

$$X(f^H \circ p_H) = (dp_H(X) f^H) \circ p_H.$$

One can introduce weak coordinate systems in G generalising the notion of a coordinate system of a Lie group (see Born [**5**]).

DEFINITION 2.2. A family $(x_i)_{i\in I}$ in $\mathscr{D}(G)$ is called a *weak coordinate system* of G in e with respect to $(X_i)_{i\in I}$ if $X_i x_j(e) = \delta_{ij}$ and $x_j^* = -x_j$ hold for all $i,j \in I$. We call $(x_i)_{i\in I}$ *projective* if, in addition, for all $H \in \mathfrak{H}$ there exist a neighbourhood $U^H \in \mathfrak{U}(e)$ and a canonical coordinate system $(x_i^H)_{i\in I_H}$ of G/H in e with respect to the basis $(dp_H(X_i))_{i\in I_H}$ of $\mathscr{L}(G/H)$ such that $x_i(y) = x_i^H \circ p_H(y)$ for all $y \in U_H$ and $i \in I_H$.

One can always find a projective weak coordinate system of G in e with respect to $(X_i)_{i\in I}$. Indeed, for every $H \in \mathfrak{H}$ let $(x_i^H)_{i\in I_H}$ be a canonical coordinate system of G/H in e with respect to the basis $(dp_H(X_i))_{i\in I_H}$ of $\mathscr{L}(G/H)$. For every $i \in I$ choose some $H(i) \in \mathfrak{H}$ with $i \in I_{H(i)}$. Let $x_i := x_i^{H(i)} \circ p_{H(i)}$. Then due to Born [**5**, Corollary 7] the family $(x_i)_{i\in I}$ is a projective weak coordinate system of G in e.

3. *Convolution semigroups and hemigroups*

A family $(\mu_t)_{t\geqslant 0}$ in $\mathfrak{M}^1(G)$ is called a (*continuous*) *convolution semigroup* if $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in \mathbb{R}_+$, $\mu_0 = \varepsilon_e$ and $\mathscr{T}_w\text{-}\lim_{t\downarrow 0} \mu_t = \mu_0$. Its *generating functional* is defined by

$$A(f) := \lim_{t\downarrow 0} \frac{1}{t} \int (f - f(e))\, d\mu_t$$

for all f in the maximal domain $D(A)$ of A.

A measure $\eta \in \mathfrak{M}_+(G)$ is called a *Lévy measure on* G if $\eta(\{e\}) = 0$, $\eta(\complement U) < \infty$ holds for all $U \in \mathfrak{U}(e)$ and $\int f\, d\eta < \infty$ holds for all $f \in \mathscr{D}(G)_+$ with $f(e) = 0$. Let $\mathbb{L}(G)$ be the set of Lévy measures on G. We introduce the parameter space

$$\mathbb{P}(G) := \mathbb{R}^I \times \mathbb{M}_I^+ \times \mathbb{L}(G).$$

Born [**5**, Corollary 5] has shown the following representation theorem for the

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generating functionals of convolution semigroups (in fact, on all locally compact groups).

THEOREM 3.1. *Let G be a Lie projective group, $(X_i)_{i\in I}$ be a projective basis for $\mathscr{L}(G)$ and $(x_i)_{i\in I}$ be a weak coordinate system of G in e with respect to $(X_i)_{i\in I}$.*

(1) *If $(\mu_t)_{t\geqslant 0}$ is a convolution semigroup in $\mathfrak{M}^1(G)$ with generating functional A then $\mathscr{D}(G)\subseteq D(A)$ and there exists exactly one triplet $(a,B,\eta)\in\mathbb{P}(G)$ such that*

$$A(f)=\sum_{i\in I}a_i X_i f(e)+\frac{1}{2}\sum_{i,j\in I}b_{ij}X_iX_jf(e)+\int\Big(f(y)-f(e)-\sum_{i\in I}X_if(e)\,x_i(y)\Big)\eta(dy)$$

holds for all $f\in\mathscr{D}(G)$.

(2) *For every triplet $(a,B,\eta)\in\mathbb{P}(G)$ there exists exactly one convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $\mathfrak{M}^1(G)$ such that the generating functional of $(\mu_t)_{t\geqslant 0}$ is defined at least on $\mathscr{D}(G)$ and admits on $\mathscr{D}(G)$ the representation given in* (1).

A family $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ is called a *continuous convolution hemigroup* (briefly hemigroup) if $\mu(s,r)*\mu(r,t)=\mu(s,t)$ for all $(s,r),(r,t)\in\mathbb{S}$, $\mu(t,t)=\varepsilon_e$ for all $t\in\mathbb{R}_+$, and the mapping $(s,t)\longmapsto\mu(s,t)$ from $\mathbb{S}$ into $\mathfrak{M}^1(G)$ is $\mathscr{T}_w$-continuous.

A hemigroup $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ is said to be of (*continuous*) *weak bounded variation* if for all $f\in\mathscr{C}_2(G)$ the function $F_f:(s,t)\longmapsto(T_{\mu(s,t)}-I)f(e)$ from $\mathbb{S}$ into $\mathbb{R}$ is of (continuous) bounded variation. (See Heyer and Pap [**11**].)

A family of measures $(\eta_t)_{t\geqslant 0}$ in $\mathfrak{M}_+(G)$ is called a *Lévy measure family on G* if $\eta_t-\eta_s\in\mathbb{L}(G)$ for all $(s,t)\in\mathbb{S}$, $\eta_0=0$, and the mapping $(s,t)\longmapsto\int f\,d\eta_t$ is continuous for all $f\in\mathscr{D}(G)_+$ with $f(e)=0$. Let $\mathbb{L}(\mathbb{R}_+,G)$ be the set of measures $\eta\in\mathfrak{M}^1(G\times\mathbb{R}_+)$ such that $\eta(\{e\}\times\mathbb{R}_+)=0$, $\eta(dy\times[0,t])\in\mathbb{L}(G)$ for all $t\in\mathbb{R}_+$, and the mapping $t\longmapsto\int f(y)\eta(dy\times[0,t])$ is continuous for all $f\in\mathscr{D}(G)_+$ with $f(e)=0$. Then there is a natural bijection between the set of all Lévy measure families $(\eta_t)_{t\geqslant 0}$ in $\mathfrak{M}_+(G)$ and the set $\mathbb{L}(\mathbb{R}_+,G)$ of Lévy measures η via $\eta_t(dy)=\eta(dy\times[0,t])$.

We introduce the parameter space $\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ of triplets (a,B,η), where $a:\mathbb{R}_+\longrightarrow\mathbb{R}^I$, $a(t)=(a_i(t))_{i\in I}$, is a continuous function such that $a(0)=0$ and for all $i\in I$ the coordinate function $a_i:\mathbb{R}_+\longrightarrow\mathbb{R}$ is of bounded variation, $B:\mathbb{R}_+\longrightarrow\mathbb{M}_I^+$ is increasing and continuous with $B(0)=0$, and $\eta\in\mathbb{L}(\mathbb{R}_+,G)$. With a triplet $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ we associate a *generating mapping* $A:\mathbb{R}_+\longrightarrow\mathbb{P}(G)$ via $A(t)=(a(t),B(t),\eta_t)$, $t\in\mathbb{R}_+$. For a family of functions $(g_t)_{t\geqslant 0}$ in $\mathscr{C}_2(G)$ and $(s,t)\in\mathbb{S}$ let

$$\int_s^t A(d\tau)\,(g_\tau):=\sum_{i\in I}\int_s^t X_ig_\tau(e)\,a_i(d\tau)+\frac{1}{2}\sum_{i,j\in I}\int_s^t X_iX_jg_\tau(e)\,b_{ij}(d\tau)$$
$$+\iint\limits_{G\times]s,t]}(g_\tau(y)-g_\tau(e)-\sum_{i\in I}X_ig_\tau(e)\,x_i(y))\,\eta(dy\times d\tau)$$

provided that the integrals exist.

DEFINITION 3.2. A hemigroup $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ and a triplet $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ are said to correspond to each other via weak backward evolution equation with respect to the basis $(X_i)_{i\in I}$ of $\mathscr{L}(G)$ and the weak coordinate system $(x_i)_{i\in I}$ of G in e (briefly correspond to each other) if for all $(s,t)\in\mathbb{S}$ and $f\in\mathscr{D}(G)$

$$(T_{\mu(s,t)}-I)f(e)=\int_s^t A(d\tau)\,(T_{\mu(\tau,t)}f).$$

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Now we give a criterion for a hemigroup in $\mathfrak{M}^1(G)$ to correspond to a triplet using the related hemigroups in $\mathfrak{M}^1(G/H)$. Let $(x_i)_{i\in I}$ be a projective weak coordinate system of G in e with respect to $(X_i)_{i\in I}$. Let $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$. For all $H\in\mathfrak{H}$ let the function $a^H:\mathbb{R}_+\longrightarrow\mathbb{R}^{I_H}$, $a^H(t)=(a_i^H(t))_{i\in I_H}$, be defined by

$$a_i^H(t):=a_i(t)+\int_G (x_i^H\circ p_H(y)-x_i(y))\,\eta(dy\times[0,t]).$$

(The integral exists since $x_i^H\circ p_H-x_i\in\mathscr{C}_e(G)$.) Moreover, let the matrix-valued function $B^H:\mathbb{R}_+\longrightarrow\mathbb{M}_{I_H}$ be defined by

$$B^H(t):=(b_{ij}(t))_{i,j\in I_H}.$$

Finally, let the measure $\eta^H\in\mathfrak{M}_+((G/H)\times\mathbb{R}_+)$ be defined by

$$\eta^H(dy\times[0,t]):=\eta_t^H(dy)$$

where $\eta_t^H:=p_H(\eta_t)$ with $\eta_t(dy):=\eta(dy\times[0,t])$. Clearly $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$.

PROPOSITION 3.3. *Let G be a Lie projective group, $(X_i)_{i\in I}$ be a projective basis for $\mathscr{L}(G)$ and $(x_i)_{i\in I}$ be a projective weak coordinate system of G in e with respect to $(X_i)_{i\in I}$. For all $H\in\mathfrak{H}$ let $(x_i^H)_{i\in I_H}$ be a coordinate system of G/H in e with respect to the basis $(dp_H(X_i))_{i\in I_H}$ of $\mathscr{L}(G/H)$ such that $x_i(y)=x_i^H\circ p_H(y)$ for all $y\in U_H$ and $i\in I_H$.*

Then a hemigroup $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ corresponds to a triplet $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ with respect to the basis $(X_i)_{i\in I}$ of $\mathscr{L}(G)$ and the weak coordinate system $(x_i)_{i\in I}$ of G if and only if for all $H\in\mathfrak{H}$ the hemigroup $(p_H(\mu(s,t)))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G/H)$ corresponds to the triplet $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$ (defined above) with respect to the basis $(dp_H(X_i))_{i\in I_H}$ of $\mathscr{L}(G/H)$ and the coordinate system $(x_i^H)_{i\in I_H}$ of G/H.

Proof. Obviously we have for all $(s,t)\in\mathbb{S}$, $H\in\mathfrak{H}$ and $f^H\in\mathscr{D}(G/H)$

$$(T_{p_H(\mu(s,t))}-I)f^H(e)=(T_{\mu(s,t)}-I)(f^H\circ p_H)(e).$$

Moreover

$$\begin{aligned}
&\sum_{i\in I_H}\int_s^t (dp_H(X_i)\,T_{p_H(\mu(\tau,t))}f^H)(e)\,a_i^H(d\tau)\\
&\qquad=\sum_{i\in I}\int_s^t X_i\,T_{\mu(\tau,t)}(f^H\circ p_H)(e)\,a_i(d\tau)\\
&\qquad\quad+\sum_{i\in I_H}\iint_{G\times]s,t]} X_i\,T_{\mu(\tau,t)}(f^H\circ p_H)(e)\,(x_i^H\circ p_H(y)-x_i(y))\,\eta(dy\times d\tau),\\
&\sum_{i,j\in I_H}\int_s^t (dp_H(X_i)\,dp_H(X_j)\,T_{p_H(\mu(\tau,t))}f^H)(e)\,b_{ij}^H(d\tau)\\
&\qquad=\sum_{i,j\in I}\int_s^t X_iX_j\,T_{\mu(\tau,t)}(f^H\circ p_H)(e)\,b_{ij}(d\tau),\\
&\iint_{(G/H)\times]s,t]}\Big((T_{p_H(\mu(\tau,t))}f^H)(y)-(T_{p_H(\mu(\tau,t))}f^H)(e)\\
&\qquad\quad-\sum_{i\in I_H}\int_s^t (dp_H(X_i)\,T_{p_H(\mu(\tau,t))}f^H)(e)\,x_i^H(y)\Big)\eta^H(dy\times d\tau)
\end{aligned}$$

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$$= \iint_{G\times]s,t]} \Big(T_{\mu(\tau,t)}(f^H \circ p_H)(y) - T_{\mu(\tau,t)}(f^H \circ p_H)(e) - \sum_{i\in I_H} \int_s^t X_i\, T_{\mu(\tau,t)}(f^H \circ p_H)(e)\, x_i^H \circ p_H(y) \Big) \eta(dy \times d\tau).$$

Consequently

$$\int_s^t A^H(d\tau)\,(T_{p_H(\mu(\tau,t))} f^H) = \int_s^t A(d\tau)\,(T_{\mu(\tau,t)}(f^H \circ p_H))$$

and the statement follows.

A hemigroup $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ is called a *diffusion hemigroup* if for all $T>0$ and $U\in\mathfrak{U}(e)$

$$\lim_{\substack{t-s\to 0\\ 0\leqslant s<t\leqslant T}} \frac{1}{t-s}\mu(s,t)(\complement U) = 0.$$

The distributions $\mu(s,t)$ of the left-increments $\xi_s^{-1}\xi_t$, $(s,t)\in\mathbb{S}$, of a process $(\xi_t)_{t\geqslant 0}$ with values in G, with càdlàg (left continuous with right limits) paths and with independent left-increments form a continuous convolution hemigroup in $\mathfrak{M}^1(G)$. Conversely, with every continuous convolution hemigroup $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ there is associated a process $(\xi_t)_{t\geqslant 0}$ with values in G and with càdlàg paths such that $\mu(s,t)$ is the distribution of $\xi_s^{-1}\xi_t$.

LEMMA 3.4. *Let $(\mu(s,t))_{(s,t)\in\mathbb{S}}$ be a hemigroup in $\mathfrak{M}^1(G)$ and $(\xi_t)_{t\geqslant 0}$ be an associated process with càdlàg paths and with independent left-increments. Let η_t denote the Lévy measure of $A(t)$, $t\in\mathbb{R}_+$. We consider the following assertions*:

(i) *$(\mu(s,t))_{(s,t)\in\mathbb{S}}$ is a diffusion hemigroup.*

(ii) *The paths of the process $(\xi_t)_{t\geqslant 0}$ are continuous with probability* 1.

(iii) *$\eta_t = 0$ for all $t\in\mathbb{R}_+$.*

Then (i) *implies* (ii).

If there exists an increasing mapping $A:\mathbb{R}_+ \longrightarrow \mathbb{P}(G)$ such that for every $f\in\mathscr{D}(G)$ the function

$$F_f:(s,t)\longmapsto \sqrt{|(T_{\mu(s,t)}-I)f(e)-(A(t)-A(s))(f)|}$$

from $\mathbb{S}$ into $\mathbb{R}$ is of continuous bounded variation then (ii) *and* (iii) *are equivalent.*

If the function F_f is Lipschitz continuous for all $f\in\mathscr{D}(G)$ then (i), (ii) *and* (iii) *are equivalent.*

(See Siebert [**16**, Theorem 3 and its corollary].)

4. *Convergence of arrays of probability measures*

Applying a convergence theorem for triangular arrays of probability measures on a Lie group we obtain the following result.

THEOREM 4.1. *Let G be a Lie projective group, $(X_i)_{i\in I}$ be a projective basis for $\mathscr{L}(G)$ and $(x_i)_{i\in I}$ be a projective weak coordinate system of G in e with respect to $(X_i)_{i\in I}$.*

Let $\{\mu_{n\ell}:(n,\ell)\in\mathbb{N}^2\}$ be an array in $\mathfrak{M}^1(G)$. For all $n\in\mathbb{N}$ let $k_n:\mathbb{R}_+\longrightarrow\mathbb{Z}_+$ be an increasing càd (left continuous) function with $k_n(0)=0$ and $k_n(\mathbb{R}_+)=\mathbb{Z}_+$. Let D be a dense subset of $\mathbb{R}_+$.

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Suppose that

(i) *there exists a measure* $\eta \in \mathbb{L}(\mathbb{R}_+, G)$ *such that for all* $t \in D$ *and* $f \in \mathscr{C}_e(G)$

$$\lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int f \, d\mu_{n\ell} = \int_G f(y)\, \eta(dy \times [0,t]);$$

(ii) *there exists a continuous function* $B: \mathbb{R}_+ \longrightarrow \mathbb{M}_I$, $B(t) = (b_{i,j}(t))_{i,j\in I}$, *such that for all* $t \in D$ *and* $i,j \in I$

$$\lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int x_i x_j \, d\mu_{n\ell} = b_{i,j}(t) + \int_G x_i(y)\, x_j(y)\, \eta(dy \times [0,t]);$$

(iii) *there exists a continuous function* $a: \mathbb{R}_+ \longrightarrow \mathbb{R}^I$, $a(t) = (a_i(t))_{i\in I}$, *such that for all* $t \in D$ *and* $i \in I$

$$\lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int x_i \, d\mu_{n\ell} = a_i(t);$$

(iv) *for all* $T > 0$ *and* $i \in I$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\substack{t-s\leqslant\delta \\ 0\leqslant s\leqslant t\leqslant T}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \left| \int x_i \, d\mu_{n\ell} \right| = 0.$$

Then $(a, B, \eta) \in \mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+, G)$ *and*

$$\bigstar_{\ell=k_n(s)+1}^{k_n(t)} \mu_{n\ell} \xrightarrow{\mathscr{T}_w} \nu(s,t) \quad \text{for all } (s,t) \in \mathbb{S}$$

where $(\nu(s,t))_{(s,t)\in\mathbb{S}}$ *is a hemigroup of continuous weak bounded variation in* $\mathfrak{M}^1(G)$ *corresponding to the triplet* (a, B, η) *via weak backward evolution equation with respect to the basis* $(X_i)_{i\in I}$ *of* $\mathscr{L}(G)$ *and the weak coordinate system* $(x_i)_{i\in I}$ *of* G.

Moreover, let $(\xi_t)_{t\geqslant 0}$ *be a process with càdlàg paths and with independent left-increments associated with the hemigroup* $(\nu(s,t))_{(s,t)\in\mathbb{S}}$, *and let us consider the following assertions*:

(a) $(\nu(s,t))_{(s,t)\in\mathbb{S}}$ *is a diffusion hemigroup.*

(b) *The paths of the process* $(\xi_t)_{t\geqslant 0}$ *are continuous with probability* 1.

(c) $\eta_0 = 0$.

Then (b) *and* (c) *are equivalent, and* (a) *implies* (b) *and* (c).

If we suppose that in addition

(v) *for all* $f \in \mathscr{C}_e(G)_+$ *the function* $t \longmapsto \int_G f(y)\, \eta(dy \times [0,t])$ *from* $\mathbb{R}_+$ *into* $\mathbb{R}$ *is Lipschitz continuous*;

(vi) *for all* $i \in I$ *the function* $t \longmapsto b_{i,i}(t)$ *from* $\mathbb{R}_+$ *into* $\mathbb{R}$ *is Lipschitz continuous*;

(vii) *there exists a subsequence* (n') *of* (n) *such that for all* $i \in I$ *the function*

$$(s,t) \longmapsto \limsup_{n'} \sum_{\ell=k_{n'}(s)+1}^{k_{n'}(t)} \left| \int_G x_i \, d\mu_{n'\ell} \right|$$

from $\mathbb{S}$ *into* $\mathbb{R}$ *is Lipschitz continuous*;

then (a), (b) *and* (c) *are equivalent*.

Proof. For all $H \in \mathfrak{H}$ let the function $a^H: \mathbb{R}_+ \longrightarrow \mathbb{R}^{I_H}$, the matrix-valued function $B^H: \mathbb{R}_+ \longrightarrow \mathbb{M}^+_{I_H}$ and the measure $\eta^H \in \mathfrak{M}_+((G/H) \times \mathbb{R}_+)$ be defined as in Section 3. Let

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the measure $\mu_{n\ell}^H \in \mathfrak{M}^1(G/H)$ be defined by $\mu_{n\ell}^H := p_H(\mu_{n\ell})$. We show that the conditions (i)–(v) of Heyer and Pap [**11**, Theorem 8.1] hold for the array $\{\mu_{n\ell}^H : (n,\ell) \in \mathbb{N}^2\}$, for the functions $\{k_n : n \in \mathbb{N}\}$ and the triplet (a^H, B^H, η^H).

First we note that the measures $\eta_n^H \in \mathfrak{M}_+((G/H) \times \mathbb{R}_+)$ are defined by

$$\eta_n^H(dy \times [0,t]) = \sum_{\ell=1}^{k_n(t)} \mu_{n\ell}^H(dy).$$

Clearly for $f^H \in \mathscr{C}_e(G/H)$ we have $f^H \circ p_H \in \mathscr{C}_e(G)$; hence by assumption (i) for all $t \in D$

$$\begin{aligned}\lim_{n\to\infty} \int_{G/H} f^H(y)\, \eta_n^H(dy \times [0,t]) &= \lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_{G/H} f^H \, d\mu_{n\ell}^H = \lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_G f^H \circ p_H \, d\mu_{n\ell} \\ &= \int_G f^H \circ p_H(y)\, \eta(dy \times [0,t]) = \int_{G/H} f^H(y)\, \eta^H(dy \times [0,t]).\end{aligned}$$

For $i,j \in I_H$ we have $(x_i^H \circ p_H)(x_j^H \circ p_H) - x_i x_j \in \mathscr{C}_e(G)$; hence by assumption (i) for all $t \in D$

$$\begin{aligned}&\lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_G ((x_i^H \circ p_H)(x_j^H \circ p_H) - x_i x_j)\, d\mu_{n\ell} \\ &\qquad = \int_G ((x_i^H \circ p_H(y))(x_j^H \circ p_H(y)) - x_i(y)\, x_j(y))\, \eta(dy \times [0,t])\end{aligned}$$

which together with assumption (ii) implies

$$\begin{aligned}\lim_{n\to\infty} \int_{G/H} x_i^H(y)\, x_j^H(y)\, \eta_n^H(dy \times [0,t]) &= \lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_{G/H} x_i^H \, x_j^H \, d\mu_{n\ell}^H \\ &= \lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_G (x_i^H \circ p_H)(x_j^H \circ p_H)\, d\mu_{n\ell} \\ &= b_{ij}(t) + \int_G (x_i^H \circ p_H(y))(x_j^H \circ p_H(y))\, \eta(dy \times [0,t]) \\ &= b_{ij}^H(t) + \int_{G/H} x_i^H(y)\, x_j^H(y)\, \eta^H(dy \times [0,t]).\end{aligned}$$

For $i \in I_H$ we have $x_i^H \circ p_H - x_i \in \mathscr{C}_e(G)$ hence by assumption (i) for all $t \in D$

$$\lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_G (x_i^H \circ p_H - x_i)\, d\mu_{n\ell} = \int_G (x_i^H \circ p_H(y) - x_i(y))\, \eta(dy \times [0,t])$$

which together with assumption (iii) implies

$$\begin{aligned}\lim_{n\to\infty} \int_{G/H} x_i^H(y)\, \eta_n^H(dy \times [0,t]) &= \lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_{G/H} x_i^H \, d\mu_{n\ell}^H \\ &= \lim_{n\to\infty} \sum_{\ell=1}^{k_n(t)} \int_G x_i^H \circ p_H \, d\mu_{n\ell} \\ &= a_i(t) + \int_G (x_i^H \circ p_H(y) - x_i(y))\, \eta(dy \times [0,t]) = a_i^H(t).\end{aligned}$$

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The convergence in assumption (i) is obviously uniform in $t\in[0,T]$ for all $T>0$ since for all $f\in\mathscr{C}_e(G)_+$ the functions

$$t\longmapsto\sum_{\ell=1}^{k_n(t)}\int f\,d\mu_{n\ell},\quad n\in\mathbb{N}$$

are increasing and continuous, and the limit function $t\longmapsto\int f(y)\,\eta(dy\times[0,t])$ is continuous. Hence for all $T>0$ and $f\in\mathscr{C}_e(G)_+$ we have

$$\lim_{\delta\to\mathbf{0}}\limsup_{n\to\infty}\sup_{\substack{t-s\leqslant\delta\\ \mathbf{0}\leqslant s\leqslant t\leqslant T}}\sum_{\ell=k_n(s)+1}^{k_n(t)}\int f\,d\mu_{n\ell}=0$$

(see Heyer and Pap [**11**, Lemma 5.1]). Consequently, by the inequality

$$\left|\int_{G/H}x_i^H\,d\mu_{n\ell}^H\right|\leqslant\left|\int_G x_i\,d\mu_{n\ell}\right|+\int_G|x_i^H\circ p_H-x_i|\,d\mu_{n\ell}$$

and observing that $|x_i^H\circ p_H-x_i|\in\mathscr{C}_e(G)_+$ we derive

$$\lim_{\delta\to\mathbf{0}}\limsup_{n\to\infty}\sup_{\substack{t-s\leqslant\delta\\ \mathbf{0}\leqslant s\leqslant t\leqslant T}}\sum_{\ell=k_n(s)+1}^{k_n(t)}\left|\int_{G/H}x_i^H\,d\mu_{n\ell}^H\right|=0.$$

Hence we can apply Heyer and Pap [**11**, Theorem 8.1]. We obtain $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$ for all $H\in\mathfrak{H}$ which implies $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$. Moreover

$$\mu_n^H(s,t):=\bigstar_{\ell=k_n(s)+1}^{k_n(t)}\mu_{n\ell}^H\xrightarrow{\mathscr{T}_w}\nu^H(s,t)\quad\text{for all }(s,t)\in\mathbb{S}$$

where $(\nu^H(s,t))_{(s,t)\in\mathbb{S}}$ is a hemigroup of continuous weak bounded variation in $\mathfrak{M}^1(G/H)$. For all $(s,t)\in\mathbb{S}$ the family of measures $(\nu^H(s,t))_{H\in\mathfrak{H}}$ is a projective system since for all $M,H\in\mathfrak{H}$ with $M\subseteq H$ we have

$$\begin{aligned}p_H^M(\nu^M(s,t))&=\lim_{n\to\infty}p_H^M\Bigg(\bigstar_{\ell=k_n(s)+1}^{k_n(t)}\mu_{n\ell}^M\Bigg)=\lim_{n\to\infty}\bigstar_{\ell=k_n(s)+1}^{k_n(t)}p_H^M(p_M(\mu_{n\ell}))\\&=\lim_{n\to\infty}\bigstar_{\ell=k_n(s)+1}^{k_n(t)}p_H(\mu_{n\ell})=\lim_{n\to\infty}\bigstar_{\ell=k_n(s)+1}^{k_n(t)}\mu_{n\ell}^H=\nu^H(s,t).\end{aligned}$$

By Heyer [**9**, Theorem 1.2.17] there exists a measure $\nu(s,t)\in\mathfrak{M}^1(G/H)$ such that

$$p_H(\nu(s,t))=\nu^H(s,t)\quad\text{for all }H\in\mathfrak{H}.$$

For all $(s,t)\in\mathbb{S}$ and $n\in\mathbb{N}$ let

$$\nu_n(s,t):=\bigstar_{\ell=k_n(s)+1}^{k_n(t)}\mu_{n\ell}.$$

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Then $v_{n\ell}^H(s,t) = p_H(v_{n\ell}(s,t))$ and for all $(s,t)\in\mathbb{S}$, $H\in\mathfrak{H}$ and $f^H\in\mathscr{K}(G/H)$ we have

$$\lim_{n\to\infty}\int_G f^H\circ p_H(y)\,\mu_n(s,t)\,(dy) = \lim_{n\to\infty}\int_{G/H} f^H(y)\,\mu_n^H(s,t)\,(dy)$$
$$= \int_{G/H} f^H(y)\,\nu^H(s,t)\,(dy) = \int_G f^H\circ p_H(y)\,\nu(s,t)\,(dy)$$

and the set $\{f^H\circ p_H : f^H\in\mathscr{K}(G/H),\ H\in\mathfrak{H}\}$ is dense in $\mathscr{K}(G)$; hence

$$\mu_n(s,t)\xrightarrow{\mathscr{T}_w}\nu(s,t)\quad\text{for all } (s,t)\in\mathbb{S}.$$

For $(s,t)\in\mathbb{S}$ taking sequences (s_ℓ) and (t_ℓ) in $\mathbb{R}_+$ with $s_\ell\to s$, $t_\ell\to t$ and $(s_\ell,t_\ell)\in\mathbb{S}$ we have for all $H\in\mathfrak{H}$ and $f^H\in\mathscr{K}(G/H)$

$$\lim_{\ell\to\infty}\int_G f^H\circ p_H(y)\,\nu(s_\ell,t_\ell)\,(dy) = \lim_{n\to\infty}\int_{G/H} f^H(y)\,\nu^H(s_\ell,t_\ell)\,(dy)$$
$$= \int_{G/H} f^H(y)\,\nu^H(s,t)\,(dy) = \int_G f^H\circ p_H(y)\,\nu(s,t)\,(dy).$$

Hence $(\nu(s,t))_{(s,t)\in\mathbb{S}}$ is a hemigroup in $\mathfrak{M}^1(G)$.

From Heyer and Pap [**11**, Theorem 8.1] we also conclude that for all $H\in\mathfrak{H}$ the hemigroup $(\nu^H(s,t))_{(s,t)\in\mathbb{S}}$ corresponds to the triplet $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$ with respect to the basis $(dp_H(X_i))_{i\in I_H}$ of $\mathscr{L}(G/H)$ and the coordinate system $(x_i^H)_{i\in I_H}$ of G/H. Hence by Proposition 3.3 we obtain that the hemigroup $(\nu(s,t))_{(s,t)\in\mathbb{S}}$ corresponds to the triplet (a,B,η) with respect to the basis $(X_i)_{i\in I}$ of $\mathscr{L}(G)$ and the coordinate system $(x_i)_{i\in I}$ of G.

Moreover, for all $H\in\mathfrak{H}$ the hemigroup $(\nu^H(s,t))_{(s,t)\in\mathbb{S}}$ is of continuous weak bounded variation; hence the mapping

$$(s,t)\longmapsto(T_{\nu^H(s,t)}-I)f^H(e) = (T_{\nu(s,t)}-I)\,(f^H\circ p_H)(e)$$

is of continuous bounded variation for all $f^H\in\mathscr{D}(G/H)$. Consequently the hemigroup $(\nu(s,t))_{(s,t)\in\mathbb{S}}$ is also of continuous weak bounded variation, and the proof of the first part is complete.

From Lemma 3.4 we obtain that (a) implies (c). By Heyer and Pap [**11**, Theorem 8.1] we obtain that for all $H\in\mathfrak{H}$ and $f^H\in\mathscr{D}(G/H)$ the function

$$F_{f^H}^H:(s,t)\longmapsto\sqrt{|(T_{\nu^H(s,t)}-I)f^H(e)-(A^H(t)-A^H(s))\,(f^H)|}$$

from $\mathbb{S}$ into $\mathbb{R}$ is of continuous bounded variation. It is easy to check that for all $t\in\mathbb{R}_+$ and $f^H\in\mathscr{D}(G/H)$ we have $A(t)\,(f^H\circ p_H) = A^H(t)\,(f^H)$. Thus we conclude that for every $f\in\mathscr{D}(G)$ the function

$$F_f:(s,t)\longmapsto\sqrt{|(T_{\nu(s,t)}-I)f(e)-(A(t)-A(s))\,(f)|}$$

from $\mathbb{S}$ into $\mathbb{R}$ is of continuous bounded variation; hence by Lemma 3.4 the statements (b) and (c) are equivalent.

Now (v)–(vii) implies that for all $H\in\mathfrak{H}$ the function $t\longmapsto\int_{G/H}f^H(y)\,\eta^H(dy\times[0,t])$ from $\mathbb{R}_+$ into $\mathbb{R}$ is Lipschitz continuous for all $f^H\in\mathscr{C}_e(G/H)_+$, the function $t\longmapsto\sum_{i\in I_H}b_{i,i}(t)$ from $\mathbb{R}_+$ into $\mathbb{R}$ is Lipschitz continuous, and there exists a subsequence (n') of (n) such that for all $i\in I_H$ the function

$$(s,t)\longmapsto\limsup_{n'}\sum_{\ell=k_{n'}(s)+1}^{k_{n'}(t)}\left|\int_{G/H}x_i^H\,d\mu_{n'\ell}^H\right|$$

from $\mathbb{S}$ into $\mathbb{R}$ is Lipschitz continuous.

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Thus as in the proof of Heyer and Pap [**11**, Theorem 8.1] we conclude that for all $H\in\mathfrak{H}$ the property $\eta^H=0$ implies that for every $f^H\in\mathscr{D}(G/H)$ the function $F_{f^H}^H$ is Lipschitz continuous, which implies that for every $f\in\mathscr{D}(G)$ the function F_f is Lipschitz continuous. Consequently again by Lemma 3.4 the statements (a), (b) and (c) are equivalent.

5. *Existence and uniqueness of hemigroups*

THEOREM 5.1. *Let G be a Lie projective group and let $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$. Then there exists exactly one hemigroup $(v(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ which corresponds to the triplet (a,B,η). Moreover, the hemigroup $(v(s,t))_{(s,t)\in\mathbb{S}}$ is of continuous weak bounded variation.*

Proof. First we prove the uniqueness. Suppose that there are two hemigroups $(v(s,t))_{(s,t)\in\mathbb{S}}$ and $(v'(s,t))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G)$ which correspond to the triplet (a,B,η). Then by Proposition 3.3 for all $H\in\mathfrak{H}$ the hemigroups $(p_H(v(s,t)))_{(s,t)\in\mathbb{S}}$ and $(p_H(v'(s,t)))_{(s,t)\in\mathbb{S}}$ in $\mathfrak{M}^1(G/H)$ correspond to the same triplet $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$. By Heyer and Pap [**11**, Theorem 7.2] we obtain $p_H(v(s,t))=p_H(v'(s,t))$ for all $(s,t)\in\mathbb{S}$ and $H\in\mathfrak{H}$. By Heyer [**9**, Theorem 1.2.17] we conclude that $v(s,t)=v'(s,t)$ for all $(s,t)\in\mathbb{S}$.

Now we prove the existence. For each $f\in\tilde{\mathscr{C}}_2(G)$, $t\in\mathbb{R}_+$ and $y\in G$ consider

$$N(t)f(y):=\sum_{i\in I}a_i(t)\,\tilde{X}_i f(y)+\frac{1}{2}\sum_{i,j\in I}b_{i,j}(t)\,\tilde{X}_i\tilde{X}_j f(y)+\int_G\Big(f(yz)-f(y)-\sum_{i\in I}\tilde{X}_i f(y)\,x_i(x)\Big)\eta(dy\times[0,t]).$$

Applying Born [**5**, Theorem 5] we obtain that for all $(s,t)\in\mathbb{S}$ the mapping $f\longmapsto(N(t)-N(s))(f)$ from $\tilde{\mathscr{C}}_2(G)$ into $\mathscr{C}^0(G)$ coincides on $\tilde{\mathscr{C}}_2(G)$ with the infinitesimal generator $N(s,t)$ of a unique convolution semigroup $(\tilde{\nu}_r(s,t))_{r\geqslant 0}$ in $\mathfrak{M}^1(G)$.

For all $H\in\mathfrak{H}$, $f^H\in\tilde{\mathscr{C}}_2(G/H)$, $y\in G$ and $t\in\mathbb{R}_+$ define $N_H(t)(f^H)(p_H(y)):=N(t)(f\circ p_H)(y)$. Let $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$ be defined as in Section 4. It is easy to check that

$$N_H(t)(f^H)(y)=\sum_{i\in I_H}a_i^H(t)\,\widetilde{dp_H(X_i)}f^H(y)+\frac{1}{2}\sum_{i,j\in I_H}b_{i,j}^H(t)\,\widetilde{dp_H(X_i)}\,\widetilde{dp_H(X_j)}f^H(y)+\int_{G/H}\Big(f^H(yz)-f^H(y)-\sum_{i\in I_H}\widetilde{dp_H(X_i)}f^H(y)\,x_i^H(x)\Big)\eta^H(dy\times[0,t])$$

for all $f^H\in\tilde{\mathscr{C}}_2(G)$, $t\in\mathbb{R}_+$ and $y\in G/H$.

For all $(s,t)\in\mathbb{S}$ and $H\in\mathfrak{H}$ define $N_H(s,t):=N_H(t)-N_H(s)$. Then $N_H(s,t)$ is indeed the restriction to $\tilde{\mathscr{C}}_2(G/H)$ of the infinitesimal generator of the convolution semigroup $(p_H(\tilde{\nu}_r(s,t)))_{r\geqslant 0}$ in $\mathfrak{M}^1(G/H)$. As in the proof of Heyer and Pap [**11**, Theorem 8.6] we obtain that

$$\mathop{\ast}_{\ell=[ns]+1}^{[nt]}p_H\Big(\tilde{\nu}_r\Big(\frac{\ell-1}{n},\frac{\ell}{n}\Big)\Big)\xrightarrow{\mathscr{T}_w}\nu^H(s,t)\quad\text{for all }(s,t)\in\mathbb{S}$$

where $(\nu^H(s,t))_{(s,t)\in\mathbb{S}}$ is a hemigroup of continuous weak bounded variation in $\mathfrak{M}^1(G/H)$ which corresponds to the triplet (a^H,B^H,η^H).

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For all $(s,t)\in\mathbb{S}$ the family $(v^H(s,t))_{H\in\mathfrak{H}}$ is a projective system, since for all $M,H\in\mathfrak{H}$ with $M\subseteq H$ we have

$$p_H^M(v^M(s,t)) = p_H^M\left(\lim_{n\to\infty}\bigstar_{\ell=[ns]+1}^{[nt]} p_M\left(\tilde{v}_r\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)\right)\right)$$
$$= \lim_{n\to\infty}\bigstar_{\ell=[ns]+1}^{[nt]} p_H\left(\tilde{v}_r\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)\right) = v_H(s,t).$$

By Heyer [**9**, Theorem 1.2.17] there exists a measure $v(s,t)\in\mathfrak{M}^1(G)$ such that $p_H(v(s,t)) = v^H(s,t)$. As in the proof of Theorem 4.1, one shows that $(v(s,t))_{(s,t)\in\mathbb{S}}$ is a hemigroup in $\mathfrak{M}^1(G)$ which corresponds to the triplet (a,B,η).

THEOREM 5.2. *Let G be a Lie projective group and let $(v(s,t))_{(s,t)\in\mathbb{S}}$ be a hemigroup of continuous weak bounded variation in $\mathfrak{M}^1(G)$. Then there exists exactly one triplet $(a,B,\eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ which corresponds to the hemigroup $(v(s,t))_{(s,t)\in\mathbb{S}}$.*

The triplet (a,B,η) is determined by the hemigroup $(v(s,t))_{(s,t)\in\mathbb{S}}$ in the following way:

$$\int_G f(y)\,\eta(dy\times[0,t]) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int_G f(y)\,v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)(dy),\quad f\in\mathscr{C}_e(G),$$
$$b_{i,j}(t) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int_G x_i(y)\,x_j(y)\,v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)(dy) - \sum_G x_i(y)\,x_j(y)\,\eta(dy\times[0,t]),$$
$$a_i(t) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int_G x_i(y)\,v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)(dy).$$

Moreover, we have for all $f\in\mathscr{C}_2(G)$ and $t\in\mathbb{R}_+$

$$A(t)(f) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int (f(y)-f(e))\,v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)(dy)$$

where $A:\mathbb{R}_+\longrightarrow\mathbb{P}(G)$ is the mapping associated with (a,B,η).

Proof. First we prove the uniqueness. Suppose that there are two triplets (a,B,η) and $(\tilde{a},\tilde{B},\tilde{\eta})$ in $\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G)$ corresponding to the hemigroup $(v(s,t))_{(s,t)\in\mathbb{S}}$. For $H\in\mathfrak{H}$ let (a^H,B^H,η^H) and $(\tilde{a}^H,\tilde{B}^H,\tilde{\eta}^H)$ in $\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$ be defined as in Section 3. By Proposition 3.3 we obtain that the triplets (a^H,B^H,η^H) and $(\tilde{a}^H,\tilde{B}^H,\tilde{\eta}^H)$ correspond to the same hemigroup $(p_H(v(s,t)))_{(s,t)\in\mathbb{S}}$ of continuous weak bounded variation in $\mathfrak{M}^1(G/H)$. Hence by Heyer and Pap [**11**, Theorem 8.7] we conclude that $(a^H,B^H,\eta^H) = (\tilde{a}^H,\tilde{B}^H,\tilde{\eta}^H)$ for all $H\in\mathfrak{H}$. Then obviously $B=\tilde{B}$. For all $t\in\mathbb{R}_+$ the families $(\eta_t^H)_{H\in\mathfrak{H}}$ and $(\tilde{\eta}_t^H)_{H\in\mathfrak{H}}$ are projective systems; hence by Heyer [**9**, Theorem 1.2.17] we obtain that $\eta_t=\tilde{\eta}_t$, and hence $\eta=\tilde{\eta}$, which implies also $a=\tilde{a}$.

Now we prove the existence. By the assumptions for all $f\in\mathscr{C}_2(G)$ the mapping $(s,t)\longmapsto(T_{v(s,t)}-I)f(e)$ is of continuous bounded variation. Hence for all $H\in\mathfrak{H}$ and $f^H\in\mathscr{C}_2(G/H)$ the mapping

$$(s,t)\longmapsto(T_{v(s,t)}-I)(f\circ p_H)(e) = (T_{p_H(v(s,t))}-I)f^H(e)$$

is also of continuous bounded variation. Thus $(p_H(v(s,t)))_{(s,t)\in\mathbb{S}}$ is a hemigroup in $\mathfrak{M}^1(G/H)$ of continuous weak bounded variation. Applying Heyer and Pap [**11**, Theorem 8.7] we conclude that there is a triplet $(a^H,B^H,\eta^H)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+,G/H)$ which

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corresponds to the hemigroup $(p_H(v(s,t)))_{(s,t)\in\mathbb{S}}$, and the triplet (a^H, B^H, η^H) is determined by the hemigroup $(p_H(v(s,t)))_{(s,t)\in\mathbb{S}}$ in the following way:

$$\int_{G/H} f^H(y)\,\eta^H(dy\times[0,t]) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int_{G/H} f^H(y)\,p_H\left(v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)\right)(dy),\quad f^H\in\mathscr{C}_e(G/H),$$

$$b^H_{i,j}(t) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int_{G/H} x^H_i(y)\,x^H_j(y)\,p_H\left(v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)\right)(dy) - \int_{G/H} x^H_i(y)\,x^H_j(y)\,\eta^H(dy\times[0,t]),$$

$$a^H_i(t) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int_{G/H} x^H_i(y)\,p_H\left(v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)\right)(dy).$$

Let (a, B, η) be defined as in the theorem. Then it is easy to show that the relationship between the family $(a^H, B^H, \eta^H)_{H\in\mathfrak{H}}$ and (a, B, η) is the same as in Section 4, and we have $(a, B, \eta)\in\mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+, G)$. By Proposition 3.3 we conclude that the triplet (a, B, η) corresponds to the hemigroup $(v(s,t))_{(s,t)\in\mathbb{S}}$.

Moreover, as in Heyer and Pap [**11**, Theorem 8.7], for all $H\in\mathfrak{H}$, $f^H\in\mathscr{C}_2(G/H)$ and $t\in\mathbb{R}_+$

$$A_H(t)(f^H) = \lim_{n\to\infty}\sum_{\ell=1}^{[nt]}\int (f^H(y)-f^H(e))\,p_H\left(v\left(\frac{\ell-1}{n},\frac{\ell}{n}\right)\right)(dy)$$

where $A_H:\mathbb{R}_+\longrightarrow\mathbb{P}(G)$ is the mapping associated with (a^H, B^H, η^H). It is easy to check that $A(t)(f^H\circ p_H) = A_H(t)(f^H)$; hence we obtain the expression for $A(t)$.

6. *Illustrations*

6.1. *Lie projective groups*

Examples of Lie projective groups for which our results are applicable are Moore groups which by definition are locally compact groups whose irreducible unitary representations are finite-dimensional. In particular, compact groups are Lie projective. For the structure of Moore groups see Heyer [**9**, Theorem N, p. 15].

For two classes of compact (non-Lie) groups G we shall describe Lie systems, projective bases and functions in $\mathscr{C}_k(G)$ (as in Born [**3**, **4**]).

6.2. *The infinite-dimensional torus*

Let $\mathbb{T} := [z\in\mathbb{C}:|z| = 1\}$ denote the (1-dimensional) torus group. For any infinite set I the group $\mathbb{T}^I$ is a connected and locally connected Abelian compact group which is metrisable if and only if I is countable. Since I is infinite, G admits small subgroups and hence is not a Lie group. By $\mathscr{F}(I)$ we abbreviate the system of all finite subsets of I. Then $\mathfrak{H} := \{\{1\}^J\times\mathbb{T}^{I\setminus J}: J\in\mathscr{F}(I)\}$ is a Lie system for G. For $J, K\in\mathscr{F}(I)$ the differential dp^J_K is the canonical mapping from $\mathbb{R}^J$ onto $\mathbb{R}^K$. Hence the Lie algebra $\mathscr{L}(G)$ of G is isomorphic (as Lie algebra) to $\mathbb{R}^I$, and $(X_k)_{k\in I}$ with $X_k := (\delta_{kj})_{j\in I}$ for every $k\in I$ forms a projective basis for $\mathscr{L}(G)$. (See Born [**4**].)

Now, the functions $f\in\mathscr{C}_k(G)$ enjoy the property that the set

$$\{\alpha\in I^j: \|X_\alpha f\|_0\neq 0 \text{ for all } j\leqslant k\}$$

is at most countable; hence they depend differentiably on at most countably many coordinates of $G := \mathbb{T}^I$. Since G is not a Lie group, the spaces $\mathscr{C}_k(G)$ contain but do not coincide with the spaces $\mathscr{C}^k(G)$ of functions on G that depend differentiably only on finitely many coordinates of the representation of G as a projective limit. These spaces of cylindrically differentiable functions are considered in the work of Bendikov [**1**] on Brownian convolution semigroups on G.

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6.3. *Solenoidal group*

The p-adic solenoidal group is the character group $\Sigma := \mathbb{Q}_d^\wedge = \mathrm{Hom}(\mathbb{Q}_d, \mathbb{T})$ of the additive group $\mathbb{Q}_d$ of rationals furnished with the discrete topology. Σ is a compact Abelian group which is metrisable, connected and contains a dense one-parameter subgroup. Σ is not a Lie group but is Lie projective. From Pontryagin's duality theorem one deduces that $\mathbb{Q}_d = \Sigma^\wedge = \{\chi_q^\wedge : q \in \mathbb{Q}\}$ where $\chi_q^\wedge(\chi) := \chi(q)$ for all $\chi \in \Sigma$ and $q \in \mathbb{Q}$. Given $k \in \mathbb{N}$ we consider the annihilator $N_k := A(\Sigma, \{\chi_{1/k!}^\wedge\})$ (in Σ). Then $\{N_k : k \in \mathbb{N}\}$ is a Lie system for Σ. In fact $N_k = A(\Sigma, \{\chi_{z/k!}^\wedge : z \in \mathbb{Z}\})$; hence $N_k \supseteq N_{k+1}$ for all $k \in \mathbb{N}$. Clearly, N_k is compact for every $k \in \mathbb{N}$, and the Gelfand–Raikov theorem implies that $\bigcap\{N_k : k \in \mathbb{N}\} = \{e\}$. Since $(\Sigma/N_k)^\wedge \cong A(\Sigma^\wedge, N_k) \cong \mathbb{Z}/k! \cong \mathbb{Z}$ and hence $\Sigma/N_k \cong \mathbb{Z}^\wedge \cong \mathbb{T}$, $\{N_k : k \in \mathbb{N}\}$ is a Lie projective system for Σ.

We now note that the characters $\chi_{1/k!}^\wedge$ correspond to the canonical mappings $p_k : \Sigma \longrightarrow \Sigma/N_k$, and the canonical mappings $p_m^n : \Sigma/N_n \longrightarrow \Sigma/N_m$ for $n, m \in \mathbb{N}$ with $m \leqslant n$ are given by $p_m^n(s) := s^{n!/m!}$ for all $s \in \Sigma/N_n \cong \mathbb{T}$. Consequently Σ is topologically isomorphic to $\varprojlim_{n \in \mathbb{N}} \mathbb{T}$ (with respect to the projective system $(\mathbb{N}, \mathbb{T}, p_m^n)$). On the other hand $\mathscr{L}(\Sigma) = \varprojlim_{n \in \mathbb{N}} \mathbb{R}$ (with respect to the projective system $(\mathbb{N}, \mathbb{R}, dp_m^n)$). Since for $n \geqslant m$ the differentials dp_m^n are isomorphisms of $\mathbb{R}$, we obtain $\mathscr{L}(\Sigma) = \mathbb{R}$. And since the canonical mappings $dp_n : \mathscr{L}(\Sigma) \longrightarrow \mathscr{L}(\Sigma/N_n)$ are also isomorphisms, every element of $\mathbb{R}\backslash\{0\}$ forms a projective basis of $\mathscr{L}(\Sigma)$.

Every function in $\mathscr{C}_k(\Sigma)$ is N_k-invariant for some N_k in the Lie system for Σ. And, as in Subsection 6.2, $\mathscr{C}_k(\Sigma) \supsetneqq \mathscr{C}^k(\Sigma)$. This implies in particular that Σ is not locally connected.

6.4. *A convolution hemigroup arising in atomic physics*

(See Hantsch and von Waldenfels [**8**], Born [**3**] and Heyer [**10**].) Let G be a second countable Lie projective group, $U = (U_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed G-random variables (on a probability space $(\Omega, \mathfrak{A}, P)$), $X = (X(t) : t \in \mathbb{R}_+)$ be a stochastically continuous process with independent left-increments on $(\Omega, \mathfrak{A}, P)$ with values in G, and $\Pi = (\Pi(t) : t \in \mathbb{R}_+)$ be a Poisson process on $(\Omega, \mathfrak{A}, P)$ with values in $\mathbb{Z}_+$ having a continuous intensity v, that is, Π is a stochastically continuous process with independent increments, and there exists an increasing, continuous function v on $\mathbb{R}_+$ such that

$$P[\Pi(t) - \Pi(s) = k] = \mathrm{e}^{-(v(t)-v(s))} \frac{(v(t)-v(s))^k}{k!}$$

for all $k \in \mathbb{Z}_+$ and all $(s, t) \in \mathbb{S}$. We assume that U, X and Π are independent. Moreover, let $(T_n : n \in \mathbb{Z}_+)$ be a sequence of real-valued random variables given by

$$T_0 := 0$$
$$T_n := \inf\{t \in \mathbb{R}_+ : \Pi(t) = n\}$$

whenever $n \in \mathbb{N}$. Now we consider the process Y defined by

$$Y(t) := \left(\prod_{j=1}^{\Pi(t)} X(T_{j-1})^{-1} X(T_j)\, U_j \right) X(T_{\Pi(t)})^{-1} X(t)$$
$$= \left(\prod_{j=1}^{\Pi(t)} X(T_j)\, U_j\, X(T_j)^{-1} \right) X(t)$$

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for all $t \in \mathbb{R}_+$, and the process W given by

$$W(t) := \prod_{j=1}^{\Pi(t)} U_j$$

for all $t \in \mathbb{R}_+$ (where for the latter case $\prod_{j \in \varnothing} U_j := e$ [P]).

It turns out that the following hold:

(1) W is a stochastically continuous process with independent left-increments taking values in G whose convolution hemigroup $(\mu(s,t):(s,t) \in \mathbb{S})$ is of continuous weak bounded variation and admits a generating mapping $A^W: \mathbb{R}_+ \longrightarrow \mathbb{P}(G)$ of the form

$$A^W(t)f = v(t) \int (f - f(\mathrm{e}))\, d(U_1(P))$$

for all $f \in \mathscr{D}(G)$, $t \in \mathbb{R}_+$. In fact, the corresponding triplet $(a, B, \eta) \in \mathbb{P}_{\mathrm{bv}}(\mathbb{R}_+, G)$ has the form $a_i(t) = v(t) \int x_i\, d(U_1(P))$ for $i \in I$, $B(t) = 0$, and $\eta(dy \times [0,t]) = v(t)\, U_1(P)$.

(2) Y is a stochastically continuous process with independent left-increments.

(3) If the process X is of continuous weak bounded variation and admits a generating mapping A^X then the process Y admits a generating mapping $A^X + A^W$, where A^W has the representation given in (1).

(4) More generally, let A be the generating mapping of a convolution hemigroup of continuous weak bounded variation, let $v \in \mathbb{L}(G)$, and let $v: \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an increasing, continuous function. Then A_1 given by

$$A_1(t)f := A(t)f + v(t) \int (f - f(e))\, dv$$

for all $f \in \mathscr{D}(G)$, $t \in \mathbb{R}_+$ defines the generating mapping of a unique convolution hemigroup of continuous weak bounded variation.

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